Viscous cosmology in the Kasner metric

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A Bianchi type-I metric of the Kasner form is used as input in Einstein's equations to explore which consequences thereby occur for the equation of state for the cosmic fluid. Both shear viscosity and bulk viscosity coefficients are assumed to be present. The solutions of Einstein's equations can naturally be categorized into two classes. *Either* the space becomes anisotropic, and the equation of state is determined once the Kasner parameters p_i are given. *Or* the space becomes isotropic, and the equation of state emerges in the conventional form $p/\rho=\gamma-1$ with a definite value of γ. We also calculate the rate of entropy production $\dot{\sigma}$ per particle, and find that $\dot{\sigma}$ becomes as large as of order $10^4\ \mathrm{s}^{-1}$ if we go back to the very early universe, $t\sim10^{-4}$ s. We also discuss the possibility of testing the anisotropy of the universe by means of redshift experiments. [S0556-2821(97)06218-8]

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I. INTRODUCTION

The construction of thermodynamic theories of the early universe is a topic that has attracted considerable interest in recent years. This is quite understandable, in view of the great predictive power of the thermodynamic formalism in general, and also because the whole inflationary idea as such is very suitable for a thermodynamic treatment.

The recent development is many-faceted, and may be divided into categories in various ways. The most simple approach is to let the cosmic fluid be endowed with a bulk viscosity, i.e., a viscosity coefficient which is able to describe the expansion of the thermodynamic theory to first order in the deviation from thermal equilibrium, while still being compatible with the assumption of an isotropic universe. If in addition a shear viscosity is allowed for, one becomes able to describe also an anisotropic universe. A useful review of viscous cosmology up to 1990 is given by Grøn [1]. More recent papers along the same lines are Refs. [2–13]. In these papers, the universe is for the most part taken to be isotropic. Several authors [2,7,8,10,13] deal with the idea that the energy of the vacuum decreases with cosmic expansion. Others [3,5,6] consider causal higher order thermodynamic theories, such as the Israel-Stewart theory.

A second kind of approach to modern cosmology is to explore the consequences of the idea that there occurred a creation of matter in the early universe. See, for example, Refs. [14–19]. A central topic that one desires to explain is the presence of the very large specific entropy ($\sigma\sim10^9$ in nondimensional terms, per baryon) in the universe.

A third kind of approach—the one to be studied in the present paper—is to allow for an *anisotropy* in the early universe, and explore where this assumption leads. As for the *present* universe, it is known that the anisotropy is very small. Thus Bunn *et al.* [20] made recently a statistical analysis of 4-yr data from the COBE satellite, and concluded that the present amount of shear, $(\sigma/H)_0$, is less than 3×10^{-9}. In the early universe, however, the anisotropy may have been larger. The possibility of a state of anisotropy at very early times is in our opinion a very natural idea to explore, as an attempt to explain, among other things, the large local anisotropies that we observe in the universe today in galaxies and supergalaxies [21]. Some other recent papers dealing with anisotropic cosmologies are Refs. [22–25].

We shall in the following be concerned with the simplest of all metric classes, viz. the Bianchi type-I class. The line element can then be expressed as

$$ds^2=-dt^2+R_1^2(t)dx^2+R_2^2(t)dy^2+R_3^2(t)dz^2. \tag{1}$$

This model is an anisotropic generalization of the Friedmann model with Euclidean spatial geometry. There are three expansion factors, R_1, R_2, and R_3, which are determined via Einstein's equations. Actually we shall restrict ourselves even further, and consider only the subclass in which the expansion factors take the Kasner form [26]:

$$ds^2=-dt^2+t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2. \tag{2}$$

Here p_1, p_2, and p_3 are three numbers that will be required to be constants. The Kasner universe, in the classical sense, refers to a *vacuum* space only; in such a case, the numbers p_i satisfy the equations

$$p_1+p_2+p_3=1, \quad p_1^2+p_2^2+p_3^2=1. \tag{3}$$

In the present case, where we will be concerned with a viscous fluid instead of empty space, Eqs. (3) will no longer be valid in general.

In the next section we establish the basic formalism, and give the expression for the energy-momentum tensor of the cosmic fluid in Eq. (9). In Sec. III we point out the two distinct categories naturally following from the application of Einstein's equations: one possibility being that the space is anisotropic, and that the equation of state follows from Eqs. (29) and (30); the other possibility being that the space is isotropic, and that the equation of state is as in Eq. (33). Section IV deals with the generation of entropy in the anisotropic universe, and Sec. V focuses on the calculation of the rate of specific entropy production $\dot{\sigma}$, employing, in particu-

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lar, the expressions for the viscosity coefficients as following from relativistic kinetic theory in the early part of the plasma era ($t = 1000$ s). It turns out that the value of $\dot{\sigma}$ at this instant is small. Going back to earlier times, however, for instance to $t = 2\times 10^{-4}$ s characterizing the start of the lepton era, we find that $\dot{\sigma}$ is much greater, of the order of 10^4 s^{-1}. In the final Sec. VI we give a brief discussion on the possibility of testing the anisotropy by means of redshift experiments.

II. BASIC EQUATIONS

Similarly as in earlier works [4,19] we use conventions for which $c=1$, the Minkowski metric $\eta_{\mu\nu}$ is $(-+++)$, and the sign of the curvature tensor is as in Misner *et al.* [27]. The cosmological constant Λ will be set equal to zero. The Einstein equations are conveniently written in the form

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T^{\alpha}_{\alpha}\right). \tag{4}$$

We first give the components of the Ricci tensor $R_{\mu\nu}$, assuming the metric (2): the nonvanishing components of the Christoffel symbols are

$$\Gamma^0_{ii} = p_i t^{2p_i-1}, \quad \Gamma^i_{i0} = \Gamma^i_{0i} = p_i/t \tag{5}$$

(no sum over i), and so we calculate

$$R_{00} = [p_1+p_2+p_3-(p_1^2+p_2^2+p_3^2)]t^{-2}, \tag{6}$$

$$R_{ii} = p_i(p_1+p_2+p_3-1)t^{2p_i-2}. \tag{7}$$

Consider next the energy-momentum tensor $T_{\mu\nu}$ in Eq. (4). Let $U^\mu = (U^0, U^i)$ be the four-velocity of the cosmic fluid, and let ζ and η be respectively the bulk viscosity and the shear viscosity. Since the Kasner space, in spite of being anisotropic, is a homogeneous space, ζ and η will not be dependent on position. However, because of the time-dependent geometry of the Kasner space, they can in principle depend on time. With $h_{\mu\nu}$ denoting the projection tensor,

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu, \tag{8}$$

we can then write

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p-\zeta\theta)h_{\mu\nu} - 2\eta\sigma_{\mu\nu}, \tag{9}$$

where ρ is the mass density and p the isotropic pressure, both taken in the local rest inertial frame. We have here introduced also the scalar expansion θ as the trace $\theta^\mu_\mu = U^\mu_{;\mu}$ of the expansion tensor

$$\theta_{\mu\nu} = \frac{1}{2}\,(U_{\mu;\alpha}h^\alpha_\nu + U_{\nu;\alpha}h^\alpha_\mu), \tag{10}$$

as well as the traceless shear tensor

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3}\, h_{\mu\nu}\theta, \quad \sigma^2 = \frac{1}{2}\,\sigma_{\mu\nu}\sigma^{\mu\nu}. \tag{11}$$

In the present case,

$$\theta_{00} = 0, \quad \theta_{ii} = p_i t^{2p_i-1},$$

$$\theta = (p_1+p_2+p_3)t^{-1}, \quad \text{other } \theta_{\mu\nu} = 0, \tag{12}$$

$$\sigma_{00} = 0, \quad \sigma_{ii} = \left[p_i - \frac{1}{3}\,(p_1+p_2+p_3)\right]t^{2p_i-1},$$

$$\sigma^2 = -\frac{1}{2t^2}\left[\frac{1}{3}\,(p_1+p_2+p_3)^2 - (p_1^2+p_2^2+p_3^2)\right]; \tag{13}$$

other $\sigma_{\mu\nu} = 0$ (no sum over i).

It is to be noted that there are no terms describing heat flux in the expression (9). Again, the homogeneity of the Kasner space is crucial here: any heat flux would have to be proportional to the spatial gradient of the temperature T, and must be zero in our case since T is the same everywhere, for a given time (cf. also the discussion in [4]).

Einstein's field equations (4) can now be found, for the Kasner metric (2). In the next section we intend to explore which implications result from them on the possible forms of the equation of state of the fluid.

III. EQUATION OF STATE FOR THE FLUID

To simplify the formalism somewhat, we introduce the symbol S for the sum of the p_i and Q for the sum of the p_i^2:

$$S = p_1+p_2+p_3,$$

$$Q = p_1^2+p_2^2+p_3^2. \tag{14}$$

Moreover, we introduce the symbol κ by

$$\kappa = 8\pi G. \tag{15}$$

The trace of the energy-momentum tensor becomes

$$T^\alpha_\alpha = -\rho + 3p - 3\zeta S/t. \tag{16}$$

Einstein's equations (4) can now be written in comoving coordinates as

$$S - Q + \frac{3}{2}\,\kappa t\zeta S = \frac{1}{2}\,\kappa t^2(\rho+3p), \tag{17}$$

$$p_i(1-S-2\kappa t\eta) + \frac{1}{2}\,\kappa t\left(\zeta + \frac{4}{3}\,\eta\right)S = \frac{1}{2}\,\kappa t^2(p-\rho). \tag{18}$$

Here, Eq. (17) corresponds to $\mu=\nu=0$ in Eq. (4), whereas Eq. (18) corresponds to $\mu=\nu=i(=1,2,3)$.

We can now distinguish between various cases, which we will discuss in the order of increasing complexity.

A. Vacuum space

When $p=\rho=0$ we obtain from Eqs. (17) and (18) $S=Q$, $p_i(1-S)=0$. Thus

$$S = Q = 1, \tag{19}$$

which are the constraints leading to the classical Kasner solution. It is instructive to give a geometrical interpretation of the constraints [28]: the three different p_i values are given in an xy plane as the x coordinates of the corners of an equilateral triangle inscribed in a circle of radius 2/3 centered at the point (1/3, 0).

B. Perfect fluid

When $\zeta=\eta=0$ in Eqs. (17) and (18), there are two options.

First option. $S=1$. The equation of state then becomes

$$p=\rho. \tag{20}$$

This is the characteristic equation for the relativistic Zel'dovich fluid; the velocity of sound is equal to the velocity of light. Moreover,

$$Q=1-2\kappa\rho t^2, \tag{21}$$

$$\rho\propto t^{-2}. \tag{22}$$

The present theory does not give any information about how to calculate the proportionality constant in Eq. (22). But if this constant happens to be known, we can calculate the values of the three parameters p_i on basis of the known values of S and Q. The p_i are in general different, so that the space is anisotropic. The above results agree with those obtained by Benton and Tupper [29] and Halpern [23].

Second option. $S\neq 1$. From Eqs. (17) and (18) it is clear that the space must in this case be *isotropic*:

$$p_1=p_2=p_3\equiv a=\text{const}. \tag{23}$$

Moreover,

$$\rho\propto t^{-2}, \quad p\propto t^{-2} \tag{24}$$

in order to make p_i constant, and we get

$$\rho=\frac{3a^2}{\kappa t^2}, \quad p=\frac{2a-3a^2}{\kappa t^2}. \tag{25}$$

The equation of state can thus be written as

$$\frac{p}{\rho}=\frac{2}{3a}-1, \tag{26}$$

which agrees with the conventional form $p/\rho=\gamma-1$ often assumed in cosmology. Whereas the first option above corresponded to $\gamma=2$, the second option corresponds to $\gamma=2/(3a)$.

C. Viscous fluid

When ζ and η are different from zero, we see first of all that

$$\zeta\propto t^{-1}, \quad \eta\propto t^{-1}, \tag{27}$$

$$\rho\propto t^{-2}, \quad p\propto t^{-2}. \tag{28}$$

Similarly as above, we cannot from the present theory fix the values of the constants of proportionality. We again distinguish between two options.

First option. S satisfies the equation

$$S=1-2\kappa t\eta. \tag{29}$$

The space is in general anisotropic. The three values of p_i can be calculated from Eq. (29) and the equation for Q:

$$Q=1+\frac{1}{2}\kappa t[3\zeta-4\eta-6\kappa t\zeta\eta-t(\rho+3p)], \tag{30}$$

if the four constants of proportionality in Eqs. (27) and (28) are known. Conversely, if one starts from a given set of values for p_i, Eqs. (29) and (30) can be used to derive the equation of state for the fluid.

Second option. Equation (29) is not satisfied. The space then becomes isotropic; this being analogous to the second option in the previous subsection. Let $p_1=p_2=p_3\equiv a$ [the constant a of course being in general different from that appearing in Eq. (23)]; then we get

$$\rho=\frac{3a^2}{\kappa t^2}, \tag{31}$$

$$p=\frac{2a-3a^2}{\kappa t^2}+\frac{3a\zeta}{t}. \tag{32}$$

From a comparison with Eq. (25) we see that the bulk viscosity, in contradistinction to the shear viscosity, contributes to the pressure in the fluid. From Eqs. (31) and (32) it follows that the equation of state can be written as

$$\frac{p}{\rho}=\frac{2+3\kappa t\zeta}{3a}-1. \tag{33}$$

That means that it is still possible to write the state equation in the form $p/\rho=\gamma-1$, but now with the value $(2+3\kappa t\zeta)/(3a)$ for the constant γ. When $\zeta=0$, Eq. (26) is recovered.

We close this section with some remarks emphasizing the conceptual meaning of the present theory. Recall first the line of reasoning in standard isotropic cosmology: in that case, one starts from the general metric for an isotropic and homogeneous universe, viz. the Friedmann-Robertson-Walker (FRW) metric, and derives the constraints imposed by Einstein's field equations on this metric. The equation of state for the fluid is thereafter imposed as an *external*, thermodynamic equation; there are no initial limiting conditions arising from the metric on this equation. On the basis of this set of equations, one is able to predict the behavior of the universe completely. In the present Kasner theory the line of reasoning is different: the equation of state emerges in each subcase as a *consistency condition*, after the imposition of Einstein's equations (17) and (18). One may ask what is the physical reason why the equation of state is determined from the formalism alone, i.e., why have we lost the freedom that we had previously in Friedmann cosmology to choose freely the equation of state? The answer lies in our adoption of the metric in the Kasner form (2), where the numbers p_i are taken to be constants. This initial assumption about the metric is physically more restrictive than is the standard FRW assumption in isotropic cosmology. Thus it is most appropriate in the present case to say that the equation of state, although following formally from Einstein's equations, is physically a direct consequence of the chosen metric.

Another notable property of the obtained equations of state for a viscous fluid is that they contain the viscosity coefficients; cf., for instance, the appearance of ζ in Eq. (33). This seems to be at variance with usual first order nonequilibrium thermodynamics; in such a case the equilibrium equation of state is assumed to remain valid although the fluid as a whole is out of equilibrium. Does this kind of behavior imply that there is a physical drawback of the present theory? At present, we are hardly in a position to express a firm opinion on this point. What we feel safe to conclude is that the presence of viscosity coefficients in the state equation makes the theory in some sense exceptional. Of course, this behavior is again rooted in the particular choice (2) for the metric. The correctness of the theory must ultimately be tested by comparing its predictions with experiment.

IV. GENERATION OF ENTROPY

A central issue in the study of viscous cosmology—beginning with the classical papers of Misner [30] and Weinberg [31]—has ever since then been to find a natural application of the viscosity concept to explain the large entropy per baryon observed in the universe. Usually, one restricts oneself to including only the bulk viscosity, in view of the large degree of isotropy in the present universe. Here, we shall be interested, in particular, with the anisotropy and the corresponding shear viscosity in the early universe.

Let us begin by defining an analytic nondimensional expression for the anisotropy of the Kasner metric. We follow the definition used by Grøn [32,1]: for the Bianchi type-I spaces the average expansion anisotropy parameter A is

$$A=\frac{1}{3}\sum_{i=1}^{3}\left(1-\frac{H_i}{H}\right)^2, \tag{34}$$

where H_i are the directional Hubble factors and H the average Hubble factor:

$$H_i=\frac{\dot{R}_i}{R_i}, \quad H=\frac{1}{3}\,(H_1+H_2+H_3). \tag{35}$$

In our case $R_i=t^{p_i}$, so that $H_i=p_i/t$ and $H=S/(3t)$. From Eq. (34) we then get

$$A=\frac{3Q}{S^2}-1. \tag{36}$$

It follows immediately that in the vacuum Kasner space, with $S=Q=1$, one has $A=2$.

Consider next the production of entropy. The entropy current four-vector, in the absence of heat flux, is

$$S^{\mu}=nk_B\sigma U^{\mu}, \tag{37}$$

where n is the baryon number density and σ the nondimensional entropy per baryon. Since

$$S^{\mu}_{;\mu}=\frac{2\eta}{T}\,\sigma_{\mu\nu}\sigma^{\mu\nu}+\frac{\zeta}{T}\,\theta^2 \tag{38}$$

we obtain, taking Eqs. (11) and (13) into account,

$$S^{\mu}_{;\mu}=\frac{1}{Tt^2}\left[\zeta S^2+2\eta\left(Q-\frac{1}{3}S^2\right)\right]. \tag{39}$$

Let us evaluate the left-hand side of this equation in the comoving frame of reference: since $\sqrt{-g}=t^S$ we first get

$$S^{\mu}_{;\mu}=t^{-S}(t^S nk_B\sigma)_{,0}. \tag{40}$$

Moreover, the conservation equation for baryon particle number, $(nU^{\mu})_{;\mu}=0$, implies that $(t^S n)_{,0}=0$ in the comoving frame. Altogether, the left-hand side of Eq. (40) simplifies to $k_B n\dot{\sigma}$. Finally including the anisotropy parameter from Eq. (36), we can write Eq. (39) as

$$\dot{\sigma}=\frac{3S^2}{nk_BTt^2}\left(\zeta+\frac{2}{3}\,\eta A\right). \tag{41}$$

Entropy production in the anisotropic universe ($A\neq 0$) is connected with the shear viscosity η, as we would expect.

V. NUMERICAL CONSIDERATIONS

Our considerations have so far been formal. Let us now make some estimates about the magnitudes of the viscosity coefficients, and the influence from them upon the entropy production.

The first question is how far in the history of the universe can we go back, assigning viscosity coefficients to the cosmic fluid in a meaningful way? No consensus on this issue seems so far to have been obtained. There are various opinions expressed in the literature, and one has even suggested that there occured phase transitions in the universe making it physically meaningful to introduce the viscosity concept as early as at inflationary times, $t\simeq 10^{-33}$ s. Here, let us at first be more modest and estimate the viscosities in the first part of the plasma era (also called the radiation era). For definiteness, we shall put $t=1000\ \mathrm{s}\equiv t_{\mathrm{in}}$. The physical advantage of considering this relatively late instant in the history of the universe is that the magnitudes of the physical quantities are relatively safe, and moreover that the physical conditions in the universe have become so nonextreme that conventional relativistic kinetic theory can be employed to calculate the viscosity coefficients.

According to standard cosmology [33,34], at $t=1000$ s the universe is characterized by ionized H and He in approximate equilibrium with radiation. The existence of energy dissipation is caused by the fact that the thermal equilibrium is not quite perfect. The number densities of electrons and protons are $n_e=n_p\simeq 10^{19}\ \mathrm{cm}^{-3}$, and the temperature is $T\simeq 4\times 10^8\ \mathrm{K}=3.5\times 10^{-2}$ MeV (the redshift is $z\simeq 10^8$). In view of the radiation dominance, the energy density is, in ordinary cgs units, $\rho c^2=a_r T^4$, where $a_r=\pi^2 k_B^4/(15\hbar^3 c^3)=7.56\times 10^{-15}\ \mathrm{erg\,cm^{-3}\,K^{-4}}$ is the radiation constant. The pressure is $p=\rho c^2/3$. As for the viscosity coefficients, it is here sufficient to include only the first terms in the polynomial approximations worked out by Caderni and Fabbri [35] (cf. also [4]):

$$\eta=\frac{5m_e^6c^8\zeta(3)}{9\pi^3\hbar^3e^4n_e}\,x^{-4},\tag{42}$$

$$\zeta=\frac{\pi c^2\hbar^3 n_e}{256e^4\zeta(3)}\,x^3.\tag{43}$$

Here $x=m_ec^2/k_BT$, $\zeta(3)=1.202$ being the Riemann zeta function of argument 3. When $T=4\times10^8$ K, one gets $x=14.8$. Then

$$\eta_{1000\ \mathrm{s}}\simeq2.8\times10^{14}\ \mathrm{g\ cm^{-1}\ s^{-1}},$$

$$\zeta_{1000\ \mathrm{s}}\simeq7.0\times10^{-3}\ \mathrm{g\ cm^{-1}\ s^{-1}}.\tag{44}$$

One remarkable fact is evident from these expressions: the shear viscosity is vastly greater than the bulk viscosity. Physically this is quite an important point; it means that the entropy production associated with even a slight anisotropy of the universe may easily outweigh the entropy generated by the bulk viscosity. A similar large difference between η and ζ may be verified to be present also for later times in the plasma era [4].

Let us consider successively the same options as in Sec. III, at the instant $t=1000$ s. For simplicity we replace everywhere the superscript ''1000 s'' by ''in''.

First option (anisotropic space). From Eq. (29) we obtain, in dimensional terms:

$$S_{\mathrm{in}}=1-2\kappa c^2(t\eta)_{\mathrm{in}}=1-1.0\times10^{-9},\tag{45}$$

where we have used $\kappa c^2=1.87\times10^{-27}\ \mathrm{cm\ g^{-1}}$. Moreover, from Eq. (30) we obtain, when omitting the negligibly small ζ terms,

$$\begin{aligned}Q_{\mathrm{in}}&\simeq1-\frac{1}{2}\,\kappa c^2[4t\eta+t^2(\rho c^2+3p)]_{\mathrm{in}}\\&=1-\kappa c^2(2t\eta+a_rt^2T^4)_{\mathrm{in}}\\&=1-3.6\times10^{-4}.\end{aligned}\tag{46}$$

Numerically it actually turns out that the contribution from η in this expression is negligible. The only nonvanishing influence from η thus occurs in the expression (45) for S_{in}. The anisotropy parameter A_{in} follows from Eq. (34):

$$A_{\mathrm{in}}=2-1.1\times10^{-3}.\tag{47}$$

The deviation from the anisotropy of the classical vacuum Kasner space is thus small. From Eq. (41) we obtain the rate of entropy production per particle per unit time:

$$\dot\sigma\simeq\frac{4\eta}{nk_BTt^2}.\tag{48}$$

It gives for $t=1000$ s

$$\dot\sigma_{\mathrm{in}}=2.0\times10^{-3}\ \mathrm{s^{-1}}.\tag{49}$$

The entropy production is thus quite small. If the expression (48) stayed approximately constant during the long plasma era (also called the radiation era), it would seem possible to obtain an appreciable amount of entropy (recombination of hydrogen occurs at a temperature of $T_{\mathrm{rec}}\simeq4000$ K at the time $t_{\mathrm{rec}}\simeq4\times10^5$ yr$=1.26\times10^{13}$ s). However, the expression (48) turns out to diminish rapidly with time. The following brief estimate is sufficient to show this: let us insert into Eq. (48) the numerical values corresponding to the instant of recombination. In addition to the values of T_{rec} and t_{rec} given above, we then need the baryon density, which is [33] $n_{\mathrm{rec}}\simeq4\times10^3\ \mathrm{cm^{-3}}$, and we need the value of η_{rec}. The latter quantity follows from the proportionality $\eta\propto t^{-1}$ [cf. Eq. (27)], which is a condition on which the present theory is based. Thus $\eta_{\mathrm{rec}}=\eta_{\mathrm{in}}t_{\mathrm{in}}/t_{\mathrm{rec}}$, and so we obtain from Eq. (48)

$$\frac{\dot\sigma_{\mathrm{rec}}}{\dot\sigma_{\mathrm{in}}}=\frac{n_{\mathrm{in}}}{n_{\mathrm{rec}}}\frac{T_{\mathrm{in}}}{T_{\mathrm{rec}}}\left(\frac{t_{\mathrm{in}}}{t_{\mathrm{rec}}}\right)^3\simeq1.2\times10^{-10}.\tag{50}$$

This result signifies that a drastic reduction in the rate of specific entropy production takes place during the plasma era. The value of η_{in} that we adopted in Eq. (44) is thus unable to account for the large entropy, $\sigma\simeq4\times10^9$, in the universe. It is of interest to compare this with the analogous conclusion drawn from a study of the use of the kinematically derived *bulk* viscosity in the isotropic and homogeneous universe: also in that case the theoretically calculated entropy is far too small to account for the observed value of σ [31,36,4].

Second option (isotropic space). All parameters p_i are equal $(\equiv a)$. From Eq. (31) we get, in cgs units,

$$a=\sqrt{(\kappa c^2)(\rho c^2)/3}\,t=\sqrt{\kappa c^2a_r/3}\,tT^2.\tag{51}$$

Inserting $T_{\mathrm{in}}=4\times10^8$ K for $t=1000$ s we get here $a=0.347$ which, according to Eq. (33), yields $\gamma=1.92$. There is thus apparently a deviation here from the value $\gamma=2$ which characterizes a Zel'dovich fluid. However, some care is called for, as regards the interpretation of the calculated value of γ, since our input value of T in Eq. (51) is after all not very accurate. If the correct value of a were instead $a=1/3$, we would arrive at an equation of state very close to that of a Zel'dovich fluid. Probably the most important physical point to be borne in mind here is that the influence from viscosity on the equation of state is at this instant very small.

One may expect that $\dot\sigma$, as calculated from Eq. (41), becomes larger if one instead considers earlier times. For instance, let us focus attention on the instant at which $T=10^{12}$ K. This instant is often taken to serve as some kind of limit for standard cosmological theory: when $T>10^{12}$ K, the universe is flooded with all kinds of particles and antiparticles. But when $T<10^{12}$ K, the large number of hadrons has disappeared, and the universe consists mainly of leptons, antileptons, and photons, plus a few surviving nucleons (this is the beginning of the lepton era). From [33] we quote the following values, at $T=10^{12}$ K:

$$t\simeq2\times10^{-4}\ \mathrm{s},\quad\rho\simeq5\times10^{13}\ \mathrm{g/\,cm^{-3}},$$

$$n\simeq6\times10^{29}\ \mathrm{cm^{-3}}.\tag{52}$$

We now return to the proportionalities (27) and (28). Whereas Eqs. (28) are satisfied automatically in the standard theory of the radiation dominated universe, we may make use of Eqs. (27), which are characteristic for the present kind of theory, to calculate ζ and η at $T=10^{12}$ K. Actually, we need only to observe that the constancy of the products $t\zeta$ and $t\eta$ makes S and Q *time independent*; cf. Eqs. (29) and (30). Consequently the anisotropy parameter A, as defined in Eq. (36), is time independent also. Consider then the expression (41) for $\dot{\sigma}$: it will vary with time in the same way as does the product $(nTt^3)^{-1}$. According to standard cosmology, $n\propto T^3$ and $t\propto T^{-2}$, so that $\dot{\sigma}\propto T^2$. Altogether, making use of Eq. (48) we get

$$\dot{\sigma}=2.0\times10^{-3}\left[\frac{10^{12}}{4\times10^{8}}\right]^2 \ \mathrm{s}^{-1}=1.25\times10^4 \ \mathrm{s}^{-1}. \quad (53)$$

This is quite an appreciable amount of entropy production. It lies at hand therefore to conclude that one has to go back to the violent conditions in the very early universe in order to get a viscous entropy production that is large enough to give any hope of explaining the large observed entropy in the universe. A related, but more drastic way of approach would be to assign an ''impulse'' viscosity to the early universe. For example, in Ref. [4] it was shown that in the $k=0$ FRW universe, an impulse bulk viscosity $\zeta_{\mathrm{infl}}\sim10^{60}\ \mathrm{g\ cm^{-1}\ s^{-1}}$ acting at some kind of phase transition at the end of the inflationary era corresponds to the correct entropy, $\sigma\simeq 4\times10^9$.

VI. CONCLUSIONS AND FINAL REMARKS

The main purpose of the present work has been to explore the consequences of using the Kasner form of metric, Eq. (2), as input in Einstein's equations, assuming that the cosmic fluid is endowed with a shear viscosity η as well as a bulk viscosity ζ. The expression for the energy-momentum tensor $T_{\mu\nu}$ is given in Eq. (9). The cosmological constant Λ has been set equal to zero. Central numerical quantities in the analysis are S and Q, defined in Eqs. (14). The main results emerging from Einstein's equations are expressed in Eqs. (17) and (18).

We may summarize as follows.

(1) For a perfect fluid, $\eta=\zeta=0$, there is one of two possibilities. *Either* the parameter S is equal to 1, in which case the space becomes anisotropic. The state of equation becomes $p=\rho$; i.e., the cosmic fluid becomes exactly a Zel'dovich fluid. *Or* S is different from 1, implying that the space becomes isotropic. Equation (26) gives then the equation of state for the fluid.

(2) For a viscous fluid, the requirement that the three Kasner parameters p_i be constants implies that ζ and η must vary with time according to $\zeta\propto t^{-1}$, $\eta\propto t^{-1}$. Again, there are two possibilities. *Either* S satisfies Eq. (29), implying that the space is anisotropic. For a given set of values for p_i, Eqs. (29) and (30) determine the equation of state. *Or* S does not satisfy Eq. (29); then the space is isotropic and the equation of state can be expressed in the conventional form $p/\rho=\gamma-1$, where γ is given by Eq. (33).

(3) The general expression for the rate of entropy production $\dot{\sigma}$ per baryon in the viscous Kasner universe was derived in Eq. (41). We tentatively investigated the magnitude of $\dot{\sigma}$ at the instant $t=1000$ s after the big bang, the reason behind this particular choice of t being that η and ζ are then calculable from ordinary relativistic kinetic theory. The result, given in Eq. (48), is very small. If one goes further back in the history of the universe, for instance to the instant $t=2\times10^{-4}$ s characterizing the start of the lepton era, and if one uses the requirements that η and ζ be inversely proportional to t, then the calculated entropy production becomes quite appreciable, as shown in Eq. (51).

We close our work with some remarks on the redshift problem. We consider the three coordinate directions separately: imagine first that an electromagnetic wave is traveling to us along the $-x$ direction. The equation of motion of a wave crest is, according to the metric (2), $dt=-t^{p_1}dx$. Assume that the crest leaves a galaxy located at the position x_e at time t_e, and reaches us at the origin $x=0$ at time t_0. Integrating the equation of motion we get

$$\int_{t_e}^{t_0} t^{-p_1}dt=x_e. \quad (54)$$

In the comoving frame of reference the spatial coordinates of a galaxy stay constant. Therefore, if the next wave crest leaves x_e at time $t_e+\delta t_e$, it will arrive here at a time $t_0+\delta t_0$ which is given by an expression like (54), only with the replacements $t_e\rightarrow t_e+\delta t_e$ and $t_0\rightarrow t_0+\delta t_0$ in the integration limits. Taking the difference between the two expressions we get

$$t_0^{-p_1}\delta t_0=t_e^{-p_1}\delta t_e. \quad (55)$$

The frequency ν_0 observed here is related to the emitted frequency ν_e by $\nu_0/\nu_e=\delta t_e/\delta t_0=(t_e/t_0)^{p_1}$. Since the redshift parameter z is defined as the fractional increase in wavelength, $z\equiv\lambda_0/\lambda_e-1$, we obtain in the present case

$$z_1=\frac{\nu_e}{\nu_0}-1=\left(\frac{t_0}{t_e}\right)^{p_1}-1, \quad (56)$$

where we have given a subscript 1 to z to indicate that we are dealing with the x direction.

The expressions for the redshifts x_2 and z_3 in the y and z coordinate directions are analogous. For an anisotropic space the redshift is thus seen to be dependent on direction. If the Kasner metric were realized to a good approximation in the universe, we would herewith have the possibility, at least in principle, of testing the magnitude of the anisotropy experimentally.

Naturally the presence of ζ and η in the equations of motion complicates the situation considerably. The complicating effect actually turns up already on the level of isotropic FRW theory: as long as viscosity is absent, we know from *adiabaticity* that $RT=\mathrm{const}$ (R is the scale factor). Therefore, the temperature obeys the relation $T=T_0(1+z)$ which, together with $\nu=\nu_0(1+z)$, makes the ratio $\nu/T=\mathrm{const}$. Now, the emitted radiation at time t_e obeys a Planck blackbody spectrum, and in view of the constancy of ν/T it continues to do so at all subsequent times. Once viscosity appears, however, the property of adiabaticity is lost, resulting in a distortion of the blackbody spectrum. Analogously, viscosity-induced distortions will occur in the present more complex, anisotropic case. An extensive discussion of the degree of isotropy of the universe, including the role of viscosity effects, has been given by Misner [30].

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