

Towards the theory of reheating after inflation

Lev Kofman

Institute for Astronomy, University of Hawaii, 2680 Woodlawn Drive, Honolulu, Hawaii 96822

Andrei Linde

Department of Physics, Stanford University, Stanford, California 94305-4060

Alexei A. Starobinsky

Landau Institute for Theoretical Physics, Kosygina St. 2, Moscow 117334, Russia

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Reheating after inflation occurs due to particle production by the oscillating inflaton field. In this paper we briefly describe the perturbative approach to reheating, and then concentrate on effects beyond the perturbation theory. They are related to the stage of parametric resonance, which we call *preheating*. It may occur in an expanding universe if the initial amplitude of oscillations of the inflaton field is large enough. We investigate a simple model of a massive inflaton field ϕ coupled to another scalar field χ with the interaction term $g^2\phi^2\chi^2$. Parametric resonance in this model is very broad. It occurs in a very unusual stochastic manner, which is quite different from parametric resonance in the case when the expansion of the universe is neglected. Quantum fields interacting with the oscillating inflaton field experience a series of kicks which, because of the rapid expansion of the universe, occur with phases uncorrelated to each other. Despite the stochastic nature of the process, it leads to exponential growth of fluctuations of the field χ . We call this process *stochastic resonance*. We develop the theory of preheating taking into account the expansion of the universe and back reaction of produced particles, including the effects of rescattering. This investigation extends our previous study of reheating after inflation. We show that the contribution of the produced particles to the effective potential $V(\phi)$ is proportional not to ϕ^2 , as is usually the case, but to $|\phi|$. The process of preheating can be divided into several distinct stages. In the first stage the back reaction of created particles is not important. In the second stage back reaction increases the frequency of oscillations of the inflaton field, which makes the process even more efficient than before. Then the effects related to scattering of χ particles on the oscillating inflaton field terminate the resonance. We calculate the number density of particles n_χ produced during preheating and their quantum fluctuations $\langle\chi^2\rangle$ with all back reaction effects taken into account. This allows us to find the range of masses and coupling constants for which one can have efficient preheating. In particular, under certain conditions this process may produce particles with a mass much greater than the mass of the inflaton field. [S0556-2821(97)05418-0]

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I. INTRODUCTION

According to inflationary theory, (almost) all elementary particles populating the universe were created during the process of reheating of the universe after inflation [1,2]. It makes this process extremely important. However, for many years the theory of reheating remained the least developed part of inflationary theory. Even now, when many features of the mechanism of reheating are understood, the literature on this subject is still full of contradictory statements.

The basic idea of reheating after inflation was proposed in the first paper on new inflation [3]: reheating occurs due to particle production by the oscillating scalar field ϕ . In the simplest inflationary models, this field is the same inflaton field ϕ that drives inflation at the early stages of the evolution of the universe. After inflation, the scalar field ϕ (which we will call inflaton) oscillates near the minimum of its effective potential and produces elementary particles. These particles interact with each other and eventually they come to a state of thermal equilibrium at some temperature T . This process completes when all (or almost all) the energy of the classical scalar field ϕ transfers to the thermal energy of

elementary particles. The temperature of the universe at this stage is called the reheating temperature, T_r .

A first attempt at a phenomenological description of this process was made in Ref. [4]. The authors added various friction terms to the equation of motion of the scalar field in order to imitate energy transfer from the inflaton field to matter. However, it remained unclear what kind of terms should be added and whether one should add them at the stage of slow rolling of the inflaton field, or only at the stage of rapid oscillations of the inflaton field.

The theory of reheating in application to the new inflation scenario was first developed in Refs. [5,6], and, in application to R^2 inflation, in Ref. [7]. It was based on perturbation theory, which was quite sufficient for obtaining the reheating temperature, T_r , in many realistic models. We will give a detailed description of this theory and develop it even further in a forthcoming publication [8]. However, perturbation theory has certain limitations, which have been realized only very recently. In particular, the mechanism of decay of the inflaton field to the vector fields discussed in [5] is efficient only at an intermediate stage of reheating in the new inflation model considered. The decay of the inflaton field to fermions

described in [6] typically is important only at very late stages of reheating. In many inflationary models neither of these mechanisms gives a correct description of the first stages of the process.

Indeed, recently it was understood [1] that in many inflationary models the first stages of reheating occur in a regime of a broad parametric resonance. To distinguish this stage from the subsequent stages of slow reheating and thermalization, we called it *preheating*. The energy transfer from the inflaton field to other boson fields and particles during preheating is extremely efficient. As we pointed out in [1], reheating never completes at the stage of parametric resonance; eventually the resonance becomes narrow and inefficient, and the final stages of the decay of the inflaton field and thermalization of its decay products can be described by the elementary theory of reheating [5,6,8]. Thus, the elementary theory of reheating proves to be very useful even in the theories where reheating begins at the stage of parametric resonance. However, it should be applied not to the original coherently oscillating inflaton field, but to the products of its decay, as well as to the part of the inflaton field which survived preheating. The short stage of explosively rapid preheating in the broad resonance regime may have long-lasting effects on the subsequent evolution of the universe. It may lead to specific nonthermal phase transitions in the early universe [9,10] and to topological defect production, it may make possible novel mechanisms of baryogenesis [11,12], and it may change the final value of the reheating temperature T_r .

The theory of parametric resonance in application to particle production by oscillating external fields was developed more than 20 years ago, see, e.g., [13]. The methods used in this theory were developed mainly for the case of narrow parametric resonance. A first attempt to apply this theory to reheating after inflation was made by Dolgov and Kirilova [14] and by Traschen and Brandenberger [15] for the narrow resonance regime in the context of the new inflation. In [14] it was conjectured that the parametric resonance in an expanding universe cannot lead to efficient reheating. The authors of Ref. [15] came to an important conclusion that parametric resonance in new inflation can be efficient. However, their investigation of parametric resonance was not quite correct; see Sec. IV of this paper.

In any case, at the moment we do not have any consistent inflationary models based on the new inflation scenario. The step towards the general theory of reheating in chaotic inflation was rather nontrivial. Indeed, the effective potential in new inflation is anomalously flat near $\phi=0$. As a result of this fine-tuned property of the effective potential, the Hubble constant at the end of inflation in this scenario is much smaller than the mass of the oscillating scalar field. Therefore the effects related to the expansion of the universe are not very destructive for the development of the resonance, which may be rather efficient even if the resonance is narrow. Narrow resonance can be rather efficient in chaotic inflation as well, in the context of conformally-invariant theories of the type of $\lambda\phi^4$. In such theories the expansion of the universe does not interfere with the development of the resonance, and therefore preheating may be efficient even if the resonance is rather narrow [1,16–19]. However, generally the effective potential is quadratic with respect to ϕ near the

minimum of the potential, which breaks the conformal invariance. As we will show in this paper, for the simplest models of inflation, such as the theory of a massive inflaton field ϕ with quadratic effective potential and interaction $g^2\phi^2\chi^2$, preheating is efficient only if the resonance is extremely broad. The theory of a broad parametric resonance in an expanding universe is dramatically different from the theory of a narrow resonance.

The basic features of the theory of a broad parametric resonance were outlined in [1], where the theory of preheating was developed in the context of the chaotic inflation scenario, taking into account back reaction of created particles and the expansion of the universe. This issue was studied later by many other authors, and a lot of very interesting results on parametric resonance and particle production have been obtained [16–31]. Of all these papers one is especially relevant to our investigation. Khlebnikov and Tkachev [30] performed a detailed three-dimensional numerical lattice simulation of broad parametric resonance in an expanding universe, taking into account the back reaction of produced particles, including, in particular, their rescattering. Their method (see also [24,28,29]) is based on solving numerically the classical equations for fluctuations of all interacting fields. It is presumably the best way to perform computer simulations of preheating.

From the point of view of analytical investigation of preheating in the broad resonance regime we should mention Ref. [21], where this regime was investigated for the case of a nonexpanding universe, and some of the results of Ref. [1] concerning this regime were obtained by a different method. However, after our paper [1] there was not much progress in analytical investigation of the broad resonance regime in an expanding universe. This is not very surprising, because the analytical investigation of preheating including back reaction is very difficult; one must describe a system of particles far away from equilibrium in the regime where effective coupling becomes strong because of anomalously large occupation numbers of boson particles produced by parametric resonance. But the main problem was related to the very unusual nature of broad parametric resonance in an expanding universe. As we will show in this paper, instead of staying in a particular resonance band, each growing mode scans many stability/instability bands within a single oscillation of the inflaton field, so the usual concept of separate resonance bands becomes inadequate. It is a stochastic process, during which the number of produced particles changes in a chaotic way. On average, the number of produced particles grows exponentially, but at some moments their number may decrease; a process which would be impossible at the classical level. We call this process *stochastic resonance*. The standard methods developed for investigation of parametric resonance simply do not apply here, so it was necessary to develop a new, more general approach.

The main purpose of the present paper is to develop the theory of preheating with an account taken of the expansion of the universe and the back reaction of created particles, including the effects of their rescattering. We will give here a detailed derivation of the results of Ref. [1], and describe recent progress in the understanding of physical processes which occur soon after the end of inflation.

We will begin our paper with discussion of the evolution

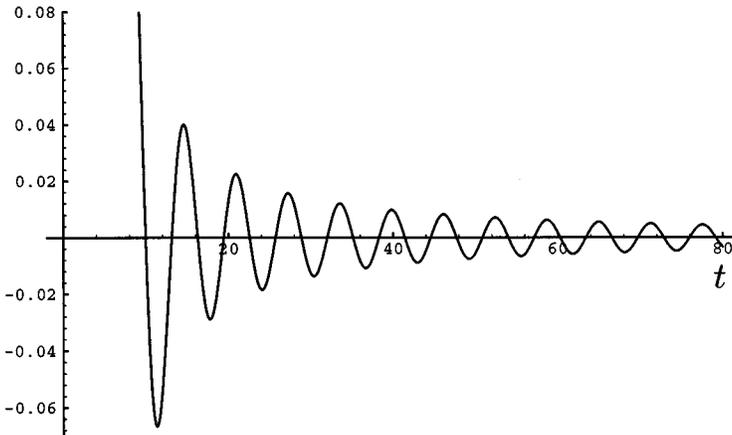


FIG. 1. Oscillations of the field ϕ after inflation in the theory $m^2\phi^2/2$. The value of the scalar field here and in all other figures in this paper is measured in units of M_p ; time is measured in units of m^{-1} .

of the scalar fields after inflation neglecting the effects of reheating, see Sec. II. Section III contains an introduction to the elementary theory of reheating [5,6,8]. We will then develop the theory of particle production due to parametric resonance following [1]. First of all, in Sec. IV we introduce the theory of reheating due to parametric resonance and discuss the relation between this theory and the elementary theory of reheating. Both theories are very simple, but the transition from one to the other is quite nontrivial; it is very difficult to understand the theory of parametric resonance using the elementary theory of reheating as a starting point, and, conversely, perturbation theory is not simply a limiting case of a weak parametric resonance. A more detailed discussion of all these issues will be contained in our forthcoming paper [8].

In Sec. V we discuss the difference between the narrow and broad resonance regimes. Section VI is devoted to a qualitative description of the development of broad resonance in an expanding universe. We describe the effect of stochastic resonance and illustrate this effect by solving the resonance equations numerically, taking into account the expansion of the universe. We find that it is much easier to perform the investigation in terms of the number of created particles, which is an adiabatic invariant, rather than in terms of wildly oscillating quantities such as $\langle\chi^2\rangle$ which are studied in many publications on preheating. In particular, in some cases $\langle\chi^2\rangle$ continues to grow even after the resonance ceases to exist and the number of χ particles remains constant. In Sec. VII we develop analytic methods for the description of broad resonance. These methods are especially appropriate for the investigation of stochastic resonance. They are applicable in those cases where the standard approach based on the investigation of Mathieu or Lamé equations fails.

Section VIII contains a discussion of the back reaction of the χ particles created by parametric resonance on the effective potential of the inflaton field. In Sec. IX we describe the process of reheating in the broad resonance regime with an account taken of the change of the frequency of oscillations of the inflaton field due to its interaction with the χ particles produced during preheating. In Sec. X we discuss the process of rescattering of χ particles and the production of ϕ particles in this process. We also consider some modifications of the picture of the second stage of reheating with an account taken of rescattering. We calculate the number of par-

ticles produced during reheating and the amplitude of perturbations $\langle\chi^2\rangle$. In Sec. XI we investigate the possibility of a copious production of particles with mass much greater than the inflaton mass. Finally, in Sec. XII we give a summary of our results and discuss their possible implications.

II. EVOLUTION OF THE INFLATON FIELD

During inflation the leading contribution to the energy-momentum tensor is given by the inflaton scalar field ϕ with the Lagrangian

$$L(\phi) = \frac{1}{2} \phi_{,i} \phi^{,i} - V(\phi), \quad (1)$$

where $V(\phi)$ is the effective potential of the scalar field ϕ . The evolution of the (flat) FRW universe is described by the Einstein equation

$$H^2 = \frac{8\pi}{3M_p^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad (2)$$

where $H = \dot{a}/a$. The Klein-Gordon equation for $\phi(t)$ is

$$\ddot{\phi} + 3H \dot{\phi} + V_{,\phi} = 0. \quad (3)$$

For sufficiently large initial values of $\phi > M_p$, the ‘‘friction’’ term $3H\dot{\phi}$ in Eq. (3) dominates over $\ddot{\phi}$ and the potential term in Eq. (2) dominates over the kinetic term. This is the inflationary stage, where the universe expands quasiexponentially, $a(t) = a_0 \exp(\int dt H(t))$. For definiteness, we will consider here the simplest models of chaotic inflation: $V(\phi) = \frac{1}{2} m \phi^2$ [2]. In these models inflation occurs at $\phi \geq M_p$. Density perturbations responsible for large-scale structure formation in these models are produced at $\phi \sim 3 - 4 M_p$. With a decrease of the field ϕ below M_p the ‘‘friction’’ term $3H\dot{\phi}$ becomes less and less important, and inflation terminates at $\phi \sim M_p/2$.

When making numerical estimates one should take into account that at the last stages of inflation the friction term is still non-negligible, and therefore during the first oscillation the amplitude of the field rapidly drops down.

For the quadratic potential $V(\phi) = \frac{1}{2} m \phi^2$ the amplitude after the first oscillation becomes only $0.04 M_p$, i.e., it drops by a factor of ten during the first oscillation; see Fig. 1. Later

on the solution for the scalar field ϕ asymptotically approaches the regime

$$\begin{aligned} \phi(t) &= \Phi(t) \sin mt, \\ \Phi(t) &= \frac{M_p}{\sqrt{3\pi mt}} \sim \frac{M_p}{2\pi\sqrt{3\pi N}}. \end{aligned} \quad (4)$$

Here $\Phi(t)$ is the amplitude of oscillations; N is the number of oscillations since the end of inflation. For simple estimates which we will make later one may use

$$\Phi(t) \approx \frac{M_p}{3mt} \approx \frac{M_p}{20N}. \quad (5)$$

The scale factor averaged over several oscillations grows as $a(t) \approx a_0(t/t_0)^{2/3}$. Oscillations of ϕ in this theory are sinusoidal, with the decreasing amplitude $\Phi(t) = (M_p/3)(a_0/a(t))^{3/2}$. The energy density of the field ϕ decreases in the same way as the density of nonrelativistic particles of mass m : $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \sim a^{-3}$. Hence the coherent oscillations of the homogeneous scalar field correspond to the matter dominated effective equation of state with vanishing pressure.

Reheating occurs when the amplitude of oscillations of the inflaton field ϕ decreases much faster than in Eq. (4), and its energy density is transferred to the energy density of other particles and fields.

III. OSCILLATIONS AND DECAY OF THE SCALAR FIELD

In the present section, we will discuss the elementary theory of reheating developed in [5,6]; see also [2]. A more detailed discussion of this theory will be contained in [8]. We will consider a basic model describing the inflaton scalar field ϕ interacting with a scalar field χ and a spinor field ψ :

$$\begin{aligned} L &= \frac{1}{2}\phi_{,i}\phi^{,i} - V(\phi) + \frac{1}{2}\chi_{,i}\chi^{,i} - \frac{1}{2}m_\chi^2(0)\chi^2 + \frac{1}{2}\xi R\chi^2 \\ &+ \bar{\psi}(i\gamma^i\partial_i - m_\psi(0))\psi - \frac{1}{2}g^2\phi^2\chi^2 - h\bar{\psi}\psi\phi. \end{aligned} \quad (6)$$

Here g , h , and ξ are small coupling constants, R is the space-time curvature, and $V(\phi)$ is the effective potential of the field ϕ . We will suppose here, for generality, that the effective potential has a minimum at $\phi = \sigma$, and near the minimum it is quadratic with respect to the field ϕ : $V(\phi) \sim \frac{1}{2}m^2(\phi - \sigma)^2$. Here m^2 is the effective mass squared of the field ϕ . After the shift $\phi - \sigma \rightarrow \phi$, the effective potential acquires the familiar form $\frac{1}{2}m^2\phi^2$, and the Lagrangian acquires an interaction term which is linear with respect to the field ϕ : $\Delta L = -g^2\sigma\phi\chi^2$. This term vanishes in the case without spontaneous symmetry breaking, where $\sigma = 0$. The masses of the χ particles and ψ after the shift become $m_\chi = \sqrt{m_\chi^2(0) + g^2\sigma^2}$ and $m_\psi = m_\psi(0) + h\sigma$. In this section we will consider the case $m \gg m_\chi$, m_ψ . We will assume that after inflation $H \ll m$. This condition is always satisfied during the last, most important stages of reheating.

We will study now the oscillation of the scalar field near the minimum of its effective potential. The energy density of the oscillating field (after the shift $\phi - \sigma \rightarrow \phi$) is

$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2$. If we ignore for a moment the effects associated with particle creation, the field ϕ after inflation oscillates near the point $\phi = 0$ with the frequency $k_0 = m$. The amplitude of oscillation decreases as $a^{-3/2}$ due to the expansion of the universe, and the energy of the field ϕ decreases in the same way as the density of nonrelativistic particles of mass m : $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \sim a^{-3}$. A homogeneous scalar field oscillating with frequency m can be considered as a coherent wave of ϕ particles with zero momenta and with particle density $n_\phi = \rho_\phi/m$. In other words, n_ϕ oscillators of the same frequency m , oscillating coherently with the same phase, can be described as a single homogeneous wave $\phi(t)$. Note that if we consider time intervals larger than the typical oscillation time m^{-1} , the energy density of the oscillating field, and the number density of the particles n_ϕ will be related to its amplitude Φ in a simple way:

$$\rho_\phi = \frac{1}{2}m^2\Phi^2, \quad (7)$$

$$n_\phi = \frac{1}{2}m\Phi^2. \quad (8)$$

Now we will consider effects related to the expansion of the universe and to particle production. For a homogeneous scalar field in a universe with a Hubble constant H , the equation of motion with nongravitational quantum corrections is

$$\ddot{\phi} + 3H(t)\dot{\phi} + [m^2 + \Pi(\omega)]\phi = 0. \quad (9)$$

Here $\Pi(\omega)$ is the flat space polarization operator for the field ϕ with four-momentum $k_i = (\omega, 0, 0, 0)$, $\omega = m$.

The real part of $\Pi(\omega)$ gives only a small correction to m^2 , but when $\omega \geq \min(2m_\chi, 2m_\psi)$, the polarization operator $\Pi(\omega)$ acquires an imaginary part $\text{Im } \Pi(\omega)$. We will assume that $m^2 \gg H^2$, $m^2 \gg \text{Im } \Pi$. The first condition is automatically satisfied after the end of inflation; the second is usually also true. We have $\Phi(t) = \Phi_0 a^{-3/2}(t) = \Phi_0 \exp(-\frac{3}{2} \int dt H(t))$. Neglecting for simplicity the time-dependence of H and $\text{Im } \Pi$ due to the expansion of the universe, we obtain a solution of Eq. (9) that describes damped oscillations of the field near the point $\phi = 0$:

$$\begin{aligned} \phi &= \Phi(t) \exp(imt) \approx \phi_0 \exp(imt) \\ &\times \exp\left[-\frac{1}{2}\left(3H + \frac{\text{Im } \Pi(m)}{m}\right)t\right]. \end{aligned} \quad (10)$$

From unitarity it follows that [32]

$$\text{Im } \Pi = m\Gamma, \quad (11)$$

where $\Gamma = \Gamma(\phi \rightarrow \chi\chi) + \Gamma(\phi \rightarrow \psi\psi)$ is the total decay rate of ϕ particles. (In a more general case one should calculate not only the imaginary part of the polarization operator, but the imaginary part of the effective action [5].) Thus Eq. (10) implies that the amplitude of oscillations of the field ϕ decreases as $\exp[-\frac{1}{2}(3H + \Gamma)t]$ due to particle production which occurs during the decay of the inflaton field.

Note that under the condition $m \gg H$, the polarization operator Π and the decay rates Γ do not depend on the curvature of the universe (and thus on time) and coincide with their flat-space limits. In particular, the probability of decay

of a ϕ particle into a pair of scalar χ particles or spinor ψ particles for $m \gg m_\chi, m_\psi$ is given by the expressions [2]

$$\Gamma(\phi \rightarrow \chi\chi) = \frac{g^4 \sigma^2}{8\pi m}, \quad \Gamma(\phi \rightarrow \psi\psi) = \frac{h^2 m}{8\pi}. \quad (12)$$

For a phenomenological description of the damping of oscillations of the scalar field ϕ (10) one may add an extra friction term $\Gamma \dot{\phi}$ to the classical equation of motion of the field ϕ , instead of adding the term proportional to the imaginary part of the polarization operator:

$$\ddot{\phi} + 3H(t)\dot{\phi} + \Gamma\dot{\phi} + m^2\phi = 0. \quad (13)$$

This phenomenological equation together with relation (11) for Γ reproduces the damped oscillator solution (10) of Eq. (9). The idea that one can describe effects of reheating by adding friction terms to the equation of motion goes back to one of the first papers on reheating [4]. At first the physical origin of such terms as well as their value remained obscure. Some authors added various auxiliary friction terms to the equations of the inflaton field in order to slow down its motion and make inflation longer; see, e.g., [4,33]. From the derivation of expression (11) for Γ it follows, however, that the simple phenomenological equation (13) is valid only at the stage of rapid oscillations of the field ϕ near the minimum of $V(\phi)$. This equation cannot be used to investigate the stage of slow rolling of the field ϕ during inflation.

According to Eq. (10), the field amplitude $\Phi(t)$ obeys the equation

$$\frac{1}{a^3} \frac{d}{dt} (a^3 \Phi^2) = -\Gamma \Phi^2. \quad (14)$$

If one multiplies it by m , one obtains the following equation for the number density (8) of the coherently oscillating ϕ particles:

$$\frac{d}{dt} (a^3 n_\phi) = -\Gamma a^3 n_\phi. \quad (15)$$

This equation has a simple interpretation. It shows that the total comoving number density of particles $\sim a^3 n_\phi$ exponentially decreases with the decay rate Γ . Similarly, one obtains the following equation for the total energy of the oscillating field ϕ :

$$\frac{d}{dt} (a^3 \rho_\phi) = -\Gamma a^3 \rho_\phi. \quad (16)$$

The decay products of the scalar field ϕ are ultrarelativistic (for $m \gg m_\chi, m_\psi$), and their energy density decreases due to the expansion of the universe much faster than the energy of the oscillating field ϕ . Therefore, reheating in our model ends only when the Hubble constant $H \sim 2/3t$ becomes smaller than Γ , because otherwise the main portion of the energy remains stored in the field ϕ . Therefore the age of the universe when reheating completes is given by $t_r \sim \frac{2}{3}\Gamma^{-1}$. At that stage the main part of the matter in the universe becomes ultrarelativistic. The age of the universe with the energy den-

sity ρ is $t = \sqrt{M_p} / \sqrt{6\pi\rho}$ [2]. This, together with the condition $t_r \sim \frac{2}{3}\Gamma^{-1}$, gives the energy density at time t_r :

$$\rho(t_r) \approx \frac{3\Gamma^2 M_p^2}{8\pi}. \quad (17)$$

If thermodynamic equilibrium sets in quickly after the decay of the inflaton field, then the matter acquires a temperature T_r , which is defined by the equation

$$\rho(t_r) \approx \frac{3\Gamma^2 M_p^2}{8\pi} \approx \frac{\pi^2 N(T_r)}{30} T_r^4. \quad (18)$$

Here $N(T)$ is the number of relativistic degrees of freedom at the temperature T ; one should take 1 for each scalar, two for each massless vector particle, etc. [2]. In realistic models one may expect $N(T_r) \sim 10^2 - 10^3$, which gives the following estimate of the reheating temperature:

$$T_r \approx 0.2 \sqrt{\Gamma M_p}. \quad (19)$$

Note that T_r does not depend on the initial value of the field ϕ ; it is completely determined by the parameters of the underlying elementary particle theory.

Here we should make an important comment. In the absence of fermions, the only contribution to the decay rate would be $\Gamma(\phi \rightarrow \chi\chi) = g^4 \sigma^2 / 8\pi m$. Note that this term disappears in the theories without spontaneous symmetry breaking, where $\sigma = 0$. This does not necessarily mean that there is no reheating at all in such theories. Indeed, decay is possible not only in the presence of a constant field σ but in the presence of a large oscillating field $\phi(t)$ as well. In what follows we will study parametric resonance and reheating in models with $\sigma = 0$, or $\sigma \ll \Phi$, where Φ is the amplitude of the oscillations. However, when reheating proceeds and Φ becomes small one may expect perturbation theory to work well. To get an estimate for the decay rate at $\sigma = 0$ let us simply write Φ instead of σ in Eq. (12): $\Gamma(\phi \rightarrow \chi\chi) \sim g^4 \Phi^2 / 8\pi m$. The problem with this term is that Φ^2 decreases as t^{-2} in the expanding universe, whereas the Hubble constant decreases only as t^{-1} . Therefore the decay rate never catches up with the expansion of the universe, and reheating never completes. Reheating can be complete only if Γ decreases more slowly than t^{-1} . Typically this requires either spontaneous symmetry breaking ($\sigma \neq 0$) or coupling of the inflaton field to fermions with $m_\psi < m/2$. If both of these conditions are violated, the inflaton field ϕ never decays completely. Such fields may be responsible for the dark matter of the universe, but it requires certain fine-tuning of the parameters. Normally, an incomplete decay of the inflaton field implies that the universe at the age of 10 billion years is cold, empty, and unsuitable for life. We should emphasize that this may happen even if the coupling constant g^2 is very large. Thus the requirement that reheating is complete imposes important constraints on the structure of the theory.

The elementary theory of reheating described above is simple and intuitively appealing. It proves to be very successful in describing reheating after inflation in many realistic inflationary models. That is why we are going to develop this theory even further in [8]. However, in some cases

where the amplitude of the oscillating field is sufficiently large, reheating occurs in a different way, in the regime of parametric resonance.

IV. PARAMETRIC RESONANCE AND LIMITS OF APPLICABILITY OF PERTURBATION THEORY

A. Perturbation theory versus narrow resonance

In the investigation performed above we made a natural assumption that the decay probability Γ of the scalar field ϕ can be calculated by ordinary methods of quantum field theory describing the decay $\phi \rightarrow \chi\chi$. However, if many χ particles have already been produced, $n_k > 1$, then the probability of decay becomes greatly enhanced due to effects related to Bose statistics. This may lead to explosive particle production.

For simplicity, we consider here the interaction between the *classical* inflaton field ϕ and the *quantum* scalar field $\hat{\chi}$ with the Lagrangian (6). The Heisenberg representation of the quantum scalar field $\hat{\chi}$ is

$$\hat{\chi}(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k (\hat{a}_k \chi_k(t) e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_k^+ \chi_k^*(t) e^{i\mathbf{k}\cdot\mathbf{x}}), \quad (20)$$

where \hat{a}_k and \hat{a}_k^+ are annihilation and creation operators. For a flat Friedmann background with scale factor $a(t)$ the temporal part of the eigenfunction with comoving momentum \mathbf{k} obeys the equation

$$\ddot{\chi}_k + 3\frac{\dot{a}}{a}\dot{\chi}_k + \left(\frac{\mathbf{k}^2}{a^2} + m_\chi^2(0) - \xi R + g^2\phi^2 \right) \chi_k = 0. \quad (21)$$

[The physical momentum $\mathbf{p} = \mathbf{k}/a(t)$ coincides with \mathbf{k} for Minkowski space, where $a = 1$.] Equation (21) describes an oscillator with a variable frequency ω due to the time-dependence of $a(t)$ and the background field $\phi(t)$. Until the last section of this paper we will suppose that the effective mass of the field χ vanishes for $\phi = 0$: $m_\chi(0) = 0$. In Sec. XI we will investigate the opposite case, $m_\chi(0) \gg m$.

As in the previous section, consider the simplest potential $V(\phi) \sim \frac{1}{2}m^2(\phi - \sigma)^2$ (to mimic the situation with spontaneous symmetry breaking) and make the shift $\phi - \sigma \rightarrow \phi$, after which the effective potential becomes $\frac{1}{2}m^2\phi^2$, and the interaction term $-\frac{1}{2}g^2\phi^2\chi^2$ transforms to $-\frac{1}{2}g^2\phi^2\chi^2 - g^2\sigma\phi\chi^2 - \frac{1}{2}g^2\sigma^2\chi^2$. We shall analyze the general equation (21) in different regimes.

Suppose first that the amplitude of oscillations ϕ is much smaller than σ , and neglect for a moment the expansion of the universe, taking $a = 1$ in Eq. (21). Then one can write the equation for modes (quantum fluctuations) of the field χ with physical momentum \mathbf{k} in the form

$$\ddot{\chi}_k + (k^2 + g^2\sigma^2 + 2g^2\sigma\Phi \sin mt) \chi_k = 0, \quad (22)$$

where $k = \sqrt{\mathbf{k}^2}$, and Φ stands for the amplitude of oscillations of the inflaton field. This equation describes an oscillator with a periodically changing frequency $\omega_k^2(t) = k^2 + g^2\sigma^2 + 2g^2\sigma\Phi \sin mt$. This periodicity may lead to para-

metric resonance for modes with certain values of k . The simplest way to describe this important effect is to make a change of variables $mt = 2z - \pi/2$, which reduces Eq. (22) to the well-known Mathieu equation [34]:

$$\chi_k'' + (A_k - 2q \cos 2z) \chi_k = 0. \quad (23)$$

Here $A_k = 4[(k^2 + g^2\sigma^2)/m^2]$, $q = 4g^2\sigma\Phi/m^2$, $z = mt/2$, and prime denotes differentiation with respect to z . The properties of the solutions of the Mathieu equation are well represented by its stability/instability chart which can be found, e.g., in [34]. An important feature of solutions of Eq. (23) is the existence of an exponential instability $\chi_k \propto \exp(\mu_k^{(n)}z)$ within the set of resonance bands of frequencies $\Delta k^{(n)}$ labeled by an integer index n . This instability corresponds to exponential growth of occupation numbers of quantum fluctuations $n_k(t) \propto \exp(2\mu_k^{(n)}z)$ that may be interpreted as particle production. In a state with a large number of Bose particles the estimates for Γ obtained in the previous subsection do not apply, and one should use much more elaborate methods of investigation based on the theory of parametric resonance.

In the case under consideration, $g\Phi \ll g\sigma \ll m$, the theory of parametric resonance is well known [35]. Indeed, in this case one has $q \ll 1$, and the resonance occurs only in some narrow bands near $A_k \simeq l^2$, $l = 1, 2, \dots$. Each band in momentum space has width of order $\Delta k \sim q^l$, so for $q < 1$ the widest and most important instability band is the first one, $A_k \sim 1 \pm q = 1 \pm 4g^2\sigma\Phi/m^2$.

The factor μ_k which describes the rate of exponential growth for the first instability band for $m^2 \gg g^2\sigma^2$ is given by [34]

$$\mu_k = \sqrt{(q/2)^2 - (2k/m - 1)^2}. \quad (24)$$

Thus resonance occurs for $k = (m/2)(1 \pm q/2)$. The index μ_k vanishes at the edges of the resonance band and takes its maximal value $\mu_k = q/2 = 2g^2\sigma\Phi/m^2$ at $k = m/2$. The corresponding modes χ_k grow at a maximal rate $\exp(qz/2)$, which in our case is given by $\exp(qmt/4) = \exp(g^2\sigma\Phi t/m)$.

The growth of the modes χ_k leads to the growth of the occupation numbers of the created particles $n_k(t)$. Indeed, the number density n_k of particles with momentum \mathbf{k} can be evaluated as the energy of that mode $\frac{1}{2}|\dot{\chi}_k|^2 + \frac{1}{2}\omega_k^2|\chi_k|^2$ divided by the energy ω_k of each particle:

$$n_k = \frac{\omega_k}{2} \left(\frac{|\dot{\chi}_k|^2}{\omega_k^2} + |\chi_k|^2 \right) - \frac{1}{2}. \quad (25)$$

When the modes χ_k grow as $\exp(qz/2)$, the number of χ particles grows as $\exp(qz)$, which in our case is equal to $\exp(qmt/2) = \exp(2g^2\sigma\Phi t/m)$.

The fact that the resonance occurs near $k = m/2$ has a simple interpretation: In the limit $g\sigma \ll m$ the effective mass of the χ particles is much smaller than m . Therefore one decaying ϕ particle creates two χ particles with momentum $k \sim m/2$. This picture is very similar to the process of decay $\phi \rightarrow \chi\chi$ discussed in the previous section, but the results are absolutely different. Indeed, in perturbation theory the amount of produced particles did not depend on the number of particles produced earlier, and the rate of production for

our model was given by $\Gamma(\phi \rightarrow \chi\chi) = g^4 \sigma^2 / 8\pi m$. Thus the decay rate Γ^{-1} was suppressed by the factor g^4 , which made the decay very slow in the weak coupling limit. By contrast, in the regime of parametric resonance the rate of production of χ particles is proportional to the amount of particles produced earlier (which is why we have exponential growth), and the rate of the process is given by an absolutely different expression $\mu_k m \sim g^2 \sigma \Phi / m$, which is greater than Γ for $\Phi > (g^2 / 8\pi) \sigma$.

Thus, before going any further we should understand how these two processes are related to each other, and why we did not find the effect of parametric resonance in the investigation performed in the previous section. Is the perturbation theory discussed there a limiting case of the narrow resonance regime or is it something else?

The reason we missed the effect of parametric resonance is rather delicate. In our calculations of the imaginary part of the polarization operator we assumed that the χ particles produced by the oscillating scalar field ϕ are normal particles on the mass shell, $k_\chi^2 = m_\chi^2$. This is what one would get solving Eq. (22) in any finite order of perturbation theory with respect to the interaction term $2g^2 \chi \sigma \Phi \sin mt$. However, if one solves the equation for the fluctuations of the field χ (22) *exactly*, one finds exponentially growing modes χ_k . This creates a new channel of decay of the scalar field ϕ .

Note that exponentially growing modes occupy a very small portion of momentum space in the narrow resonance limit. This means that the fluctuations of the field χ for almost all k are normal fluctuations with $k_\chi^2 = m_\chi^2$. In this case our calculation of the imaginary part of the polarization operator does apply. If the resulting value of Γ appears to be smaller than $2\mu_k m \sim qm$, then the perturbative decay of the scalar field may coexist with the parametric resonance. One may consider several different possibilities. In the beginning the scalar field ϕ can be expected to oscillate with amplitude $\Phi > (g^2 / 8\pi) \sigma$. In this regime parametric resonance leads to the exponential growth of modes χ_k , as we discussed above. However, gradually the field ϕ loses its energy, and its amplitude Φ becomes smaller than $(g^2 / 8\pi) \sigma$. In this regime the amplitude of the field Φ decays exponentially within a time Γ^{-1} which is smaller than the typical time necessary for parametric resonance to occur. One may say that the perturbative decay makes the energy eigenstate (the mass) of the field ϕ “wide,” with width Γ , and when this width exceeds the width of the resonance band $\sim qm/2$, the resonance terminates. Starting from this moment perturbation theory takes over, and the description of reheating should be given along the line of the elementary theory described in the previous section.

Thus, the standard effect of scalar field decay described by the elementary theory of reheating [5,6,8] and preheating due to parametric resonance are two *different* effects. In an expanding universe there exist other reasons for evolving from parametric resonance to perturbative decay.

First of all, during the expansion of the universe the field ϕ decreases not only because of its decay, but because of the “friction term” $3H\dot{\phi}$ in the equation of motion for the field ϕ . Thus one should compare qm with the effective decay rate $3H + \Gamma$: Parametric resonance occurs only for $qm \geq 3H + \Gamma$. Note that for $\Gamma > H$ perturbative decay leads to

reheating even neglecting parametric resonance. Therefore to check whether parametric resonance appears at the time when perturbative decay is inefficient, i.e., in the case $\Gamma < H$, it is enough to consider the condition $qm \geq 3H$.

Another important mechanism which can prevent parametric resonance from being efficient is the redshift of momenta k away from the resonance band. The total width of the first band is given by qm ; the width of the part where the resonance is efficient is somewhat smaller; one can estimate it as $qm/2$. The time Δt during which a given mode remains within this band depends on the equation of state of matter, and typically can be estimated by qH^{-1} . During this time the number of particles in growing modes increases as $\exp(q^2 m / 2H)$. This leads to efficient decay of inflatons only if $q^2 m \geq H$. In the narrow resonance limit $q \ll 1$ this is a stronger condition than the condition $qm \geq 3H$.

In general, it is possible that exponential growth during the time Δt is small, but $\Delta t \ll H^{-1}$ and therefore resonance still plays some role in reheating. However, this is a rather exceptional situation. Therefore typically the set of conditions for the resonance to be efficient can be formulated as

$$qm \geq \Gamma, \quad (26)$$

$$q^2 m \geq H. \quad (27)$$

In the model considered above these conditions yield

$$\Phi > \frac{g^2}{32\pi} \sigma, \quad (28)$$

$$\Phi \geq \frac{m\sqrt{mH}}{4g^2\sigma}. \quad (29)$$

Thus parametric resonance can be efficient at a sufficiently large Φ , but reheating never ends in the regime of parametric resonance. As soon as the amplitude of oscillations becomes sufficiently small, parametric resonance terminates, and reheating can be described by the elementary theory developed in [5,6,8]. Typically the reheating temperature is determined by these last stages of this process. Therefore one should not calculate the reheating temperature simply by finding the endpoint of the stage of parametric resonance, as many authors do. The role of the stage of preheating is to prepare a different setup for the last stage of reheating. It changes the reheating temperature, and it may lead to interesting effects such as nonthermal symmetry restoration and new mechanisms of baryogenesis. However, reheating never ends in the regime of parametric resonance; it does not make much sense to calculate the reheating temperature at the end of the stage of preheating.

The expansion of the universe and the inflaton decay are not the only mechanisms which could prevent the development of resonance. As we will show, back reaction of created particles may change the parameters A_k and q . There will be no resonance if the χ particles decay with decay rate $\Gamma > \mu_k m$, or if within the time $\sim (\mu_k m)^{-1}$ they are taken away from the resonance band because of their interactions. Also, there is no explosive reheating if the decay products include fermions since the fermion occupation numbers cannot be large because of the Pauli principle. This happens, for

example, in many inflationary models based on supergravity where inflaton decay is often accompanied by gravitino production [36].

If reheating never *ends* in a state of narrow parametric resonance, one may wonder whether reheating may *begin* in a state of narrow resonance. As we are going to show, in most cases inflation begins in a state of broad parametric resonance; the resonance typically ceases to exist as soon as it becomes narrow. But before analyzing this issue, we will take one last look at the model which we studied above.

B. Processes at $\phi \sim \sigma$

In our investigation of the simple model with spontaneous symmetry breaking ($\sigma \neq 0$) we assumed that the amplitude of oscillations of the scalar field ϕ is very small, $\Phi \ll \sigma$. Therefore we retained only the quadratic part of the effective potential, $V(\phi) \sim (\phi - \sigma)^2$. However, in realistic models of spontaneous symmetry breaking this condition is satisfied only at the end of parametric resonance. Indeed, let us consider a theory with spontaneous symmetry breaking with the usual potential $(\lambda/4)(\phi^2 - \sigma^2)^2$. Then after spontaneous symmetry breaking and the corresponding shift $\phi - \sigma \rightarrow \phi$ the theory at $\phi \ll \sigma$ can be represented as a theory of a massive scalar field with a mass $m^2 = 2\lambda\sigma^2$ interacting with the field χ which acquires mass $m_\chi^2 = g^2\sigma^2$. In this respect, it coincides with the toy model studied in the previous subsection. However, there are some important differences.

First of all, the process $\phi \rightarrow \chi\chi$ is possible only if $m > 2m_\chi$. This was one of the conditions which we used in our investigation: we assumed that $m \gg m_\chi$, i.e., $\lambda \gg g^2$. However, in this case the interaction $(\lambda/4)\phi^4$ which we did not take into account so far may become more important than the interaction $(g^2/2)\phi^2\chi^2$ which we considered. As a result, the production of ϕ particles may be more efficient than the production of χ particles.

In order to investigate this possibility let us study for a moment a model with the effective potential $(\lambda/4)(\phi^2 - \sigma^2)^2$ in the limit $\lambda \gg g^2$, i.e., neglecting the interaction $(g^2/2)\phi^2\chi^2$. We will assume here that in the beginning the field ϕ was at the top of the effective potential. At that time its effective mass squared was negative, $m^2(0) = -\lambda\sigma^2$. This fact alone, independent of any parametric resonance, leads to the production of particles of the field ϕ . The main point here is that all modes with $k < \sqrt{\lambda}\sigma$ grow exponentially, which breaks the homogeneity of the oscillating scalar field. This is an interesting effect, which has some nontrivial features, especially if one takes the expansion of the universe into account. We will return to its discussion elsewhere. However, this effect does not last long because away from the maximum of the effective potential its curvature becomes positive.

When the amplitude of the oscillations of the field ϕ near $\phi = \sigma$ becomes smaller than σ , the field begins oscillating near its minimum with a frequency $m \approx \sqrt{2\lambda}\sigma$. The parametric resonance with ϕ -particle production in this regime can be qualitatively understood if the equation for the fluctuations $\delta\phi_k$ is approximately represented as a Mathieu equation. The modes ϕ_k grow in essentially the same way as the modes in the second instability band of the Mathieu equation with $A_k = 4k^2/m^2 + 4$, $q = 6(\Phi/\sigma)$. For $q \geq 1$, we are in the

broad resonance regime, and there is a significant production of ϕ particles. However, for $q \ll 1$, i.e., for $\Phi \ll \sigma/6$, the parametric resonance in the second band becomes very inefficient. (One can obtain the same result by a more accurate investigation of parametric resonance in this situation in terms of the Lamé equation, but this is not our purpose here.)

Thus, we are coming to the following picture of parametric resonance in this model. In the beginning of the rolling of the field ϕ down to the minimum of the effective potential, the leading source of particle production is associated with the tachyonic mass of the field ϕ . Soon after that, the leading mechanism is the decay of a coherently oscillating field ϕ into ϕ particles. This mechanism remains dominant until the amplitude of the field Φ becomes much smaller than σ , after which the decay $\phi \rightarrow \chi\chi$ studied in the previous section becomes more important. (This process becomes somewhat more complicated if the back reaction of the produced particles is taken into account.) Finally, when the amplitude of the oscillations Φ becomes smaller than $(g^2/32\pi)\sigma$, or when it becomes smaller than $m\sqrt{mH}/4g^2\sigma$, whichever comes first, the parametric resonance ceases to exist, and the decay $\phi \rightarrow \chi\chi$ is described by the elementary theory of reheating based on perturbation theory.

We should note that the χ -particle production in this model for $\lambda \gg g^2$ was first studied in [15]. However, as we just mentioned, at $\Phi \sim \sigma$ this process is subdominant as compared to the ϕ -particle production, which was not studied in [15]. The process of χ -particle production is more efficient than ϕ -particle production only for $\Phi \ll \sigma$. In this regime our results differ from those obtained in [15] by the factor Φ/σ in the exponent. This difference is very significant because it leads to a much less efficient reheating, which shuts down as soon as Φ becomes sufficiently small.

The models studied in the last two subsections can be considered as a good laboratory where one can study different features of parametric resonance. However, in our investigation so far we did not discuss the question of initial conditions for resonance in these models. Indeed, after 15 years of investigation we still have not found any simple mechanism which will put the inflaton field on the top of the potential at $\phi = 0$ in the new inflation scenario. Also, the shape of the potential required for new inflation (extremely flat near the origin) is rather artificial. As soon as we consider generic initial conditions for the scalar field ϕ in more realistic inflationary models, such as chaotic inflation in the theory with a simple potential $m^2\phi^2/2$, the theory of parametric resonance becomes different in many respects from the simple theory described above.

V. BROAD RESONANCE VERSUS NARROW RESONANCE IN MINKOWSKI SPACE

In the chaotic inflation scenario one does not impose any *a priori* conditions on the initial value of the scalar field. In many models of chaotic inflation the initial amplitude of oscillations of the field ϕ can be as large as M_p , i.e., much greater than any other dimensional parameters such as σ . Therefore in the remaining part of the paper we will concentrate on the simplest chaotic inflation model without symmetry breaking with the effective potential $V(\phi) = (m^2/2)\phi^2$,

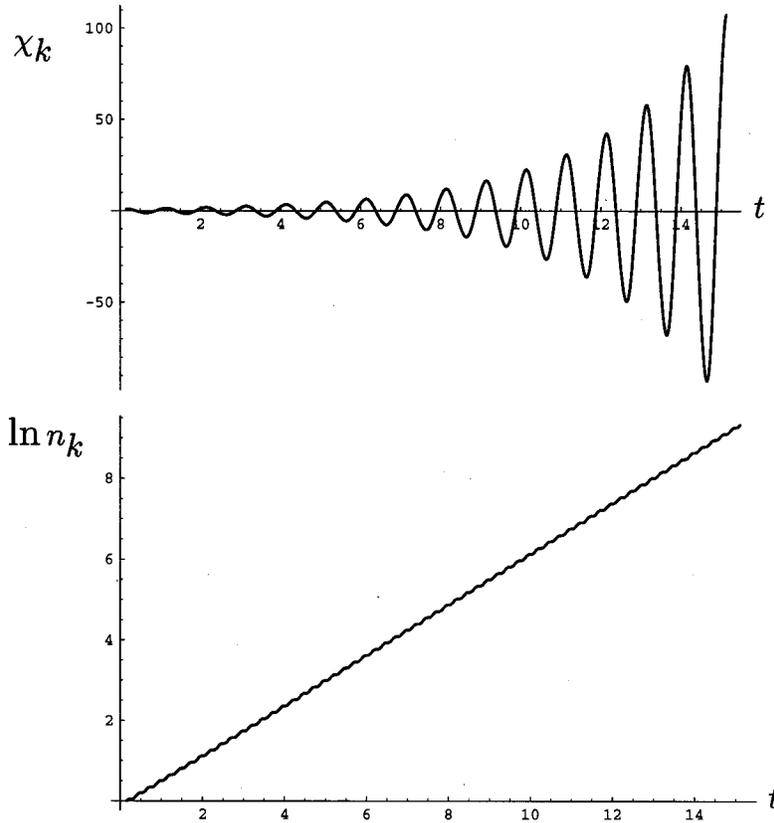


FIG. 2. Narrow parametric resonance for the field χ in the theory $m^2\phi^2/2$ in Minkowski space for $q \sim 0.1$. Time is shown in units of $m/2\pi$, which is equal to the number of oscillations of the inflaton field ϕ . For each oscillation of the field $\phi(t)$ the growing modes of the field χ oscillate one time. The upper figure shows the growth of the mode χ_k for the momentum k corresponding to the maximal speed of growth. The lower figure shows the logarithm of the occupation number of particles n_k in this mode; see Eq. (25). As we see, the number of particles grows exponentially, and $\ln n_k$ in the narrow resonance regime looks like a straight line with a constant slope. This slope divided by 4π gives the value of the parameter μ_k . In this particular case $\mu_k \sim 0.05$, exactly as it should be in accordance with the relation $\mu_k \sim q/2$ for this model.

and the interaction term $-\frac{1}{2}g^2\phi^2\chi^2$. In this case instead of Eq. (22) one has

$$\ddot{\chi}_k + (k^2 + g^2\Phi^2 \sin^2(mt))\chi_k = 0. \quad (30)$$

This equation describes an oscillator with a periodically changing frequency $\omega^2(t) = k^2 + g^2\Phi^2 \sin^2 mt$. One can write it as a Mathieu equation [Eq. (23)] with $A_k = k^2/m^2 + 2q$, $q = g^2\Phi^2/4m^2$, $z = mt$.

For $g\Phi < m$ we have a narrow resonance with $q \ll 1$. In this regime the resonance is more pronounced in the first resonance band, for modes with $k^2 \sim m^2(1 - 2q \pm q)$. The modes χ_k with momenta corresponding to the center of the resonance at $k \sim m$ grow as $e^{qz/2}$, which in our case equals $e^{\mu_k mt} \sim \exp(g^2\Phi^2 t/8m)$, and the number of χ particles grows as $e^{2\mu_k z} \sim e^{qz} \sim \exp(g^2\Phi^2 t/4m)$. This process can be interpreted as a resonance with decay of two ϕ particles with mass m to two χ particles with momenta $k \sim m$. We show the results of the numerical solution of Eq. (30) for the fastest growing mode χ_k in the narrow resonance regime in Fig. 2. Typically, the rate of development of the parametric resonance does not differ much from the rate of the growth of the leading mode χ_k ; see a discussion of this issue in the next section.

On the other hand, for oscillations with a large amplitude Φ the parameter $q = g^2\Phi^2/4m^2$ can be very large. In this regime the resonance occurs for a broad range of values of k , the parameter μ_k can be rather large, and reheating becomes extremely efficient. The resonance occurs for modes with $k^2/m^2 = A - 2q$, i.e., above the line $A = 2q$ on the stability-instability chart for the Mathieu equation [1]. The standard

methods of investigation of narrow parametric resonance do not work here. The difference between these two regimes can be easily grasped by comparing solutions of Eq. (30) for small and for large q ; see Figs. 2 and 3.

The time evolution is shown in units $m/2\pi$, which corresponds to the number of oscillations N of the inflaton field ϕ . The oscillating field $\phi(t) \sim \Phi \sin mt$ is zero at integer and half-integer values of the variable $mt/2\pi$. This allows us to identify particle production with time intervals when $\phi(t)$ is very small.

During each oscillation of the inflaton field ϕ , the field χ oscillates many times. Indeed, the effective mass $m_\chi(t) = g\phi(t)$ is much greater than the inflaton mass m for the main part of the period of oscillation of the field ϕ in the broad resonance regime with $q^{1/2} = g\Phi/2m \gg 1$. As a result, the typical frequency of oscillation $\omega(t) = \sqrt{k^2 + g^2\phi^2(t)}$ of the field χ is much higher than that of the field ϕ . Within one period of oscillation of the field ϕ the field χ makes $O(q^{1/2})$ oscillations. That is why during the most of this interval it is possible to talk about an adiabatically changing effective mass $m_\chi(t)$. Therefore, in the broad resonance regime the amplitude of χ_k is minimal at the points where the frequency is maximal, $|\chi_k| \propto \omega(t)^{-1/2}$, i.e., at $\phi(t) = \Phi$, and it increases substantially near the points at which $\phi(t) = 0$.

For very small $\phi(t)$ the change in the frequency of oscillations $\omega(t)$ ceases to be adiabatic. The standard condition necessary for particle production is the absence of adiabaticity in the change of $\omega(t)$:

$$\frac{d\omega}{dt} \gtrsim \omega^2. \quad (31)$$

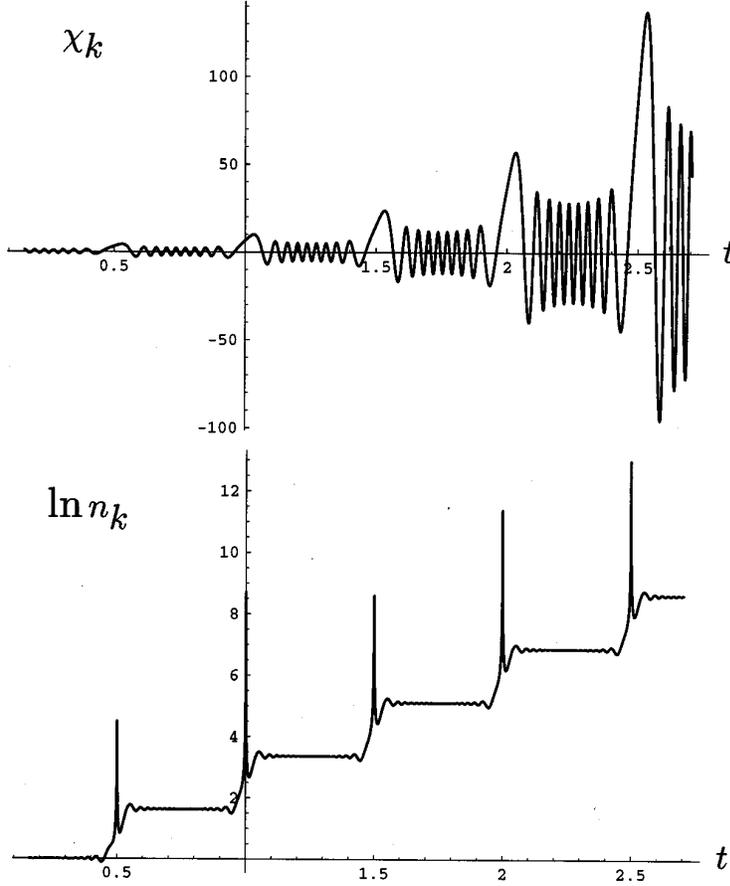


FIG. 3. Broad parametric resonance for the field χ in Minkowski space for $q \sim 2 \times 10^2$ in the theory $m^2 \phi^2/2$. For each oscillation of the field $\phi(t)$ the field χ_k oscillates many times. Each peak in the amplitude of the oscillations of the field χ corresponds to a place where $\phi(t) = 0$. At this time the occupation number n_k is not well defined, but soon after that time it stabilizes at a new, higher level, and remains constant until the next jump. A comparison of the two parts of this figure demonstrates the importance of using proper variables for the description of preheating. Both χ_k and the integrated dispersion $\langle \chi^2 \rangle$ behave erratically in the process of parametric resonance. Meanwhile n_k is an adiabatic invariant. Therefore, the behavior of n_k is relatively simple and predictable everywhere except the short intervals of time when $\phi(t)$ is very small and the particle production occurs. In our particular case, the average rate of growth of n_k is close to the maximal possible rate for our model, $\mu_k \sim 0.3$.

One should note that for a narrow resonance this condition is not necessary, because even a small variation of $\omega(t)$ may be exponentially accumulated in the course of time. However, for a broad resonance one should expect a considerable effect during each oscillation, which implies that the condition (31) should be satisfied. To find the time interval Δt_* and the typical momenta k_* when and where it may happen let us remember that for small ϕ one has $\phi \approx m\Phi$. Therefore our condition (31) implies that

$$k^2 \leq (g^2 \phi m \Phi)^{2/3} - g^2 \phi^2. \quad (32)$$

Let us consider those momenta k^2 which satisfy condition (32) as a function of $\phi(t)$. This condition becomes satisfied for small k when the field $\phi(t)$ becomes smaller than $\sqrt{m\Phi/g}$. The maximal range of momenta for which particle production occurs corresponds to $\phi(t) = \phi_*$, where

$$\phi_* \approx \frac{1}{2} \sqrt{m\Phi/g} \approx \frac{1}{3} \Phi q^{-1/4}. \quad (33)$$

The maximal value of momentum for particles produced at that epoch can be estimated by $k_{\max} = \sqrt{gm\Phi/2}$. In the main part of the interval $|\phi| \leq 2\phi_*$ the range of momentum remains smaller but the same order of magnitude as k_{\max} . Thus one may estimate a typical value of momentum of particles produced at that stage as $k_*/2$, where

$$k_* = \sqrt{gm\Phi} = \sqrt{2mq}^{1/4}. \quad (34)$$

This simple estimate practically coincides with the result of a more detailed and rigorous investigation which will be performed in Sec. VII. This is a very important result [1], which we are going to use throughout the paper.¹ This result implies, in particular, that in the broad resonance regime $m \ll k_* \leq g\Phi$.

Each time the field ϕ approaches the point $\phi = 0$, it spends time

$$\Delta t_* \sim \frac{2\phi_*}{\dot{\phi}} \sim \frac{1}{\sqrt{gm\Phi}} \sim k_*^{-1} \quad (35)$$

in the domain $|\phi| \leq \phi_*$. During that time $k_* \sim m_\chi = g\phi_*$, so that $\omega \sim k_*$. This estimate of Δt_* tells us that particle production in the broad resonance regime occurs within a time of order of the period of one oscillation of the field χ , $\Delta t_* \sim \omega^{-1}$, in agreement with the uncertainty principle. One can easily identify these short intervals in Fig. 3.

In the semiclassical regime when the frequency $\omega_k(t)$ is changing adiabatically, n_k is a constant which coincides with an adiabatic invariant. To appreciate the usefulness of the introduction of the adiabatic invariant n_k , one should com-

¹In this paper we will use both physical momenta and comoving momenta. Our definition of k_* refers to physical momentum.

pare the evolution of the modes χ_k with the evolution of the occupation numbers corresponding to each of these modes shown in Figs. 2 and 3. As we see, in the narrow resonance regime χ_k vigorously oscillates, whereas $\ln n_k$ grows like a straight line. In the broad resonance regime the field amplification occurs near the points $\phi(t)=0$ where the process is not adiabatic. The occupation number n_k , being an adiabatic invariant, changes only during these short time intervals, when the number of particles is not well defined.

Analytical investigation of the broad resonance regime in the context of the theory of reheating was first reported in [1]; see also [20]. Now we are going to perform a much more detailed investigation of this regime.

VI. STOCHASTIC RESONANCE IN AN EXPANDING UNIVERSE

To understand why the broad resonance regime is so important for the theory of reheating in an expanding universe, let us remember that resonance in an expanding universe appears only for $q^2 m \geq H$, which in our case reads

$$g\Phi \geq 2m \left(\frac{H}{m} \right)^{1/4}. \quad (36)$$

In the simplest inflationary models including the model which we consider now the value of the Hubble constant at the end of inflation is of the same order as the inflaton mass m , but somewhat smaller. Indeed, as we already mentioned, during the first oscillation the amplitude of the field Φ is of order $M_p/20$, which gives the Hubble constant $H \sim \sqrt{2\pi/3} m \Phi / M_p \sim 0.1m$. Since dependence of the resonance condition on H is very weak ($H^{1/4}$), one may conclude that the regime of explosive reheating after inflation will occur only if the amplitude of oscillation satisfies the condition $\Phi > m/g$. Thus explosive decay ends at $\Phi \leq m/g$, i.e., at $q \leq 1/4$.

This means that preheating in this model cannot begin for $\Phi < m/g$, which would correspond to the narrow resonance regime. Narrow resonance may be important at the late stages of preheating [1], but at that stage one should take into account back reaction of the particles produced at the previous stage of broad parametric resonance, so the theory of the narrow resonance at the end of preheating is much more complicated than the one contained in the previous subsection.

In fact, efficient preheating often requires extremely large initial values of q . Indeed, the amplitude of the scalar field decreases during the expansion of the universe much faster than $H^{1/4}$, so for not very large initial values of q the condition (36) becomes violated before the resonance has enough time to transfer the energy of the oscillating field ϕ into the energy of χ particles. As we will show in Sec. IX, in the model under consideration preheating is efficient only if the initial value of q at the end of inflation is very large, $q_0 \geq 10^3$.

In the models with extremely large q the expansion of the universe makes preheating very peculiar: instead of a regular resonance process we encounter a rather unusual effect which we call *stochastic resonance*.

Let us first look at the results of the numerical study of the development of broad resonance in an expanding universe, and try to interpret them. Note that at this stage we do not consider the effects of back reaction and rescattering of particles; we will discuss these effects later. Our main strategy here is to study a general picture step by step, and then correct it later, because otherwise the physical interpretation of the processes which occur during preheating will remain obscure.

First of all, let us consider Eq. (21) for the mode χ_k in an expanding universe with $m_\chi^2 = 0$, $\xi = 0$ in the asymptotic regime when $a = (t/t_0)^{2/3}$, and $\Phi(t) = M_p / \sqrt{3\pi} m t$. Strictly speaking, the last two conditions are satisfied only for sufficiently large t . However, if we begin counting time from the end of inflation, taking for definiteness $t_0 = \pi/2m$ (which formally corresponds to the time after a quarter of one oscillation of the field ϕ), then we will have an approximation which is sufficiently good for our purposes. With these definitions, the initial values of the field ϕ and the parameter q in our calculations are given by

$$\phi_0 = \frac{2M_p}{\pi\sqrt{3}\pi} \sim \frac{M_p}{5}, \quad q_0^{1/2} = \frac{gM_p}{\pi\sqrt{3}\pi m} \sim \frac{gM_p}{10m}. \quad (37)$$

On the other hand, if one wants to investigate the situation numerically, one can simply solve a combined system of equations for $a(t)$, $\Phi(t)$, and $\chi_k(t)$. We will not do it here because our main goal is to develop an analytical approach to the theory of preheating.

The investigation of parametric resonance in an expanding universe can be simplified if instead of χ_k one introduces the function $X_k(t) = a^{3/2}(t)\chi_k(t)$, which is given by $(t/t_0)\chi_k(t)$ in our case. Then instead of Eq. (21) we have a much simpler equation

$$\ddot{X}_k + \omega_k^2 X_k = 0, \quad (38)$$

where

$$\omega_k^2 = \frac{k^2}{a^2(t)} + g^2 \Phi^2 \sin^2 m t + \Delta, \quad (39)$$

and $\Delta = m_\chi^2 - \frac{3}{4}(\dot{a}/a)^2 - \frac{3}{2}(\ddot{a}/a) - \xi R$. This term is usually very small. Indeed, we will consider here the case of light χ particles, such that $m_\chi \ll k_*$, in which case one can simply neglect m_χ . Also, soon after the end of inflation one has $H^2 = (\dot{a}/a)^2 \sim \ddot{a}/a \ll m^2$. As a result, typically one can neglect the term Δ altogether. Equation (38) describes an oscillator with a variable frequency $\omega_k^2(t)$ due to the time dependence of the background field $\phi(t)$ and $a(t)$. As an initial condition one should take the positive-frequency solution $X_k(t) \simeq e^{-i\omega_k t} / \sqrt{2\omega_k}$.

The series of three figures in this section shows different stages of development of the fastest growing mode χ_k in the broad resonance regime in an expanding universe in the theory $m^2 \phi^2/2$ for an initial value of the parameter $q \sim 3 \times 10^3$. Note that during the expansion of the universe the amplitude of scalar field oscillations decreases approximately as t^{-1} . Therefore in order to illustrate the *relative* growth of the fluctuations of the field χ with respect to the

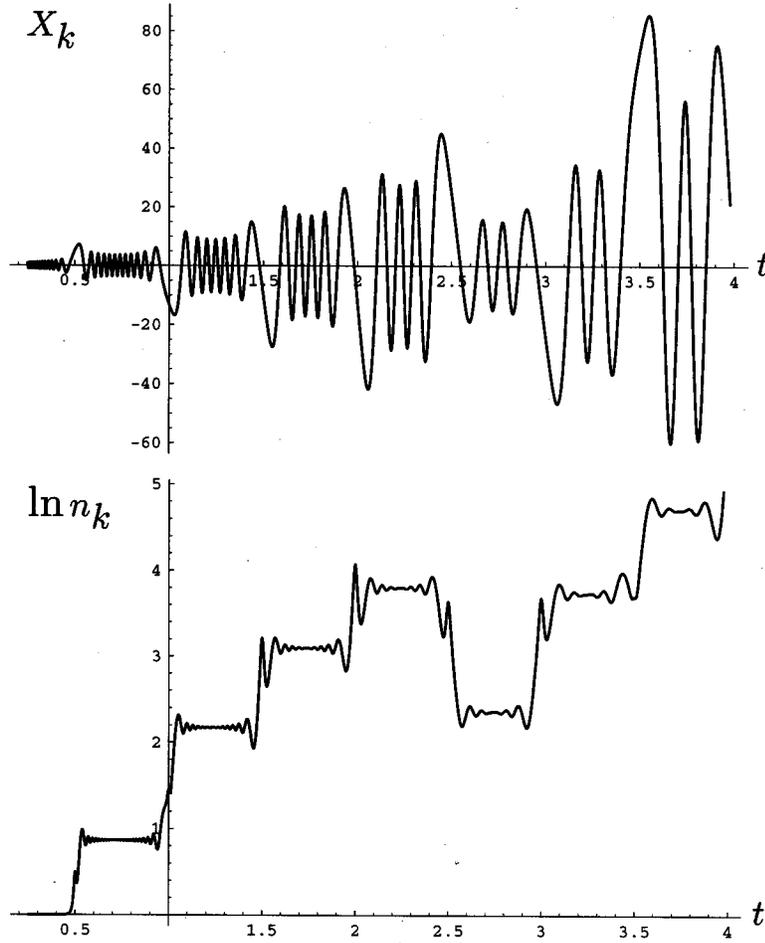


FIG. 4. Early stages of parametric resonance in the theory $m^2\phi^2/2$ in an expanding universe with scale factor $a \sim t^{2/3}$ for $g = 5 \times 10^{-4}$, $m = 10^{-6}M_p$. According to our conventions (37), initial value of the parameter q in this process is $q_0 \sim 3 \times 10^3$. Note that the number of particles n_k in this process typically increases, but it may occasionally decrease as well. This is a distinctive feature of stochastic resonance in an expanding universe. A decrease in the number of particles is a purely quantum mechanical effect which would be impossible if these particles were in a state of thermal equilibrium.

amplitude of the oscillating field ϕ we show not the growing mode χ_k itself, but its rescaled value $X_k = \chi_k(t/t_0)$, where t_0 corresponds to the beginning of the calculation. One can construct an adiabatic invariant for Eq. (38), which has an interpretation of the comoving occupation number of particles n_k in the mode k in an expanding universe:

$$n_k = \frac{\omega_k}{2} \left(\frac{|\dot{X}_k|^2}{\omega_k^2} + |X_k|^2 \right) - \frac{1}{2}. \quad (40)$$

Note that this function does not have any factors inversely proportional to the volume a^3 . These factors will appear when we calculate the number density of particles in physical (not comoving) coordinates.

In the beginning we have parametric resonance very similar to the one studied in the previous section; compare Fig. 3 and Fig. 4. As before, one can identify the periods when χ -particle production is most efficient with the intervals when the field ϕ becomes small. An important difference is that because of the gradual decrease in amplitude of the field ϕ the effective mass of the field χ and, correspondingly, the frequency of its oscillations decrease in time. As a result, in the beginning within each half of a period of oscillation of the field ϕ the field χ_k oscillates many times, but then it oscillates more and more slowly.

To understand the rather peculiar behavior of X_k and n_k in this process, let us check in which resonance band our process develops. The number of the band in the theory of the Mathieu equation is given by $n = \sqrt{A}$. In our case reheating occurs for $A \sim 2q$, i.e., $n \sim \sqrt{2q} \sim g\Phi/\sqrt{2}m$. Suppose we have an inflationary theory with $m \sim 10^{-6}M_p$, and let us take as an example $g \sim 10^{-1}$. Then after the first oscillation of the field, according to Eq. (4), we have $\Phi(t) \sim M_p/20$, which corresponds to $q \sim 10^8/16$. This gives the band number $n \sim 3 \times 10^3$. After another oscillation the amplitude of the field drops by a factor of 2, and the band number decreases by a factor of 2 as well, down to $n \sim 1.5 \times 10^3$.

In other words, even during a single oscillation the field does not remain in the same zone of the Mathieu equation. Instead of that it jumps over 10^3 different instability bands. The theory of a broad resonance in Minkowski space is much less explored than the theory of a narrow resonance, but the theory of a broad resonance in an expanding universe proves to be even more complicated. The standard method of investigation of resonance using the Mathieu equation in a single resonance band completely fails here.

Still not everything is lost. Indeed, as we have found in the previous section, in the broad resonance regime particle production occurs only in a small vicinity of $\phi = 0$, corresponding to integer and half-integer $N = mt/2\pi$. Nothing depends on the exact way the field ϕ behaves at all other mo-

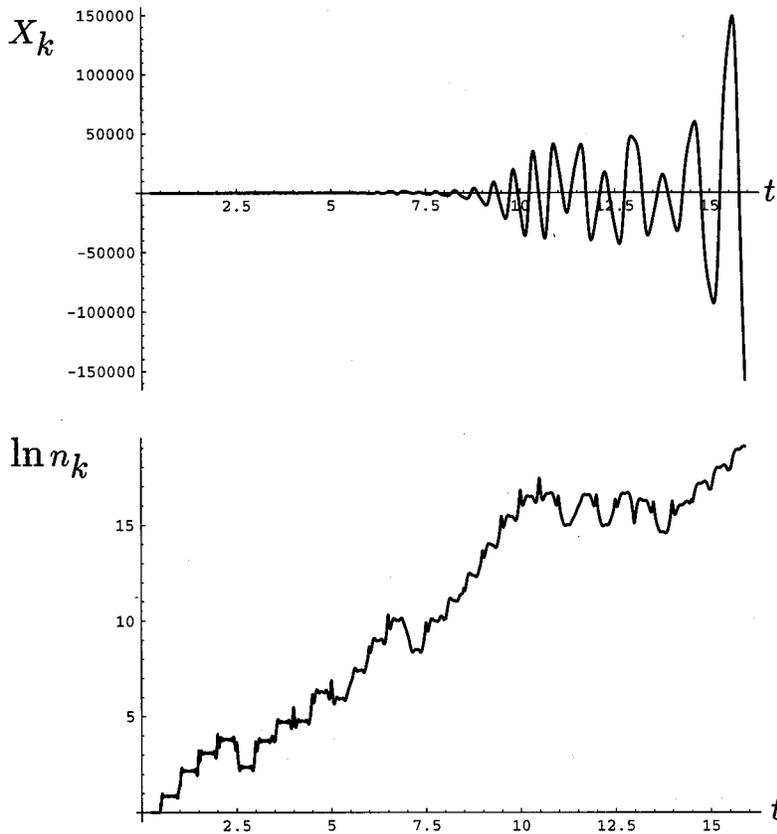


FIG. 5. The same process as in Fig. 4 during a longer period of time. The parameter $q = g^2 \Phi^2 / 4m^2$ decreases as t^{-2} during this process, which gradually makes the broad resonance more and more narrow. As before, we show time t in units of $2\pi/m$, which corresponds to the number of oscillations of the inflaton field.

ments. In this sense the description of the process of particle production at $\phi=0$ is very robust with respect to change in the shape of the potential $V(\phi)$ and of the equation describing the field χ , insofar as it does not alter the behavior of either field at the stage when $\phi(t)$ approaches zero. Therefore some (but not all) of the results related to the Mathieu equation can be useful for investigation of broad parametric resonance in an expanding universe even though the regime we are going to investigate is fundamentally different.

One of the most important differences between broad resonance in Minkowski space and in an expanding universe can be understood by inspecting the behavior of the *phase* of the functions χ_k near the points where $\phi(t)=0$. Indeed, Fig. 3 shows that near all points where $\phi=0$ the phases of χ_k are equal. The physical meaning of this effect is very simple: In order to open a swinging door by a small force one should apply it periodically, “in resonance” with the motion of the door.

However, in an expanding universe such a regime is impossible, not only because of the redshift of the momentum k/a , but mainly because the frequency of oscillations of the field χ_k is proportional to Φ , which decreases in time. The frequency of oscillations of the modes χ_k changes dramatically with each oscillation of the field ϕ . Therefore for large q the phases of the field χ_k at successive moments when $\phi(t)=0$ are practically uncorrelated with each other. Using our analogy, one may say that the door is vibrating with a large and ever changing frequency, so it is very difficult to push it at a proper moment of time, and successfully repeat it

many times in a row. That is why at some moments the amplitude of the field χ_k decreases; see Fig. 4.

This could suggest that broad parametric resonance in an expanding universe is simply impossible. Fortunately, this is not the case, for two main reasons. First of all, as we are going to show in the next section, even though the phases of the field χ_k at the moment when $\phi(t)=0$ in an expanding universe with $q \gg 1$ are practically unpredictable, in 75% of all events the amplitude of χ_k grows at that time. Moreover, even if it were not the case, and the amplitude would grow only in 50% of all events, the total number of χ particles would still grow exponentially. Indeed, as we will see, during each “creative moment” $\phi(t)=0$ in the broad resonance regime the number of particles at each mode may either decrease by a factor of ~ 10 , or grow by a factor of ~ 10 . Thus if we begin with 10 particles in each of the two modes, after the process we get 1 particle in the first mode and 100 particles in the second. Therefore the total number of particles in this example grows by more than a factor of 5. The theory of this effect is very similar to the theory of self-reproduction of an inflationary universe, where in most points the inflaton field rolls down, but those parts of the universe where it jumps up continue growing exponentially [2].

As a result, parametric resonance does take place. However, in order to describe it some new methods of investigation of parametric resonance should be developed. We will do this in the next section.

Stochastic resonance occurs only during the first part of the process, when the effective parameter q is very large and the resonance is very broad. Gradually the amplitude of the

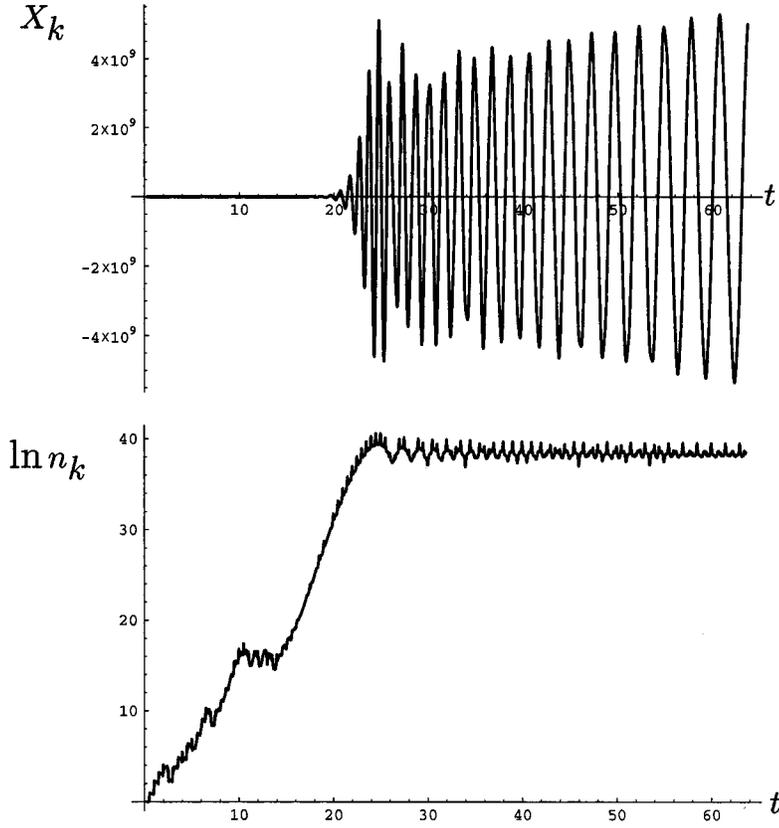


FIG. 6. The same process during a longer time, which is shown in the units $mt/2\pi$, corresponding to the number of oscillations N . The figures show the growth of the mode X_k for the momentum k corresponding to the maximal speed of growth of n_k . In this particular case $k \sim 4m$. Towards the end of this period, after approximately 25 oscillations of the inflaton field, the resonance ceases to exist, and the occupation number n_k becomes constant.

field ϕ decreases, which makes q smaller. Expansion of the universe slows down, the field stays in each resonance band for a longer time, and eventually the standard methods of investigation based on the Mathieu equation become useful again. As we will show in Sec. VII E, stochastic resonance ends and the standard methods become useful after the first $q_0^{1/2}/\sqrt{2\pi}$ oscillations, which may happen even before the effective parameter q decreases from $q_0 \gg 1$ to $q \sim 1$; see Eq. (79). One of the manifestations of this effect is a short plateau for $\ln n_k$ which appears in Fig. 5 for $10 \leq t \leq 15$. This plateau corresponds to the time when the stochastic resonance ends, and the mode X_k appears in the region of stability, which divides the second and the first instability band of the Mathieu equation; see Fig. 7.

To get a better understanding of this effect one should continue our calculations for a longer period of time, see Fig. 6. At $t > 15$ the process does not look like a broad resonance anymore, but the amplitude still grows exponentially at a rather high rate until the amplitude of the field Φ becomes smaller than m/g , which corresponds to $q \sim 1/3 - 1/4$. Soon after that the resonance ceases to exist and the amplitude stabilizes at some constant value.

The time t_f and the number of oscillations N_f at the end of parametric resonance in an expanding universe can be estimated by finding the moment when $g\Phi \approx gM_p/3mt$ is equal to m :

$$t_f \approx \frac{gM_p}{3m^2}, \quad N_f \approx \frac{gM_p}{6\pi m}. \quad (41)$$

As one can check, this estimate for our case ($m = 10^{-6}M_p$, $g = 5 \times 10^{-4}$) gives $N_f \sim 26.5$, which is in good agreement

with the results of our computer calculations shown in Fig. 6. A small disagreement (about 10%) appears because our criterion for the end of the resonance $g\Phi \sim m$ was not quite precise: the resonance ends somewhat earlier, at $g\Phi \sim 1.1m$.

This more exact result can be deduced from Fig. 7, which shows that the first instability band for $k=0$ extends from $q \sim 0.8$ to $q \sim 1/3$. Therefore the growth of all modes with

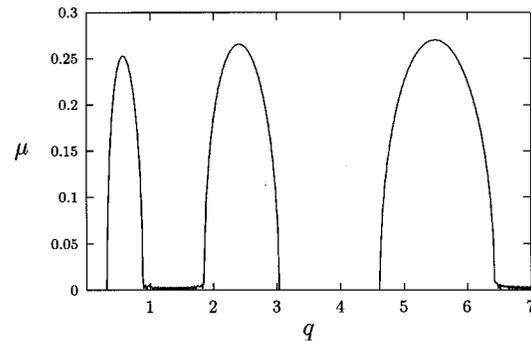


FIG. 7. The structure of the resonance bands for the Mathieu equation along the line $A=2q$, which correspond to excitations with $k=0$ in our model. The modes with small k are especially interesting because the momenta of the excitations are redshifted during the expansion of the universe. A small plateau at $10 \leq t \leq 15$ on Fig. 5 corresponds to the time where stochastic resonance ceases to exist, all modes are redshifted to small k , and the system spends some time in the interval with $1 \leq q \leq 2$, which is outside the instability zone. The last stage of the resonance shown in Fig. 6 corresponds to the resonance in the first instability band with $q < 1$.

$k \ll m$ terminates not at $g^2 \Phi^2 / 4m^2 \sim 1/4$, but slightly earlier, at $g^2 \Phi^2 / 4m^2 \sim 1/3$.

At the time $t \sim t_f/2$ one has $q \sim 1$. During the time from $t_f/2$ to t_f the resonance occurs in the first resonance band, the resonance is not very broad and there are no stochastic jumps from one resonance band to another. At the time just before $t_f/2$ there was no resonance; the field was in the stability band between $q=1$ and $q=2$; see Fig. 7.

An interesting effect which is shown in Fig. 6 is a slow growth of the amplitude X_k which continues even after the resonance terminates and n_k becomes constant. This happens because the momentum of each mode gradually becomes smaller due to the expansion of the universe, and this leads to a growth of χ_k even though n_k does not change. This is one of the examples which shows that in order to describe parametric resonance one should use proper variables such as n_k , because otherwise one may get the incorrect idea that the resonance continues even for $t > 25$.

If one ignores a small island of stability near $t \sim 12$, one may conclude that during the main part of the process the slope of the curve $\ln n_k$ remains almost constant. In our case this corresponds to the exponential growth of the occupation number n_k with an effective parameter $\mu_k \sim 0.13$. This fact will be very useful for us later, when we will calculate the number of particles produced during the parametric resonance. Such a calculation is our main goal. It is also necessary in order to verify whether one should modify our resonance equations due to the presence of χ particles. As we will see, no modifications are needed for theories with $g \lesssim 3 \times 10^{-4}$. However, for greater values of g (and in particular for the case of $g \sim 5 \times 10^{-4}$ discussed above) the resonance ends in a somewhat different way; see Sec. IX B.

In order to illustrate the stochastic nature of the resonance in this theory, we will present here at sample of results for the resonance for several different values of the coupling constant g in the interval from 0.9×10^{-4} to 10^{-3} . One might expect the results to change monotonically as g changes in this interval. However, this is not the case. The table contains the results concerning the initial momentum k (in units of m) corresponding to the fastest growing mode, the total increase of the number of particles $\ln n_k$ at the end of the resonance for this mode, the average value μ for this mode, and the time t_f (the number of oscillations of the field ϕ) at the end of the resonance:

g	k	μ	t_f	$\ln n_k$
0.9×10^{-4}	1.5	0.1	5	6
10^{-4}	2	0.14	5	9
1.1×10^{-4}	0.5	0.17	5.5	12
1.2×10^{-4}	1.5	0.12	6	9
1.3×10^{-4}	1	0.13	6.5	11
1.4×10^{-4}	2	0.12	7	11
1.5×10^{-4}	0.5	0.18	7	17
2×10^{-4}	3.5	0.12	11	16
3×10^{-4}	0.5	0.14	14	27
5×10^{-4}	4	0.13	24	40
10^{-3}	6	0.12	48	75

Thus we see that the leading mode in this interval of the

coupling constant has initial momentum comparable to m and slightly smaller than the typical initial width of the resonance $k_*/2$, which changes from $2m$ to about $5m$ for g changing from 10^{-4} to 10^{-3} . The reason why k is usually (though not always) somewhat smaller than $k_*/2$ is very simple. The resonance is broad only during the first half of the time. Narrow parametric resonance which appears during the second part of preheating typically is more efficient for smaller k . We should note that for $g \gtrsim 3 \times 10^{-4}$, at the last stage of preheating one should take into account back reaction of produced particles, which makes the narrow resonance stage very short; see Sec. IX B. In such a case the resonance has the width $k_*/2$ in terms of the value of the momentum k at the beginning of preheating.

Of course, investigation of the leading growing mode is insufficient: One should integrate over all modes with all possible k , which we are going to do later. However, the number of particles n_k is exponentially sensitive to k . Therefore the main contribution to the integral will be given by the trajectories close to the leading one. It is similar to what happens, e.g., in the theory of tunneling, where one first finds the optimal trajectory corresponding to the minimum of action, and calculates e^{-S} along this trajectory. Similarly, one can calculate the rate of growth of the total number of χ particles by finding the leading trajectory and calculating the average value of μ along the trajectory.

The table clearly demonstrates that the effective values of μ and especially the final number of particles n_k produced by the resonance are extremely sensitive to even very small modifications of g , and change in a rather chaotic way even when g changes by only 10%. That is why we call this process ‘‘stochastic resonance.’’ We see from the table that for $g \sim 10^{-3}$ the occupation numbers n_k become incredibly large. It will be shown in Sec. IX that for $g \sim 10^{-4}$ back reaction of created particles is not very important, but for $g \gtrsim 3 \times 10^{-4}$ back reaction becomes crucial, because it does not allow the resonance to produce an indefinitely large number of particles. To investigate these issues we should first develop the theory of stochastic resonance, and then take into account back reaction.

VII. ANALYTIC THEORY OF STOCHASTIC RESONANCE

In this section we are going to develop a new method to study the time evolution of the eigenfunctions $\chi_k(t)$ in the most interesting case of broad resonance. This method is based on the crucial observation made in the previous sections: In the broad resonance regime the evolution of the modes $\chi_k(t)$ is adiabatic and the number of particles does not grow in the intervals when $|\phi(t)| > \phi_*$. The number of particles changes only in the short intervals when $|\phi(t)| \lesssim \phi_* \ll \Phi$.

The quantum field theory of particle creation in a time varying background is naturally formulated in terms of adiabatic (semiclassical) eigenfunctions. This formalism is introduced in the next subsection. Then we will find the change of the particle number density from a single kick, when $\phi(t)$ crosses zero at some time t_j . For this purpose it is enough to consider the evolution of $\chi_k(t)$ in the interval when $\phi^2(t)$ is very small, so it can be represented by its quadratic part $\propto (t-t_j)^2$. This process looks like wave propagation in a

time dependent parabolic potential. We can combine the action of the subsequent parabolic potentials to find the net effect of the particle creation. Using our formalism, we consider a toy model of broad resonance in Minkowski space, and broad resonance in an expanding universe, which turns out to have a stochastic nature.

A. Adiabatic representation of the eigenfunctions

The semiclassical, or adiabatic evolution of the eigenfunction $\chi_k(t)$ can be represented in a specific mathematical form. For this we adopt a physically transparent method to treat Eq. (38) for an arbitrary time dependence of the classical background field which was originally developed by Zel'dovich and Starobinsky [37] for the problem of particle creation in a varying gravitational field.

Let us represent solutions of Eq. (38) as products of its solution in the adiabatic approximation, $\exp(\pm i \int f dt \omega)$, and some functions $\alpha(t)$ and $\beta(t)$:

$$a^{3/2} \chi_k(t) \equiv X_k(t) = \frac{\alpha_k(t)}{\sqrt{2\omega}} e^{-i \int^t \omega dt} + \frac{\beta_k(t)}{\sqrt{2\omega}} e^{+i \int^t \omega dt}. \quad (42)$$

An additional condition on the functions α and β can be imposed by taking the derivative of Eq. (42) as if α and β were time-independent. Then Eq. (42) is a solution of Eq. (38) if the functions α_k, β_k satisfy the equations

$$\dot{\alpha}_k = \frac{\dot{\omega}}{2\omega} e^{+2i \int^t \omega dt} \beta_k, \quad \dot{\beta}_k = \frac{\dot{\omega}}{2\omega} e^{-2i \int^t \omega dt} \alpha_k. \quad (43)$$

In terms of classical waves of the χ field, quantum effects occur due to departure from the initial positive-frequency solution; therefore the initial conditions at $t \rightarrow 0$ are $\alpha_k = 1$, $\beta_k = 0$. Normalization gives $|\alpha_k|^2 - |\beta_k|^2 = 1$.

The coefficients $\alpha_k(t)$ and $\beta_k(t)$ in our case coincide with the coefficients of the Bogoliubov transformation of the creation and annihilation operators, which diagonalizes the Hamiltonian of the χ field at each moment of time t . The particle occupation number is $n_k = |\beta_k|^2$; see Eq. (40). The vacuum expectation value for the particle number density per comoving volume is

$$\langle n_\chi \rangle = \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 |\beta_k|^2. \quad (44)$$

In this section we will calculate β_k , n_k , and $\langle n_\chi \rangle$ in the nonperturbative regime of broad resonance, where all of these values can be very large.

It is instructive to return in the framework of this formalism to the simpler perturbative regime which we discussed earlier in Sec. III. Assuming $|\beta_k| \ll 1$, from Eqs. (43) one can obtain an iterative solution:

$$\beta_k \approx \frac{1}{2} \int_0^t dt' \frac{\dot{\omega}}{\omega} \exp\left(-2i \int_0^{t'} dt'' \omega(t'')\right). \quad (45)$$

Using $\omega(t) = \sqrt{(k/a)^2 + g^2 \Phi^2 \sin^2 mt}$, we can evaluate Eq. (45) containing an oscillating integrand by the method of stationary phase [7]. In the case of the massive scalar field decaying via the interaction $g^2 \sigma \phi \chi^2$, the dominant contribu-

tion is given by the integration near the moment t_k , where $a(t_k) = 2k/m$. As we already mentioned, this corresponds to the creation of a pair of massless χ particles with momentum $k = \frac{1}{2} a(t_k) m$ from an inflaton with mass (energy) m at the instant t_k of the resonance between the mode k and the background field. The decay rate of the inflaton field calculated with this method can be described by Eq. (12).

For the interaction $\phi \phi \rightarrow \chi \chi$, the process in the regime $|\beta_k| \ll 1$ can be interpreted as creation of a pair of χ particles with momentum $k = a(t_k) m$ from a pair of massive inflatons with energy m each. The decay rate of the massive inflaton field in this case rapidly decreases with the expansion of the universe as $(1/a^4)(d/dt)(a^4 \rho_\chi) \propto a^{-6}$. Therefore a complete decay of the massive inflaton field in the theory with the $\phi^2 \chi^2$ interaction is impossible. One should have additional terms such as $g^2 \sigma \phi \chi^2$ or $h \bar{\psi} \psi \phi$. This is a very important conclusion which we already discussed in Sec. III.

B. Interpretation of parametric resonance in terms of successive scattering on parabolic potentials

We suggest a new analytic method to solve approximately the basic equations (21) and (38) for the eigenfunctions χ_k which correspond to the χ particles created by the oscillating inflaton field $\phi(t)$. This method is rather general; it can be applied to many models of preheating. One may also apply it to the idealized case when the universe does not expand and back reaction is not taken into account. In the cases where the equation for the modes χ_k can be reduced to an equation with periodic coefficients (including the Mathieu equation), our method accurately reproduces the solution of this equation, and gives us an interesting insight into the physics of parametric resonance. This method is rather powerful; it enables one to investigate some features of the regime of broad parametric resonance which, to the best of our understanding, have not been known before.

In the realistic situation which we study in this paper, when the expansion of the universe as well as the back reaction are taken into account, in some models (e.g., nonconformal theory) the equation for the modes χ_k cannot be considered as an equation with periodic coefficients, and the analysis based on standard stability/instability charts is not applicable. This is the situation where our method will be especially useful.

Let us consider the general equation (38). As we noticed, the eigenfunction $X_k(t)$ has adiabatic evolution between the moments t_j , $j = 1, 2, 3, \dots$, where the inflaton field is equal to zero $\phi(t_j) = 0$ (i.e., twice within a period of inflaton oscillation). The nonadiabatic changes of $X_k(t)$ occur only in the vicinity of t_j . Therefore we expect that the semiclassical solution (42) of Eq. (38) is valid everywhere but around t_j . Let the wave $X_k(t)$ have the form of the adiabatic solution (42) before the scattering at the point t_j :

$$X_k^j(t) = \frac{\alpha_k^j}{\sqrt{2\omega}} \exp\left(-i \int_0^t \omega dt\right) + \frac{\beta_k^j}{\sqrt{2\omega}} \exp\left(+i \int_0^t \omega dt\right), \quad (46)$$

the coefficients α_k^j and β_k^j are constant for $t_{j-1} < t < t_j$. Then after the scattering, $X_k(t)$, within the interval $t_j < t < t_{j+1}$, has the form

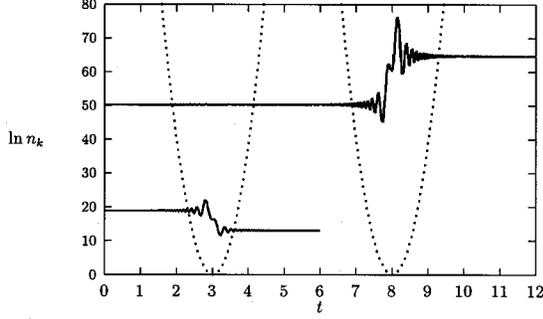


FIG. 8. The change of the comoving particle number n_k due to scattering at the parabolic potential, calculated from Eq. (50). The dotted lines show the sequence of the parabolic potentials $g^2\phi^2(t) \approx g^2\Phi^2 m^2(t-t_j)^2$ where scattering occurs. Time is given in units of $2\pi/\kappa$. The number of particles can either increase or decrease at the scattering, depending on the phase of the incoming wave.

$$X_k^{j+1}(t) = \frac{\alpha_k^{j+1}}{\sqrt{2\omega}} \exp\left(-i \int_0^t \omega dt\right) + \frac{\beta_k^{j+1}}{\sqrt{2\omega}} \exp\left(+i \int_0^t \omega dt\right), \quad (47)$$

and the coefficients α_k^{j+1} and β_k^{j+1} are constant for $t_j < t < t_{j+1}$.

Equations (46) and (47) are essentially the asymptotic expressions for the incoming waves (for $t < t_j$) and for the outgoing waves (for $t > t_j$), scattered at the moment t_j . Therefore the outgoing amplitudes α_k^{j+1} , β_k^{j+1} can be expressed through the incoming amplitudes α_k^j , β_k^j with help of the reflection R_k and transmission D_k amplitudes of scattering at t_j :

$$\begin{pmatrix} \alpha_k^{j+1} e^{-i\theta_k^j} \\ \beta_k^{j+1} e^{+i\theta_k^j} \end{pmatrix} = \begin{pmatrix} 1 & R_k^* \\ D_k & D_k^* \end{pmatrix} \begin{pmatrix} \alpha_k^j e^{-i\theta_k^j} \\ \beta_k^j e^{+i\theta_k^j} \end{pmatrix}. \quad (48)$$

Here $\theta_k^j = \int_0^{t_j} \omega(t) dt$ is the phase accumulated by the moment t_j .

Now we specify the scattering at the moment t_j . The interaction term $g^2\phi^2(t)$ in Eq. (38) has a parabolic form around all the points t_j : $g^2\phi^2(t) \approx g^2\Phi^2 m^2(t-t_j)^2 \equiv k_*^4(t-t_j)^2$, where the current amplitude of the fluctuations Φ is defined in Eq. (4), and the characteristic momentum $k_* = \sqrt{g\Phi m}$. In the general case k_* depends on time via the time dependence of $\Phi \propto a^{-3/2}$. Figure 8 illustrates two possible outcomes of the scattering of the wave $X_k(t)$ on the parabolic potential near zeros of the function $g^2\phi^2(t)$. Depending on the phase of the incoming wave, the corresponding number of particles may either decrease or grow.

First, let us consider the mode equation around a single parabolic potential. In the vicinity of t_j the general equation (38) is transformed to the equation

$$\frac{d^2 X_k}{dt^2} + \left(\frac{k^2}{a^2} + g^2\Phi^2 m^2(t-t_j)^2 \right) X_k = 0. \quad (49)$$

For simplicity we introduce a new time variable $\tau = k_*(t-t_j)$ and a scaled momentum $\kappa = k/ak_*$. Notice that $\kappa^2 = (A_k - 2q)/2\sqrt{q}$. In general, k_* and κ depend on t_j through $a(t_j)$, and should be marked by the index j , which we drop for the moment. Then Eq. (49) for each j is reduced to the simple equation

$$\frac{d^2 X_k}{d\tau^2} + (\kappa^2 + \tau^2) X_k = 0. \quad (50)$$

The asymptote of this equation, which corresponds to the incoming wave, matches the form (46). The asymptote corresponding to the outgoing wave matches the form (47). Therefore the reflection R_k and transmission D_k amplitudes of scattering at t_j are essentially the reflection and transmission amplitudes of scattering at the parabolic potential. Thus the problem is reduced to the well-known problem of wave scattering at a (negative) parabolic potential [38], which we consider in the next subsection.

C. Particle creation by parabolic potentials

A general analytic solution of Eq. (50) is the linear combination of the parabolic cylinder functions [39]: $W(-\kappa^2/2; \pm\sqrt{2}\tau)$. The reflection R_k and transmission D_k amplitudes for scattering on the parabolic potential can be found from these analytic solutions:

$$R_k = -\frac{ie^{i\varphi_k}}{\sqrt{1+e^{\pi\kappa^2}}}, \quad (51)$$

$$D_k = \frac{e^{-i\varphi_k}}{\sqrt{1+e^{-\pi\kappa^2}}}, \quad (52)$$

where the angle φ_k is

$$\varphi_k = \arg\Gamma\left(\frac{1+i\kappa^2}{2}\right) + \frac{\kappa^2}{2} \left(1 + \ln\frac{2}{\kappa^2}\right). \quad (53)$$

The angle φ depends on the momentum k . Notice the following properties of these coefficients: $R_k = -iD_k e^{-(\pi/2)\kappa^2}$; $|R_k|^2 + |D_k|^2 = 1$. Substituting Eqs. (51) and (52) into Eq. (48), we can obtain the evolution of α_k^j , β_k^j amplitudes from a single parabolic scattering in terms of the parameters of the parabolic potential and the phase θ_k^j only.

The mapping of α_k^j , β_k^j into α_k^{j+1} , β_k^{j+1} reads as

$$\begin{pmatrix} \alpha_k^{j+1} \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} \sqrt{1+e^{-\pi\kappa^2}} e^{i\varphi_k} & ie^{-(\pi/2)\kappa^2+2i\theta_k^j} \\ -ie^{-(\pi/2)\kappa^2-2i\theta_k^j} & \sqrt{1+e^{-\pi\kappa^2}} e^{-i\varphi_k} \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix}. \quad (54)$$

Since the number density of χ particles with momentum \mathbf{k} is equal to $n_k = |\beta_k(t)|^2$, from Eq. (54) one can calculate the number density of outgoing particles $n_k^{j+1} = |\beta_k^{j+1}|^2$ after the scattering on the parabolic potential out of $n_k^j = |\beta_k^j|^2$ incoming particles:

$$n_k^{j+1} = e^{-\pi\kappa^2} + (1 + 2e^{-\pi\kappa^2})n_k^j - 2e^{-(\pi/2)\kappa^2} \sqrt{1 + e^{-\pi\kappa^2}} \sqrt{n_k^j(1 + n_k^j)} \sin\theta_{\text{tot}}^j, \quad (55)$$

where the phase $\theta_{\text{tot}}^j = 2\theta_k^j - \varphi_k + \arg\beta_k^j - \arg\alpha_k^j$.

Before we apply the formalism (48) and (55) to specific models, we shall analyze these generic equations. Although we did not specify yet the phase θ_{tot}^j , we already can learn a lot from the form (55). First of all, the number of created particles is a steplike function of time. The value of n_k^j is a constant between two successive scatterings at points t_j and t_{j+1} . The number of particles is changed exactly at the instances t_j in a steplike manner, in full agreement with the exact numerical solution; see Fig. 4. The effect of particle creation is significant if $\pi\kappa^2 \lesssim 1$, otherwise the exponential term $e^{-\pi\kappa^2}$ suppresses the effect of particle accumulation. This gives us the important general criterion for the width of the resonance band [1]:

$$\kappa^2 = \frac{A - 2q}{2\sqrt{q}} \leq \pi^{-1}, \quad (56)$$

where $A = k^2/a^2m^2 + 2q$, $q = g^2\Phi^2/4m^2$. Equivalently, one can write this condition in the form

$$\frac{k^2}{a^2} \leq k_*^2/\pi = gm\Phi/\pi. \quad (57)$$

This estimate of the resonance width $k \leq k_*/\sqrt{\pi}$ practically coincides with the estimate $k \leq k_*/2$ (34) derived in Sec. V by elementary methods.

Next, let us consider the large occupation number limit, $n_k \gg 1$. From Eq. (55) we derive

$$n_k^{j+1} \approx (1 + 2e^{-\pi\kappa^2} - 2\sin\theta_{\text{tot}}^j e^{-(\pi/2)\kappa^2} \sqrt{1 + e^{-\pi\kappa^2}}) n_k^j. \quad (58)$$

The factor in the right-hand side of this equation depends on the coupling constant g through $\kappa^2 \propto g^{-1}$. This dependence has the structure $\exp(-1/g)$, which is a nonanalytic function of g at $g=0$. Therefore the number of particles generated in the broad resonance regime cannot be derived using a perturbative series with respect to coupling parameter g . Thus formula (58) clearly manifests the nonperturbative nature of the resonance effects.

The growth index μ_k is defined by the formula

$$n_k^{j+1} = n_k^j \exp(2\pi\mu_k^j). \quad (59)$$

Comparing Eqs. (59) and (58) we find

$$\mu_k^j = \frac{1}{2\pi} \ln(1 + 2e^{-\pi\kappa^2} - 2\sin\theta_{\text{tot}}^j e^{-(\pi/2)\kappa^2} \sqrt{1 + e^{-\pi\kappa^2}}). \quad (60)$$

The first two terms in Eq. (58) correspond to the effect of spontaneous particle creation, which always increases the number of particles. The last term corresponds to induced particle creation, which can either increase or decrease the number of particles. At first glance it looks paradoxical that

the number of particles n_k^j created from the time-varying external field cannot only increase but sometimes decrease, i.e., the growth index μ_k can be not only positive but sometimes negative. Indeed, it is well known that if the $|in\rangle$ -state of the quantum field χ corresponds to n particles, then the number of particles in the $|out\rangle$ -state due to the interaction with the external field will always be greater than n . This is how to resolve the paradox: the particles created from the vacuum by the time-varying external field are not in the n -particle $|in\rangle$ -state but are in the squeezed $|in\rangle$ -state. In this case the interference of the wave functions can lead to a decrease of the particle number.

The whole effect of the particle production crucially depends on the interference of the wave functions, i.e., the phase correlation/anticorrelation between successive scatterings at the parabolic potentials. The maximal value of μ is reached for positive interference when $\sin\theta_{\text{tot}} = -1$ and is equal to $\mu = (1/\pi)\ln(1 + \sqrt{2}) \approx 0.28$; see also [1,20]. The typical value of μ corresponds to $\sin\theta_{\text{tot}} = 0$ and is equal to $\mu = (1/2\pi)\ln 3 \approx 0.175$. The value of μ is negative for negative interference when $\sin\theta_{\text{tot}} = 1$. Therefore the behavior of the resonance essentially depends on the behavior of the phase θ_k^j as a function of k for different time intervals j ; see Fig. 8. In the case of a fixed amplitude of the background field $\Phi(t) = \text{const}$ and $a(t) = \text{const}$, the phases θ_k^j do not depend on time but only on k . In this case we expect the existence of separate stability and instability k bands. However, this separation is washed out as soon as the phases θ_k^j are significantly varying with time due to changes in the parameters of the background field, for instance, in $\Phi(t)$ and $a(t)$.

Now we estimate the net effect of particle creation after a number of oscillation of the inflaton field. Equations (48) and (55) are recurrence relations for the α_k^j and β_k^j coefficients and for the number of particles n_k^j after successive actions of the parabolic potentials centered at t_1, t_2, \dots . To find the number of particles created up to the moment t_j , one has to repeat the formulas j times for the initial values $\alpha_k^0 = 1$, $\beta_k^0 = 0$, $n_k^0 = 0$ and a random initial phase θ_k^0 .

After a number of inflaton oscillations, the occupation number of χ particles is

$$n_k(t) = \frac{1}{2} \exp\left(2\pi \sum_j \mu_k^j\right) \approx \frac{1}{2} \exp\left(2m \int^t dt \mu_k(t)\right), \quad (61)$$

where we convert the sum over μ_k^j to an integral over $\mu_k(t)$. In some cases the index $\mu_k(t)$ does not depend on time. In a more general case one can replace $\mu_k(t)$ by an effective index μ_k^{eff} defined by the relation $\int^t dt \mu_k(t) = \mu_k^{\text{eff}} t$, which, for brevity, we will write simply as $\mu_k t$. Then the total number density of created particles is given by

$$n_\chi(t) = \frac{1}{(2\pi a)^3} \int d^3k n_k(t) = \frac{1}{4\pi^2 a^3} \int dk k^2 e^{2m\mu_k t}. \quad (62)$$

The function μ_k has a maximum $\mu_{\text{max}} \equiv \mu$ at some $k = k_m$. The integral (62) can be evaluated by the steepest descent method:

$$n_\chi(t) \approx \frac{1}{4\pi^2 a^3} \frac{k_m^2 e^{2\mu mt}}{\sqrt{2\pi mt \mu_k''}} \approx \frac{1}{8\pi^2 a^3} \frac{\Delta k k_m^2 e^{2\mu mt}}{\sqrt{\pi \mu mt}}, \quad (63)$$

where μ_k'' is the second derivative of the function μ_k at $k=k_m$ which we estimated as $\mu_k'' \sim 2\mu/\Delta k^2$, Δk being the width of the resonance band. Thus the effect of particle creation is defined by the leading value of the growth index μ , by the leading momentum k_m and by the width of the resonance band Δk . In practice typically $k_m \sim \Delta k \sim k_*/2$, so we can use an estimate

$$n_\chi(t) \sim \frac{k_*^3}{64\pi^2 a^3 \sqrt{\pi \mu mt}} e^{2\mu mt}. \quad (64)$$

In order to calculate $n(t)$ one should find the values of the parameters μ and k_* .

In what follows in this section we will apply the general formalism of successive parabolic potentials first to the toy model without the expansion of the universe, where $a(t)=\text{const}$ and $\Phi(t)=\text{const}$, in the case of broad resonance, $q \gg 1$. We will find the resonance zones and the number of particles which would be created in such a model. Then we consider a realistic case with the expansion of the universe taken into account. It turns out that the resonance in an expanding universe is very different from that without expansion.

D. Broad parametric resonance without expansion of the universe

Let us apply the general formalism of the previous subsection to the toy model neglecting the expansion of the universe. This is equivalent to taking $a(t)=1$. Thus, we will study the evolution of the eigenfunctions in the case with fixed values of the background parameters and without back reaction of created particles. In this case Eq. (21) is reduced to the standard Mathieu equation (23) with $A_k = k^2/m^2 + 2q$, $q = g^2 \phi^2/4m^2$, $z = mt$.

As we saw in Sec. VI, for the realistic situation with the expansion of the universe the Mathieu equation is applicable only at the last stages of the resonance when $q \leq 1$. For $q \gg 1$ this equation has only a heuristic meaning for our problem.

For the Mathieu equation with a large value of q (which is a constant in this subsection) we have the broad resonance regime. In this case the parameters κ^2 and φ_k of matrix (54) are time-independent, i.e., they are the same for different j . The phase θ_k^j is simple: $\theta_k^j = \theta_k j$. Here $\theta_k = \int_{t_{j-1}}^{t_j} dt \omega_k$ is the phase accumulating between two successive zeros of $\phi(t)$, i.e., within one half of a period of the inflaton oscillations, π/m , so that $\theta_k = \int_0^{\pi/m} dt \omega_k$. To find α_k^j and β_k^j we have to apply the same matrix (54) j times. We are mainly interested in the regime with a large number of created particles, $n_k^j = |\beta_k^j|^2 \gg 1$. In this regime $|\alpha_k^j| \approx |\beta_k^j|$, so α_k^j and β_k^j are distinguished by their phases only. In this case there is a simple solution of the matrix Eq. (54) for an arbitrary j :

$$\alpha_k^j = \frac{1}{\sqrt{2}} e^{(\pi\mu_k + i\theta_k)j}, \quad (65)$$

$$\beta_k^j = \frac{1}{\sqrt{2}} e^{i\vartheta} e^{(\pi\mu_k - i\theta_k)j}, \quad (66)$$

where ϑ is a constant phase. In principle, it is possible to construct not only the asymptotic solution (65), (66), but the general solution which starts with $\beta_k^0 = 0$. However, the general solution very quickly converges to the simple solution (65), (66), which contains all the physically relevant information. From Eq. (66) the number of particles created by the time $t \approx \pi j/m$ is

$$n_k = \frac{1}{2} e^{2\pi\mu_k j} = \frac{1}{2} e^{2\mu_k m t}, \quad (67)$$

where μ_k from Eqs. (65), (66) is indeed the growth index. Substituting the solution (65), (66) into Eq. (54), we get a complex equation for the parameters μ_k and θ_k :

$$e^{(\pi\mu_k + i\theta_k)} = \sqrt{1 + e^{-\pi\kappa^2}} e^{-i\varphi_k} + i e^{-(\pi/2)\kappa^2 - i\vartheta}. \quad (68)$$

Along side the solution (65), (66), there is another asymptotic solution of the matrix equation (54):

$$\alpha_k^j = \frac{1}{\sqrt{2}} e^{(\pi\mu_k + i\theta_k + i\pi)j}, \quad (69)$$

$$\beta_k^j = \frac{1}{\sqrt{2}} e^{i\vartheta} e^{(\pi\mu_k - i\theta_k - i\pi)j}, \quad (70)$$

with the condition

$$-e^{(\pi\mu_k + i\theta_k)} = \sqrt{1 + e^{-\pi\kappa^2}} e^{+i\varphi_k} + i e^{-(\pi/2)\kappa^2 - i\vartheta}. \quad (71)$$

Excluding the phase ϑ from the complex equations (68) and (71), it is easy to find a single equation for the growth index μ_k valid for both solutions:²

$$e^{\pi\mu_k} = |\cos(\theta_k - \varphi_k)| \sqrt{1 + e^{-\pi\kappa^2}} + \sqrt{(1 + e^{-\pi\kappa^2}) \cos^2(\theta_k - \varphi_k) - 1}. \quad (72)$$

In the instability bands, the parameter μ_k in Eq. (72) should be real. Therefore, the condition for the momentum k to be in the resonance band is $\cos(\theta_k - \varphi_k) \geq 1/\sqrt{(1 + e^{-\pi\kappa^2})}$, or

$$|\tan(\theta_k - \varphi_k)| \leq e^{-(\pi/2)\kappa^2}. \quad (73)$$

To further analyze the constraints on the width (73) and strength (72) of the resonance, we should find how the phases θ_k and φ_k depend on the momentum k . The angle φ_k as a function of k is defined by Eq. (53). For the phase θ_k we have

²Notice that the number of particles calculated with Eq. (72) is in agreement with the general formula (58). From the definition of $\theta_{k,\text{tot}}$ and the solutions α_k^j and β_k^j we have $\theta_{k,\text{tot}} = \varphi_k - \vartheta$. Therefore from the complex equations (68) and (71) we have additionally that $\cos(\theta_{k,\text{tot}}) = \sqrt{1 + e^{-\pi\kappa^2}} \sin(\theta_k - \varphi_k)$.

$$\begin{aligned}\theta_k &= \int_0^{\pi/m} dt \sqrt{k^2 + g^2 \phi^2(t)} \approx \frac{2g\Phi}{m} + \frac{\kappa^2}{2} \left(\ln \frac{g\Phi}{m\kappa^2} + 4\ln 2 + 1 \right) \\ &= 4\sqrt{q} + \frac{k^2}{4\sqrt{q}m^2} \left(\ln \left[4q \left(\frac{m}{k} \right)^2 \right] + 4\ln 2 + 1 \right).\end{aligned}\quad (74)$$

To obtain these estimates we used the condition that $\kappa^2 \ll g\Phi/m$ for the resonant modes. In Eq. (74) we presented θ_k in two equivalent forms: first in terms of the physical parameters g , Φ , and κ , and second in terms of the parameters q and k . Combining Eqs. (74) and (53) for the phases θ_k and φ_k , we can find how $\theta_k - \varphi_k$ depends on k . The leading term in $\theta_k - \varphi_k$ for large values of q is the term $2g\Phi/m = 4\sqrt{q}$ which does not depend on k . Substituting $\theta_k - \varphi_k$ into Eq. (73) we get the equation for the width of the resonance explicitly in terms of k for a given parameter q . Equation (73) transparently shows the presence of a sequence of stability/instability bands as a function of k . Typical half-width of a resonance band is $k^2 \sim 0.1k_*^2$. Substituting $\theta_k - \varphi_k$ into Eq. (72), we find the strength of the resonance as a function of k . The effect of amplification is not a monotonic function of q . The strongest amplification is realized for discrete values of the parameter q : $q = (n\pi/4)^2$, where n is an integer. For this case μ_k has a maximum at $k=0$. We can illustrate our results graphically for this case, since the function $\arg\Gamma[(1+i\kappa^2)/2]$ involved in the expression for φ_k (53) has a particularly simple form for $\kappa^2 \ll 1$:

$$\arg\Gamma\left(\frac{1+i\kappa^2}{2}\right) \approx -0.982\kappa^2. \quad (75)$$

Then we have

$$\theta_k - \varphi_k \approx 4\sqrt{q} + \frac{k^2}{8\sqrt{q}m^2} (\ln q + 9.474). \quad (76)$$

The function μ_k derived with the formulas (72) and (76) is plotted in Fig. 9 for $q = (64\pi/4)^2$. We also plot μ_k derived numerically from the Mathieu equation (30). We conclude that the predictions of the analytic theory developed here for the Mathieu equation with large q are rather accurate.

E. Stochastic resonance in an expanding universe

Let us consider the creation of χ particles by harmonic oscillations of the inflaton field in an expanding universe. Because of the expansion of the universe, there are few complications in Eq. (21) for the modes χ_k in an expanding universe in comparison with the Mathieu equation. The effect of the term $3H\dot{\chi}$ can be eliminated by using $X_k = a^{3/2}\chi_k$; see Eq. (38). The redshift of momenta $k \rightarrow k/a(t)$ should be taken into account, especially at the latest stages. The most important change is the time dependence of the parameter $q = g^2\Phi^2/4m^2$: $q \propto t^{-2} \propto N^{-1}$. For the broad resonance case where $q \gg 1$, this parameter significantly varies within a few inflaton oscillations; hence, the concept of the static stability-instability chart of the Mathieu equation cannot be utilized in this important case.

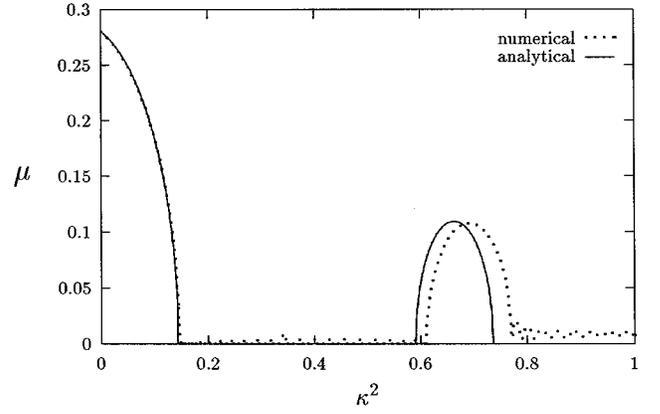


FIG. 9. The characteristic exponent μ_k of the Mathieu equation (30) as a function of $\kappa^2 \equiv k^2/k_*^2$, for $q = 32\pi^2$. The dotted curve is obtained from a numerical solution. Two instability bands are shown. The solid curve for these instability bands was derived analytically with Eqs. (72) and (76) where the simple approximation (75) was used. The numerical and analytical results are in a perfect agreement for the first band where the approximation (75) is accurate. By improving expansion (75), one can reach similar agreement for the higher bands as well.

Surprisingly, the most interesting case when the parameter q is large and time-varying can also be treated analytically by the method of successive parabolic scatterings. Indeed, the matrix mapping for the α_k^j and β_k^j developed in subsections B and C is also valid in the case of an expanding universe. Let us consider the phase accumulating between two successive zeros of the inflaton field:

$$\begin{aligned}\theta_k^j &= \int_{t_j}^{t_{j+1}} dt \sqrt{k^2/a + g^2\phi^2(t)} \\ &\approx \frac{2g\Phi}{m} + \frac{\kappa^2}{2} \left(\ln \frac{g\Phi}{m\kappa^2} + 4\ln 2 + 1 \right) \\ &\approx \frac{gM_p}{5mj} + O(\kappa^2),\end{aligned}\quad (77)$$

where we use Eq. (4) for the amplitude of oscillations, Φ , as a function of the number of oscillations, $N \approx 2j$. If the initial value $gM_p/10m \approx \sqrt{q_0}$ is large, then variation of the phase $\delta\theta_k^j$ between successive scatterings due to the j dependence is $\delta\theta_k^j \approx gM_p/20mj^2$, or in terms of the number of oscillations

$$\delta\theta_k \approx \frac{gM_p}{20mN^2} = \frac{\sqrt{q}}{2N^2}. \quad (78)$$

The crucial observation is the following: for large initial values of q , the phase variation $\delta\theta_k$ is much larger than π for all relevant k . Therefore, all the phases θ^j in Eqs. (48) and (55) in this case can be considered to be random numbers. For given q , the phases are random for the first

$$N_{\text{stoch}} \approx \frac{q^{1/4}}{\sqrt{2\pi}} \quad (79)$$

oscillations. For example, for $q_0 = 10^6$ the phases are random for the first dozen oscillations, and for $q_0 = 10^9$, neglecting back reaction effects, the phases would be random for the first hundred oscillations. During this time each mode experiences chaotic behavior in the standard terms of the theory of chaotic systems [40]; a small change in the values of parameters and/or initial conditions can lead to large changes in the final results.

We will show in Sec. IX B that the back reaction of created particles leads to an exponentially rapid decrease of q down to $q \sim 1/4$ at the last moments of preheating. This means that the parameter q in this regime remains very large and phases remain random until the very last stages of preheating.

The stochastic character of the phases, θ_k^j , significantly simplifies the analysis of the matrix equation (48). Indeed, since there is no memory of the phases, each mapping can be considered as independent of the previous ones.

As we see in Eq. (58), the number of created particles depends on the phase $\theta_{\text{tot}} = \varphi_k + 2\theta_k^j + \arg\beta_k - \arg\alpha_k$. In principle, from the matrix equation (48) one can derive a series of equations which allow one to express the phases $\arg\beta_k$, $\arg\alpha_k$, and eventually θ_{tot} through the random phase θ_k^j .

For qualitative analysis we simply assume that θ_{tot} is a random phase. As a result the number of particles n_k^{j+1} obeys the recursion equation

$$n_k^{j+1} \approx (1 + 2e^{-\pi\kappa_j^2} - 2\sin\hat{\theta} e^{-(\pi/2)\kappa_j^2} \sqrt{1 + e^{-\pi\kappa_j^2}}) n_k^j, \quad (80)$$

where $\hat{\theta}$ is a random phase in the interval $(0, 2\pi)$, and κ_j^2 is slowly changing with j as $\kappa_j^2 = k^2/a^2 gm\Phi \propto j^{-1/3}$.

Equation (80) defines the number of particles at an arbitrary moment as a function of the random phase. Therefore, n_k^j is a random variable which can either increase or decrease depending on the realization of the phase. Qualitatively, each mapping corresponds to one of the two possibilities depicted in Fig. 8. Therefore, the whole process of particle creation is the superposition of elementary processes where n_k jumps up or down. This explains the random behavior of n_k in Fig. 5. On average the number of particles is amplified with time, i.e., n_k increases more often than it decreases.

Stochastic resonance is different in many aspects from the usual broad parametric resonance of the Mathieu equation, considered in the previous subsection. Let us investigate the basic features of the stochastic resonance. First, the structure of Eq. (80) does not imply the existence of separate stability or instability bands. Indeed, the loss of the phase interference is related to any k within the broad interval $k \leq k_*$, where the coefficients of the mapping (80) are not exponentially suppressed. Therefore, as one can see by comparison of Figs. 9 and 10, the stochastic resonance is significantly broader (almost by an order of magnitude) than each of the stability zones of the Mathieu equation, $\Delta k \sim k_*$. It makes stochastic resonance much more stable with respect to possible mechanisms which, in principle, could terminate parametric resonance. For instance, the conclusion that the $g^2\phi^2\chi^2$ interaction can terminate broad parametric resonance in Minkowski

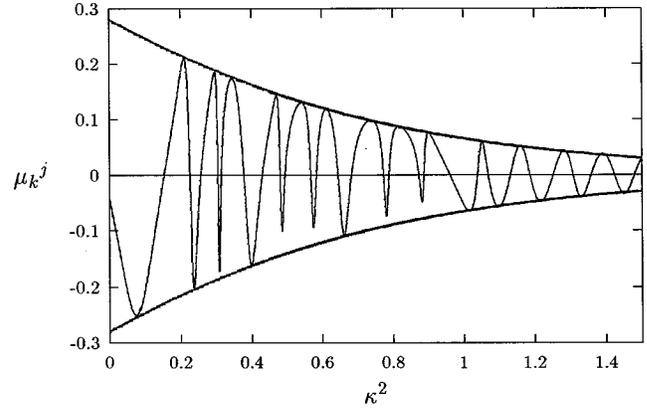


FIG. 10. The characteristic exponent μ_k of the mode Eq. (38) in an expanding universe as a function of κ^2 for the initial value of the parameter $q = (32\pi)^2 \approx 10^4$, obtained from a numerical solution. The curve is obtained at the time after the first 5 oscillations, which corresponds to μ_k^j with $j = 10$. The envelope of the curve is obtained from Eq. (81) by taking there $\sin\hat{\theta} = \pm 1$. We see that there is a complete agreement between the analytical prediction of the amplitude of μ_k [Eq. (81)] and the results of the numerical investigation. Contrary to the static case of Fig. 9, the resonance is much broader, there are no distinguished stability/instability bands, and for certain values of momenta the function μ_k^j is negative. During the stochastic resonance regime, this function changes dramatically with every half period of the inflaton oscillations. Comparison of Figs. 9 and 10 shows that it is incorrect to use the structure of the resonance bands of the static Mathieu equation for investigation of the stage of stochastic resonance, unless one is only looking for a very rough estimate of μ .

space-time [29] cannot be easily generalized to the case of an expanding universe, where the broad resonance is stochastic and much wider.

Second, the exponent μ_k is also a random variable:

$$\mu_k^j = \frac{1}{2\pi} \ln(1 + 2e^{-\pi\kappa_j^2} - 2 \sin \hat{\theta} e^{-(\pi/2)\kappa_j^2} \sqrt{1 + e^{-\pi\kappa_j^2}}). \quad (81)$$

The functional form of μ_k for stochastic resonance is different from that for broad parametric resonance. It changes with every half period of the inflaton oscillations. An example of μ_k calculated at intermediate stage of stochastic resonance (for $j = 10$ with the initial value of the parameter $q = 10^4$) is plotted in Fig. 10.

Equation (81) implies that for $\pi\kappa^2 \ll 1$ the value of μ_k^j is positive (i.e., the number of particles *grows*) for $\pi/4 < \hat{\theta} < 3\pi/4$. This occurs for one quarter of all possible values of $\hat{\theta}$, in the range of $-\pi < \hat{\theta} < \pi$. Therefore, positive and negative occurrences of μ_k for $\kappa \leq 1/\sqrt{\pi}$ appear in the proportion 3:1, so that the probability for the number of particles to increase is three times higher than the probability of its decreasing; see Sec. VI. Computer simulations of this process confirm this result. However, there will be also a ‘‘natural selection effect’’: among all modes χ_k there will be some modes for which positive occurrences of μ_k appear more often than in the proportion 3:1, and these modes will give the dominant contribution to the total number of produced particles. The typical mean value of the characteristic expo-

ment is $\mu_k \sim 0.13$, but the actual number is very sensitive to even a very small change of parameters; see the table in Sec. VI. Based on the central limit theorem, we expect that the statistics of the random variable n_k obey the log-normal distribution in the regime of the stochastic resonance.

From Eq. (81) one could expect that the suppression of particle production occurs not at $\kappa^2 > \pi^{-1}$, but at $\kappa^2 > 2\pi^{-1}$. However, the situation is more complicated. As soon as the second term under the logarithm becomes small, the probability for the number of particles to increase becomes equal to the probability of its decreasing, so the process of particle production becomes much less efficient.

The stochastic resonance occurs for N_{stoch} oscillations of the inflaton field defined by Eq. (79). When the parameter q decreases because of the expansion of the universe and becomes smaller than order 1, which happens for $N > N_{\text{stoch}}$, the resonance becomes very similar to the usual parametric resonance with $q \leq 1$. However, at some stage it may become necessary to correct this description by taking into account back reaction of the created particles.

VIII. RESONANCE, BACK REACTION, AND RESCATTERING

Until now we have treated the field χ as a test field in the presence of the background fields $\phi(t)$ and $a(t)$ which have independent dynamics. We found the effect of the resonant amplification of $\chi_k(t)$, which corresponds to the exponentially fast creation of n_χ particles. As we have seen, the resonance in an expanding universe in the beginning may be very broad, then it becomes narrow, and then eventually disappears.

Because of the exponential instability of the χ field, we expect its back reaction on the background dynamics to gradually accumulate until it affects the process of resonance itself. Therefore the development of resonance is divided into two stages. At the first stage of the process, the back reaction of the created particles can be neglected. As we will see, this stage is in fact rather long, and if the initial value of q was small enough ($q_0 \lesssim 10^3$) preheating may end before the back reaction becomes important (see also [28]). However, if q_0 is greater than about 10^3 , then at some moment the description of the parametric resonance changes. We enter the second stage of preheating where the back reaction should be taken into account. In what follows we will treat the first and second stages of preheating separately.

There are several ways in which back reaction can alter the process. First of all, interaction with particles created by parametric resonance may change the effective masses of all particles and the frequency of oscillation of the inflaton field. Also, scattering of the particles off each other and their interaction with the oscillating field $\phi(t)$ (we will vaguely call both processes ‘‘rescattering’’) may lead to additional particle production and to the removal of previously produced particles from the resonance.

In our model there will be two especially important effects. First, χ particles may change the frequency m of oscillations of the field $\phi(t)$. This may increase the value of m in the mode equation, which can make the resonance narrow and eventually shut it down.

The second effect is the production of ϕ particles, which

occurs due to interaction of χ particles with the oscillating field $\phi(t)$. One can visualize this process as scattering of χ particles on the oscillating field $\phi(t)$. In each act of interaction, each χ particle takes one ϕ particle away from the homogeneous oscillating field $\phi(t)$. When many ϕ particles are produced, they may change the effective mass of the field χ , making χ particles so heavy that they no longer can be produced. Also, scattering, when it occurs for a sufficiently long time, can destroy the oscillating field $\phi(t)$ by decomposing it into separate ϕ particles.

In this section we will derive the general set of equations which describe the self-consistent dynamics of the classical homogeneous inflaton field $\phi(t)$, as well as the fluctuations of the fields χ and ϕ . We will then discuss different feedbacks of the amplified fluctuations. In particular, we will check the energy balance between the background homogeneous inflaton field $\phi(t)$, the fluctuations $\chi(t, \mathbf{x})$, and the fluctuations $\phi(t, \mathbf{x})$.

A. Self-consistent evolution of ϕ and χ fields

We can describe all of these effects within a full set of self-consistent equations. The Friedmann equation for a universe containing classical field $\phi(t)$ and particles χ and ϕ with densities ρ_χ and ρ_ϕ is

$$3H^2 = \frac{8\pi}{M_p^2} \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 + \rho_\chi + \rho_\phi \right), \quad (82)$$

where ρ_χ and ρ_ϕ are the energy densities of χ particles and ϕ particles, respectively.

The mode Eq. (38) for $X_k(\mathbf{t}) = a^{3/2}(t) \chi_k(t)$ now should include a term describing the coupling between χ and ϕ fluctuations:

$$\begin{aligned} \ddot{X}_k(t) + \left(\frac{k^2}{a^2} + g^2 \Phi^2 \sin^2 mt \right) X_k(t) \\ = - \int dt' X_k(t') \Pi_\chi(t, t'; \mathbf{k}), \end{aligned} \quad (83)$$

where the polarization operator for the field $\chi_k = a^{-3/2} X_k$ is $\Pi_\chi(t, t'; \mathbf{k}) \equiv \int d^3 x e^{i\mathbf{k}(\mathbf{x} - \mathbf{x}')} \Pi_\chi(t, t'; \mathbf{x} - \mathbf{x}')$.

We will also consider quantum fluctuations of the inflaton field $\delta\phi(t, \mathbf{x}) = \phi(t, \mathbf{x}) - \phi(t)$ which can exist on top of the homogeneous inflaton condensate $\phi(t)$. The mode equation for $\varphi_k(t) \equiv a^{3/2} \delta\phi_k(t)$ is

$$\ddot{\varphi}_k(t) + \left(\frac{k^2}{a^2} + m^2 \right) \varphi_k(t) = - \int dt' \varphi_k(t') \Pi_\phi(t, t'; \mathbf{k}), \quad (84)$$

where $\Pi_\phi(t, t'; \mathbf{k})$ is a corresponding polarization operator for the field $\delta\phi_k(t) \equiv a^{-3/2} \varphi_k(t)$. The equation for the homogeneous condensate $\phi(t)$ is

$$\ddot{\phi}(t) + 3H\dot{\phi}(t) + m^2\phi(t) = -\Gamma_\phi(t) = -\Pi_\phi^1(t)\phi(t). \quad (85)$$

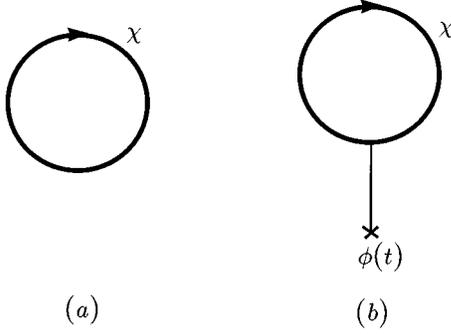


FIG. 11. The one-loop diagrams for the contribution of χ particles to the effective action of the field ϕ (a) and to its derivative $\Gamma_\phi(t)$ (b). The thick line corresponds to the Green function of the χ particles in the external field $\phi(t)$.

Here $\Gamma_\phi(t)$ is the tadpole diagram, representing the derivative of the effective action of the field ϕ (not the decay rate). The one-loop diagram representing $\Gamma_\phi(t)$ is shown in Fig. 11. The thick line corresponds to the exact solution of the classical equation of motion of the field χ in the external field ϕ .

To get an expression for the polarization operator of the field ϕ , one should differentiate the effective action twice with respect to the scalar field $\delta\phi$. The result can be represented as a sum of two polarization operators shown in Fig. 12. Π_ϕ^1 can be identified with the contribution of the fluctuations of the field χ to the mass squared of the field ϕ : $\Delta m^2 = g^2 \langle \chi^2 \rangle$. Note that it is directly related to Γ_ϕ : $\Gamma_\phi = -\Pi_\phi^1$. The polarization operator Π_ϕ^2 has a more complicated structure; it contains an external scalar field $\phi(t)$ in each of its vertices due to the interaction $g^2 \delta\phi \phi(t) \chi^2$.

The self-consistent dynamics described by Eqs. (82)–(85) is rather complicated and not very well investigated. There

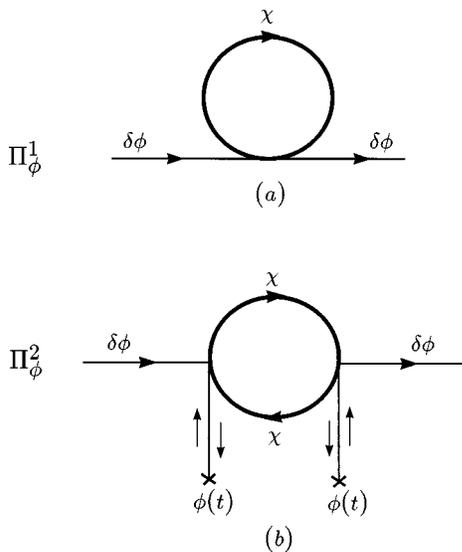


FIG. 12. Two diagrams for the polarization operator of the field ϕ . Thin and thick lines represent the fields ϕ and χ , respectively. The first diagram corresponds to the Hartree approximation which takes into account the contribution of $\langle \chi^2 \rangle$. The contributions of these two diagrams to the effective mass of ϕ particles can be comparable to each other.

are several different approximations which can be used to solve these equations in the context of preheating. We will describe them in this section.

B. Hartree approximation

The simplest way to take into account the back reaction of the amplified quantum fluctuations χ is to use the Hartree approximation,

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi + g^2\langle\chi^2\rangle\phi = 0, \quad (86)$$

where the vacuum expectation value for χ^2 is

$$\langle\chi^2\rangle = \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 |X_k(t)|^2. \quad (87)$$

Quantum effects contribute to the effective mass m_ϕ of the inflaton field as follows: $m_\phi^2 = m^2 + g^2\langle\chi^2\rangle$. The Hartree approximation corresponds to the first of the two diagrams of Fig. 12.

Initially, we have no fluctuations $\varphi_k(t)$, and we can use Eq. (38) for the modes X_k . One can express $\langle\chi^2\rangle$ in terms of the $\alpha_k(t)$ and $\beta_k(t)$ coefficients describing the resonance:

$$\begin{aligned} \langle\chi^2\rangle &= \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 \frac{1}{\omega} \\ &\times \left(|\beta_k|^2 + \text{Re} \left[\alpha_k \beta_k^* \exp \left(-2i \int_0^t \omega dt \right) \right] \right). \end{aligned} \quad (88)$$

This formal expression may need to be renormalized. The WKB expansion of the solution of Eqs. (43) provides a natural scheme of regularization [37]. However, in our case the coefficients α_k and β_k of the Bogoliubov transformation appear due to particle production (as opposed to vacuum polarization), so the integral in Eq. (88) is finite and does not require further regularization.

Let us estimate $\langle\chi^2\rangle$ from Eq. (88) using the results of the previous section. For the resonant creation of χ particles we have $|\beta_k|^2 \equiv n_k \approx \frac{1}{2} e^{2\mu_k m t}$, $\text{Re} [\alpha_k \beta_k^* \exp(-2i \int \omega dt)] \approx |\beta_k|^2 \cos(2 \int \omega dt - \arg \alpha_k + \arg \beta_k)$. For $\omega \approx g\phi(t) = g\Phi \sin mt$ the phase in this expression is equal to $(2g\Phi/m) \cos mt$ plus a small correction $O(\kappa^2)$. Due to this small correction, the term $(2g\Phi/m) \cos mt$ acquires a numerical factor $C < 1$ after the integration $\int d^3k$:

$$\langle\chi^2\rangle \approx \frac{1 + C \cos(2g\Phi \cos mt/m)}{2\pi^2 a^3} \int_0^\infty dk k^2 \frac{1}{\omega} n_k. \quad (89)$$

In the broad resonance case when $\phi > \phi_*$ (i.e., for most of the time), one has $k/a \ll g\phi$, $\omega \approx g|\phi(t)|$, and, therefore,

$$\langle\chi^2\rangle \approx \left(1 + C \cos \frac{2g\Phi \cos mt}{m} \right) \frac{n_\chi}{g|\phi(t)|}. \quad (90)$$

This means in particular that in the broad resonance regime the effective mass squared of the background field $\phi(t)$ in the Hartree approximation

$$m_\phi^2 = m^2 + \left(1 + C \cos\left(\frac{2g\Phi}{m} \cos mt\right) \right) \frac{gn_\chi}{|\phi(t)|}, \quad (91)$$

oscillates with two frequencies. One is the frequency of oscillation of $|\phi(t)|$, which is equal to $2m$. In addition, when $\phi(t) = \Phi$, the effective mass squared m_ϕ^2 oscillates with a very high frequency $\sim 2g\Phi \gg m$. The amplitudes of both oscillations are as large as the maximal value of $g^2 \langle \chi^2 \rangle$. One can easily identify both types of oscillations of $\langle \chi^2 \rangle$ in the numerical simulations of Khlebnikov and Tkachev [30].

The resulting equation for the field $\phi(t)$ looks as follows:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi + gn_\chi \left(1 + C \cos\frac{2g\Phi \cos mt}{m} \right) \frac{\phi}{|\phi|} = 0. \quad (92)$$

The last term in this equation oscillates with a frequency $\sim 2g\Phi$. In the broad resonance regime with $g\Phi \gg m$ the high-frequency oscillation of this term does not much affect the evolution of the field $\phi(t)$ because the overall sign of the term $C \cos(2g\Phi \cos mt/m)$ changes many times during each oscillation of the field ϕ . One may wonder, however, whether these high-frequency oscillations may lead to a copious production of ϕ particles. A preliminary investigation of this issue shows that the quasiperiodic change of the last term in Eq. (92) does not lead to parametric resonance, but a nonresonant particle production is possible because the effective mass changes in a very nonadiabatic way: $dm/dt \sim gm\Phi \sim k_*^2 \gg m^2$.

In the first approximation one may neglect this effect and write Eq. (92) as

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi + gn_\chi \frac{\phi}{|\phi|} = 0. \quad (93)$$

Even in this simplified form the last term of this equation looks rather unusual. It is not proportional to ϕ , which would be the case if χ particles gave a ϕ -independent contribution to the effective mass of the field ϕ . In our case this contribution is inversely proportional to $|\phi|$. As a result, the field ϕ behaves as if it were oscillating in the effective potential $gn_\chi |\phi|$.

To estimate the change in the frequency of oscillations of the field ϕ due to the term $gn_\chi(\phi/|\phi|)$ in Eq. (93), one can neglect the term $3H\dot{\phi}$ in the equation for the homogeneous field ϕ , because $H \ll m$ at the end of the first stage of preheating, when the term $gn_\chi(\phi/|\phi|)$ becomes important. Let us find when the frequency increase due to the interaction with χ particles becomes greater than the initial frequency m . In order to do this one should solve the equation $\dot{\phi} = -gn_\chi$ in the interval $0 < \phi < \Phi$. The time during which the field ϕ falls down from Φ to 0 is $\Delta t = \sqrt{2\Phi/gn_\chi}$. This time corresponds to one quarter of a period of an oscillation. This gives the following expression for the frequency of oscillations of the field ϕ in the regime when it is much greater than its bare mass squared m^2 :

$$\omega_\phi = \frac{\pi}{2\sqrt{2}} m_\phi \approx m_\phi. \quad (94)$$

Here m_ϕ is the value of the effective mass of the field ϕ at the moment when $\phi(t) = \Phi$ (the oscillations of $\langle \chi^2 \rangle$ being ignored). Therefore to estimate the change of the frequency of oscillations of the scalar field ϕ one can use the standard expression $m_\phi^2 = m^2 + g^2 \langle \chi^2 \rangle$ for the effective mass squared of the field ϕ , where by $\langle \chi^2 \rangle$ one should understand its *smallest* value per period, which appears for $\phi(t) = \Phi$. This implies that the frequency of oscillations of the inflaton field does not change until the number of χ particles grows to

$$n_\chi \approx \frac{m^2\Phi}{g} = \frac{2m^3}{g^2} q^{1/2}. \quad (95)$$

This is a very important criterion which defines the duration of the first stage of preheating where the back reaction of the created particles can be neglected.

For future reference we include here expressions for the energy density and pressure of the nonrelativistic χ particles. The contribution of χ particles to the energy density $\rho_\chi(\phi)$ of the oscillating field ϕ in terms of $\alpha_k(t)$ and $\beta_k(t)$ is given by

$$\rho_\chi(\phi) = \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 \omega |\beta_k|^2, \quad (96)$$

where $|\beta_k|^2 = n_k$. This expression does not have any high-frequency modulations which we have found for the Hartree term $(g^2/2) \langle \chi^2 \rangle \phi^2$. During the main part of each oscillation of the field ϕ , the field χ has mass much greater than the range of the integration $\sim k_*$, which means that $\omega \approx g|\phi(t)|$, and

$$\rho_\chi(\phi) = \frac{g|\phi|}{2\pi^2 a^3} \int_0^\infty dk k^2 n_k = g|\phi| n_\chi. \quad (97)$$

The contribution of χ particles to pressure in terms of $\alpha_k(t)$ and $\beta_k(t)$ is given by

$$p_\chi(\phi) = -\frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 \omega \left[\text{Re} \left(\alpha_k \beta_k^* \exp \left(-2i \int_0^t \omega dt \right) \right) + \frac{k^2}{3\omega^2} |\beta_k|^2 \right] = -g|\phi| n_\chi C \cos \frac{2g\Phi \cos mt}{m}. \quad (98)$$

The last equality holds in the nonrelativistic limit, for $\phi \gg \phi_*$. Average pressure in this regime is equal to zero, as it should be for nonrelativistic particles.

C. Is the Hartree approximation sufficient for the calculation of particle masses?

In the previous subsection we investigated the change of frequency of oscillations of the classical background field $\phi(t)$ due to its interaction with χ particles; see Eqs. (93) and (94). What about the spectra of perturbations $\delta\phi$? In order to answer this question one should calculate both diagrams shown in Fig. 12. The first of these diagrams, Fig. 12(a), gives the same contribution $\Pi_\phi^1 = g^2 \langle \chi^2 \rangle$ as the one which we already calculated when we studied oscillations of the field $\phi(t)$. As we have seen, in the situation where fluctuations $\chi_k(t)$ are amplified by resonance, even the calculation

of this simple diagram is rather nontrivial and leads to an unusual result (92). The calculation of the polarization operator Π_ϕ^2 , Fig. 12(b), is much more involved. Similar diagrams have been ignored in all previous papers on preheating. Let us try to understand, however, whether Π_ϕ^2 can be neglected as compared with Π_ϕ^1 . A positive answer to this question would imply that the Hartree approximation is sufficient not only for the investigation of the oscillations of the field $\phi(t)$, but also for finding the spectrum of perturbations of the field ϕ .

Usually when one calculates similar diagrams at high temperature, the polarization operator Π_ϕ^1 in the high-temperature limit is proportional to T^2 , whereas Π_ϕ^2 is less divergent at large momenta and therefore grows only as T . Therefore in the high-temperature approximation, the first diagram, which corresponds to the Hartree approximation, gives the leading contribution. In our case this issue should be reconsidered because the leading contribution to the diagrams is given by particles with large occupation numbers and relatively small momenta.

The back reaction of created particles becomes essential only at later stages of reheating, when, as we will see shortly, $H \ll m$. Therefore at that stage one can neglect the expansion of the universe when calculating polarization operators, and it is more convenient to perform all calculations in terms of the usual, physical (rather than comoving) momenta k and the modes $\chi_k(t)$. Therefore throughout the rest of the paper we will use physical momenta, k , p , etc. During the last stages of reheating they remain almost constant, but in order to relate them to the original physical momenta for each mode χ_k one should remember that physical momenta are redshifted as $a^{-1}(t)$.

To calculate Π_ϕ^2 one needs to know the Green function of the field χ in an external field $\phi(t)$, which is given by

$$G_\chi(x, x') = \int d^3k T[\chi_k(t)\chi_k(t')]e^{ik(x-x')}, \quad (99)$$

where T stands for time ordering. The calculation of the diagram for Π_ϕ^2 , Fig. 12(b), using this Green function for the internal lines of the field χ is rather tedious. Therefore, we will make certain simplifications. Consider the broad resonance regime $q \gg 1$ at a time when $\phi(t) \gg \phi_*$. At this stage there is no particle production, and the adiabatic form (42) can be used for the eigenfunction $\chi_k(t)$. Consider a time interval $\Delta t < m^{-1}$ near the time when the inflaton field $\phi(t)$ reaches its maximum, Φ . During this short interval, one can neglect the expansion of the universe and the change of the field $\phi(t)$, i.e., one may take $\phi(t) \approx \Phi$. The Green function in the space-time representation consists of two parts. The first part is similar to the standard Green function in Minkowski space in the fixed background field ϕ . The second part contains the high frequency modulation $e^{i\omega(t+t')}$. Both terms are of the same order. One can show that in this regime the first term in the expression for the Green function (99) has a simple form in the momentum representation:

$$G(k) = \frac{i}{k^2 - m_\chi^2} + 2\pi n_k \delta(k^2 - m_\chi^2). \quad (100)$$

Here $m_\chi = g\phi(t)$, and k is a physical momentum. The first term in this equation is the standard Green function for quantum fluctuations in the vacuum. The second term is proportional to the occupation number $n_k = |\beta_k|^2$ of the χ particles.

The second part of the full Green function containing the modulation $e^{i\omega(t+t')}$ does not have a simple interpretation in the momentum representation. Omitting this part does not affect the order-of-magnitude estimate of the polarization operator. This can be most easily seen for the diagram Fig. 12(a), where the calculations are much simpler. Indeed, with the complete Green function (99) one can immediately reproduce the result (92) for the diagram in Fig. 12(a). Meanwhile, if one uses Eq. (100), then in the large n_k limit one gets the first, nonoscillating term in the brackets of Eq. (91):

$$\Pi_\phi^1 \approx \frac{g^2}{(2\pi)^4} \int d^4p 2\pi \delta(p^2 - m_\chi^2) n_p = \frac{gn_\chi}{|\phi(t)|}. \quad (101)$$

The part of the Green function containing the modulation $e^{i\omega(t+t')}$ in this case gives us the second (rapidly oscillating) term in Eq. (91).

Thus, whereas in the first approximation one can interpret the growing modes of the field χ during parametric resonance as normal particles on the mass shell with the standard Green function (100), this interpretation in general is not quite adequate and may lead to the loss of some terms such as the oscillating term discussed above. Still we correctly reproduced the most important part of the polarization operator Π_ϕ^1 .

Let us try to estimate the polarization operator Π_ϕ^2 using the simple Green function (100) for $|\phi(t)| \approx \Phi$. The general structure of the polarization operator is given by

$$\Pi_\phi^2(k) \sim -i \frac{g^4 \Phi^2}{(2\pi)^4} \int d^4p G(p)G(p-k \pm q). \quad (102)$$

The sign of q depends on whether the external field $\phi(t)$ brings the momentum $q_0 = m, \mathbf{q} = 0$ to the two vertices of the polarization operator or takes this momentum away.

It is not our purpose now to perform a complete calculation of Π_ϕ^2 in this paper because we do not need to know the exact spectrum of perturbations $\delta\phi$. Our main goal here is to find out whether or not Π_ϕ^2 may contain terms comparable to the Hartree operator Π_ϕ^1 . And indeed, if one calculates, for example, the diagram where the external field $\phi(t)$ brings a momentum $q_0 = m, \mathbf{q} = 0$ to the first vertex and takes it away from the second vertex, one finds (ignoring factors of order 1) that this contribution to the real part of Π_ϕ^2 for $k_0 = m, \mathbf{k} = 0$ in the limit $n_p \gg 1$ has the same structure as Π_ϕ^1 :

$$\text{Re } \Pi_\phi^2 \sim - \frac{g^4 \Phi^2}{(2\pi)^3} \int \frac{n_p d^3p}{p_0(p_0^2 - m^2)} \sim - \frac{gn_\chi}{\Phi}. \quad (103)$$

Here $p_0 \equiv \omega = \sqrt{\mathbf{p}^2 + g^2 \Phi^2} \approx g\Phi$ for a typical resonant mode with $g^2 \Phi^2 \gg \mathbf{p}^2 \sim gm\Phi \gg m^2$. Thus, for $|\phi(t)| \approx \Phi$ the second polarization operator of Fig. 12 contains terms of the same order of magnitude as the value of the polarization operator in the Hartree approximation. This result indicates

that one may need to go beyond the Hartree approximation used in many papers on preheating.

This result looks paradoxical. In particular, one could argue that the Hartree approximation is closely related to the $1/N$ approximation, which is expected to give exact results in the limit $N \rightarrow \infty$. Indeed, instead of a single χ field one can take N fields χ_i with the interaction $(g^2/2N)\phi^2\chi_i^2$. The Hartree diagram is proportional to g^2 , i.e., it survives in the limit $N \rightarrow \infty$, whereas the expression for the polarization operator Π_ϕ^2 is proportional to g^4/N . That is why usually at large N one can neglect contributions like Π_ϕ^2 as compared with Π_ϕ^1 . Indeed, this would be true in our case as well if the field χ had a large ϕ -independent mass. But in the theory we are discussing now its mass squared is $(g^2/N)\phi^2$. As we have seen, when one calculates Π_ϕ^2 this mass squared appears in the denominator. As a result, the factor g^4/N in front of the diagram becomes g^2 , so that this diagram also survives in the limit $N \rightarrow \infty$ and has the same order of magnitude as the Hartree diagram in the $1/N$ approximation. This means, in particular, that without a complete calculation of Π_ϕ^2 one cannot be sure that the $1/N$ approximation gives a correct spectrum of particles in the limit $N \rightarrow \infty$ when applied to the theory of preheating.

To avoid misunderstandings we should reiterate that this problem appears in the calculations of the effective masses of the ϕ particles but not in the calculation of corrections to the equation of motion of the background field $\phi(t)$, which was our main goal in Sec. VIII.

D. Classical approximation to the self-consistent dynamics

Fluctuations of Bose fields generated from vacuum by an external field in the large occupation number limit can be considered as classical waves with Gaussian statistics; see, e.g., [41]. Therefore in the first approximation all fields χ , $\delta\phi$ can be treated as interacting classical waves. This makes it possible to study preheating by investigating a system of nonlinear classical equations or by lattice numerical simulations of the interacting classical scalar fields [24,28–30].

The Fourier decomposition of the Klein-Gordon equations of the interacting fields can be reduced to mode equations. The mode equation for $X_k = a^{3/2}\chi_k$ is

$$\begin{aligned} \ddot{X}_k + \left(\frac{k^2}{a^2} + g^2\phi^2(t) \right) X_k = & - \frac{g^2\phi(t)}{(2\pi)^3 a^{3/2}} \\ & \times \int d^3k' X_{\mathbf{k}-\mathbf{k}'} \varphi_{\mathbf{k}'} - \frac{g^2}{(2\pi a)^3} \\ & \times \int d^3k' d^3k'' X_{\mathbf{k}-\mathbf{k}'+\mathbf{k}''} \varphi_{\mathbf{k}'} \varphi_{\mathbf{k}''}. \end{aligned} \quad (104)$$

The mode equation for $\delta\phi_k(t) \equiv a^{-3/2}\varphi_k(t)$ is

$$\begin{aligned} \ddot{\varphi}_k + \left(\frac{k^2}{a^2} + m^2 \right) \varphi_k = & - \frac{g^2\phi(t)}{(2\pi)^3 a^{3/2}} \\ & \times \int d^3k' X_{\mathbf{k}-\mathbf{k}'} X_{\mathbf{k}'} - \frac{g^2}{(2\pi a)^3} \\ & \times \int d^3k' d^3k'' \varphi_{\mathbf{k}-\mathbf{k}'+\mathbf{k}''} X_{\mathbf{k}'} X_{\mathbf{k}''}. \end{aligned} \quad (105)$$

The first term in the right-hand side of this equation describes rescattering of χ particles on the classical field $\phi(t)$, which leads to ϕ -particle production. The second term describes scattering of ϕ particles and χ particles. Corrections to the effective mass of the modes ϕ_k appear as a result of the iterative solution of the system of equations which we now present.

The equation for the oscillating background field $\phi(t)$ looks as follows:

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + m^2\phi = & - \frac{g^2\phi}{(2\pi)^3 a^3} \\ & \times \int d^3k' X_{\mathbf{k}'}^2 - \frac{g^2}{(2\pi)^3 a^{9/2}} \\ & \times \int d^3k' d^3k'' \varphi_{\mathbf{k}''-\mathbf{k}'} X_{\mathbf{k}'} X_{\mathbf{k}''}. \end{aligned} \quad (106)$$

The first term on the right-hand side of this equation is proportional to the polarization operator Π_ϕ^1 , which is shown in Fig. 12(a). The second term describes rescattering, which is related to the imaginary part of the polarization operator Π_ϕ^2 , Fig. 12(b). Neglecting this term, one reproduces Eq. (86) with the term containing $\int d^3k |X_{\mathbf{k}}|^2$ playing the role of the induced mass. Thus the classical approximation reproduces the Hartree approximation, but it also takes into account effects related to rescattering.

In the beginning one can neglect $\varphi_k(t)$ and the corresponding integral terms in Eq. (104). Later, the fluctuations $X_k(t)$ are amplified by the resonance and give rise to $\varphi_k(t)$ fluctuations via the integral terms in Eq. (105). When the amplitude of fluctuations $\varphi_k(t)$ grows significantly, they begin to contribute to the integral terms of Eq. (104). We will show (see Sec. XB) that the amplitude $\varphi_k(t)$ grows with time as $e^{2\mu mt}$. Therefore the number of particles corresponding to $\delta\phi$ fluctuations grows as $e^{4\mu mt}$, i.e., much faster than n_χ . The interaction terms in Eqs. (104) and (105) can be interpreted as scattering of χ particles on the inflaton field. Because of the very fast generation of $\delta\phi$ fluctuations, $|\delta\phi|^2 \propto e^{4\mu t}$, the process of rescattering can be very important. However, it is not so easy to evaluate its full significance for the efficiency of the resonance. For example, if the particles ϕ produced during rescattering have small momenta k , they cannot be distinguished from the homogeneous oscillating scalar field, and therefore they do not make any difference to the development of the resonance, see the

discussion of this issue in Sec. X. Therefore we need to know not only how many $\delta\phi$ particles are produced, but also whether they are “hard” particles with large momenta or “soft” particles with small momenta. We will return to this question in Sec. X.

IX. TWO STAGES OF PREHEATING, RESCATTERING BEING NEGLECTED

Previously, we were mainly following the evolution of each particular mode χ_k . Now we will study their integral effect in an expanding universe.

As we have found in the previous section, the development of broad parametric resonance can be divided into two stages. In the first stage $n_\chi < m^2\Phi/g$, back reaction of the particles χ can be neglected, and the frequency of oscillations of the field ϕ is determined by its mass m . (We will argue later that at this stage their scattering also does not lead to any important effects.) In the second stage $n_\chi > m^2\Phi/g$, and the frequency of oscillations of the field ϕ becomes determined not by its bare mass, but by its interaction with χ particles. Now we will study the first and second stage of broad parametric resonance.

We begin with the first stage when the back reaction of created particles can be neglected. Then we consider the second stage where back reaction is important assuming a certain hierarchy of the feedback effects: effective mass of the inflaton is changed first, and rescattering may become important afterwards. In this section we will neglect rescattering. In the next section we will discuss rescattering and the validity of the assumption mentioned above.

A. The first stage of preheating: no back reaction and no rescattering

In the first stage of preheating one can ignore the back reaction of created particles on the frequency of oscillations of the field $\phi(t)$. As we have found in Sec. IX A, this stage ends at the moment t_1 when

$$n_\chi(t_1) \approx \frac{m^2\Phi(t_1)}{g}. \quad (107)$$

In the next section we will show that the effects related to rescattering also do not alter the development of the resonance during this stage. In this section we will estimate the duration of the first stage t_1 , the number of inflaton oscillations N_1 at the time $t=t_1$, the number of created particles $n_\chi(t_1)$, the energy density of these particles $\rho_\chi(t_1)$ and the value of $\langle\chi^2(t_1)\rangle$. We will use symbols Φ , q , and k_* without any indices for the running (time-dependent) values of the amplitude of the field $\phi(t)$, of the q factor, and of $\sqrt{gm\Phi(t)}$, whereas, for example, q_0 will correspond to the value of q at the beginning of preheating, and q_1 will correspond to its value in the end of the first stage of preheating.

One can use Eq. (64) to estimate n_χ . First one should determine which fluctuations χ_k are amplified during the entire period of the resonance. The fluctuations amplified by the broad resonance have physical momenta $k \leq k_*/2 \sim \sqrt{gm\Phi}/2$; see Eq. (34). [More precisely, one may expect $k \leq k_*/\sqrt{\pi}$, see Eq. (56).] Then the amplitude Φ in

this expression decreases as about $M_p/3mt$. Therefore, the resonance width decreases as $k \sim t^{-1/2}$, whereas redshift of the momenta of previously produced particles occurs as $a^{-1} \sim t^{-2/3}$, i.e., somewhat faster. [In terms of comoving momenta k , the resonance width grows as $k \approx a(t)\sqrt{gm\Phi}/2 \propto t^{1/6}$.] This means that those modes which have been amplified at the first stages of the process continue to be amplified later on. There are modes which were outside of the resonance band in the very beginning, but entered the resonance band later. However, after a time $\sim (2\mu m)^{-1}$ the fluctuations which have been amplified from the very beginning will be exponentially larger than the “newcomers.” Therefore the modes which do not enter the resonance band from the beginning typically give a subdominant contribution to the net effect.

Thus, with reasonably good accuracy, during the first stage of preheating one may consider only those fluctuations which have been amplified from the very beginning.³ This is important because it means that in all integrals one should consider only momenta which initially, when $a(t_0)=1$, $\Phi(t)=\Phi_0$, were in the interval

$$k(t_0) \leq k_*(t_0)/2 \approx \sqrt{gm\Phi_0}/2 \approx mq_0^{1/4}/\sqrt{2}, \quad (108)$$

where $q_0 = g^2\Phi_0^2/4m^2$.

The most important element of our calculations is the exponentially growing occupation number of particles with $k < k_m$: $n(t) \propto e^{2\mu mt}$. Here μ is an effective index which describes an average rate of growth for modes with $k \leq k_*$; see Sec. VII C. In our model μ depends on g , but not very strongly; see the table in Sec. VI. Typically it is in the range 0.1–0.2. For definiteness, in our estimates we will use $\mu=0.13$ which we have found numerically for a certain range of values of the coupling constant g ; see the table in Sec. VI. As we will see, in the context of our approach an error in our estimate of μ , say of 10%, does not create an exponentially large error in the final result (contrary to the remark of [17]); it only leads to an error of 10% in the calculation of the duration of the first stage of preheating. Our final results will be even less sensitive to the value of the subexponential factor in Eq. (64).

Substituting Eq. (108) into Eq. (64), we find

$$n_\chi(t) \approx \frac{(gm\Phi_0)^{3/2}}{64\pi^2 a^3 \sqrt{\pi\mu m(t-t_0)}} e^{2\mu m(t-t_0)}, \quad (109)$$

where t_0 is the beginning of the inflaton oscillation. The convention we used in Sec. VI is that $t_0 = \pi/2m$, which gives

³The total duration of the first stage of preheating in our model typically is about $10^2 m^{-1}$. If one compares the redshifted value $k_*(t_0)/a(t)$ of the physical momentum which was equal to $k_*(t_0) \sim \sqrt{gm\Phi_0}$ at the beginning of preheating, and the running value of $k_*(t) \sim \sqrt{gm\Phi(t)}$, one finds out that in the beginning these two quantities coincide, whereas at the end of the first stage of preheating the running value of k_* is greater than the redshifted one by only a factor ~ 2 . Therefore at the end of the first stage and at the beginning of the second stage of preheating instead of calculating the redshifted value of $k_*(t_0)/2$ one may simply use the condition $k \leq k_*(t)/4$ for the growing modes.

$\Phi_0 \approx M_p/5$ and $q_0 = 10^{10} g^2$. Our choice is also very close to the convention of Ref. [30]. (This particular choice is not going to be important because the total duration of the process is much greater than t_0 .) With this choice of t_0 we have $a(t) = (2mt/\pi)^{2/3}$. For $t \gg t_0$ one has⁴

$$n_\chi(t) \approx 10^{-4} \frac{(gmM_p)^{3/2}}{(mt)^{5/2} \mu^{1/2}} e^{2\mu mt}. \quad (110)$$

Now we have to substitute Eq. (110) and $\Phi(t) \approx M_p/3mt$ into Eq. (107). The result can be transformed into an equation for t_1 :

$$t_1 \approx \frac{1}{4\mu m} \ln \frac{10^6 m (mt_1)^3}{g^5 M_p}. \quad (111)$$

An approximate solution of Eq. (111) for $\mu \approx 0.13$ is $t_1 \approx (1/4\mu m) \ln(10^{12} m/g^5 M_p)$ [1]. As we will see soon, this is a good estimate not only for the duration of the first stage of preheating, but for the duration of the whole process, because the second stage of preheating typically is rather short.

For a realistic value $m \approx 10^{-6} M_p$ in chaotic inflation in the theory $m^2 \phi^2/2$, our estimate gives [9]

$$t_1 \approx \frac{5}{4\mu m} \ln \frac{15}{g}. \quad (112)$$

For instance, for $\mu = 0.13$ and $g = 0.1$ one has $t_1 \approx 50m^{-1}$; for $g = 10^{-2}$ one has $t_1 \approx 70m^{-1}$; for $g = 10^{-3}$ one has $t_1 \approx 90m^{-1}$, etc.

The value of the field $\Phi_1 \equiv \Phi(t_1)$ at the end of this first stage is given by

$$\Phi_1 \approx \frac{M_p}{3mt_1} = \frac{4\mu M_p}{3} \ln^{-1} \frac{10^{12} m}{g^5 M_p}. \quad (113)$$

Another important quantity is the value of the parameter $q = g^2 \Phi^2/4m^2$ at the end of the first stage:

$$q_1^{1/2} = \frac{g\Phi_1}{2m} = \frac{2g\mu M_p}{3m} \ln^{-1} \frac{10^{12} m}{g^5 M_p}. \quad (114)$$

To find the typical occupation numbers at the end of the first stage of reheating, let us remember that $n_\chi = (1/2\pi^2) \int k^2 dk n_k$, and that integration typically goes from 0 to the physical momentum $\sim k_*/2$. This gives an estimate

$$n_k \approx \frac{48\pi^2 n_\chi}{k_*^3}. \quad (115)$$

The occupation numbers of χ particles $n_k(t_1)$ by the end of that stage can be estimated as $n_k(t_1) \approx 3 \times 10^2 g^{-2} q_1^{-1/4}$; see Eqs. (61) and (111).

Using the results of this section, for different values of the coupling constant g one can estimate the initial value q_0 of the parameter q , its value q_1 at the end of the first stage of preheating, the value Φ_1 , and the number of oscillations N_1 which the field ϕ makes from the end of inflation to the end of the first stage. In the table below we give somewhat rounded numbers:

g	q_0	q_1	Φ_1/M_p	N_1
10^{-3}	10^4	3	3.5×10^{-3}	15
10^{-2}	10^6	550	5×10^{-3}	11
10^{-1}	10^8	10^5	7×10^{-3}	8

The energy density at the end of the first stage is given by

$$\frac{m^2 \Phi_1^2}{2} \approx \frac{8\mu^2 m^2 M_p^2}{9} \ln^{-2} \frac{10^{12} m}{g^5 M_p}. \quad (116)$$

It is worth comparing the frequency of the inflaton oscillations m with the Hubble parameter at that time:

$$H(t_1) \approx m \sqrt{\pi/3} \frac{2\Phi_1}{M_p} \approx m \frac{8\mu}{3} \ln^{-1} \frac{10^{12} m}{g^5 M_p}. \quad (117)$$

For instance, for $\mu = 0.13$, $g = 10^{-2}$, $m = 10^{-6} M_p$, one has

$$H(t_1) \sim 10^{-2} m. \quad (118)$$

Thus, at the last stages of preheating (though not at the beginning) one can, in the first approximation, neglect the expansion of the universe.

At that time, when $g^2 \langle \chi^2 \rangle \approx m^2$, the total energy density [on the right-hand side of Eq. (82)] becomes approximately equally distributed between the interaction energy $V_\chi(\phi) = g\Phi_1 n_\chi = m^2 \Phi_1^2$ and the potential energy density $m^2 \Phi_1^2/2$ of the field ϕ . The kinetic energy of χ particles can be estimated as $\langle (\nabla \chi)^2 \rangle \approx k_*^2 \langle \chi^2 \rangle \approx g\Phi_1 m \langle \chi^2 \rangle \approx m^2 \Phi_1^2 (m/g\Phi_1) \approx m^2 \Phi_1^2 q_1^{-1/2}$.

If preheating does not end with the end of the first stage, i.e., if $q_1 \gg 1/4$, then the kinetic energy remains small: $\langle (\nabla \chi)^2 \rangle \approx m^2 \Phi_1^2 q_1^{-1/2} \ll g^2 \Phi_1^2 \langle \chi^2 \rangle \approx m^2 \Phi_1^2$. However, if at the end of the first stage $q_1 \sim 1$, then at that time a considerable fraction of the energy of the inflaton field will have been transformed into the kinetic energy of the χ particles: $\langle (\nabla \chi)^2 \rangle \approx m^2 \Phi_1^2 q_1^{-1/2} \approx m^2 \Phi_1^2$.

Let us find the range of values of the coupling constant g for which preheating ends during the first stage and for which investigation of back reaction is not necessary. Without taking account of the back reaction preheating ends at the time t_f when $g\Phi(t)$ drops down to m , which gives $t_f \approx gM_p/3m^2$ (see Sec. VI). Therefore, preheating ends in the first stage if $t_f \leq t_1$, i.e., if

$$g \lesssim \frac{4m}{\mu M_p} \ln \frac{15}{g}. \quad (119)$$

⁴Equation (110) is a starting point for our further estimates. To derive this equation we used the theory of successive parabolic scatterings. However, the general structure of Eq. (110) can be easily understood even without any use of this theory. As we already mentioned, the value of μ can be obtained by solving the Mathieu equation numerically in an expanding universe; see Sec. VI. One can make a simple estimate of μ even without using a computer. Indeed, we know that the parameter μ along the line $A=2q$ changes from 0 to 0.28 [1]. An average of these numbers, 0.14, provides an excellent approximation to the true value of μ .

For our values of parameters this gives the condition $g \lesssim 3 \times 10^{-4}$. In our convention, this corresponds to an initial value $q_0 \lesssim 10^3$.

In this regime the total number density of χ particles created during preheating is given by

$$n_\chi \simeq \frac{m^4}{gM_p} \exp \frac{2g\mu M_p}{3m}, \quad (120)$$

and the χ fluctuations at the end of this stage are given by

$$\langle \chi^2 \rangle \simeq \frac{m^3}{gM_p} \exp \frac{2g\mu M_p}{3m}. \quad (121)$$

Equation (119) implies that for $g \approx 3 \times 10^{-4}$ this quantity should coincide with the value of $\langle \chi^2 \rangle$ at the end of the first stage of preheating, $\langle \chi^2 \rangle = m^2/g^2$. Thus, for $g \approx 3 \times 10^{-4}$ one has

$$\sqrt{\langle \chi^2 \rangle} \approx 3 \times 10^{16} \text{ GeV}. \quad (122)$$

The possibility to obtain enormously large fluctuations of the field χ is one of the most remarkable features of preheating. For comparison, if the field χ were in a state of thermal equilibrium, the dispersion of its fluctuations would be given by $\sqrt{\langle \chi^2 \rangle} = T/2\sqrt{3}$ [42]. Therefore in order to obtain $\sqrt{\langle \chi^2 \rangle} \sim 3 \times 10^{16}$ GeV one would need to have $T \gtrsim 10^{17}$ GeV, which is practically impossible in the context of inflationary cosmology. Here such fluctuations can be generated prior to thermalization due to the resonance at the stage of preheating. Fluctuations (122) change the effective masses of particles interacting with the field χ . The simplest way to study this possibility is to add to our model another scalar field η with a potential describing symmetry breaking, for example, $V(\eta, \chi) = \lambda[(\eta - \eta_0)^2 + \eta^2 \chi^2]$. For sufficiently small λ this addition does not affect preheating and does not change any of our results concerning $\langle \chi^2 \rangle$. It is obvious that the generation of perturbations $\langle \chi^2 \rangle$ leads to symmetry restoration in this model for $\eta_0 \lesssim \sqrt{\langle \chi^2 \rangle}$ on a scale up to $\eta_0 \sim 10^{16}$ GeV [9,10]. Such effects may have important cosmological implications [43].

Thus, we can distinguish between different scenarios depending on the coupling constant g .

For $g \ll 3 \times 10^{-4}$ the broad resonance ends during the first stage. In this case parametric resonance is not efficient enough to transfer a significant part of the energy of the inflaton field to the energy of χ particles. The most important part of the process of preheating in such theories is described by the elementary theory of reheating [5,6,8].

For $g \sim 3 \times 10^{-4}$, at the end of the first stage $q_1 \sim 1/4$, and the energy becomes approximately equally distributed between the energy of the oscillating scalar field ϕ and the energy of χ particles produced by its oscillations.

For $g > 3 \times 10^{-4}$ the broad resonance continues after the end of the first stage. To investigate the further development of the resonance one should study quantum effects which could be produced by the χ fluctuations interacting with the oscillating field $\phi(t)$.

Before doing so, let us remember that the presence of the interaction $g^2 \phi^2 \chi^2$ typically leads to radiative corrections to the effective potential of the type $(g^4 \phi^4 / 32\pi^2) \ln \phi$. For

$g \gtrsim 10^{-3}$ this term becomes greater than the term $m^2 \phi^2 / 2$ for $\phi \sim 4M_p$, when the density perturbations determining the structure of our part of the universe were produced. Thus one may argue that in models of the type considered above g should be smaller than 10^{-3} . If g belongs to the narrow interval between 3×10^{-4} and 10^{-3} , reheating ends soon after the end of the first stage, and the effects of back reaction are only marginally important. For the description of preheating in such theories it is sufficient to use the simple estimates obtained in this section.

However, in supersymmetric theories radiative corrections from bosons and fermions have a tendency to cancel each other. In such theories the coupling constant g can be much greater than 10^{-3} . Therefore, we will continue to consider all possible values of the coupling constant g without assuming that $g < 10^{-3}$.

B. The second stage of preheating, neglecting rescattering

We defined the second stage of preheating, $t > t_1$, as the stage when the frequency of inflaton oscillations due to the feedback of amplified χ fluctuations is no longer m but is determined by the back reaction of χ particles. In this section we will investigate preheating neglecting rescattering. In the next section we will discuss the validity of this assumption. The frequency of the inflaton oscillations during this stage was derived in Sec. VIII B and given by Eq. (94). Since this frequency is much greater than the bare mass m , the second stage is much shorter than the first one. Indeed, at this stage each oscillation takes a time which is much shorter than $2\pi m^{-1}$, whereas the number of particles, as before, grows as $e^{4\pi\mu N}$, where N is the number of oscillations. Therefore the number of particles can grow exponentially within a time which is much shorter than H^{-1} . This implies that one can neglect expansion of the universe and the corresponding decrease of the total energy density of matter during the second stage of preheating.

Let us consider the inflaton oscillations $\phi(t)$ during the second stage. From Eq. (93) in the limit $H \ll m_\phi$ we have

$$\ddot{\phi} + m^2 \phi + g n_\chi \text{sgn} \phi = 0, \quad (123)$$

where $\text{sgn} \phi$ is ± 1 depending on the sign of the value ϕ , $n_\chi(t)$ is a function of time, the expansion of the universe is neglected, and $m^2 \phi \ll g^2 n_\chi \text{sgn} \phi$. The solution of this equation $\phi(t)$ consists of a sequence of segments of parabolas with opposite orientation that are symmetric relative to the t axis and match at $\phi = 0$. The equation for the modes $\chi_k(t)$ will contain the square of $g\phi(t)$ instead of $g^2 \Phi^2 \sin^2 mt$. Thus the behavior of $\chi_k(t)$ for $\phi(t)$ determined by Eq. (123) is somewhat different from the behavior of χ_k as described by the Mathieu equation. Nevertheless, this is not a real problem here.

Indeed, if one does not take back reaction into account, then, according to our investigation in Sec. VI, the system spends half of the time in the broad resonance regime, and another half of the time in the regime with $q \sim 1$, so this regime is very important. However, let us consider the effects of back reaction. The parameter $q = g^2 \Phi^2 / 4m_\phi^2$ at the second stage can be estimated using the ‘‘effective mass’’ (or, more exactly, the frequency of oscillations of the field

ϕ) $m_\phi^2 \sim g n_\chi / \Phi \sim g^2 \langle \chi^2 \rangle$ (94). This gives $q \sim g \Phi^3 / 4 n_\chi$. The end of the resonance, as before, occurs at $q \sim 1/4$, see below. The number of χ particles grows exponentially, so during the previous oscillation one had $q \sim e^{4\pi\mu} / 4 \sim 1$, and during the previous oscillation q was much greater than 1. Therefore during all the time except the last one or two oscillations the parameter q was very large, the resonance was very broad, and it could be described by the theory of stochastic resonance. This theory is very robust; it depends only on the speed of the field ϕ near $\phi=0$. Thus the difference between the Mathieu equation and the equation for the modes χ_k in the field $\phi(t)$ satisfying Eq. (123) in this context becomes unimportant.

On the other hand, at the time when q decreases, the structure of the first resonance band becomes important. We investigated this issue by solving equations for the modes χ_k numerically. We found that if the field $\phi(t)$ obeys Eq. (123), the structure of the first resonance band for χ_k at small momenta is very similar to that of the Mathieu equation. Therefore, the second stage of preheating in this case ends when

$$q = \frac{g^2 \Phi^2}{4 m_\phi^2} \simeq \frac{g^2 \Phi^2}{4 g^2 \langle \chi^2 \rangle} \simeq \frac{g \Phi^3}{4 n_\chi} \sim 1/4, \quad (124)$$

just as before. This happens at some moment t_2 when

$$g \Phi_2 \simeq m_\phi(t_2), \quad \Phi_2 \simeq \sqrt{\langle \chi^2 \rangle_2}, \quad n_\chi(t_2) \simeq \frac{g |\Phi_2|^3}{4}. \quad (125)$$

At this time the total energy density becomes approximately equally distributed between the kinetic energy of χ particles $\sim (g m_\phi \Phi / 8) \langle \chi^2 \rangle$ and the energy $\sim g \Phi n_\chi$ of their interaction with the field ϕ (which includes the potential energy of the field ϕ). This energy should be equal to the total energy of the system at the time t_1 , which is given by $3 m^2 \Phi_1^2 / 2$. The final value of the inflaton field at the end of resonance is

$$\Phi_2 \simeq \Phi_1 q_1^{-1/4}. \quad (126)$$

Thus, Φ_2 is somewhat smaller than Φ_1 for $q_1 > 1$:

$$\Phi_2 \simeq \sqrt{\langle \chi^2 \rangle_2} \simeq \left(\frac{8 \mu m M_p}{3g} \ln^{-1} \frac{10^{12} m}{g^5 M_p} \right)^{1/2}. \quad (127)$$

To find the typical occupation numbers of the modes with $k \sim k_*$ at the end of the second stage of reheating, let us remember that $n_k \simeq 48 \pi^2 n_\chi / k_*^3$. This corresponds to enormously large occupation numbers [1]

$$n_k(t_2) \simeq 10^2 g^{-2}. \quad (128)$$

This result indicates potential problems with the perturbative investigation of preheating at the end of its second stage. Adding extra internal lines of the diagrams may introduce enormous factors $n_k \simeq 10^2 g^{-2}$, which may cancel extra degrees of g^2 which appear in the higher order corrections.

In order to calculate the duration of the second stage let us note that $n_\chi(t_2) \simeq n_\chi(t_1) e^{4\pi\mu N_2}$. One can show that $n_\chi(t_2) / n_\chi(t_1) \simeq 4 q_1^{1/4}$. Therefore, the duration of the second stage is

$$N_2 \simeq \frac{1}{4\pi\mu} \ln 4 q_1^{1/4}. \quad (129)$$

Using the table of values of q_1 given in the previous subsection, one can conclude that the second stage may take from 2 oscillations (for $g = 10^{-3}$) to about 10 oscillations (for $g = 10^{-1}$).

Numerical estimates of $\Phi_2 \sim \sqrt{\langle \chi^2 \rangle_2}$ show that it can be in the range of 10^{15} to 10^{16} GeV. As an example, for $g = 10^{-2}$, which corresponds to $q_0 \simeq 10^6$, one has $\Phi_2 \sim \sqrt{\langle \chi^2 \rangle_2} \simeq 10^{16}$ GeV. An interesting feature of Eq. (127) is the inverse dependence of $\sqrt{\langle \chi^2 \rangle_2}$ on the value of the coupling constant.

Note that in addition to the high-frequency oscillations with frequency $\sim g \Phi$ discussed in Sec. VIII B, the amplitude of fluctuations $\sqrt{\langle \chi^2 \rangle}$ experiences oscillations with a frequency $2m$. At the end of the second stage these two frequencies coincide. In all our estimates we calculated the *minimal* value of $\sqrt{\langle \chi^2 \rangle}$ which occurs when $|\phi(t)| \simeq \Phi$. It was important for us because this is the time which determines the frequency of oscillations of the field $\phi(t)$. Near $\phi(t) = 0$ the amplitude of fluctuations $\sqrt{\langle \chi^2 \rangle}$ is greater than at $|\phi(t)| \simeq \Phi$, but close to the end of the second stage of preheating this difference becomes less significant.

The results of numerical calculations of $\sqrt{\langle \chi^2 \rangle}$ performed in [30] are in agreement with our estimates for $g \leq 3 \times 10^{-4}$ but give a few times greater value of $\sqrt{\langle \chi^2 \rangle_2}$ for larger g . The difference can be interpreted as a result of rescattering of χ particles during the second stage of preheating.

X. RESCATTERING

Theoretical considerations contained in [24–27,30] and numerical lattice simulations of preheating [24,28–30] indicate that there is another effect which should be incorporated into the preheating scenario. In the context of the model investigated in this paper, one should consider the generation of inflaton fluctuations $\delta\phi$ due to the interaction of χ particles with the oscillating inflaton field $\phi(t)$, and subsequent interaction between χ and $\delta\phi$ fluctuations. We already discussed in Sec. VIII D the possibility to describe this process by equations for classical waves. One may also represent the classical scalar field as a condensate of ϕ particles with zero momentum, and interpret ϕ -particle production as a result of rescattering of χ particles and the ϕ particles in the condensate [24,25,29,30]. This ‘‘particlelike’’ interpretation of the interaction allows one to use the concept of cross section of the interacting particles, and the Boltzmann equation for the occupation numbers.

The theory of this process is rather complicated, and its interpretation in terms of the rescattering of elementary particles is not universally valid; see Sec. X B. Still we can formulate the following apparently general results. First, there is a significant generation of rapidly growing fluctuations $\delta\phi \propto e^{2\mu m \phi t}$ due to the interaction between χ particles

and the oscillating field $\phi(t)$. Second, the generation of large fluctuations of $\delta\phi$ can terminate the resonant creation of χ particles only at the end of the second stage of reheating. In this section we will try to justify these statements.

A. Generation of ϕ particles by rescattering

To evaluate the effects of rescattering we will use here the ‘‘particlelike’’ interpretation of rescattering. First, one should make an estimate of the cross section σ for the scattering of χ particles with an effective mass $g\phi(t)$ and a typical physical momentum $\simeq k_*/2 = \sqrt{gm\Phi}/2$ on ϕ particles of mass m with zero initial momentum which constitute the oscillating field $\phi(t)$. The effective mass of the field χ is time dependent. This makes investigation of their scattering rather complicated. However, in the broad resonance regime during the main part of the oscillation [for $|\phi| > \phi_* \simeq \frac{1}{3}\Phi q^{-1/4}$ (33)], the field χ changes adiabatically. During this time, the effective mass of the field χ also changes adiabatically, so one may consider χ particles as ordinary particles with an effective mass $g\phi(t)$. We will also consider the oscillating scalar field $\phi(t)$ as a collection of particles ϕ with an effective mass m_ϕ and number density $n_\phi = m_\phi \Phi^2/2$.

We will suppose now that in such situation one can use the standard result for the cross section for elementary particles ϕ and χ with constant masses:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{|p_\phi| \mathcal{M}^2}{64\pi^2 E_\phi E_\chi (E_\phi + E_\chi) |v_\phi - v_\chi|}. \quad (130)$$

Here all energies E_ϕ, E_χ and velocities v_ϕ, v_χ are given in the center-of-mass (c.m.) frame and refer to the initial state, except for p_ϕ which refers to the final state. \mathcal{M}^2 is the square of the matrix element, which is given by g^4 [32].

During most of an oscillation one has $|\phi| > \phi_* \simeq \frac{1}{3}\Phi q^{-1/4}$, and $m_\chi = g\phi \gg k_* \sim \sqrt{gm\Phi}$. In this case both the ϕ particles and χ particles are nonrelativistic. If one goes to the c.m. frame one finds that the ϕ particles have a small speed $v_\phi \approx \frac{1}{2}\sqrt{m_\phi/g\phi} \gg v_\chi$. Thus $E_\phi = m_\phi$, $E_\chi \approx g\phi$. For $g\phi \gg m_\phi$ the absolute value of the momentum of the ϕ particles does not change after scattering, $|p_\phi| \approx (m_\phi/2)\sqrt{m_\phi/g\phi} \ll m_\phi$. This gives, after the integration of Eq. (130) over $d\Omega$, a single particle cross section $\sigma_1 \sim g^4/16\pi E_\chi^2 = g^2/16\pi\phi^2$.

Now one should take into account that the actual cross section will be much greater because the scattering occurs not in a vacuum, but in a state which already contains many bosons ϕ and χ . There are many χ particles from the resonance and many inflaton particles ϕ . Naively one would expect that the cross section should be proportional to the product of the occupation numbers n_p^ϕ and n_k^χ in the final state. However, the corresponding terms disappear in the collision integral in the Boltzmann equation, which takes into account all the channels of scattering. Therefore in the investigation of enhancement of the cross section due to the large occupation numbers of particles in the final state, one should consider terms proportional either to n_p^χ or n_k^ϕ , but not to $n_p^\phi n_k^\chi$. In the beginning of the process $n_\chi \gg n_\phi$, and the cross sec-

tion σ_1 should be multiplied by $n_p^\chi \simeq 48\pi^2 n_\chi(t)/k_*^3$. This gives, for $\phi(t) \simeq \Phi$, $\sigma \sim 3\pi g^2 n_\chi/k_*^3 \Phi^2$.

Using this result, one can estimate the time for each χ particle to experience one scattering with a ϕ particle belonging to the oscillating field $\phi(t)$: $\tau = 1/\sigma n_\phi v_\phi \simeq 0.5(\Phi^2/n_\chi)$. In particular, at the end of the first stage, $n_\chi \simeq m_\phi^2 \Phi_1/g$, which yields

$$\tau \simeq m_\phi^{-1} q_1^{1/2}. \quad (131)$$

For $g \sim 10^{-3}$ this time is of the same order as the time of one oscillation of the field ϕ ; see the table in Sec. IX. However, just one oscillation before the end of the first stage the density of particles was much smaller and rescattering was inefficient. For $g \gtrsim 10^{-2}$ this time is much greater than the time of one oscillation, which means that rescattering occurs only during the second stage of preheating.

In the ‘‘particlelike’’ picture the number of χ particles does not change in each act of interaction (apart from its growth due to the resonance), but each collision releases one ϕ particle from the homogeneously oscillating field $\phi(t)$. Since the scattering time for each χ particle $\tau \propto n_\chi(t)$, one may conclude that the number of free ϕ particles grows with time as $n_\phi \propto 5n_\chi^2/\Phi^2 m_\phi \propto e^{4\mu m_\phi t}$. However, the true dependence is more complicated because during each interaction the χ particles will slow down. This affects their subsequent interactions.

B. On the validity of the ‘‘particlelike’’ interpretation of rescattering

In the previous subsection we considered rescattering of particles during time intervals when $\phi(t) > \phi_*$. At that stage χ particles are nonrelativistic. In contrast, during the short time intervals $\Delta t_* \simeq k_*^{-1}$, when $|\phi(t)| < \phi_*$, χ particles are ultrarelativistic, and their effective mass $g\phi$ is very small comparing to their typical momenta $\sim k_*/2$. If one uncritically repeats the calculation of the rescattering for the case of ultrarelativistic χ particles in the time interval Δt_* , one obtains a much higher cross section and a much shorter rescattering time $\tau \simeq m_\phi^2/3\pi^2 g^2 n_\chi$ than that of the nonrelativistic case of the previous subsection.

However, within the very short time interval $\Delta t_* \simeq q^{-1/4} m_\phi^{-1}$, one cannot use the standard methods of calculation [32] developed for the investigation of processes which begin at $t = -\infty$ and end at $t = +\infty$. The uncertainty principle tells us that during the time Δt_* one cannot specify the energy of particles with an accuracy better than k_* . Therefore during the short interval Δt_* one cannot tell the difference between a ϕ particle with momentum $k=0$, belonging to the classical field $\phi(t)$, and a free ϕ particle with momentum $k < k_*$, i.e., one cannot tell whether scattering occurred or not. This question can be answered only by observing the system for a longer time, comparable to m_ϕ^{-1} , but during the main part of such intervals the effective mass of each ϕ particle is large, and cross section is much smaller than the cross section which one would obtain by naive application of the S matrix approach during a small interval Δt_* . In other words, we cannot use the standard formalism of particle scattering to describe scattering around zeros of the inflaton field. Another element missing in this formalism

is that the field χ is not in an n_χ -particle quantum state, but is in the squeezed state. (We have discussed already one of the nontrivial consequences of this fact, namely the high-frequency modulation of $\langle\chi^2\rangle$.) Thus one may wonder whether one can trust the results of our calculations for the more safe situation when $\phi > \phi_*$, and what we can say about the contribution of the intervals with $\phi < \phi_*$ to the net rescattering effect?

Here we will outline a possible way to answer this question. Let us consider the self-consistent set of equations (104) and (105) for the interacting fields in the classical approximation. Equation (105) describes the evolution of the $\delta\phi_k(t)$ fluctuation. Let us concentrate on the first integral term in Eq. (105), assuming for the moment that the second term is subdominant until $\delta\phi_k(t)$ increases sufficiently. What we obtain is the equation for the forced oscillations of $\delta\phi_k(t)$. The homogeneous part of this inhomogeneous linear differential equation has a simple Green function $\propto \sin\Omega_{\mathbf{k}}(t-t')$, where $\Omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m_\phi^2$. Then the solution of Eq. (105) with only the first integral term is

$$\begin{aligned} \delta\phi_k(t) = & -\frac{g^2}{(2\pi)^3\Omega_{\mathbf{k}}} \int_0^t dt' \sin\Omega_{\mathbf{k}}(t-t') \phi(t') \\ & \times \int d^3k' \chi_{\mathbf{k}-\mathbf{k}'}(t') \chi_{\mathbf{k}'}(t') + \text{H.c.} \end{aligned} \quad (132)$$

Here, as before, k is a physical momentum. This solution expresses the function $\delta\phi_k(t)$ via the known functions $\phi(t)$ describing the inflaton oscillations, see Eq. (4), and the functions $\chi_k(t)$; see Sec. VII. Equation (132) takes into account all the regimes of $\phi(t)$, as well as the resonant amplification of χ_k . In particular, from this it follows that the amplitude $\varphi_k(t)$ grows with time as $e^{2\mu m_\phi t}$, because the amplitude χ_k grows as $e^{\mu m_\phi t}$. Therefore the number of particles corresponding to $\delta\phi$ fluctuations is proportional to $n_\chi^2(t) \propto e^{4\mu m_\phi t}$, i.e., grows much faster than n_χ . Another specific prediction which follows from Eq. (132) is that the random field $\delta\phi(t, \mathbf{x})$ at the early stages of its generation will have non-Gaussian statistics in contrast to the random Gaussian field $\chi_{t, \mathbf{x}}$.

Let us further investigate the solution (132). The inner integral $\int d^3k' \chi_{\mathbf{k}-\mathbf{k}'}(t') \chi_{\mathbf{k}'}(t')$ is time-dependent. It is convenient to change the order of integration of $\int dt'$ and $\int d^3k'$. Then the right-hand side of the solution (132) will contain terms such as

$$\begin{aligned} & \frac{g^2\Phi e^{i\Omega_{\mathbf{k}}t'}}{(2\pi)^3\Omega_{\mathbf{k}}} \int d^3k' \int_0^t dt' \frac{\beta_{\mathbf{k}'}\beta_{\mathbf{k}-\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}-\mathbf{k}'}(t')\omega_{\mathbf{k}'}(t')}} \\ & \times \exp\left(-i\Omega_{\mathbf{k}}t' + im_\phi t' - i \int_0^{t'} dt'' \omega_{\mathbf{k}-\mathbf{k}'}(t'') \right. \\ & \left. + i \int_0^{t'} dt'' \omega_{\mathbf{k}'}(t'')\right), \end{aligned} \quad (133)$$

where $\omega_{\mathbf{k}}^2(t) = \mathbf{k}^2 + g^2\phi^2(t)$. During each half of the oscillation $\beta_{\mathbf{k}}$ is constant; see Sec. VII A. (Note that the coefficients $\beta_{\mathbf{k}}$ corresponding to the classical waves will be dimensionless if one uses discrete modes \mathbf{k} .)

It is easy to see that different choices of signs in Eq. (133) correspond to different channels of scattering between $\delta\phi$, χ , and ϕ waves. The terms (133) correspond to the generation of $\delta\phi$ fluctuations due to the scattering of χ and ϕ waves. Obviously, one can leave in the inner integral \int_0^t only the segment $\int_{t_j}^t$ (where $t < t_j + \pi/m_\phi$) from the most recent cycle of the inflaton oscillation, when $\beta_{\mathbf{k}'}$ is the largest. During this interval $\beta_{\mathbf{k}'}$ is constant. Therefore to further investigate the inner integral $\int_{t_j}^t$, we shall consider the variation of the phase of the exponent in Eq. (133) $\theta \simeq -\Omega_{\mathbf{k}}t' + m_\phi t' - \int_{t_j}^t dt'' \omega_{\mathbf{k}-\mathbf{k}'}(t'') + \int_{t_j}^t dt'' \omega_{\mathbf{k}'}(t'')$ within this time interval $t - t_j < \pi/m_\phi$, which describes the interference of the four interacting waves $\phi(t)$, $\delta\phi_{\mathbf{k}}$, $\chi_{\mathbf{k}'}$, and $\chi_{\mathbf{k}-\mathbf{k}'}$. Earlier we estimated the integral $\int_{t_j}^t dt'' \omega_{\mathbf{k}'}(t'') \simeq (2g\Phi/m_\phi) \cos m_\phi t' + O(\kappa^2)$; see Eq. (77). The crucial observation is that for the process $\chi_{\mathbf{k}'}\phi_0 \rightarrow \delta\phi_{\mathbf{k}}\chi_{\mathbf{k}'-\mathbf{k}}$ the large terms $(2g\Phi/m_\phi) \cos m_\phi t'$ in the expression for θ are cancelled and the phase θ does not oscillate within each half of the period, $t - t_j < \pi/m_\phi$. As a result, the integral $\int dt$ cannot be reduced to the usual delta-function $\delta(-\Omega_{\mathbf{k}} + m_\phi - \omega_{\mathbf{k}-\mathbf{k}'} + \omega_{\mathbf{k}'})$, as one would expect in the ‘‘particle-like’’ picture. Instead, in the wave picture we will have nonvanishing contributions from the bunches of modes \mathbf{k} and \mathbf{k}' for which the phase $\theta \simeq \pi$, which corresponds to the interaction of packets of χ and $\delta\phi$ waves. In contrast to the process of rescattering, the annihilation process $\chi_{\mathbf{k}}\chi_{\mathbf{k}'} \rightarrow \delta\phi_{\mathbf{k}''}\delta\phi_{\mathbf{k}+\mathbf{k}'-\mathbf{k}''}$ and the inverse process will be suppressed because the corresponding time integrals have very rapidly oscillating exponents $\exp[\pm i(4g\Phi/m_\phi) \cos m_\phi t']$.

The analysis of Eq. (133) shows the hard component $\delta\phi$ with $k \simeq k_*$ can be generated only during the very short time intervals $\Delta t_* \simeq k_*^{-1}$ around zeros of the inflaton field. The soft component with momenta $k \ll k_*$ is generated all the time. Soft particles produced at $|\phi| > \phi_*$ have very small momenta in the range of $0 < k < m$. It makes sense to talk about such particles as free ϕ particles removed from the coherently oscillating field $\phi(t)$ only at time intervals $\tau \gg m_\phi^{-1}$. An estimate of the soft component from Eq. (133) at the beginning of the process is $\langle\phi^2\rangle_{\text{soft}} \simeq g^2 n_\chi^2 / m_\phi^4$, whereas for the hard component one has $\langle\phi^2\rangle_{\text{hard}} \simeq \langle\phi^2\rangle_{\text{soft}} / \sqrt{q}$. Since $\delta\phi$ grows very fast, one has to be careful with the range of validity of the solution (132). Indeed, Eq. (132) is only the first term in the iterative solution of Eq. (105). As soon as $\delta\phi$ grows, we have to consider the iterative solutions of both Eqs. (104) and (105). We have to take into account the corrections to X_k due to the X and φ coupling on the right-hand side of Eq. (104) as well as the second bilinear term on the right-hand side of Eq. (105). Because of the exponential growth in the number of particles, these corrections to the simple solution (132) very quickly become important, which makes further investigation rather complicated.

One should note that in addition to rescattering, there may exist other mechanisms of ϕ -particle production. For example, let us consider fluctuations $\delta\phi$ with effective mass squared $g^2\langle\chi^2\rangle$. As we already emphasized, this term is time-dependent. First of all, it experiences quasiperiodic high-frequency modulation, which, as we already noted in

Sec. VIII B, may serve as an additional source of ϕ particles. In addition, the term $g^2\langle\chi^2\rangle$ oscillates with period π/m_ϕ . During each oscillation it changes from its minimal value $g\Phi n_\chi$ [for $|\phi(t)|=\Phi$] to a much greater value $\sim 3g\Phi n_\chi q^{1/4}$ [for $|\phi(t)|=\phi_*$]. This leads to a significant periodic change in the properties of ϕ particles, which is especially pronounced when $|\phi(t)|\leq\phi_*$. A preliminary investigation of this issue indicates the possibility of a parametric resonance with ϕ -particle production.

Our main purpose here was not to give the final analysis of this issue but rather to outline different approaches to the problem of rescattering and ϕ -particle production, which should provide a proper framework for future investigation.

C. Rescattering and the end of preheating

Can rescattering kill the resonance? In Sec. X A we found that rescattering can be rather efficient at the second stage of preheating. What can we say about the influence of rescattering on the development of parametric resonance?

The simplest idea would be to estimate the effective mass of the χ particles induced by the fluctuations $\langle\phi^2\rangle$: $\Delta m_\chi^2\sim g^2\langle\phi^2\rangle$. However, this would not be quite correct. Indeed, the whole process of χ -particle production occurs in the interval $|\phi|\leq\phi_*$ during the time $t_*\sim(gm_\phi\Phi)^{-1/2}=k_*^{-1}$; see Eq. (35). If oscillations of the modes $\delta\phi$ occur during a longer time, then from the point of view of the creation of χ particles they cannot be distinguished from the oscillations of the field $\phi(t)$, and therefore they do not harm the development of stochastic resonance. We called such modes ‘‘soft,’’ and the modes with $k\geq k_*/4$ ‘‘hard.’’

Fluctuations of the scalar field ϕ can be harmful to the development of the resonance if they can considerably alter the motion of the field ϕ in the interval $|\phi|\leq\phi_*$. The only fluctuations which can change the direction of their motion during the short time $t_*\sim k_*^{-1}$ are the modes with $k\geq 2\pi k_*\gg k_*$. This effect does not seem to be very important. At the time when the homogeneous mode $\phi(t)$ enters the interval $|\phi|>\phi_*$, it has a kinetic energy $\dot{\phi}^2/2\sim m_\phi^2\Phi^2/2$. In order to alter the motion of the field ϕ the ‘‘hard’’ fluctuations $\delta\phi$ should (occasionally) have comparable (and opposite) speed, and therefore they should have a kinetic energy comparable to $m_\phi^2\Phi^2/2$. Thus, the resonance disappears only after the kinetic energy of ϕ particles with momenta $k\gg k_*$ becomes comparable to the total energy of the oscillating field $\phi(t)$. This could happen only at the very end of preheating.

However, there is another mechanism which may harm the resonance. Each mode χ_k ‘‘probes’’ space on a length scale $\Delta l\sim 2\pi k^{-1}$. If the field $\delta\phi$ is homogeneous on this scale, it acts as a homogeneous background for the mode χ_k . On the other hand, if $\delta\phi$ is inhomogeneous on this scale, then the field χ_k has an integrated interaction with all inhomogeneities of the field $\delta\phi$ on the scale $\Delta l\sim 2\pi k^{-1}$, i.e., it interacts with the contribution to $\langle\phi^2\rangle$ from the modes with momenta greater than k . This corresponds to the appearance of an ‘‘effective mass squared’’ $\Delta m_\chi^2\sim g^2\langle\phi^2\rangle$, but only the modes with momenta greater than k should be taken into account in this calculation. Thus, from the point of view of

the development of parametric resonance, one can introduce a new notion of an effective mass squared $\Delta m_\chi^2(k)\sim g^2\langle\phi^2\rangle_k$, where the index k means that we take into account only the modes with momenta greater than k .

If the effective mass squared $\Delta m_\chi^2(k)$ becomes greater than k^2 , the equation of motion for such modes χ_k changes considerably. This effect kills the resonance for the mode χ_k if $\Delta m_\chi^2(k)$ becomes greater than the width of the resonance. The resonance for the leading modes with $k\sim k_*/4$ ends when $\Delta m_\chi^2(k_*)\sim g^2\langle\phi^2\rangle_{\text{hard}}$ becomes greater than $k_*^2/4$.

The difference between the total value of $\langle\phi^2\rangle$ and $\langle\phi^2\rangle_{\text{hard}}\equiv\langle\phi^2\rangle_{k_*/4}$ can be quite significant. The number of ϕ particles produced in each scattering is equal to the number of χ particles, each ϕ particle taking away some portion of the momentum k of the corresponding χ particle. If this portion is small, $\delta\phi$ fluctuations corresponding to these particles have momenta much smaller than $k_*/4$. Therefore, they do not give any contribution to the effective mass $\Delta m_\chi^2(k\sim k_*/4)$, so they do not hurt the resonance at such momenta. If in the first collision a χ particle with momentum $k\sim k_*/4$ gives a significant portion of its energy to a ϕ particle, then it loses its energy, and in subsequent collisions it will produce only harmless $\delta\phi$ fluctuations with $k\ll k_*/4$.

Thus, one may argue that if rescattering is efficient, the number of ‘‘hard’’ ϕ particles produced by χ particles should be similar to the initial number of χ particles with momenta $\sim k_*/4$, i.e., $n_\phi^{\text{hard}}\leq n_\chi$, whereas the total number of ϕ particles produced by rescattering may be much greater. At the second stage of reheating, when $g^2\langle\chi^2\rangle\gg m_\phi^2$, one can use an estimate

$$\langle\delta\phi^2\rangle\approx\frac{1}{2\pi^2}\int\frac{k^2dk}{\sqrt{k^2+g^2\langle\chi^2\rangle}}n_k^\phi. \quad (134)$$

If $\langle\delta\phi^2\rangle$ is dominated by soft fluctuations with $k^2\ll g^2\langle\chi^2\rangle$, then at the second stage of the resonance one should expect a strong anticorrelation between oscillations of $\langle\chi^2\rangle$ and $\langle\delta\phi^2\rangle$. This prediction is in agreement with the numerical results of [30].

Now let us concentrate on the ‘‘hard’’ fluctuations with typical momenta $\sim k_*/4$. They can hamper the resonance if they make the field χ massive, with an induced effective mass squared $\Delta m^2\sim g^2\langle\delta\phi^2\rangle_{\text{hard}}$ comparable to the square of the typical momentum of χ particles $k\sim k_*/4$:

$$g^2\langle\delta\phi^2\rangle_{\text{hard}}\geq gm_\phi\Phi/16. \quad (135)$$

Suppose that a fraction γ of all energy $m_\phi^2\Phi^2/2$ is transferred to the kinetic energy $(k^2/2)\langle\delta\phi^2\rangle_{\text{hard}}$ of ‘‘hard’’ fluctuations,

$$\frac{gm_\phi\Phi}{32}\langle\delta\phi^2\rangle_{\text{hard}}\approx\gamma\frac{m_\phi^2\Phi^2}{2}. \quad (136)$$

This gives

$$g^2\langle\delta\phi^2\rangle_{\text{hard}}\approx 16\gamma gm_\phi\Phi. \quad (137)$$

Comparison of Eqs. (137) and (135) shows that $\gamma\geq 1/256$, i.e., the resonance may slow down and eventually terminate

only when the oscillating field ϕ transfers at least $\sim 1/256$ of its energy to the ‘‘hard’’ fluctuations ϕ . The total energy of all ϕ particles will be somewhat greater than that. These particles get their kinetic energy from the kinetic energy of χ particles $\sim (gm_\phi\Phi/8)\langle\chi^2\rangle$, so one may expect that the resonance terminates only after $(gm_\phi\Phi/32)\langle\chi^2\rangle$ becomes greater than $1/256(m_\phi^2\Phi^2/2)$. This can only occur close to the end of preheating. Let $\sqrt{\langle\chi^2\rangle}_r$ and Φ_r be the values of χ fluctuations and amplitude of the background field at the moment t_r when the parametric resonance is terminated by rescattering. Taking into account that at the second stage of preheating $m_\phi^2 \simeq g^2\langle\chi^2\rangle$ one finds that at the end of preheating

$$\sqrt{\langle\chi^2\rangle}_r \geq \Phi_r/16. \quad (138)$$

Note also that $\sqrt{\langle\chi^2\rangle}_r \leq \Phi_r$, because this would correspond to the result which we obtained in Sec. IX B neglecting rescattering. In our subsequent calculations we will use the estimate $\sqrt{\langle\chi^2\rangle}_r \sim 10^{-1}\Phi_r$. This value is somewhat smaller than $\sqrt{\langle\chi^2\rangle}_2 \simeq \Phi_2$ which we obtained in Sec. IX B neglecting rescattering. However, the difference between these two values is in fact not very large because $\Phi_r > \Phi_2$.

We are going to find $\sqrt{\langle\chi^2\rangle}_r$ and Φ_r , which should replace our previous estimates for $\sqrt{\langle\chi^2\rangle}_2$ and Φ_2 at the end of the second stage neglecting rescattering. Again we will use energy conservation. At the end of the first stage the energy density was equal to the potential energy density $m^2\Phi_1^2/2$ of the inflaton field plus the energy of its interaction $g\Phi_1 n_\chi \sim m^2\Phi_1^2$, where m is the bare inflaton mass. At the end of the resonance (at the second stage), with an account taken of rescattering, the kinetic energy of the χ particles remains small, so the whole energy $\sim 3m_\phi^2\Phi_1^2/2$ transforms to the energy density of interaction between χ particles and the field ϕ , $\rho_\chi = g\Phi_r n_\chi \sim g^2\langle\chi^2\rangle_r \Phi_r^2 \sim 10^{-2}g^2\Phi_r^4$. Note that ρ_χ includes the energy of the oscillating scalar field $\phi(t)$. Energy conservation implies that $\Phi_r \sim 3.5\sqrt{m}\Phi_1/g \sim 2.5\Phi_1 q_1^{-1/4}$. However, Φ_r obviously cannot be greater than Φ_1 . This means that rescattering can terminate the resonance either if $\sqrt{\langle\chi^2\rangle}_r \geq 10^{-1}\Phi_r$, in which case we essentially recover the previous results of Sec. IX B, or if $q_1 \geq 10^2$. In the last case one has $\sqrt{\langle\chi^2\rangle}_r \sim 0.35\sqrt{m}\Phi_1/g$, which yields

$$\sqrt{\langle\chi^2\rangle}_r \sim \left(\frac{\mu m M_p}{6g} \ln^{-1} \frac{10^{12}m}{g^5 M_p} \right)^{1/2}. \quad (139)$$

This estimate should replace Eq. (127) derived without account taken of rescattering. In particular, for $g = 10^{-2}$, which corresponds to $q_0 = 10^6$, and $q_1 \sim 550$, we get $\sqrt{\langle\chi^2\rangle}_r \approx 2.5 \times 10^{15}$ GeV. To compare this result to the result of [30] one should note that the definition of q_0 in [30] differs slightly from ours, so it is better to compare our results for a given g rather than for a given q_0 . In particular, one should compare their results for $q_0 = 10^6$ with our results for $g = 10^{-2}$: $\sqrt{\langle\chi^2\rangle}_r \approx 3 \times 10^{15}$ GeV. This result agrees, to within a factor of 2, with the results of the lattice simulation of [30].

One should not overemphasize the significance of this agreement. The theory of the last stages of preheating is extremely complicated, and there are many points in which

our rough estimates could be improved. One should remember also that we are discussing stochastic resonance, which is extremely sensitive to even minor changes of parameters; see the table in Sec. VI [44]. From this perspective it is even somewhat surprising that one can describe many features of this process by analytical methods with rather good accuracy.

Strictly speaking, the condition which we derived does not imply that the resonance is completely terminated. The leading modes, which have been amplified from the very beginning, stop growing when the effective mass of the field χ becomes greater than $k \sim k_*/4$. However, the subleading modes still continue their growth until the effective mass becomes greater than $k_*/2$. This process is very inefficient, but $\langle\chi^2\rangle$ continues slowly growing for a while. Moreover, $\langle\chi^2\rangle$ may grow a little even when the resonance is completely terminated and new particles are no longer produced. Indeed, due to the decay of the field $\phi(t)$, the effective mass of the χ particles becomes smaller, and therefore $\langle\chi^2\rangle$ may become greater even if n_χ remains constant. These effects are not very significant, but they make it difficult to clearly recognize the end of parametric resonance by looking at the behavior of $\langle\chi^2\rangle$. That is why throughout this paper, alongside the dispersion of the fluctuations which is studied in most papers on preheating, we use the number density of particles to investigate the resonance.

An estimate of the density of χ particles at the end of the resonance can be obtained by multiplying $\langle\chi^2\rangle_r$ by $g\Phi_r \sim 16g\sqrt{\langle\chi^2\rangle}_r$. It is given by

$$n_\chi \sim 0.4g^{-1/2} \left(\mu m M_p \ln^{-1} \frac{10^{12}m}{g^5 M_p} \right)^{3/2}. \quad (140)$$

It is useful to compare this number with the number of ϕ particles n_ϕ in the oscillating field $\phi(t)$ which survive the rescattering. To distinguish the particles ϕ in the oscillating field and the free ϕ particles created by rescattering, we will denote the number of particles in the classical field as n_ϕ^c . At the end of the resonance it is given by $m_\phi\Phi_r^2/2$, where m_ϕ is the effective mass $g\sqrt{\langle\chi^2\rangle}_r \simeq 0.1g\Phi_r$. Meanwhile $n_\chi \sim g\Phi_r\langle\chi^2\rangle_r \sim 10^{-2}g\Phi_r^3$. Therefore,

$$n_\chi \sim 10^{-1}n_\phi^c. \quad (141)$$

Equation (141) says that at the end of the resonance χ particles need to rescatter only 10 times to destroy the coherent oscillations of the classical field, i.e., to decompose it into separate ϕ particles. Therefore one may expect that at the end of the resonance or very soon after it χ particles may destroy the classical field $\phi(t)$ completely, in agreement with [30]. This means that the final stage of decay of the homogeneously oscillating classical scalar field in our model is determined not by resonance but by rescattering.

The decay of the classical scalar field $\phi(t)$ is not the end of the story, but rather the beginning of a new stage of reheating. As we pointed out in [1], it does not make much sense to calculate the reheating temperature at this stage of the process. Indeed, from the point of view of the energy stored in the ϕ particles, it is not very important whether it is in the form of ϕ fluctuations or in the form of a coherently

oscillating field ϕ . According to our estimates, the kinetic energy of χ particles may constitute only about 10^{-2} of the total energy at the end of parametric resonance. This estimate may be too pessimistic, but even if the true energy is much higher, the main fraction of energy after the end of the resonance remains stored in the energy of ϕ particles, and the energy of their interaction with χ particles. The total energy of χ fluctuations at large t decreases as a^{-4} , whereas the energy of ϕ fluctuations as well as the energy of the oscillating field $\phi(t)$ at large t decreases as a^{-3} . Even if the total energy of the oscillating field $\phi(t)$ and of ϕ particles were very small after preheating, eventually it would again dominate the energy density of the universe. Equation (141) gives us additional information: the number of ϕ particles after preheating is at least ten times greater than the number of χ particles. If these particles do not decay, they will always dominate the energy density of the universe, which is unacceptable. Therefore when preheating ends one should apply the elementary (perturbative) theory of reheating [5,6] to describe the decay of the remnants of the classical oscillating field $\phi(t)$ as well as the decay of the large amount of ϕ particles created by rescattering. We will return to the theory of this process in a subsequent publication [8].

XI. PRODUCTION OF SUPERHEAVY PARTICLES DURING PREHEATING

One of the most interesting effects which may become possible during preheating is the copious production of particles which have a mass greater than the inflaton mass m . This question is especially interesting in the context of the theory of GUT baryogenesis, which may occur in a rather unusual way if superheavy particles with mass M a few times heavier than m can be produced [11]. Such processes are impossible in perturbation theory and in the theory of narrow parametric resonance. However, we are going to show that superheavy χ particles with mass $M \gg m$ can be produced in the regime of a broad parametric resonance.

In order to study this regime let us return to Sec. V, where we made a simple derivation of the width of the resonance band; see Eq. (32). The only modification which should be made to this equation in the case where the field χ has a ϕ -independent mass $m_\chi(0) \equiv M$ is to add it to k^2 on the left-hand side (LHS) of the equation:

$$k^2 + M^2 \lesssim (g^2 \phi m_\phi \Phi)^{2/3} - g^2 \phi^2. \quad (142)$$

As before, the maximal range of momenta for which particle production occurs corresponds to $\phi(t) = \phi_*$, where $\phi_* \approx \frac{1}{2} \sqrt{m_\phi \Phi / g}$. The maximal value of momentum for particles produced at that epoch can be estimated by $k_{\max}^2 + M^2 = g m_\phi \Phi / 2$. The resonance becomes efficient for

$$g m_\phi \Phi \gtrsim 4M^2. \quad (143)$$

Thus, the inflaton oscillations may lead to a copious production of superheavy particles with $M \gg m$ if the amplitude of the field Φ is large enough, $g \Phi \gtrsim 4M^2 / m$.

However, in an expanding universe Φ and m_ϕ are time-dependent. One should not only have a very large field at the

very beginning of the process; one should continue to have $g m_\phi \Phi \gtrsim 4M^2$ until the end of preheating.

During the second stage of preheating both m_ϕ and Φ change very rapidly, but their product remains almost constant because the energy density of the field ϕ , which is proportional to $m_\phi^2 \Phi^2 / 2$, practically does not change until the very end of preheating. Therefore it is sufficient to check that $g m_\phi \Phi \gtrsim 4M^2$ at the end of the first stage of preheating. One can represent this criterion in a simple form:

$$M \lesssim \frac{m}{\sqrt{2}} q_1^{1/4} \approx m \left(\frac{g \mu M_p}{3m} \ln^{-1} \frac{10^{12} m}{g^5 M_p} \right)^{1/2}. \quad (144)$$

For example, one may take $M = 2m$ and $g \approx 0.007$, which corresponds to $q_0 = 10^6$ in the normalization of Ref. [30]. In this our condition (144) is satisfied, and an investigation with an account taken of rescattering shows a relatively insignificant suppression of $\langle \chi^2 \rangle$, approximately by a factor of 3. Our investigation suggests that for $g \gg 10^{-2}$ this process should not be suppressed at all. Equation (144) shows that for sufficiently large g one can produce superheavy particles with $M \gg m$. For example, production of χ particles with $M = 10m$ is possible for $g \gtrsim 0.065$.

In fact, suppression of superheavy particle production may be even less significant. Indeed, the resonance becomes strongly suppressed if it occurs only for $k^2 \ll k_*^2 / 4 \sim g m_\phi \Phi / 4$. As a result, the condition for the efficient preheating (143) can be slightly relaxed: $g m_\phi \Phi \gtrsim 2M^2$. This small modification implies that heavy particle production is not strongly suppressed for

$$M \lesssim m g_1^{1/4} \approx m \left(\frac{2g \mu M_p}{3m} \ln^{-1} \frac{10^{12} m}{g^5 M_p} \right)^{1/2}.$$

For $M = 10m$ this leads to a rather mild condition $g \gtrsim 0.036$.

We conclude that at least in our simple model, the production of superheavy particles is possible. However, with an increase of g the total number of produced particles becomes smaller; see Eq. (140). It would be most interesting to investigate this issue in realistic models of elementary particles and to apply the results to the theory of baryogenesis.

XII. DISCUSSION

In this paper we discussed the theory of preheating for the simple model of a massive inflaton field ϕ interacting with another scalar field χ . As we have seen, the theory of preheating is very complicated even in such a simple model. Our main purpose was not to answer all questions related to the theory of preheating, but to develop an adequate framework in which these questions should be investigated.

In the beginning particle production occurs in the regime of a broad parametric resonance, which gradually becomes narrow and then terminates. If the resonance is narrow from the very beginning, or even if it is not broad enough, it remains inefficient. We have found that broad resonance in an expanding universe is actually a stochastic process. The theory of this process, which can be called stochastic resonance, or stochastic amplification, is dramatically different from the theory of parametric resonance in Minkowski

space. Therefore one cannot simply apply the standard methods of investigation of parametric resonance in Minkowski space; it was necessary to develop new analytical methods for the investigation of stochastic resonance in an expanding universe. We have found the typical width of the resonance $\sim k_*/2$ and the typical rate of the exponential growth of the number of produced particles in this regime. An important feature of our formalism of investigation of the broad resonance regime is its robustness with respect to modification of the form of the effective potential. Our methods should apply not only to theories with the potential $m^2\phi^2/2$, but to any potential $V(\phi)$ when the resonance is broad.

One should note that the main reason why broad resonance has a stochastic nature is the expansion of the universe. In the conformally invariant theories such as the theory $(\lambda/4)\phi^4 + (g^2/2)\phi^2\chi^2$ with $g^2 \gg \lambda$ the resonance is broad but not stochastic because expansion of the universe does not interfere with its development [19]. In realistic theories where the inflaton field ϕ has mass m the conformal invariance is broken and one could expect that the broad resonance becomes stochastic as soon as the amplitude of the oscillations of the field ϕ becomes smaller than $m/\sqrt{\lambda}$. Indeed, for $\Phi \lesssim m/\sqrt{\lambda}$ the resonance is described by the model of a massive inflaton field considered in this paper. A more detailed investigation of this question shows that in models with $g^2 \gg \lambda$ the resonance becomes stochastic even earlier, at $\Phi \lesssim (g/\sqrt{\lambda})(\pi^2 m^2/3\lambda M_p)$ [19].

In our investigation of preheating we took into account the interaction of the oscillating inflaton field ϕ with the particles produced during preheating. We have found, in particular, that the correction to the effective mass squared of the oscillating field ϕ is proportional to $gn_\chi/|\phi|$, and the equation of motion of the field ϕ acquires a term $\sim gn_\chi(\phi/|\phi|)$. This term experiences quasiperiodic oscillations with a very high frequency $\sim 2g\Phi$, which do not much affect the motion of the field $\phi(t)$ but may serve as an additional source of ϕ particles.

We have found that if the coupling constant g^2 in the interaction term $(g^2/2)\phi^2\chi^2$ is small ($g \lesssim 3 \times 10^{-4}$), the resonance terminates at the stage when the back reaction of produced particles is unimportant. For larger values of g^2 the resonance terminates due to a combined effect of the growth of the effective mass of the inflaton field and rescattering, which in its turn increases the effective mass of χ particles, making them heavy and hard to produce. We made an estimate of the number of χ particles produced during preheating and their quantum fluctuations $\langle \chi^2 \rangle$ with all back reaction effects taken into account.

Traditionally, the only purpose of the theory of reheating was to obtain the value of the reheating temperature. From this point of view the theory of preheating for the simple model which we studied in this paper does not change the situation. For $g \lesssim 3 \times 10^{-4}$ the total energy density of produced particles is exponentially small. Similarly, it remains extremely small even for large g if χ particles have mass M much greater than about $10m$. In the case when M is small and $g \gtrsim 3 \times 10^{-4}$, the χ -particle production is very efficient. However, we have found that even in this case after preheating one has many more ϕ particles than χ particles. If χ particles are massless, or if they can easily decay, their con-

tribution to the energy density of the universe rapidly decreases. Therefore, after preheating the main contribution to the energy density of the universe is again given by the ϕ particles. The only difference is that prior to preheating these particles constitute the oscillating classical inflaton field $\phi(t)$, whereas after preheating they acquire various spatial momenta and become decoherent. Thus, as we already pointed out in [1], it does not make much sense to calculate the reheating temperature immediately after preheating. One should study the subsequent decay of the ϕ particles. The theory of this decay is described by the elementary theory of reheating [5,6,8]. So why should one study extremely complicated nonperturbative effects which may happen at the stage of parametric resonance, if in the end they will not greatly change our old estimates of the reheating temperature?

We believe that the investigation of nonperturbative effects in the very early universe is worth the trouble. In fact, the complex nature of this process makes it especially interesting. Indeed, a few years ago the standard picture of the evolution of the universe included a remarkable stage of explosive expansion (inflation) in the vacuumlike state, which is responsible for its large-scale structure, and a rather dull stage of decay of the inflaton field, which is responsible for the matter content of the universe. The processes which could happen during the later stage were typically ignored.

Now we see that the stage of reheating deserves a more detailed investigation. Explosive processes far away from thermal equilibrium could impact the further evolution of the universe. As we know, the appearance of baryon asymmetry requires the absence of thermal equilibrium, so it is only natural to investigate the possibility of baryogenesis at the stage of reheating; see, e.g., [5,21,11].

Particles produced by the resonance have energies which are determined by the properties of the resonance bands. Typically this energy is much smaller than the temperature which would appear if the particles were instantaneously thermalized. Meanwhile, the total number of particles produced by parametric resonance is much greater than the number of particles in thermal equilibrium with the same energy density. Fluctuations associated with these particles can be anomalously large. For example, we have found that for certain values of coupling constants in our model $\sqrt{\langle \chi^2 \rangle}$ may become of the order of 10^{16} GeV, and $\sqrt{\langle \phi^2 \rangle}$ may become even greater than $\sqrt{\langle \chi^2 \rangle}$. In models describing several interacting scalar fields such anomalously large fluctuations may lead to specific nonthermal phase transitions in the early universe on the scale of 10^{16} GeV [9,10]. As we pointed out in [9], the investigation of such phase transitions in the theory of a single self-interacting field ϕ is rather involved because one needs to separate the effects related to the oscillations from the effects related to the fluctuations of the same field. Therefore an optimal way to study nonthermal phase transitions is to investigate the models where the fluctuations produced during preheating restore symmetry for the field which does not oscillate during the oscillations of the inflaton field; see Sec. IX. We will return to the discussion of this effect in a separate publication [42].

Unlike fluctuations in thermal equilibrium, the nonthermal fluctuations produced by a parametric resonance often exhibit a non-Gaussian nature. In particular, ‘‘fluctuations of

fluctuations'' can be very large. This means that in some regions of the universe one can find fluctuations at a level much greater than its average value. This effect may play an important role in the theory of topological defect production. Indeed, even if the average level of fluctuations is smaller than the critical level which leads to monopole production, they may be produced in the rare islands where the level of the fluctuations is anomalously high. Note that in order to avoid cosmological problems and burning of neutron stars by the monopole catalysis of baryon decay, the density of the primordial monopoles should be suppressed by 20 to 30 orders of magnitude. It was easy to achieve such suppression for the usual thermal fluctuations which appear after reheating, but for the nonthermal fluctuations produced by resonance the situation may be quite different.

There is an additional reason which makes the investigation of preheating so interesting. The theory of particle production in the early universe was one of the most challenging problems of theoretical cosmology in the early 1970s. However, powerful methods of investigation developed at that time produced rather modest results: particle creation could be efficient only near the cosmological singularity, at densities comparable with M_p^4 . This process could not considerably increase the total number of particles in the universe.

Now we see that in the context of inflationary cosmology *all* particles populating our part of the universe have been created due to quantum effects soon after the end of inflation. The investigation of these effects sometimes requires the development of new theoretical methods involving quantum field theory, cosmology, the theory of parametric resonance, the theory of stochastic processes, and nonequilibrium quantum statistics.

In a situation where nonperturbative effects play an important role, and the number of produced particles grows exponentially, one could expect that the only reliable tool for the investigation of preheating would be numerical simulations. Fortunately, one can go very far by developing analytical methods. For sufficiently small values of the coupling constant ($g \lesssim 3 \times 10^{-4}$) these methods allow us to make a very detailed investigation of preheating. For higher values of the coupling constant one can describe preheating analytically during most of the process. At the last stage of preheating the situation becomes too complicated, and numerical methods become most adequate. Even in these cases analytical methods allow us to obtain estimates of the same order of magnitude as the results of numerical calculations, and sometimes this agreement is even much better. Taking into account all of the uncertainties involved in the analytical investigation of stochastic resonance as well as in the computer simulations, this agreement looks rather encouraging. It remains a challenge to develop a complete analytical theory of preheating, and to apply it to realistic inflationary models with many interacting fields.

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