Energy-momentum tensor for cosmological perturbations

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We study the effective energy-momentum tensor (EMT) for cosmological perturbations and formulate the gravitational back-reaction problem in a gauge-invariant manner. We analyze the explicit expressions for the EMT in the cases of scalar metric fluctuations and of gravitational waves and derive the resulting equations of state. The formalism is applied to investigate the back-reaction effects in chaotic inflation. We find that for long wavelength scalar and tensor perturbations, the effective energy density is negative and thus counteracts any preexisting cosmological constant. For scalar perturbations during an epoch of inflation, the equation of state is de Sitter-like. [S0556-2821(97)02118-8]

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I. INTRODUCTION

It is well known [1–3] that gravitational waves propagating in some background space-time have an effect on the dynamics of this background. A convenient way to describe the back reaction of the fluctuations on the background is in terms of an effective energy-momentum tensor (EMT). In the short wave limit, when the typical wavelength of gravitational waves is small compared with the curvature of the background space-time, they act as a radiative fluid with an equation of state $p=\frac{1}{3}\rho$ (where p and ρ denote pressure and energy density, respectively). The presence of a large amount of gravity waves in the early Universe can lead to important consequences for cosmology. For example, it can speed up nucleosynthesis and lead to a higher fraction of helium, resulting in constraints on models producing a too large amplitude of gravitational waves in the early Universe (see, e.g., Ref. [4] and references quoted therein).

In most models of the early Universe, scalar-type metric perturbations are more important than gravity waves. On length scales smaller than the Hubble radius, the amplitude of scalar fluctuations increases in time, and, in most models, scalar perturbations are responsible for seeding structure in the Universe. In this paper, we study the back-reaction problem for both scalar and tensor perturbations (cosmological perturbations and gravitational waves, respectively). We derive the effective EMT which describes the back reaction and apply the result to calculate EMT for both long- and short-wavelength fluctuations in particular models for the evolution of the Universe.

One of the main puzzles to be solved is the problem of gauge invariance of the effective EMT. As is well known (see, e.g., Ref. [5] for a comprehensive review), cosmological perturbations transform nontrivially under coordinate transformations (gauge transformations). However, the answer to the question ''how important are perturbations for the evolution of a background'' must be independent of the choice of gauge, and hence the back-reaction problem must be formulated in a gauge-invariant way.

In a recent Letter [6], we demonstrated how the back-reaction problem can be set up in a gauge-invariant manner. We applied the result to estimate the magnitude of back-reaction effects in the chaotic inflationary Universe scenario. In this paper, we study in more detail the effective EMT of cosmological perturbations. In particular, we derive the equation of state satisfied by this EMT. As we show, the back-reaction effects of gravity waves and of scalar fluctuations decouple. In the short-wavelength limit, we recover the result $p=\frac{1}{3}\rho$ for gravity waves. In the long-wavelength limit, scalar fluctuations about a de Sitter background have an equation of state $p\approx-\rho$ with $\rho<0$.

The study of back-reaction effects for gravitational waves goes back a long way. Following pioneering work of Brill and Hartle [7], Isaacson [8] defined an effective EMT for gravitational waves which was shown to be gauge invariant for high-frequency waves after averaging over both space and time. This prescription only makes sense, however, when considering fluctuations on scales much smaller than those characterizing the background. In applications to physics of the very early Universe the fluctuations of interest have wavelengths larger than the Hubble radius and a frequency smaller than the expansion rate. Hence, Isaacson's procedure for defining an effective EMT is inapplicable.

Back-reaction effects for density inhomogeneities have been considered only recently, and even then without addressing questions of gauge dependence. The focus of the early work of Futamase and co-workers [9] and of Seljak and Hui [10] was on effects of inhomogeneities on local observables such as the expansion rate of the Universe. For a recent study of this issue in the context of Newtonian cosmology, the reader is referred to the work of Buchert and Ehlers [11]. The focus of our work, on the other hand, is to formulate the back-reaction problem in general relativity in a gauge-invariant manner.

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The outline of this paper is as follows. In the following section we formulate some useful properties of diffeomorphism transformations. The back-reaction problem is set up in Sec. III and then in the next section we recast the back-reaction problem in terms of gauge-invariant variables. In Sec. V we first demonstrate that the contributions of scalar and tensor fluctuations to the effective EMT do not interfere in the leading approximation. We then study in detail the effective EMT for scalar perturbations, focusing on the equations of state obtained in the long- and short-wavelength limits. In Sec. VI we derive the effective EMT for gravitational waves. As an application, we consider the back reaction of cosmological perturbations in the chaotic inflationary Universe scenario. We summarize our results in Sec. VIII.

II. GAUGE TRANSFORMATIONS

The gauge group of general relativity is the group of diffeomorphisms. A diffeomorphism corresponds to a differentiable coordinate transformation. The coordinate transformation on the manifold $\mathcal{M}$ can be considered as generated by a smooth vector field ξ^α. Let us take some coordinate system x^α on $\mathcal{M}$ in which, for instance, some arbitrary point P of that manifold has coordinates x_P^α. The solution of the differential equation

$$\frac{d\chi^\alpha(P;\lambda)}{d\lambda}=\xi^\alpha[\chi(P;\lambda)], \tag{1}$$

with initial conditions

$$\chi^\alpha(P;\lambda=0)=x_P^\alpha, \tag{2}$$

defines the parametrized integral curve $x^\alpha(\lambda)=\chi^\alpha(P;\lambda)$ with the tangent vector $\xi^\alpha(x_P)$ at P. Therefore, given the vector field ξ^α on $\mathcal{M}$ we can define an associated coordinate transformation on $\mathcal{M}$ as, for instance, $x_P^\alpha \stackrel{\xi}{\rightarrow} \tilde{x}_P^\alpha=\chi^\alpha(P;\lambda=1)$ for any given P. Assuming that ξ^α is small one can use the perturbative expansion for the solution of Eq. (1) to obtain [12]

$$\tilde{x}_P^\alpha=\chi^\alpha(P;\lambda=1)=x_P^\alpha+\xi^\alpha(x_P)+\tfrac{1}{2}\xi^\alpha_{,\beta}\xi^\beta+O(\xi^3), \tag{3}$$

which we can write in short-hand notation as

$$\chi_P^\alpha(P;\lambda=1)=(e^{\xi^\beta\partial/\partial x^\beta}x^\alpha)_P. \tag{4}$$

Thus the general coordinate transformation $x \stackrel{\xi}{\rightarrow} \tilde{x}$ on $\mathcal{M}$ generated by the vector field $\xi^\alpha(x)$ can be written as

$$x^\alpha \rightarrow \tilde{x}^\alpha=e^{\xi^\beta\partial/\partial x^\beta}x^\alpha. \tag{5}$$

Conversely, given any two coordinate systems x^α and $\tilde{x}^\alpha$ on $\mathcal{M}$ which are not too distant, we can find the vector field $\xi^\alpha(x)$ which generates the coordinate transformations $x\rightarrow\tilde{x}$ in the sense (5). Of course, in such a way we cannot cover all possible coordinate transformations [13]. However, the class of transformations described above is wide enough for our purposes.

The variables which describe physics on the manifold $\mathcal{M}$ are tensor fields Q. Under coordinate transformations $x\rightarrow\tilde{x}$ the value of Q at the *given point P of the manifold* transforms according to the well-known law

$$Q(x_P)\rightarrow\tilde{\mathcal{Q}}(\tilde{x}_P)=\left(\frac{\partial\tilde{x}}{\partial x}\right)_P\cdots\left(\frac{\partial x}{\partial\tilde{x}}\right)_P Q(x_P). \tag{6}$$

Note that both sides in this expression refer to the same point of manifold which has different coordinate values in different coordinate frames, that is, $\tilde{x}_P=x_P$. The question about the transformation law for the tensor field Q can be formulated in a different way. Namely, given two different points $\mathcal{P}$ and $\tilde{\mathcal{P}}$, which have the same coordinate values in different coordinate frames, that is, $\tilde{x}_{\tilde{\mathcal{P}}}=x_{\mathcal{P}}$, we could ask how to express the components of the tensor in the coordinate frame $\tilde{x}$ at the point $\tilde{\mathcal{P}}$ (denoted by $\tilde{\mathcal{Q}}_{\tilde{\mathcal{P}}}$) in terms of Q and its derivatives given in the frame x at the point $\mathcal{P}$. The answer to this question is found with the help of Lie derivatives with respect to the vector field ξ which generates the appropriate coordinate transformation $x\rightarrow\tilde{x}$ according to Eq. (5). Its infinitesimal form is given in some books on general relativity (see, e.g., Ref. [14]):

$$\tilde{\mathcal{Q}}(\tilde{x}_{\tilde{\mathcal{P}}}=x_0)=[Q-\mathcal{L}_\xi Q+O(\xi^2)](x_{\mathcal{P}}=x_0), \tag{7}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector field ξ. Transformation (7) is the infinitesimal form of the gauge transformations of the diffeomorphism group. The finite form of it is obtained by exponentiating Eq. (7):

$$Q(x)\rightarrow\tilde{\mathcal{Q}}(x)=(e^{-\mathcal{L}_\xi}Q)(x)=Q(x)-(\mathcal{L}_\xi Q)(x)+\tfrac{1}{2}(\mathcal{L}_\xi\mathcal{L}_\xi Q)(x)+O(\xi^3). \tag{8}$$

Equations (7) and (8) are tensor equations, where for notational convenience, tensor indices have been omitted. Note that, despite the fact that the transformation law (8) is the consequence of transformation law (6), they are different in the following respect: transformation law (6) is well defined for any tensor given only at the point P, while Eq. (8) is defined only for tensor *fields.*

In the derivations which follow below we make substantial use of some properties of the Lie derivatives and elements of functional calculus. Therefore, for the convenience of the reader we would like to recall some basic useful facts from functional analysis and from the theory of Lie derivatives. Readers not interested in these formal considerations may skip to the next section.

Let us consider a tensor field $G(x)$ (e.g., Riemann tensor or Einstein tensor) which is formed from the metric tensor $g(x)$ and its derivatives $\partial g/\partial x, \ldots$, that is,

$$G(x)\equiv G[\partial/\partial x,g(x)]. \tag{9}$$

Applying the operator $\exp(-\mathcal{L}_\xi)$ to $G(x)$, we obtain the value of this tensor, denoted $\tilde{G}(x)$ at the point $\tilde{\mathcal{P}}$, whose coordinates in the new coordinate frame are $\tilde{x}(x)=x$. On the other hand, $\tilde{G}(x)$ in the frame $\tilde{x}$ can also be calculated from the metric tensor $\widetilde{g}(x)=[\exp(-\mathcal{L}_\xi)g](x)$ and its derivatives $\partial\widetilde{g}(x)/\partial x, \ldots$, according to the prescription (9). The results should coincide and, therefore, one can conclude (see Ref. [15] for an explicit proof of this nontrivial fact) that

$$(e^{-\mathcal{L}_\xi}G)(x)=G\left[\frac{\partial}{\partial x},(e^{-\mathcal{L}_\xi}g)(x)\right], \tag{10}$$

that is, the Lie derivative can be taken through the derivatives $\partial/\partial x$ without ''changing'' them in expressions where these derivatives are used to build the tensors (e.g., Riemann tensor) out of the other tensors (e.g., metric tensor).

We will be interpreting functions $G(x)=G[\partial/\partial x,g(x)]$ defined on the manifold $\mathcal{M}$ as the parametrized set of functionals G_x defined on the space of functions $g(x')$ according to the formula

$$G_x\equiv G[\partial/\partial x,g(x)]=\int G[\partial/\partial x',g(x')]\delta(x-x')dx', \tag{11}$$

where $\delta(x-x')$ is the Dirac δ function. Then the functional derivative $\delta G_x/\delta g(x')$ can be defined in the standard way:

$$\delta G_x=\int[\delta G_x/\delta g(x')]\delta g(x')dx', \tag{12}$$

where δG_x is the change of the functional G_x under an infinitesimal variation of $g(x')$: $g(x')\rightarrow g(x')+\delta g(x')$.

If, for instance, $G_x=g(x)$, then

$$\delta G_x=\delta g(x)=\int\delta(x-x')\delta g(x')dx', \tag{13}$$

and comparing this formula with Eq. (12), we deduce that

$$\frac{\delta G_x}{\delta g(x')}=\delta(x-x'). \tag{14}$$

As another example, consider $G_x=\partial^2 g(x)/\partial x^2$. Using the definitions (11) and (12), one gets

$$\frac{\delta G_x}{\delta g(x')}=\frac{\partial^2}{\partial x'^2}\delta(x-x'). \tag{15}$$

In the following, the functional derivative $F_{xx'}=\delta G_x/\delta g(x')$ will be treated as an operator which acts on the function $f(x')$ according to the rule:

$$F_{xx'}*f(x')=\int F_{xx'}f(x')dx'. \tag{16}$$

We will also use DeWitt's condensed notation [16] and assume that continuous variables (t,x^i) are included in the indices, e.g., $A^\alpha(x^i,t)=A^{(\alpha,x^i,t)}=A^a$, where a is used as the collective variable to denote (α,x^i,t). In addition, we adopt as a natural extension of the Einstein summation rule that ''summation'' over repeated indices also includes integration over appropriate continuous variables: e.g.,

$$A^aB_a=A^{(\alpha,x,t)}B_{(\alpha,x,t)}=\sum_\alpha\int A^\alpha(x,t)B_\alpha(x,t)dxdt. \tag{17}$$

We shall write functional derivatives using the following short-hand notation:

$$\frac{\delta G_x}{\delta g(x')}\equiv\frac{\delta G}{\delta g^{a'}}\equiv G_{,a'}. \tag{18}$$

For instance, the useful formula

$$\mathcal{L}_\xi G(x)=\int d^4x'\,\frac{\delta G(x)}{\delta g(x')}\mathcal{L}_\xi g(x'), \tag{19}$$

which follows from Eq. (10), in condensed notation, takes the form

$$\mathcal{L}_\xi G=G_{,a}(\mathcal{L}_\xi g)^a, \tag{20}$$

where in addition we omitted all ''irrelevant'' indices.

III. BACK-REACTION PROBLEM FOR COSMOLOGICAL PERTURBATIONS

We consider a homogeneous, isotropic Universe with small perturbations. This means we can find a coordinate system (t,x^i) in which the metric $(g_{\mu\nu})$ and matter (φ) fields, denoted for brevity by the collective variable q^a, can be written as

$$q^a=q_0^a+\delta q^a, \tag{21}$$

where the background field q_0^a is defined as a homogeneous part of q^a on the hypersurfaces of constant time t and, therefore, q_0^a depends only on the time variable t (we recall that the variables t and x^i are included in the index a). The perturbations δq^a depend on both time and spatial coordinates, and by assumption they are small:

$$|\delta q^a|\ll q_0^a. \tag{22}$$

From our definition of the background component q_0^a it follows that the spatial average of δq^a vanishes:

$$\langle\delta q^a\rangle=\lim_{V\to\infty}\frac{\int_V\delta q^a d^3x}{\int_V d^3x}=0. \tag{23}$$

Spatial averaging is defined with respect to the background metric, not with respect to the perturbed metric as was done in Ref. [10]. Our definition is the appropriate one when establishing what ''perturbations'' are and when constructing the general back-reaction framework. The averaging of Ref. [10] is appropriate when discussing the ''expected'' values of physical quantities for real observers in systems with fluctuations on scales smaller than the Hubble radius.

The Einstein equations

$$G_{\mu\nu}-8\pi GT_{\mu\nu}:=\Pi_{\mu\nu}=0 \tag{24}$$

can be expanded in a functional power series in δq^a about the background q_0^a, if we treat $G_{\mu\nu}$ and $T_{\mu\nu}$ as functionals of q^a,

$$\Pi(q_0^a)+\Pi_{,a}|_{q_0^a}\delta q^a+\tfrac{1}{2}\Pi_{,ab}|_{q_0^a}\delta q^a\delta q^b+O(\delta q^3)=0. \tag{25}$$

To lowest order the background q_0^a should obey the Einstein equations

$$\Pi(q_0^a)=0 \tag{26}$$

and the fluctuations δq^a satisfy the linearized Einstein equations

$$\Pi_{,a}(q_0)\,\delta q^a=0. \tag{27}$$

By definition, the spatial average of the linear term in δq^a in Eq. (25) vanishes. The term quadratic in δq^a, however, does not. Therefore, the spatial averaging of Eq. (25) leads to higher order corrections in the equations describing the behavior of the homogeneous background mode. Thus the ''corrected'' equations which take into account the back reaction of small perturbations on the evolution of the background are

$$\Pi(q_0^a)=-\tfrac{1}{2}\langle\Pi_{,ab}\delta q^a\delta q^b\rangle. \tag{28}$$

At first sight, it seems natural to identify the quantity on the right-hand side of Eq. (28) as the effective EMT of perturbations which describes the back reaction of fluctuations on the homogeneous background. However, it is not a gauge-invariant expression and, for instance, does not vanish for ''metric perturbations'' induced by a coordinate transformation in Minkowski space-time.

In the next section we will rewrite the back-reaction equations in a manifestly gauge-invariant form. First, however, we want to show that the physical content of Eq. (28) is independent of the gauge chosen to do the calculation in, provided that we take into account that the background variables change to second order under a gauge transformation.

The coordinate transformation (7) induces (to second order in perturbation variables) the following diffeomorphism transformation of a variable q:

$$q=q_0+\delta q\rightarrow e^{-\mathcal{L}_\xi}(q_0+\delta q)=q_0+\delta q-\mathcal{L}_\xi q_0-\mathcal{L}_\xi\delta q + \tfrac{1}{2}\mathcal{L}_\xi^2 q_0. \tag{29}$$

Hence, to linear order, the change in δq is

$$\delta q\rightarrow\delta\widetilde{q}=\delta q-\mathcal{L}_\xi q_0, \tag{30}$$

while, to second order, the background variable transforms nontrivially as

$$q_0\rightarrow\widetilde{q}_0=q_0-\langle\mathcal{L}_\xi\delta q\rangle+\tfrac{1}{2}\langle\mathcal{L}_\xi^2 q_0\rangle, \tag{31}$$

where $\langle\xi\rangle=0$ has been assumed.

In order to prove that Eq. (28) is independent of gauge, we must show that

$$\Pi(q_0)=-\tfrac{1}{2}\langle\Pi_{,ab}\delta q^a\delta q^b\rangle\Leftrightarrow\Pi(\widetilde{q}_0)=-\tfrac{1}{2}\langle\Pi_{,ab}\delta\widetilde{q}^a\delta\widetilde{q}^b\rangle \tag{32}$$

(to second order in perturbation variables). Making use of Eqs. (30) and (31), we obtain

$$\Pi(\widetilde{q}_0)=\Pi(q_0)-\Pi_{,a}\langle\mathcal{L}_\xi\delta q^a\rangle+\tfrac{1}{2}\Pi_{,a}\langle\mathcal{L}_\xi^2 q_0^a\rangle \tag{33}$$

and

$$\begin{aligned}-\tfrac{1}{2}\langle\Pi_{,ab}\delta\widetilde{q}^a\delta\widetilde{q}^b\rangle=&-\tfrac{1}{2}\langle\Pi_{,ab}\delta q^a\delta q^b\rangle\\&+\langle\Pi_{,ab}\mathcal{L}_\xi q_0^a\delta q^b\rangle\\&-\tfrac{1}{2}\langle\Pi_{,ab}\mathcal{L}_\xi q_0^a\mathcal{L}_\xi q_0^b\rangle.\end{aligned} \tag{34}$$

In order to show that the extra terms on the right-hand sides of Eqs. (33) and (34) cancel out in Eq. (32), we make use of Eq. (19) in the equation

$$\langle(e^{-\mathcal{L}_\xi}-1)\Pi(q_0+\delta q)\rangle=0. \tag{35}$$

Expanding this equation to second order in perturbations, we obtain the identity

$$\begin{aligned}&\tfrac{1}{2}\langle\Pi_{,a}\mathcal{L}_\xi^2 q_0^a\rangle+\tfrac{1}{2}\langle\Pi_{,ab}\mathcal{L}_\xi q_0^a\mathcal{L}_\xi q_0^b\rangle-\langle\Pi_{,a}\mathcal{L}_\xi\delta q^a\rangle\\&-\langle\Pi_{,ab}\mathcal{L}_\xi q_0^a\delta q^b\rangle=0,\end{aligned} \tag{36}$$

which completes the proof that the extra terms mentioned above cancel out.

IV. GAUGE-INVARIANT FORM OF THE BACK-REACTION EQUATIONS

Although we have shown in the previous section that the physical content of Eq. (28) should be the same in all coordinate systems, it is useful for many purposes to recast this equation in an explicitly gauge-invariant way. In particular, this will allow us to define a gauge-invariant EMT for cosmological perturbations.

We start by writing down the metric for a perturbed spatially flat Friedmann-Robertson-Walker (FRW) universe:

$$\begin{aligned}ds^2=&(1+2\phi)dt^2-2a(t)(B_{,i}-S_i)dx^idt\\&-a^2(t)[(1-2\psi)\delta_{ij}+2E_{,ij}+F_{i,j}+F_{j,i}+h_{ij}]\\&\times dx^idx^j,\end{aligned} \tag{37}$$

where $a(t)$ is the scale factor, and where the three-scalars ϕ, ψ, B, and E characterize scalar metric perturbations. The symbols S_i and F_i are transverse three-vectors (giving vector fluctuations), and h_{ij} is a traceless transverse three-tensor (gravity waves).

We will now attempt to construct gauge-invariant variables Q from the gauge-dependent quantities q. To do that let us take the set of four quantities X^μ (not necessarily a four-vector) and form a Lie operator with X^μ (denoted by $\mathcal{L}_X$), treating X^μ formally as a four-vector. Later on, after we specify the properties which X^μ should satisfy if we want it to help build gauge-invariant variables, we will then construct X^μ explicitly out of the metric perturbation variables.

However, let us leave X^μ unspecified for a moment. Using $\mathcal{L}_X$ we define the new variable Q according to the prescription

$$Q = e^{\mathcal{L}_X} q. \tag{38}$$

If we attempt the same construction after performing a gauge transformation (with parameter ξ), then taking into account that as a result of this transformation $X \to \widetilde{X}$ and

$$q \to \widetilde{q} = e^{-\mathcal{L}_\xi} q, \tag{39}$$

we obtain

$$\widetilde{Q}(x) = e^{\mathcal{L}_{\widetilde{X}}} e^{-\mathcal{L}_\xi} q(x). \tag{40}$$

If we demand that Q is gauge invariant, that is, $\widetilde{Q} = Q$, then comparing Eqs. (38) and (40), we arrive at the condition

$$e^{\mathcal{L}_{\widetilde{X}}} = e^{\mathcal{L}_X} e^{\mathcal{L}_\xi}, \tag{41}$$

which imposes strong restrictions on the transformation law $X \to \widetilde{X}$. From Eq. (41) and by using the Baker-Campbell-Hausdorff formula for the products of exponentials of operators, one can easily find that the condition for gauge invariance of Q implies that under a diffeomorphism generated by ξ,

$$X^\mu \to \widetilde{X}^\mu = X^\mu + \xi^\mu + \tfrac{1}{2}[X,\xi]^\mu + \cdots, \tag{42}$$

where the ellipsis denotes terms of cubic and higher order.

Note that Q is a gauge-invariant variable characterizing both the background

$$Q_0^a = q_0^a + \langle \mathcal{L}_X \delta q^a \rangle + \tfrac{1}{2} \langle \mathcal{L}_X^2 q_0^a \rangle \tag{43}$$

and the linearized perturbations

$$\delta Q^a = \delta q^a + \mathcal{L}_X q_0^a. \tag{44}$$

Making use of the gauge-invariant variable Q we can recast the back-reaction problem (28) in a manifestly gauge-invariant form. If q satisfies the Einstein equations, it follows from our basic identity (10) that

$$e^{\mathcal{L}_X} \Pi(q) = \Pi(e^{\mathcal{L}_X} q) = \Pi(Q) = 0. \tag{45}$$

Expanding the above equation to second order in δQ and taking the spatial average of the result yields

$$\Pi(Q_0) = -\tfrac{1}{2} \langle \Pi_{,ab} \delta Q^a \delta Q^b \rangle = -\tfrac{1}{2} \langle G_{,ab} \delta Q^a \delta Q^b \rangle + 4\pi G \langle T_{,ab} \delta Q^a \delta Q^b \rangle, \tag{46}$$

which is the desired gauge-invariant form of the back-reaction equation. Reinserting tensor indices, the above equation can be rewritten as

$$G_{\mu\nu}(Q_0) = 8\pi G [T_{\mu\nu}(Q_0) + \tau_{\mu\nu}(\delta Q)], \tag{47}$$

where

$$\tau_{\mu\nu}(\delta Q) \equiv -\frac{1}{16\pi G} \langle \Pi_{\mu\nu,ab} \delta Q^a \delta Q^b \rangle \tag{48}$$

can be interpreted as the gauge-invariant effective EMT for cosmological perturbations.

At this point, however, we must return to the question of what X^μ is. It is a question of linear algebra to find the linear combinations of the perturbation variables of Eq. (37) that have (to linear order) the required transformation properties: namely,

$$X^\mu \to X^\mu + \xi^\mu. \tag{49}$$

The solution we will use is

$$X^\mu = [a(B - a\dot{E}), -E_{,i} - F_i], \tag{50}$$

where an overdot denotes a derivative with respect to the time variable t. But this choice is not unique. There is a four-parameter family of possible choices labeled by real parameters α, $\overline{\alpha}$, γ, and σ. The component X^0 is

$$X^0 = \alpha a(B - a\dot{E}) + (1-\alpha)\left(\frac{a}{\dot{a}}\psi\right), \tag{51}$$

the X^i components have a traceless piece

$$X^i_{\mathrm{Tr}} = -\sigma F_i + (1-\sigma)\int_0^t dt' \frac{1}{a} S_i(t') \tag{52}$$

and a trace

$$X^i_L = X_{L,i}, \tag{53}$$

with the function X_L defined as

$$X_L = -\gamma E + (1-\gamma)\int_0^t dt' \frac{1}{a}\left[\frac{1}{a} X^0(\overline{\alpha};t') - B(t')\right]. \tag{54}$$

Finally,

$$X^i = X^i_{\mathrm{Tr}} + X^i_L. \tag{55}$$

Demanding regularity in the limit where the expansion rate vanishes forces $\alpha = \overline{\alpha} = 1$. In this case, the dependence on γ drops out of Eq. (54) and we are left with a one-parameter degeneracy of X^μ labeled by σ.

V. ENERGY-MOMENTUM TENSOR FOR SCALAR PERTURBATIONS

In this and the following section, we calculate the effective EMT for scalar and tensor perturbations, respectively. Since vector modes decay in an expanding Universe, we shall in the following take them to be absent.

In models such as the inflationary Universe scenario, scalar and tensor modes are statistically independent Gaussian random fields. In this case, the effective EMT (46) separates into two independent pieces, the first due to the scalar perturbations, the second due to the tensor modes,

$$\tau_{\mu\nu}(\delta Q) = \tau^{\mathrm{scalar}}_{\mu\nu}(\delta Q) + \tau^{\mathrm{tensor}}_{\mu\nu}(\delta Q). \tag{56}$$

Note that if we neglect vector perturbations then it is enough to consider only scalar modes in X^μ. Hence, the contribution to the effective EMT, Eq. (48), coming from the term $\mathcal{L}_X q_0^a$ appearing in δQ^a is a contribution to $\tau_{\mu\nu}^{\rm scalar}$ alone.

It is easy to verify that for our choice of X^μ, namely,

$$X^\mu=[a(B-a\dot{E}),-E,_i], \tag{57}$$

the variables δQ^a of Eq. (44) for the metric perturbations correspond to Bardeen's gauge-invariant variables [17]. In fact, the application of Eq. (44) yields the gauge-invariant metric tensor

$$\delta g_{\mu\nu}^{(gi)}=\delta g_{\mu\nu}+\mathcal{L}_X g_{\mu\nu}^{(0)}, \tag{58}$$

and from the time-time and diagonal spatial components, respectively, we immediately obtain the two gauge-invariant variables:

$$\phi^{(gi)}\equiv\Phi=\phi+[a(B-a\dot{E})]^{\cdot},$$

$$\psi^{(gi)}\equiv\Psi=\psi-\dot{a}(B-a\dot{E}). \tag{59}$$

For the remaining components of the metric tensor, the gauge-invariant combinations vanish.

Hence, calculating the general gauge-invariant effective EMT, Eq. (48), reduces to calculating

$$\tau_{\mu\nu}=-\frac{1}{16\pi G}\langle\Pi_{\mu\nu,ab}\delta q^a\delta q^b\rangle \tag{60}$$

in longitudinal gauge $(B=E=0)$, in which

$$ds^2=(1+2\phi)dt^2-a^2(t)(1-2\psi)\delta_{ij}dx^i dx^j. \tag{61}$$

For many types of matter (scalar fields included), T_{ij} is diagonal to linear order in δq. In this case, it follows from the linearized Einstein equations that

$$\phi=\psi. \tag{62}$$

Thus, in the longitudinal gauge the variable ϕ entirely characterizes the metric perturbations. We shall consider scalar field matter, in which case the linear matter fluctuations are described by $\delta\varphi$. The motivation for our choice of matter follows since we have applications of our formalism to inflationary cosmology in mind. The linearized Einstein equations also relate $\delta\varphi$ and ϕ. In fact, there is a single gauge-invariant variable characterizing linearized scalar fluctuations.

We will now calculate $\tau_{\mu\nu}^{(2)}$ for scalar perturbations [metric (61)]. The contribution of the gravitational part to the effective EMT can be easily calculated with the help of the formulas [see, e.g., in [2], p. 965, Eq. (35.5g) a and b]

$$\begin{aligned}R_{\mu\nu}^{(2)}=\tfrac{1}{2}R_{\mu\nu,ab}\delta g^a\delta g^b=\tfrac{1}{2}[\tfrac{1}{2}\delta g^{\alpha\beta}_{\ |\mu}\delta g_{\alpha\beta|\nu}+\delta g^{\alpha\beta}(\delta g_{\alpha\beta|\mu\nu}\\ +\delta g_{\mu\nu|\alpha\beta}-\delta g_{\alpha\mu|\nu\beta}-\delta g_{\alpha\nu|\mu\beta})+\delta g_\nu^{\ \alpha|\beta}(\delta g_{\alpha\mu|\beta}\\ -\delta g_{\beta\mu|\alpha})-(\delta g^{\alpha\beta}_{\ \ |\beta}-\tfrac{1}{2}\delta g^{|\alpha})(\delta g_{\alpha\mu;\nu}+\delta g_{\alpha\nu|\mu}\\ -\delta g_{\mu\nu|\alpha})],\end{aligned} \tag{63}$$

if we substitute in these expressions the metric (61). In this formula the vertical bar denotes covariant derivatives with respect to the background metric. Expanding the energy-momentum tensor for a scalar field,

$$T_{\mu\nu}=\varphi_{,\mu}\varphi_{,\nu}-g_{\mu\nu}[\tfrac{1}{2}\varphi^{,\alpha}\varphi_{,\alpha}-V(\varphi)] \tag{64}$$

to second order in $\delta\varphi$ and δg and combining it with the result for $G_{\mu\nu}^{(2)}$ we obtain, from Eq. (48),

$$\begin{aligned}\tau_{00}=\frac{1}{8\pi G}[&+12H\langle\phi\dot{\phi}\rangle-3\langle(\dot{\phi})^2\rangle+9a^{-2}\langle(\nabla\phi)^2\rangle]\\ &+\tfrac{1}{2}\langle(\delta\dot{\varphi})^2\rangle+\tfrac{1}{2}a^{-2}\langle(\nabla\delta\varphi)^2\rangle+\tfrac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle\\ &+2V'(\varphi_0)\langle\phi\delta\varphi\rangle,\end{aligned} \tag{65}$$

and

$$\begin{aligned}\tau_{ij}=a^2\delta_{ij}\Big\{&\frac{1}{8\pi G}[(24H^2+16\dot{H})\langle\phi^2\rangle+24H\langle\dot{\phi}\phi\rangle+\langle(\dot{\phi})^2\rangle\\ &+4\langle\phi\ddot{\phi}\rangle-\tfrac{4}{3}a^{-2}\langle(\nabla\phi)^2\rangle]+4\dot{\varphi}_0^{\,2}\langle\phi^2\rangle+\tfrac{1}{2}\langle(\delta\dot{\varphi})^2\rangle\\ &-\tfrac{1}{6}a^{-2}\langle(\nabla\delta\varphi)^2\rangle-4\dot{\varphi}_0\langle\delta\dot{\varphi}\phi\rangle-\tfrac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle\\ &+2V'(\varphi_0)\langle\phi\delta\varphi\rangle\Big\},\end{aligned} \tag{66}$$

where H is the Hubble expansion rate, and where we have used the fact that $\phi=\psi$ for theories in which δT_{ij} is diagonal at linear order.

Before discussing the long- and short-wavelength limits of the equation of state satisfied by the cosmological perturbations, let us briefly recall a few crucial points from the theory of linear fluctuations (see Ref. [5] for a comprehensive overview).

The simplest way to derive the equations of motion satisfied by the gauge-invariant perturbation variables is to go to longitudinal gauge $(B=E=0)$, in which the gauge-invariant variables Φ and Ψ coincide with the metric fluctuations ϕ and ψ. The equations of motion (27),

$$\delta G^\mu_\nu=8\pi G\,\delta T^\mu_\nu, \tag{67}$$

derived in this gauge coincide with the equations for Bardeen's gauge-invariant variables.

For scalar field matter, as mentioned above, $\phi=\psi$. In this case, the 00 and ii perturbation equations combine into a second order differential equation for Φ which on scales larger than the Hubble radius and for a time-independent background equation of state has the solution

$$\phi(\mathbf{x},t)\simeq c(\mathbf{x}) \tag{68}$$

(modulo the decaying mode). The $0i$ equations give a constraint relating ϕ and $\delta\varphi$:

$$\dot{\phi}+H\phi=4\pi G\dot{\varphi}_0\delta\varphi. \tag{69}$$

Our aim is to work out $\tau_{\mu\nu}$ in an inflationary Universe. In most models of inflation, exponential expansion of the Universe results because φ_0 is rolling slowly, i.e.,

$$\dot{\varphi}_0 \simeq -\frac{V'}{3H}, \tag{70}$$

where a prime denotes the derivative with respect to the scalar matter field. The $\dot{\phi}$ term in Eq. (69) for the nondecaying mode of perturbations is proportional to a small slow roll parameter and, therefore, can be neglected. Thus from Eqs. (69) and (70), we obtain

$$\delta\varphi = -\frac{2V}{V'}\phi. \tag{71}$$

When considering the contributions of long-wavelength fluctuations to $\tau_{\mu\nu}$, we can neglect all terms in Eqs. (65) and (66) containing gradients and $\dot{\phi}$ factors. Because of the "slow-rolling" condition (70), the terms proportional to $\dot{\varphi}_0^2$ and $\dot{H}$ are negligible during inflation (but they become important at the end of inflation). Hence, in this approximation,

$$\tau_{00} \simeq \tfrac{1}{2}V''\langle\delta\varphi^2\rangle + 2V'\langle\phi\delta\varphi\rangle \tag{72}$$

and

$$\tau_{ij} \simeq a^2\delta_{ij}\left\{\frac{3}{\pi G}H^2\langle\phi^2\rangle - \frac{1}{2}V''\langle\delta\varphi^2\rangle + 2V'\langle\phi\delta\varphi\rangle\right\}. \tag{73}$$

Making use of Eq. (71), this yields

$$\rho_s \equiv \tau_0^0 \cong \left(2\frac{V''V^2}{V'^2} - 4V\right)\langle\phi^2\rangle \tag{74}$$

and

$$p_s \equiv -\tfrac{1}{3}\tau_i^i \cong -\rho^{(2)}. \tag{75}$$

Thus, we have shown that the long-wavelength perturbations in an inflationary Universe have the same equation of state $p_s = -\rho_s$ as the background.

One of the main results which emerges from our analysis is that

$$\rho_s < 0 \tag{76}$$

for the long-wavelength cosmological perturbations in all realistic inflationary models. Thus, the effective $\tau_{\mu\nu}$ counteracts the cosmological constant driving inflation. Note that the same sign of ρ emerges when considering the vacuum state EMT of a scalar field in a fixed background de Sitter spacetime (see, e.g., Refs. [18,19]).

For short-wavelength fluctuations ($k \gg aH$), both ϕ and $\delta\varphi$ oscillate with a frequency $\propto k$. In this case [see Eq. (69)],

$$\phi \sim 4\pi G\frac{ia}{k}\dot{\varphi}_0\delta\varphi. \tag{77}$$

Hence, it follows by inspection that all terms containing ϕ in Eqs. (65) and (66) are suppressed by powers of Ha/k compared to the terms without dependence on ϕ and, therefore,

$$\tau_{00} \simeq \tfrac{1}{2}\langle(\delta\dot{\varphi})^2\rangle + \tfrac{1}{2}a^{-2}\langle(\nabla\delta\varphi)^2\rangle + \tfrac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle \tag{78}$$

and

$$\tau_{ij} = a^2\delta_{ij}\{\tfrac{1}{2}\langle(\delta\dot{\varphi})^2\rangle - \tfrac{1}{6}a^{-2}\langle(\nabla\delta\varphi)^2\rangle - \tfrac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle\}, \tag{79}$$

which is a familiar result.

VI. ENERGY-MOMENTUM TENSOR FOR GRAVITATIONAL WAVES

In the case of gravitational waves the metric takes the form

$$ds^2 = dt^2 - a^2(t)(\delta_{ik} + h_{ik})dx^i dx^k, \tag{80}$$

where h_{ik} is defined as the transverse traceless part of the metric perturbations and, therefore, the components $\delta g_{ik} \equiv h_{ij}$ are gauge invariant themselves.

In the absence of matter fluctuations, the effective EMT $\tau_{\mu\nu}$ is given by the first term in Eq. (46). Making use of the second variation of $R_{\mu\nu}$ given by Eq. (63) and the equation of motion for gravitational waves

$$\ddot{h}_{ij} + 3\frac{\dot{a}}{a}\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} = 0, \tag{81}$$

we obtain

$$8\pi G\tau_{00} = \frac{\dot{a}}{a}\langle\dot{h}_{k\ell}h_{k\ell}\rangle + \tfrac{1}{8}\left(\langle\dot{h}_{k\ell}\dot{h}_{k\ell}\rangle + \frac{1}{a^2}\langle h_{k\ell,m}h_{k\ell,m}\rangle\right) \tag{82}$$

and

$$8\pi G\tau_{ij} = \delta_{ij}a^2\left\{\frac{3}{8a^2}\langle h_{k\ell,m}h_{k\ell,m}\rangle - \tfrac{3}{8}\langle\dot{h}_{k\ell}\dot{h}_{k\ell}\rangle\right\} + \tfrac{1}{2}a^2\langle\dot{h}_{ik}\dot{h}_{kj}\rangle + \tfrac{1}{4}\langle h_{k\ell,i}h_{k\ell,j}\rangle - \tfrac{1}{2}\langle h_{ik,\ell}h_{jk,\ell}\rangle. \tag{83}$$

The first expression can be interpreted as the effective energy density of gravitational waves

$$\rho_{gw} = \tau_0^0. \tag{84}$$

The relation between τ_{ij} and the quantity which we could naturally interpret as an effective pressure is not so straightforward as it looks at first glance. The problem is that the energy-momentum tensor for the gravity waves $\tau_{\mu\nu}$ is not conserved itself, that is, $\tau^\mu_{\nu|\mu} \neq 0$. This is not surprising since the gravitational perturbations "interact" with the background and only the total EMT must be conserved,

$$[T^\mu_\nu(Q_0) + \tau^\mu_\nu(\delta Q)]_{|\mu} = 0. \tag{85}$$

as a consequence of Bianchi identities for background, $G^\mu_\nu(Q_0)_{|\mu} = 0$. In fact, expanding the exact conservation law $T^\mu_{\nu;\mu} = 0$ in perturbations to second order and averaging the resulting equation we obtain

$$T^\beta_\alpha(Q_0)_{|\beta} = -\langle{}^{(2)}\Gamma^\beta_{\beta\gamma}\rangle T^\gamma_\alpha + \langle{}^{(2)}\Gamma^\gamma_{\alpha\beta}\rangle T^\beta_\gamma. \tag{86}$$

Deriving Eq. (86) we took into account that in our case (when we have only gravity waves) matter perturbations are absent. The energy-momentum tensor for the background is diagonal and isotropic, that is, $T^0_0=\rho^{(0)}$ and $T^i_k=-p^{(0)}\delta^i_k$. Also, we will consider only isotropic fields of gravitational waves. In such a case the conservation law (85), taking into account Eq. (86), can be written down explicitly in a familiar form as

$$\dot{\rho}_{gw}+3H(\rho_{gw}+p_{gw})=0, \tag{87}$$

where

$$p_{gw}=-\frac{1}{3}\tau^i_i-\frac{1}{3H}\langle {}^{(2)}\Gamma^\beta_{\beta 0}\rangle(\rho^{(0)}+p^{(0)}) \tag{88}$$

can be interpreted as the pressure of gravitational waves.

The second term in Eq. (88) does not contribute to the pressure only in a de Sitter universe in which $p^{(0)}=-\rho^{(0)}$. In this case the term $-\frac{1}{3}\tau^i_i$ itself can be interpreted as a pressure. A similar but a bit more complicated analysis can be done for the scalar perturbations, for which one-third of the trace of the spatial part of the EMT can be interpreted as a pressure also only in a de Sitter universe.

As for scalar perturbations, we will now study the equation of state for gravity waves both in the short- and long-wavelength limits in various models for the evolution of the Universe. First, assuming that the field of gravity waves is isotropic and by averaging the diagonal elements of Eq. (83) we obtain the contribution of τ to the pressure

$$-\frac{8\pi G}{3}\tau^i_i=\frac{7}{24a^2}\langle h_{k\ell,m}h_{k\ell,m}\rangle-\frac{5}{24}\langle \dot{h}_{k\ell}\dot{h}_{k\ell}\rangle. \tag{89}$$

For fluctuations with wavelength smaller than the Hubble radius, the first term on the right-hand side of Eq. (82) and the second term in Eq. (88) are negligible and the time average of the temporal and spatial gradient terms are the same. Hence,

$$p_{gw}=\frac{1}{3}\rho_{gw}=\frac{1}{8\pi G}\,\frac{1}{12a^2}\langle\langle h_{k\ell,m}h_{k\ell,m}\rangle\rangle \tag{90}$$

where $\langle\langle\ \rangle\rangle$ indicate that in addition to spatial average, a time average over a period $T\ll H^{-1}$ has been taken. As expected, short-wavelength gravitational waves behave like radiation, independent of the evolution of the background, and their energy density decays as $a^{-4}(t)$.

In the case of long-wavelength gravitational waves the calculations are less straightforward. First, we consider a de Sitter background:

$$a(t)=e^{H(t-t_0)}, \tag{91}$$

where t_0 is a reference time. Let us take an isotropic field of gravity waves with comoving wave numbers k. For the non-decaying mode of long-wavelength gravity waves ($k/a\ll H$), the solution of Eq. (81) is

$$h_{ij}=A_k\epsilon_{ij}\left[1+\frac{1}{2}\left(\frac{k}{aH}\right)^2+O\left(\left(\frac{k}{aH}\right)^3\right)\right]e^{i\vec{k}\vec{x}}, \tag{92}$$

where A_k is a constant (related to the spectrum of gravity waves) and ϵ_{ij} is the polarization tensor. Substituting this solution in formula (82) we obtain the following expression for the energy density of the gravity waves to lowest order in k:

$$\rho_{gw}\simeq-\frac{1}{8\pi G}\,\frac{7}{8}\,\frac{k^2}{a^2}\langle|A_k|^2\epsilon_{ij}\epsilon^{ij}\rangle \tag{93}$$

and correspondingly from Eq. (89) we derive that the pressure is

$$p_{gw}\simeq-\tfrac{1}{3}\rho_{gw}. \tag{94}$$

Note that the whole contribution to the pressure in this case comes from the first term in Eq. (88) . The energy density of long-wavelength gravity waves decays as $\sim a^{-2}$.

In a radiation-dominated Universe the scale factor increases as

$$a(t)=\left(\frac{t}{t_0}\right)^{1/2}, \tag{95}$$

and the solution for long-wavelength gravity waves ($k\ll Ha$) is

$$h_{ij}=A_k\epsilon_{ij}\left[1-\frac{1}{6}\left(\frac{k}{aH}\right)^2+O\left(\left(\frac{k}{aH}\right)^3\right)\right]e^{i\vec{k}\vec{x}}.$$

Inserting these results into Eqs. (82) and (83) yields

$$\rho_{gw}\simeq-\frac{1}{8\pi G}\,\frac{5}{24}\,\frac{k^2}{a^2}\langle|A_k|^2\epsilon_{ij}\epsilon^{ij}\rangle \tag{96}$$

and

$$p_{gw}\simeq-\tfrac{1}{3}\rho_{gw}. \tag{97}$$

In this case both of the terms in Eq. (88) give comparable contributions to the pressure, namely, the contribution of the first term there is $p_1=\frac{21}{5}p_{gw}$ and the contribution of the second term is negative and equal to $p_2=-\frac{16}{5}p_{gw}$. As in the previous case the energy density decays as a^{-2}.

VII. BACK REACTION IN INFLATIONARY UNIVERSE

As an application of the formalism developed in this paper, we will study the effects of back reaction in inflationary cosmology. To be specific, we consider a single field chaotic inflation model [20] and take the inflaton potential to be

$$V(\varphi)=\tfrac{1}{2}m^2\varphi^2. \tag{98}$$

Furthermore, we specify an initial state at a time t_i in which the homogeneous inflaton field has the value $\varphi_0(t_i)$ and the fluctuations are minimal.

We will focus on the contribution of long-wavelength modes to the effective EMT, which by assumption vanishes at the initial time t_i. Provided that $\varphi_0(t_i)$ is sufficiently large, the slow-rolling conditions are satisfied and exponential expansion will commence. At this point, quantum vacuum fluctuations begin to generate perturbations on scales which ''leave'' the Hubble radius (i.e., whose physical wavelength becomes larger than the Hubble radius). As time proceeds, more modes leave the Hubble radius, and hence the contribution to the effective EMT builds up. We wish to estimate the magnitude of the resulting effective EMT.

As discussed in Sec. V, scalar metric perturbations in this model are characterized by a single function ϕ_k (where k stands for the comoving wave number). From the constraint equation (71) it follows that

$$\delta\varphi_k \simeq -\varphi_0 \phi_k . \tag{99}$$

Hence, the dominant terms in the effective energy-momentum tensor $\tau_{\mu\nu}$ [see Eqs. (74) and (75)] are proportional to the correlator $\langle \phi^2 \rangle$. The amplitudes of ϕ_k are known from the theory of linear cosmological perturbations. Using the results for ϕ_k valid during inflation [5] we obtain

$$\langle \phi^2(t) \rangle = \int_{k_i}^{k_t} \frac{dk}{k} |\phi_k|^2 = \frac{m^2 M_P^2}{32\pi^4 \varphi_0^4(t)} \int_{k_i}^{k_t} \frac{dk}{k} \left[\ln \frac{H(t)a(t)}{k} \right]^2 , \tag{100}$$

where t denotes physical time, and M_P is the Planck mass. The integral runs over all modes with scales larger than the Hubble radius, i.e.,

$$k < k_f(t) = H(t)a(t), \tag{101}$$

but smaller than the Hubble radius at the initial time t_i, i.e.,

$$k > k_i = H(t_i)a(t_i). \tag{102}$$

The infrared cutoff k_i is a consequence of our choice of initial state.

For the potential $V(\varphi)$ considered here, the scale factor is given by

$$a(t) = a(t_i) \exp\left(\frac{2\pi}{M_P^2} [\varphi_0^2(t_i) - \varphi_0^2(t)] \right). \tag{103}$$

The intregral over k in Eq. (100) can be calculated explicitly, giving

$$\langle \phi^2(t) \rangle \sim \frac{m^2 M_P^2}{32\pi^4 \varphi_0^4(t)} \left[\frac{2\pi \varphi_0^2(t_i)}{M_P^2} \right]^3 . \tag{104}$$

Making use of Eq. (74), we finally obtain the fractional contribution of scalar perturbations ρ_s to the total energy density

$$\frac{\rho_s(t)}{\rho_0} \simeq -\frac{3}{4\pi} \frac{m^2 \varphi_0^2(t_i)}{M_P^4} \left[\frac{\varphi_0(t_i)}{\varphi_0(t)} \right]^4 , \tag{105}$$

where ρ_0 is the background energy density of the homogeneous scalar field φ_0. In situations in which the ratio (105) becomes of the order 1, back reaction becomes very important.

Several consequences can be derived from Eq. (105). First of all, back reaction may lead to a shortening of the period of inflation. Without back-reaction, inflation would end when $\varphi_0(t) \sim M_P$ (see, e.g., [20]). Inserting this value into Eq. (105), one can expect that if

$$\varphi_0(t_i) > \varphi_{\rm br} \sim m^{-1/3} M_P^{4/3} , \tag{106}$$

then the back reaction will become important before the end of inflation and may shorten the period of inflation. It is interesting to compare this value with the scale

$$\varphi_0(t_i) \sim \varphi_{\rm sr} = m^{-1/2} M_P^{3/2} , \tag{107}$$

which emerges in the scenario of stochastic chaotic inflation [21,22] as the ''self-reproduction'' scale beyond which quantum fluctuations dominate the evolution of $\varphi_0(t)$. Notice that $\varphi_{\rm sr} \gg \varphi_{\rm br}$ since $m \ll M_P$. Hence, even in the case when self-reproduction does not take place, back-reaction effects can be very important.

Alternatively, we can fix $\varphi_0(t_i)$ and use the expression (105) to determine at which value of φ_0 [denoted by $\varphi_0(t_f)$] back reaction becomes important, which presumably implies that inflation will end at that point. The result is

$$\varphi_0(t_f) \sim \frac{m^{1/2} \varphi_0(t_i)^{3/2}}{M_P} . \tag{108}$$

We will conclude this section with an analysis of the back reaction of scalar perturbations on the evolution of the homogeneous component of scalar field $\varphi_0(t)$. The equation for the scalar field $\varphi_0(t)$ taking into account the back reaction of perturbations can be obtained if we start with the exact Klein-Gordon equation

$$\Box_{g_0+\delta g}(\varphi_0 + \delta\varphi) + V'(\varphi_0 + \delta\varphi) = 0, \tag{109}$$

expand it to second order in perturbations $\delta g, \delta\varphi$, and take the average. The result is

$$(\ddot{\varphi}_0 + 3H\dot{\varphi}_0)(1 + 4\langle \phi^2 \rangle) + V' + \tfrac{1}{2} V''' \langle \delta\varphi^2 \rangle - 2\langle \phi \delta\ddot{\varphi} \rangle - 4\langle \dot{\phi} \delta\dot{\varphi} \rangle - 6H\langle \phi \delta\dot{\varphi} \rangle + 4\dot{\varphi}_0 \langle \dot{\phi} \phi \rangle - \frac{2}{a^2} \langle \phi \nabla^2 \delta\varphi \rangle = 0. \tag{110}$$

For long-wavelength perturbations in the inflationary Universe, the term containing spatial derivatives of $\delta\varphi$ is negligible. Hence, for potential (98), Eq. (110) becomes (see [15] for a detailed analysis)

$$\ddot{\varphi}_0 + 3\frac{\dot{a}}{a}\dot{\varphi}_0 = -\frac{m^2}{1 + 2\langle \phi^2 \rangle} \varphi_0 . \tag{111}$$

We conclude that the back reaction of perturbations on the evolution of φ_0 becomes very important when $\langle \phi^2 \rangle \sim 1$, at the same time as they become important for the evolution of the background geometry of space-time.

VIII. CONCLUSIONS

We have defined a gauge-invariant effective energy-momentum tensor (EMT) of cosmological perturbations which allows us to describe the back reaction of the perturbations on the evolution of background space-time. Our formalism can be applied both to scalar and tensor perturbations and applies independent of the wavelength of the fluctuations.

In particular, our analysis applies to cosmological perturbations produced during a period of inflation in the very early Universe. In this case, we have worked out the specific form of the effective EMT for both density perturbations and gravitational waves. The contribution of long-wavelength scalar fluctuations to the energy density has a negative sign and thus counteracts the cosmological constant (note that this effect is a purely classical one, in contrast with the quantum-mechanical effects counteracting the cosmological constant discussed in Refs. [18,19,23]). The equation of state of the effective EMT is the same as that of the background. The contribution of long-wavelength gravitational waves to the energy density also has a negative sign, and in this case the equation of state is $p=-1/3\rho$. Note that an instability of de Sitter space to long-wavelength fluctuations was also discovered in Ref. [24].

Applied to the chaotic inflationary Universe scenario, we found that the back reaction of the generated perturbations on the evolution of the background can become very important before the end of the inflation even if we start at an energy scale below the ‘‘self-reproduction scale.’’

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[1] L. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon, Oxford, 1975).
[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).
[3] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[4] R. L. Davis, Phys. Lett. **161B**, 285 (1985); D. Bennett, Phys. Rev. D **33**, 872 (1986); R. Brandenberger, A. Albrecht, and N. Turok, Nucl. Phys. **B277**, 605 (1986).
[5] V. Mukhanov, H. Feldman, and R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[6] V. Mukhanov, L. R. W. Abramo, and R. Brandenberger, Phys. Rev. Lett. **78**, 1624 (1997).
[7] D. Brill and J. Hartle, Phys. Rev. **135**, 1 (1964), B271–278.
[8] R. Isaacson, Phys. Rev. **166**, 2 (1968), 1263–1271 and 1272–1280. See also M. A. H. MacCallum and A. H. Taub, Commun. Math. Phys. **30**, 153 (1973) and Y. Choquet-Bruhat, *ibid.* **12**, 16 (1969).
[9] T. Futamase, Phys. Rev. Lett. **61**, 2175 (1988); T. Futamase, Mon. Not. R. Astron. Soc. **237**, 187 (1989). See also S. Bildhauer and T. Futamase, Gen. Relativ. Gravit. **23**, 1251 (1991).
[10] U. Seljak and L. Hui, in Proceedings of Clusters, Lensing and the Future of the Universe, College Park, Maryland, 1995 (unpublished).
[11] T. Buchert and J. Ehlers, Astron. Astrophys. **320**, 1 (1997).
[12] A. H. Taub, J. Math. Phys. (N.Y.) **2**, 787 (1961).
[13] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonega, ‘‘Perturbations of spacetime: gauge transformations and gauge invariance at second order and beyond,’’ Report No. gr-qc/9609040.
[14] J. Stewart, *Advanced General Relativity* (Cambridge University Press, New York, 1990).
[15] L. Raul W. Abramo, Ph.D. thesis, Brown University, 1997.
[16] B. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965.)
[17] J. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[18] L. Ford, Phys. Rev. D **31**, 710 (1985).
[19] E. Mottola, Phys. Rev. D **31**, 754 (1985); **33**, 1616 (1986); **33**, 2136 (1986); P. Mazur and E. Mottola, Nucl. Phys. **B278**, 694 (1986); J. Traschen and C. Hill, Phys. Rev. D **33**, 3519 (1986).
[20] A. Linde, Phys. Lett. **129B**, 177 (1983).
[21] A. A. Starobinsky, in *Current Topics in Field Theory, Quantum Gravity and Strings*, edited by H. J. de Vega and N. Sánchez, Lecture Notes in Physics Vol. 246 (Springer-Verlag, Berlin, 1982.)
[22] A. S. Goncharov, A. D. Linde, and V. F. Mukhanov, Int. J. Mod. Phys. A **2**, 561 (1987); A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland, 1990); Phys. Scr. **T36**, 30 (1991).
[23] N. Tsamis and R. Woodard, Phys. Lett. B **301**, 351 (1993); , Nucl. Phys. **B474**, 235 (1996).
[24] L. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982).