Remarks on nonperturbative $O(1/m_c^2)$ corrections to $\Gamma(\overline{B}\to X_s\gamma)$

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(Received 6 March 1997)

We present an estimate of certain higher-order corrections to the contribution of the charm triangle loop in the inclusive $\overline{B}\to X_s\gamma$ decay rate recently discussed by Voloshin. We find that these corrections are minute and hence the result found by Voloshin, although small, is quite robust. [S0556-2821(97)05115-1]

PACS number(s): 12.39.Hg

The concepts of heavy quark universality, symmetry, and effective field theory have been exhaustively applied to inclusive B decays [1]. By these means, the inclusive $\overline{B}\to X_c l\nu$ and $\overline{B}\to X_s\gamma$ decays can be related to the underlying (perturbatively computable) $b\to cl\nu$ and $b\to s\gamma$ quark decays with only $O(\Lambda_{\rm QCD}^2/m_b^2)$ soft physics and b-quark binding corrections. These results have been brought into question by a recent paper of Voloshin [2], where a nonperturbative correction to the inclusive rate for $\overline{B}\to X_s\gamma$ is identified which scales as $\Lambda_{\rm QCD}^2/m_c^2$.

The appearance of nonperturbative corrections apparently missed in the heavy quark effective theory (HQET) treatment of inclusive B decays is unsettling and deserves a more thorough discussion. In this Brief Report we show how, in the unphysical limit where $m_c^2\gg m_b\Lambda_{\rm QCD}$, it is possible to understand how the Voloshin correction arises in the context of HQET. In this limit, in fact, this correction is formally less than terms of $O(m_c^2/m_b^2)$ and hence is *under control*. In real life, however, $m_c^2\sim m_b\Lambda_{\rm QCD}$ and, in principle, Voloshin's correction is subject to considerable uncertainty. In practice, however, we show that all corrections to Voloshin's result are quite negligible, so that indeed his computation provides the dominant contribution.

The corrections of $O(\Lambda_{\rm QCD}^2/m_c^2)$ identified by Voloshin [2] arose by considering the contribution of the gluon-photon penguin graph, shown in Fig. 1. After applying a Fierz transformation to the $\overline{s_L}\gamma_\mu c_L\overline{c_L}\gamma^\mu b_L$ four-quark operator in the underlying effective Hamiltonian, this graph is proportional to the famous AVV triangle diagram. Because of the Glashow-Iliopoulos-Maiani (GIM) mechanism, there is no anomaly in the axial vertex, and the result is necessarily nonsingular in $(k_\gamma+k_g)^2$. In the limit of vanishing gluon momentum considered by Voloshin [2], this contribution must scale as $V_{is}^*V_{ib}/m_i^2$ for the up-type quarks i running in the loop. In his calculation, Voloshin ignores the up quark loop because of its miniscule Cabibbo-Kobayashi-Mashawa (CKM) matrix elements and because, in no sense, can m_u be considered larger than the typical gluon momentum in the problem. Since $V_{cs}^*V_{cb}\simeq -V_{ts}^*V_{tb}$ and $m_t^2\gg m_c^2$, the c quark loop in Fig. 1 dominates and, indeed, the dominant effective operator for the $b\to s\gamma g$ process scales as $1/m_c^2$. This is the origin of Voloshin's result for the effective Lagrangian for this process:

$$\mathcal{L}_{\rm Vol}^{b\to s\gamma g}=\frac{eQ_c}{8\pi^2}\sqrt{2}G_FV_{cs}^*V_{cb}\left(\overline{s_L}\gamma^\mu\frac{\lambda^a}{2}b_L\right)\times\frac{ig}{3m_c^2}G_a^{\nu\lambda}\partial_\lambda\widetilde{F}_{\mu\nu}. \tag{1}$$

Other observations regarding effective Lagrangians of this type have been made previously. The effect on the exclusive $B\to K^*\gamma$ decay has been studied in Ref. [3], while the authors of Ref. [4] have studied the effect of the $b\to s\gamma g$ process on the photon energy spectrum.

The interference of the above term with the leading perturbative contribution for the $b\to s\gamma$ process, given by the effective Lagrangian [5]

$$\mathcal{L}_{b\to s\gamma}=\frac{e}{16\pi^2}\frac{4G_F}{\sqrt{2}}V_{ts}^*V_{tb}C_7(\mu)m_b\overline{s_L}\sigma_{\mu\nu}b_RF^{\mu\nu}, \tag{2}$$

in the operator product expansion (OPE) for the inclusive rate yields an effective amplitude for the $b\to s\gamma$ process

$$T_{b\to s\gamma}=-\frac{\alpha}{32\pi^4}G_F^2m_b^5\frac{1}{27}{\rm Re}(V_{cs}^*V_{cb}V_{ts}V_{tb}^*C_7)\times\left(\frac{\overline{bg}\sigma_{\mu\nu}(\lambda_a/2)G_a^{\mu\nu}b}{2m_c^2}\right). \tag{3}$$

This produces a correction to the $\overline{B}\to X_s\gamma$ rate:

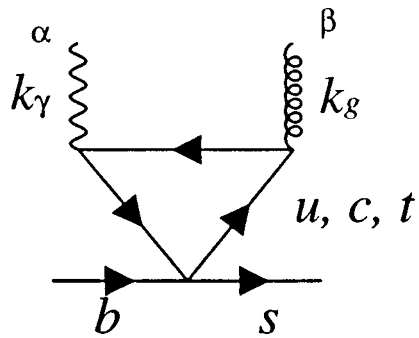

FIG. 1. Triangle diagram for the $b\to s\gamma g$ process. We have omitted a second diagram with the gluon and photon interchanged.

$$\frac{\delta\Gamma(\overline{B}\to X_s\gamma)}{\Gamma(\overline{B}\to X_s\gamma)}=\frac{1}{27C_7}\frac{\mu_g^2}{m_c^2}\simeq-0.025. \tag{4}$$

The numerical result above follows by using for the coefficient C_7 of the leading operator mediating the $\overline{B}\to X_s\gamma$ transition, $C_7\simeq-0.3$ [6], and using the standard evaluation for the strength of the chromomagnetic interaction of the b quark inside the B hadron [7]:

$$\mu_g^2=\tfrac{1}{2}\langle B|\overline{b}g\sigma_{\tau\nu}G_a^{\tau\nu}\frac{\lambda_a}{2}b|B\rangle=\tfrac{3}{4}(M_B^{*2}-M_B^2)\simeq 0.4\ \mathrm{GeV}^2. \tag{5}$$

Although Eq. (4) is a small correction to the inclusive rate, its sensitivity to the scale m_c^2 brings into question the expectations from HQET. To understand what is going on, it is useful to examine more fully the gluon-photon penguin graph considered by Voloshin. Because the AVV graph has been analyzed in detail by Adler [8], it is straightforward to consider corrections to the effective Lagrangian (1) or, equivalently, the amplitude (3).

The triangle graph in question is proportional to the tensor

$$I_{\mu\alpha\beta}=I(k_\gamma,k_g)\{\epsilon_{\mu\nu\alpha\beta}(k_g\cdot k_\gamma)(k_\gamma^\nu-k_g^\nu) -(\epsilon_{\nu\mu\alpha\tau}k_{\gamma\beta}-\epsilon_{\nu\mu\beta\tau}k_{g\alpha})k_\gamma^\tau k_g^\nu\}, \tag{6}$$

where the invariant function, $I(k_\gamma,k_g)$, is given by

$$I(k_\gamma,k_g)=24\int_0^1 x\,dx\int_0^{1-x}y\,dy \times\frac{1}{[m_i^2-k_g^2x(1-x)-2xyk_g\cdot k_\gamma]}, \tag{7}$$

with m_i being the mass of the quark in the loop. In the limit where the gluon 4-momentum vanishes, Eq. (7) reduces to the $1/m_c^2$ factor alluded to earlier. In fact, the gluon 4-momentum is never vanishing. It is of order of the typical momentum of the B meson constituents, which is $O(\Lambda_{\mathrm{QCD}})$.[1] Obviously, for the u-quark loop, it makes no sense to consider the $k_g^\mu\to 0$ limit and so the scale $1/m_u^2$ never enters the problem. Given the very small magnitude of $V_{us}^*V_{ub}$ relative to $V_{cs}^*V_{cb}$, it is perfectly sensible—as Voloshin [2] does—to neglect the u-quark loop altogether.

The situation is also clear for the t-quark loop. In this case m_t is really much larger than all other scales. Given that the $k_g^\mu\to 0$ limit is appropriate, the photon-gluon penguin graph involving the t-loop will contribute an effective interaction like that of Eq. (1), but scaled by $1/m_t^2$. This interaction, however, gives a negligibly small correction to the $\overline{B}\to X_s\gamma$ rate. For the c-quark loop, however, the effective $k_g^\mu\to 0$ limit which yields the $b\to s\gamma g$ Lagrangian of Eq. (1) does not appear so safe.

It is certainly possible to drop the $k_g^2x(1-x)$ term in the denominator of Eq. (7) relative to m_c^2, since $m_c^2\gg\Lambda_{\mathrm{QCD}}^2$. However, in the B rest frame $|\mathbf{k}_\gamma|\sim m_b/2$, thus the $2xyk_g\cdot k_\gamma$ term in Eq. (7) is of $O(m_b\Lambda_{\mathrm{QCD}})$, which is not really small compared to m_c^2. Voloshin's result [Eq. (1)] neglects this term altogether.

Before evaluating corrections to Voloshin's formula, it is useful to make some general remarks. The approximation of also dropping the $2xyk_g\cdot k_\gamma$ terms in Eq. (7) for the c-quark loop is only tenable in a world where $m_c^2\gg\Lambda_{\mathrm{QCD}}m_b$. If this were the case, the Voloshin correction to the $B\to X_s\gamma$ inclusive rate, given in Eq. (4), would not violate, per se, the heavy quark expansion, since

$$\frac{\delta\Gamma(\overline{B}\to X_s\gamma)}{\Gamma(\overline{B}\to X_s\gamma)}\sim\frac{\mu_g^2}{m_c^2}\sim\frac{\Lambda_{\mathrm{QCD}}^2}{m_c^2}\ll\frac{m_c^2}{m_b^2}. \tag{8}$$

On the other hand, dropping the $2xyk_g\cdot k_\gamma$ term is *never* a good approximation in the heavy m_b limit ($m_b\to\infty$, with m_c fixed) envisaged in the HQET. In this limit, the effective Lagrangian (1) scaled by $1/m_c^2$ never arises, since the proper limit for Eq. (7) is not $1/m_c^2$, but $-6/k_g\cdot k_\gamma$. Indeed, in this limit the photon-gluon penguin contribution vanishes identically because of the GIM mechanism, $V_{is}^*V_{ib}=0$.

A class of systematic corrections to Voloshin's result can be computed by retaining the full function $I(k_\gamma,k_g)$ in the photon-gluon penguin graph. To be more precise, since $m_c^2\gg\Lambda_{\mathrm{QCD}}^2$, it suffices to consider

$$I(k_\gamma,k_g)=24\int_0^1 x\,dx\int_0^{1-x}y\,dy\frac{1}{(m_c^2-2xyk_\gamma\cdot k_g)}. \tag{9}$$

Expanding the denominator in powers of $k_\gamma\cdot k_g/m_c^2$ identifies a progressive set of higher dimensional operators which contribute to the $b\to s\gamma g$ Lagrangian. This expansion effectively replaces the $G_a^{\nu\lambda}\partial_\lambda\tilde{F}_{\mu\nu}/m_c^2$ operator in Eq. (1) by

$$\frac{1}{m_c^2}G_a^{\nu\lambda}i\partial_\lambda\tilde{F}_{\mu\nu}\to\frac{1}{m_c^2}G_a^{\nu\lambda}i\partial_\lambda\tilde{F}_{\mu\nu} +\frac{4}{15m_c^4}(i\partial_\alpha i\partial_\lambda\tilde{F}_{\mu\nu})(iD^\alpha G_a^{\nu\lambda}) +\frac{3}{35m_c^6}(i\partial_\alpha i\partial_\beta i\partial_\lambda\tilde{F}_{\mu\nu})(iD^\alpha iD^\beta G_a^{\nu\lambda})+\cdots. \tag{10}$$

The interference of these higher dimensional terms with the leading order operator (2) for the $b\to s\gamma$ process in the operator product expansion formula for the inclusive rate provides the desired set of corrections to the Voloshin result.

In practice, it is simpler to compute these corrections by first calculating the correlator of the leading order contribution to $b\to s\gamma$, Eq. (2), with the full c-quark photon-gluon penguin graph and then expanding the result in powers of the gluon 4-momentum. A tedious but straightforward calculation of the correlator depicted in Fig. 2 yields the following formula for the effective amplitude for the $b\to s\gamma$ process:

[1] To be more precise, as we will argue later, $|\mathbf{k}_g|=O(\Lambda_{\mathrm{QCD}})$.

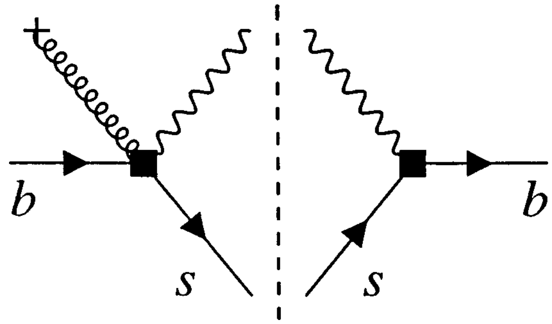

FIG. 2. The cut diagram which, together with its complex conjugate, yields the contribution of $\mathcal{L}_{b\to s\gamma g}$ to the inclusive $\overline{B}\to X_s\gamma$ rate.

$$T_{b\to s\gamma}=-\frac{\alpha}{32\pi^4}G_F^2 m_b^5\frac{1}{27}\mathrm{Re}\times(V_{cs}^* V_{cb}V_{ts}V_{tb}^* C_7)\overline{bg}\sigma_\nu{}^\alpha G_a^{\nu\lambda}\frac{\lambda_a}{2}bJ_{\alpha\lambda}. \quad (11)$$

Here $J_{\alpha\lambda}$ involves an integral of $I(k_\gamma,k_g)$ over the photon and s-quark phase space and is given by

$$J_{\alpha\lambda}=-\frac{48\pi}{m_b^2}\int\frac{d^3s}{(2\pi)^3 2s^0}\frac{d^3k_\gamma}{(2\pi)^3 2k_\gamma^0}(2\pi)^4\times\delta^4(P-s-k_\gamma)k_{\gamma\alpha}k_{\gamma\lambda}I(k_\gamma,k_g), \quad (12)$$

where P is the b quark 4-momentum.

The tensor $J_{\alpha\lambda}$ can be expanded in terms of scalar functions which depend on the invariants which are left over after the phase space integrations (k_g^2, $P\cdot k_g$, and P^2):

$$J_{\alpha\lambda}=J_1\eta_{\alpha\lambda}+J_2P_\alpha P_\lambda+J_3k_{g\alpha}k_{g\lambda}+J_4(k_{g\alpha}P_\lambda+k_{g\lambda}P_\alpha). \quad (13)$$

The scalar functions J_i are easily identified by contracting $J_{\alpha\lambda}$ with the gluon and/or the b quark 4-momentum. They involve combinations of the phase space integrals of $I(k_\gamma,k_g)$, $(k_\gamma\cdot k_g)I(k_\gamma,k_g)$, and $(k_\gamma\cdot k_g)^2 I(k_\gamma,k_g)$. These integrals are readily computed in a power series in the gluon momentum. We will illustrate this with the phase space integral of $I(k_\gamma,k_g)$, which we perform in the b rest frame:

$$\langle I(k_\gamma,k_g)\rangle=\int\frac{d^3s}{(2\pi)^2 2s^0}\frac{d^3k_\gamma}{(2\pi)^3 2k_\gamma^0}(2\pi)^4\delta^4(P-s-k_\gamma)\times 24\int_0^1 x\,dx\int_0^{1-x}y\,dy\frac{1}{m_c^2-2xyk_g\cdot k_\gamma}$$
$$=-\frac{3}{2\pi}\int_0^1 x\,dx\int_0^{1-x}y\,dy\times\frac{1}{xym_b|\mathbf{k}_g|}\ln\frac{1-m_b(E_g+|\mathbf{k}_g|)xy/m_c^2}{1-m_b(E_g-|\mathbf{k}_g|)xy/m_c^2}. \quad (14)$$

The gluon energy in the above is typically much smaller than the gluon momentum. Roughly speaking, the B matrix elements involving the gluon energy are of order the difference in kinetic energy of the b quark before and after absorbing a soft gluon. Hence the gluon energy is of order $E_g\sim\Delta T_b\sim\Lambda_{\rm QCD}^2/m_b$, while $|\mathbf{k}_g|\sim\Lambda_{\rm QCD}$—the gluons are predominantly spacelike. Dropping the gluon energy E_g relative to $|\mathbf{k}_g|$ and expanding the logarithm in powers of $m_b\Lambda_{\rm QCD}/m_c^2$ yields a rapidly converging power series for $\langle I(k_\gamma,k_g)\rangle$:

$$\langle I(k_\gamma,k_g)\rangle=\frac{1}{8\pi m_c^2}\left(1+\frac{1}{140}\frac{m_b^2|\mathbf{k}_g|^2}{m_c^4}+\frac{1}{6930}\frac{m_b^4|\mathbf{k}_g|^4}{m_c^8}+\cdots\right). \quad (15)$$

The higher-order terms in the series involve integrals of $x^n y^n$, which fall off asymptotically as $\sqrt{\pi}/n^{3/2}4^{n+1}$.

Using the decomposition (13) in Eq. (1) and retaining only the terms that do not vanish by the equations of motion gives for the $b\to s\gamma$ amplitude the expansion

$$T_{b\to s\gamma}=-\frac{\alpha}{32\pi^4}G_F^2 m_b^5\frac{1}{27}\mathrm{Re}(V_{cs}^*V_{cb}V_{ts}V_{tb}^*C_7)\times\overline{b}(J_1\sigma_{\nu\lambda}+J_3\sigma_{\nu\alpha}k_g^\alpha k_{g\lambda}+J_4\sigma_{\nu\alpha}k_g^\alpha P_\lambda)gG_a^{\nu\lambda}\frac{\lambda_a}{2}b. \quad (16)$$

Calculations analogous to those that led to Eq. (15) lead to the following expressions for the leading behavior of the scalar functions J_1, J_3, and J_4 in the b rest frame:[2]

$$J_1=\frac{1}{2m_c^2}\left(1+\frac{3}{700}\frac{m_b^2|\mathbf{k}_g|^2}{m_c^4}\right),$$
$$J_3=\frac{1}{2m_c^2}\left(-\frac{3}{350}\frac{m_b^2}{m_c^4}\right), \quad (17)$$
$$J_4=\frac{1}{2m_c^2}\left(\frac{2}{15}\frac{1}{m_c^2}\right).$$

Although the J_4 term in Eq. (16) is nominally linear in the gluon 4-momentum, it actually gives a correction of $O(\Lambda_{\rm QCD}^2/m_c^2)$ upon using Faraday's law for the gluon fields ($\mathbf{k}\times\mathbf{E_a}=E_g\mathbf{B_a}$), since $E_g\sim\Lambda_{\rm QCD}^2/m_b$. Thus the leading correction to Voloshin's result for $T_{b\to s\gamma}$ given in Eq. (3) arises only from the J_1 and J_3 terms and involves operators quadratic in the gluon 4-momentum. Using Eq. (17), one identifies this correction as

[2]The J_2 term drops out because of the equations of motion.

$$\delta T_{b\to s\gamma} = -\frac{\alpha}{32\pi^4} G_F^2 m_b^5 \frac{1}{27} \mathrm{Re}(V_{cs}^* V_{cb} V_{ts} V_{tb}^* C_7) \times \left(-\frac{3m_b^2}{1400 m_c^6} \langle B| g\bar{b} \frac{\lambda_a}{2} [\sigma_{\nu\lambda}(iD)^2 G_a^{\nu\lambda} + \sigma^{\nu\alpha}\{iD_\lambda, iD_\alpha\} G_a^{\nu\lambda}] b|B\rangle \right). \tag{18}$$

Although one cannot relate the above matrix element to a property of the B mesons, as was done for the leading matrix element, we expect it to be of $O(\Lambda_{\rm QCD}^4)$. However, because of the tiny numerical coefficient, even though $m_b^2\Lambda_{\rm QCD}^2 \sim m_c^4$, the above correction is totally negligible. So Voloshin's result (4) is robust, at least as far as these corrections go.

There are, of course, many other corrections to Voloshin's result. However, these are all of $O(\Lambda_{\rm QCD}^4/m_c^4)$, down by a factor of $\Lambda_{\rm QCD}^2/m_c^2$ relative to the leading term calculated by Voloshin. Some of these arise by considering the full photon-gluon penguin graph, e.g., the J_4 term in Eq. (18). Others come from terms where $\mathcal{L}_{\rm Vol}^{b\to s\gamma g}$ is correlated with itself in the OPE for the total width. Yet others come from photon-gluon penguin graphs in which two or more soft gluons are emitted. Although all of these contributions involve unknown matrix elements, because $\Lambda_{\rm QCD}^2 \ll m_c^2$ it is reasonable to expect that they also should yield quite small corrections to Eq. (4).

Our analysis gives some confidence that the $1/m_c^2$ nonperturbative corrections to the $\bar{B}\to X_s\gamma$ process calculated by Voloshin [2] are an accurate estimate of these corrections. Unfortunately, these interesting corrections are numerically small for this particular decay. Their existence, however, raises the interesting question of whether other $1/m_c^2$ corrections may give more sizable contributions to other processes. We hope to return to this issue at a later date.

We are grateful to Mark Wise for useful discussions on these matters. He and his collaborators have arrived at similar conclusions to ours [9]. We thank them for discussing their results prior to publication. We also would like to thank Misha Voloshin for a helpful discussion. One of us (S.N.) is grateful for the hospitality of the UCLA Department of Physics and Astronomy, where part of this work was completed. This work was supported in part by Department of Energy Grant No. DOE-FG03-91ER40662, Task C.

[1] Nonperturbative contributions to the $\bar{B}\to X_s\gamma$ rate have been discussed in the context of HQET by A. F. Falk, M. Luke, and M. J. Savage, Phys. Rev. D **49**, 3367 (1994). For reviews of heavy quark effective theory, see, e.g., M. B. Wise, in Proceedings, Radiative Corrections, Gatlinburg, 1994 (unpublished), hep-ph/9411264; T. Mannel, talk given at the Workshop on Heavy Quarks at Fixed Target (HQ 96), St. Goar, Germany, 1996 (unpublished), hep-ph/9611411.

[2] M. B. Voloshin, Phys. Lett. B **397**, 275 (1997).

[3] A. Khodjamirian, R. Rückl, G. Stoll, and D. Wyler, Report Nos. WUE-ITP-97-001, MPI-PhT/97-5, Zurich Uni-31/96, Technion-ph-96-24, hep-ph/9702318 (unpublished).

[4] A. Ali and C. Greub, Phys. Lett. B **361**, 46 (1995).

[5] B. Grinstein, R. Springer, and M. Wise, Nucl. Phys. B **339**, 269 (1990).

[6] K. Chetyrkin, M. Misiak, and M. Münz, Report Nos. ZU-TH 24/96, TUM-HEP-263/96, MPI/PhT/96-123, hep-ph/9612313 (unpublished).

[7] See, e.g., A. F. Falk and M. Neubert, Phys. Rev. D **47**, 2965 (1993).

[8] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[9] See Z. Ligeti, L. Randall, and M. Wise, Phys. Lett. B **402**, 178 (1997).