Nonequilibrium QCD: Interplay of hard and soft dynamics in high-energy multigluon beams

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A quantum-kinetic formulation of the dynamical evolution of a high-energy nonequilibrium gluon system at finite density is developed to study the interplay between quantum fluctuations of high-momentum (hard) gluons and the low-momentum (soft) mean color field that is induced by the collective motion of the hard particles. From the exact field equations of motion of QCD, a self-consistent set of approximate quantum-kinetic equations are derived by separating hard and soft dynamics and choosing a convenient axial-type gauge. This set of master equations describes the momentum space evolution of the individual hard quanta, the space-time development of the ensemble of hard gluons, and the generation of the soft mean field by the current of the hard particles. The quantum-kinetic equations are approximately solved to order $g^2(1+g\overline{A})$ for a specific example, namely, the scenario of a high-energy gluon beam along the light cone, demonstrating the practical applicability of the approach. [S0556-2821(97)00817-5]

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I. INTRODUCTION AND SUMMARY

The physics of high-density QCD becomes an increasingly popular object of research, both from the experimental, phenomenological interest, and from the theoretical, fundamental point of view. Presently, and in the near future, the collider facilities such as the DESY ep collider HERA (ep, eA?), Fermilab Tevatron ($p\overline{p}$, pA), BNL Relativistic Heavy Ion Collider (RHIC), and CERN Large Hadron Collider (LHC) ($p\overline{p}$, AA) are able to probe new regimes of dense quark-gluon matter at very small Bjorken x or/and at large A, with rather different dynamical properties. The common feature of high-density QCD matter that can be produced in these experiments is an expected novel exhibition of the interplay between the high-momentum (short-distance) perturbative regime and the low-momentum (long-wavelength) nonperturbative physics. For example, with HERA and Tevatron experiments, one hopes to gain insight into problems concerning the saturation of the strong rise of the proton structure functions at small Bjorken x, possibly due to color-screening effects that are associated with the overlapping of a large number of small-x partons. Another example is the anticipated formation of a quark-gluon plasma in RHIC and LHC heavy ion collisions, where multiple parton rescattering and cascading may generate a high-density environment, in which the collective motion of the quanta can give rise to non-Abelian long-wavelength excitations and screening of color charges.

In any case, the study of coherent low-momentum excitations in QCD, that are generated by, and interacting with, the high-momentum partonic color charges, is of fundamental interest in several respects. First, it provides insight into the basic features of non-Abelian multiparticle dynamics and a step towards a rigorous description of parton transport properties in a dense environment. Secondly, it may help to resolve current problems encountered in perturbative QCD, for instance the absence of static magnetic color screening [1], the problem of infrared renormalons [2] connected with the resummation of perturbation theory in the small-x regime, or the problem of confinement associated with collective ''glue'' behavior of nonperturbative gluons [3]. Interesting progress in these areas is continuously being made, and consistent schemes have emerged to perform calculations of the parton evolution at very small x [4], at very large density [5,6], and for high-temperature QCD of a quark-gluon plasma [7].

Most progress in the context of bulk multiparton dynamics at high density has been made by studying ''hot QCD'' with a thermally equilibrated quark-gluon system at very high temperature T. ''Hot QCD'' has the attractive advantage that the parton density is homogeneous and isotropic in momentum, and its exact form $\propto T^3$ is known, since $T \gg \Lambda \approx 200$ MeV is the only energy scale in the problem. For this academic scenario, inconsistencies of former perturbative calculations have been resolved by gauge-invariant resummation techniques [8] as studied in various applications [9], and moreover, a self-consistent kinetic theory has been formulated [10].

The present paper, extending previous work of Ref. [11], is to be viewed in this very context: it takes the ''hot QCD'' developments as inspirational guideline, but aims to describe the opposite physics extreme, namely a highly nonequilibrium,[1] nonuniform, and nonisotropic parton system. Specifically, the attempt is made to derive from first principles a self-consistent kinetic description for a *nonequilibrium scenario of a gluon beam directed along the light cone*, that is, a high-density system of gluons, moving with very large energies $k_0 \simeq k_z \gg k_\perp \gg \Lambda$ along a beam direction

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[1]The term ''nonequilibrium'' is used in the sense of statistical many-body physics, describing a quantum system far off the state of maximum entropy and thermal equilibrium. Such a nonequilibrium system may in general be spatially inhomogenous and anisotropic in momentum, in contrast to a homogeneous, thermal ensemble, or translation invariant system in vacuum.

(the k_z axis), as it would be typical for the initial stage of a high-energy collider experiment (an extreme example is a collision of two heavy nuclei at the LHC, involving many thousands of gluons coming down the beam pipe). For simplicity the quark degrees of freedom are ignored, but are straightforward to include.

As illustrated schematically in Fig. 1, the initial multigluon state is imagined as a highly Lorentz contracted sheet of bare gluons, characterized by a very large momentum scale Q (e.g., in an ultrarelativistic nuclear collision, the typical momentum transfer of hard scatterings that materialize the gluons out of the colliding beam nuclei). Hence the typical energy and longitudinal momentum of the initial gluons is $\sim Q$. The subsequent evolution of these bare quanta is, at leading order α_s, well known to lead to a rapid multiplication and diffusion of gluons through real and virtual radiation, corresponding to bremsstrahlung and Coulomb-field regeneration, respectively [12]. As a consequence, the typical gluon momenta both in longitudinal and transverse directions, decrease [see Fig. 1(a)]. As long as the average transverse momentum is sufficiently large, $k_\perp \geqslant \mu \sim 1-2$ GeV, $\alpha_s(\mu^2) \ll 1$, a perturbative description of the evolution of the gluon density G is appropriate, but when $k_\perp < \mu$, nonperturbative dynamics is expected to take over, governed by the collective infrared behavior of a large number of long-wavelength gluons. If the number density of low-momentum gluons below μ is large, their dynamics may approximately be described classically [5,13] in terms of a coherent mean field $\bar{A}$ [see Fig. 1(b)].

Given this heuristic picture, the near-at-hand rationale is therefore to subdivide the dynamical development of the gluon ensemble into a perturbative quantum evolution in the short-distance regime $Q^2 \geqslant k_\perp^2 \geqslant \mu^2$, and a nonperturbative, but classical, mean field in the long-wavelength regime $k_\perp^2 < \mu^2$. The corresponding degrees of freedom are referred to as *hard gluons* for $k_\perp \geqslant \mu$, whereas excitations with $k_\perp < \mu$ represent the *soft mean field*.

Because the hard gluons have small transverse extent $\lambda \sim 1/k_\perp \leqslant 0.2$ fm (for $\mu = 1$ GeV), they can be considered, locally in space-time, as incoherent self-interacting quanta, if the interparticle distance is significantly larger than λ. On the other hand, when the typical transverse momenta drop below μ, the gluons begin to act coherently, and collectivity arises, because the motion taking place over a distance scale $1/\mu$ or larger, involves coherently a large number of hard particles, which gives rise to an average soft color field. The crucial point of this *hard-soft separation* is that the over long distances $\lambda > 1/\mu$, the soft mean field represents the average gluon motion, but at short distances $\lambda \leqslant 1/\mu$ the hard gluons may be described approximately as in free space. Certainly, such a rigid division of hard and soft physics in terms of a single parameter μ, is at his point an arbitrary and idealizing definition. However, the arbitrariness can in principle be removed by considering the variation with respect to μ, as in the usual renormalization-group framework. This interesting task is beyond the scope of this paper, and remains to be addressed in the future.

The nonquilibrium scenario of a light cone beam of gluons along the light cone has two major advantages over the opposite thermal equilibrium extreme, the isotropic quark-

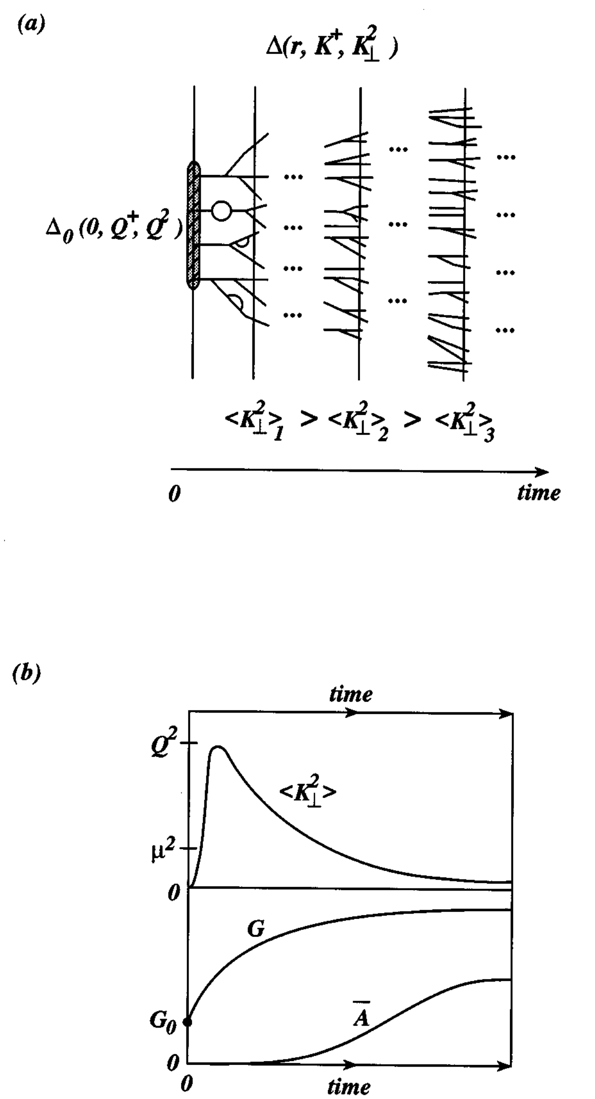

FIG. 1. Nonequilibrium scenario of gluon beam along the light cone. (a) The initial multigluon state, prepared at time $t_0 = 0$ at the hard scale Q with initial condition $\Delta_0(0,Q^+,Q^2)$, develops forward in time which is described by the evolution of the gluon propagator $\hat{\Delta}(r,K^+,K_\perp^2)$ being a function of both space-time $r^\mu = (t,\vec{r})$ and momentum $K^\mu = (E,\vec{K})$. The gluons, propagating with large $K^+ = E + K_z \gg K_\perp$ along the z axis, are accompanied by real and virtual radiation which causes a diffusion in both transverse direction $r_\perp$ and transverse momentum $K_\perp$ as time goes on: The emission of gluons increases the multiplicity and decreases the average transverse momenta $\langle K_\perp^2 \rangle$ at given light cone time $r^- = t - z$ and light cone position $r^+ = t + z$. (b) Top: Qualitative picture of time evolution of the typical transverse momentum $\langle K_\perp^2 \rangle$ of hard gluons, where the earliest emitted daughter gluons have the largest $K_\perp^2 \lesssim Q^2$ and later produced gluons have much smaller $K_\perp^2$. Eventually modes with $K_\perp^2 < \mu^2$ will be populated significantly. Bottom: Corresponding time development of the number density G of hard gluons from initial value G_0 and of the average soft field $\bar{A}$ that is induced by the population of gluons with $K_\perp^2 < \mu^2$, starting from zero initial value. Speculatively, one would expect a saturation at asymptotic times due to screening of further gluon emission by the presence of the soft mean field.

gluon plasma. First, it favors the two-scale separation between hard and soft physics. Second, it allows to choose an axial-type gauge which eliminates to large extent the problems of nonlinearities and of ghost degrees of freedom that are encountered in usual covariant gauges. The two-scale separation arises naturally here, because Lorentz contraction and time dilation along the beam direction plus the limited transverse momenta, force the hard gluon fluctuations and self-interactions to be highly localized to short distances, and separates their quantum motion from the low-momentum mean-field dynamics over comparably long distances. On the other hand, the choice of a noncovariant axial gauge, characterized by a directed four-vector n along a fixed axis, is very suggestive, because the geometry and kinematics allows to choose n parallel to the gluon momentum k_z, in which case perturbative QCD calculations formally reduce in many respects to the Abelian QED counter parts. This is not possible for an isotropic thermal system, where all possible directions of gluon motion are equally probable. Given these premises, the *quantum dynamics* is dominated by the self-interactions of the hard gluons, which make them fluctuate localized around the light cone, whereas the *kinetic dynamics* can well be described statistical-mechanically in terms of mutual interactions among them and in the presence of their generated soft mean field. As elaborated in Ref. [11], these notions are the keys to formulating a quantum-kinetic description, by combining standard techniques of parton evolution and renormalization group, with relativistic many-body transport theory.

The *main result of this study* within the outlined physics framework, is a *set of three master equations*, which couple the quantum evolution of short-distance fluctuations of the individual hard gluons, the space-time development of the gluon system as a whole, and the generation of the soft mean field: (i) an *evolution equation* for the spectral density $\hat{\rho}$ of each individual hard gluon, which determines the intrinsic gluon distribution of a hard gluon in accord with mass- and coupling-constant renormalization, and which dresses up the bare initial gluons to renormalized ''quasiparticles;'' (ii) a *transport equation* for the space-time development of the whole ensemble of these renormalized gluons with respect to their propagation in the self-generated soft mean field, as well as due to their scatterings off each other, which determines the physical gluon phase-space density G; (iii) a *Yang-Mills equation* for the generation of the soft mean field $\bar{A}$, which is induced by the effective color current of the hard, renormalized gluons, where the current is obtained from the momentum-weighted gluon phase-space density. Although this set of equations appears at first sight to be of impractical complexity, it allows in fact for a practical applicable calculation scheme, as will be demonstrated with an explicit sample calculation.

To arrive at the above master equations, three essential aspects of the problem have to be merged: first, the physics-dictated aspect of space-time, kinematics and geometry, second, the quantum field aspect of gluon excitations and self-interactions, and third, the statistical aspect of multiparticle interactions in the presence of the mean field. The nontrivial interconnection of these aspects require to *work directly at the level of equations of motion*, rather than on the level of Feynman diagrams, because the relative proportions and interactions of hard and soft quanta can only be calculated self-consistently from the equations of motion.

The strategy for deriving the above master equations follows closely the previous work of Refs. [11]. The path-integral representation of the Yang-Mills action gives an infinite set of equations of motion for the nonequilibrium n-point Green functions, which is the well known analogue of the Bogoliubov-Born-Green-Kirkwood-Yuon (BBGKY) hierarchy [14]. This hierarchy, which represents the exact theory, is truncated to a system of equations involving only the one- and two-point functions, by arguing that higher-order correlators $n\geq 3$ are comparably small. To achieve self-consistency of the truncated set of equations at the $n=2$ level, the $n\geq 3$ functions must be implicitly lumped into the one- and two-point functions. After separating hard and soft field modes, as alluded to before, the one-point function is identified with the soft average field $\bar{A}_\mu=\langle A_\mu(x)\rangle$ and the two-point function is given by the hard gluon correlator $i\hat{\Delta}_{\mu\nu}=\langle a_\mu(x)\, a_\nu(y)\rangle_P$, where A_μ and a_μ represent the soft and hard modes, respectively. The truncated set of equations of motion then involves the nonequilibrium version of the Dyson-Schwinger equation for $\hat{\Delta}$ and the classical Yang-Mills equation for the soft mean-field $\bar{A}$. The two field-equations of motion for $\hat{\Delta}$ and $\bar{A}$ can be cast into much simpler quantum-kinetic equations with the help of the Wigner-function technique and gradient expansion, and the assumption of two-scale separation implying that the long-wavelength $\bar{A}$ field is slowly varying on the short-distance scale of the hard quantum fluctuations. The result is then the above set of master equations.

A powerful theoretical framework to derive from exact field equations of motion the above approximate quantum kinetic equations, is the so-called *closed-time-path* (CTP) formalism. The CTP formalism is a general tool for treating initial value problems of irreversible multiparticle dynamics in quantum field theory. It therefore provides an appropriate language to describe the problem of nonequilibrium gluon dynamics within a well-established theoretical framework. Originally introduced by Schwinger [15] and Keldysh [16] the CTP formalism and its diverse applications is documented in great detail in the literature [17–23]. In particular, I refer to Ref. [11], where the CTP method is applied to high-energy QCD, and to Appendixes B and C.

The fundamental starting point of nonequilibrium field theory in the CTP formalism is to write down the in-in amplitude Z_P for the evolution of the initial quantum state $|\mathrm{in}\rangle$ forward in time into the remote future. As reviewed in Appendix B, this generalizes the usual quantum field theory approach based on the vacuum-vacuum transition amplitude, or in-out amplitude, to account for the *a-priori* presence of medium particles described by the density matrix $\hat{\rho}(t_0)$ and to evolve this nontrivial initial state in the presence of the medium from t_0 to t_∞ in the future. The *in-in* amplitude Z_P is graphically depicted in Fig. 2, and formally it is given by $Z_P[\mathcal{J},\hat{\rho}]=\langle \mathrm{in}|\mathrm{in}\rangle_{\mathcal{J},\hat{\rho}}$, where $\mathcal{J}=(\mathcal{J}^+,\mathcal{J}^-)$ is an external source with components on the upper $+$ and lower $-$ time branch, and $\hat{\rho}(t_0)$ denotes the initial state density matrix. From the path-integral representation of Z_P one obtains then the nonequilibrium Green functions. The convenient feature

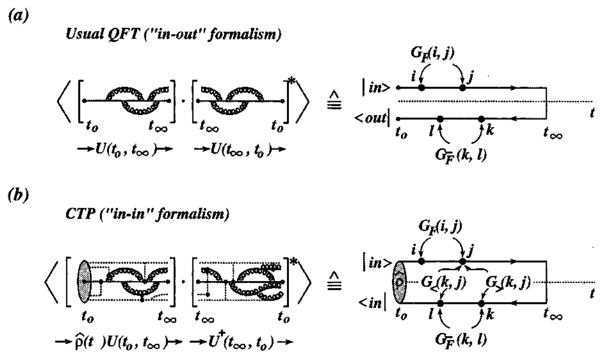

FIG. 2. Difference between the *in-out* formalism of usual quantum field theory (in free space or ''vacuum'') and the *in-in* formalism of the CTP formulation (in the presence of surrounding particles or ''medium''). (a) The in-out amplitude described by the evolution of an asymptotic $|\text{in}\rangle$ state at $t_0 \to -\infty$ to an asymptotic $|\text{out}\rangle$ state at $t_\infty \to \infty$ by means of the time evolution operator $U(t_0, t_\infty)$. Because $U(t_0, t_\infty) = U^\dagger(t_\infty, t_0)$, forward and backward evolution are identical, and there is no correlation between the two time branches. Consequently, the Feynman propagator $G_F = G_{\bar{F}}^*$ contains the full dynamics of the two-point correlations. (b) The in-in amplitude starts at t_0 with a nontrivial initial multiparticle state described by the density matrix $\hat{\rho}(t_0)$ and evolves again by means of time evolution operator $U(t_0, t_\infty)$. Due to statistical interactions among the many evolving particles acting as a medium, in this case $\hat{\rho}(t_0) U(t_0, t_\infty) \neq U^\dagger(t_\infty, t_0)$. Consequently, the statistical correlation between the two time branches have the effect that $G_F \neq G_{\bar{F}}^*$, and moreover require the introduction of additional correlation functions $G_>$ and $G_<$ to account for the cross-talk between upper and lower time branches.

of this Green function formalism on the closed-time path P is that it is formally analogous to standard quantum field theory, based on the vacuum-vacuum, or in-out amplitude $Z[\mathcal{J}] = \langle \text{out} | \text{in} \rangle_{\mathcal{J}} = \langle 0 | 0 \rangle_{\mathcal{J}}$, except for the fact that in the CTP formalism, the fields have contributions from both time branches. For details I refer to Appendix B, where the basics of the CTP formalism are summarized, in particular, how to obtain the path integral for Z_P that generates the Green functions on the closed-time path P.

The interpretation of this formal apparatus for the evolution along the closed-time path P is rather simple: If the initial state is the vacuum itself, that is, the absence of a medium generated by other particles, then the density matrix $\hat{\rho}$ is diagonal and one has $|\text{in}\rangle \to |0\rangle$. In this case the evolution along the $+$ branch is identical to the antitime ordered evolution along the $-$ branch (modulo an irrelevant phase), and space-time points on different branches cannot cross-talk. In the presence of a medium, however, the density matrix contains off-diagonal elements, and there are statistical correlations between the quantum system and the medium particles (e.g., scatterings) that lead to correlations between space-time points on the $+$ branch with space-time points on the $-$ branch. Hence, when addressing the evolution of a multiparticle system, both the deterministic self-interaction of the quanta, i.e., the time-(anti-time-)ordered evolution along the $+$ $(-)$ branch *and* the statistical mutual interaction with each other, i.e., the non-time-ordered cross-talk between the $+$ and $-$ branches, must be included in a self-consistent manner. The CTP method achieves this through the time integration along the contour P. Although for physical observables the time values are on the $+$ branch, both $+$ and $-$ branches will come into play at intermediate steps in a self-consistent calculation.

The *outline of the paper* is as follows. In Sec. II the field equations of motion for the hard gluon propagator and the soft mean field are derived from the path-integral representation of the in-in amplitude Z_P for noncovariant gauges. After separating hard and soft degrees of freedom, two key approximations are made, that allow us to cast the infinite hierarchy of exact equations of motion in terms of a truncated system of only two approximate equations, namely a Dyson-Schwinger equation for the hard gluon propagator, and a Yang-Mills equation for the soft field. In Sec. III the transition to a quantum kinetic description is worked out. This requires one further key approximation in conjunction with a clear definition of quantum and kinetic space-time regimes, such that the aforementioned two-scale separation is guaranteed. This also defines the limits for the applicability of the quantum kinetic approximation. Provided that the separability condition is satisfied, one finally arrives at the set of master equations discussed above, for which a systematic calculation scheme is proposed. In Sec. IV an explicit calculation to solve the master equations is presented for the physics scenario depicted in Fig. 1. I consider the evolution of an initial incoherent ensemble of bare gluons moving collinearly along the light cone as it proceeds in its momentum- and space-time development and generates its soft mean field. To avoid overkill of too many technical details, each section is accompanied by Appendixes. Appendix A defines the notation and conventions used throughout the paper. Appendix B reviews the basics of the CTP formalism. Appendix C discusses the application of the CTP method to QCD for noncovariant gauges. Appendix D shows the advantageous absence of ghosts in noncovariant gauges. Appendix E gives details on how to obtain from the in-in amplitude $\mathcal{Z}_P$ an approximate effective action functional from which the of motion for the hard gluon propagator and the soft field are derived. Appendix F summarizes some basic analyticity properties of the free-field propagators in the CTP formalism and discusses their relation to the gluon phase-space density.

Finally some remarks on the most closely *related work* in the literature (for an extended discussion, see the Introduction of Ref. [11]). Blaizot and Iancu [10] have in a series of papers developed a kinetic theory for ''hot QCD,'' i.e., the case of a high-temperature quark-gluon plasma. One of the key elements of their approach is the formulation of a well-defined and consistent approximation scheme. I adopt many features of this approach in the present, rather different physical context. The inclusion of the aspect of quantum evolution and renormalization is new here.

McLerran and co-workers [24], as well as Makhlin [25], have developed different approaches to calculate the quantum evolution of parton systems with light cone dominance, i.e., in a beam-type scenario as considered in the present work. The McLerran-Venugopalan model also gives a predictive estimate for the feedback effect of the coherent mean field on the hard gluon evolution that generates this field. In several respects I follow a similar route. In the present work, the fact that it embodies in addition the aspect of space-time development of the evolution is new.

Boyanovski *et al.* [26] have intensely studied the nonequilibrium evolution in scalar field theory, using extensively the techniques of the CTP formalism in conjunction with a large-N expansion. Although the focus on this paper is rather different, many of the concepts and results in their papers concerning the first-principles time evolution of the quantum system with associated particle production, dissipation, mean-field dynamics, etc., may serve as a scalar toy model for QCD.

II. INTERPLAY OF ''HARD'' AND ''SOFT'' GLUON DYNAMICS

A. The in-in amplitude for QCD in noncovariant gauges and the concept of approximation

The in-in amplitude Z_P introduced in Sec. I admits a path-integral representation which is the generating functional for the nonequilibrium Green functions defined on closed-time path P, as discussed in Appendixes B and C:

$$Z_P[\mathcal{K}]=\int \mathcal{D}\mathcal{A}\exp\{i(I[\mathcal{A},\mathcal{K}])\}, \tag{1}$$

where $\mathcal{A}_\mu^a=(\mathcal{A}_\mu^{a+},\mathcal{A}_\mu^{a-})$ has two components, living on the upper $(+)$ and lower $(-)$ time branches of Fig. 2, with $\mathcal{D}\mathcal{A}=\Pi_{\mu,a}\mathcal{D}\mathcal{A}_\mu^{a+}\mathcal{D}\mathcal{A}_\mu^{a-}$, and where $\mathcal{K}$ represents the presence of external sources. I consider here the *class of noncovariant gauges* defined by [27,28]

$$\langle n^\mu \mathcal{A}_\mu^a(x)\rangle=0, \tag{2}$$

where n^μ is a constant four-vector, being either spacelike $(n^2<0)$, timelike $(n^2>0)$, or lightlike $(n^2=0)$. The particular choice of the vector n^μ is usually dictated by the physics or computational convenience, and distinguishes *axial gauge* $(n^2<0)$, *temporal gauge* $(n^2>0)$, and *light cone gauge* $(n^2=0)$. Referring to Appendix D, the great advantage of these gauges is that the Faddeev-Popov ghosts decouple, so that in practical calculations the ghost degrees of freedom can be ignored, just as in Abelian gauge theories.

Then the action I in the exponential of Eq. (1) is given by (cf. Appendix C)

$$I[\mathcal{A},\mathcal{K}]\equiv I_{\rm YM}[\mathcal{A}]+I_{\rm GF}[n\cdot\mathcal{A}]+\mathcal{K}[\mathcal{A}], \tag{3}$$

containing the Yang-Mills action $I_{\rm YM}$, the gauge-fixing term $I_{\rm GF}$, and the initial state source term $\mathcal{K}$, containing multipoint correlations concentrated at $t=t_0$:

$$I_{\rm YM}[\mathcal{A}]=-\frac{1}{4}\int_P d^4x\mathcal{F}_{\mu\nu}^a(x)\mathcal{F}^{\mu\nu,a}(x),$$

$$I_{\rm GF}[n\cdot\mathcal{A}]=-\frac{1}{2\alpha}\int_P d^4x[n\cdot\mathcal{A}^a(x)]^2, \tag{4}$$

$$\mathcal{K}[\mathcal{A}]=\mathcal{K}^{(0)}+\int_P d^4x\mathcal{K}_\mu^{(1)a}(x)\mathcal{A}^{\mu,a}(x)+\frac{1}{2}\int_P d^4xd^4y\mathcal{K}_{\mu\nu}^{(2)ab}(x,y)\mathcal{A}^{\mu,a}(x)\mathcal{A}^{\nu,b}(y)+\cdots. \tag{5}$$

The exact knowledge of the in-in amplitude Z_P from Eq. (1), would require to the calculation of all Green functions up to infinite order, and would correspond to the full solution of QCD in nonequilibrium media. Rather than that, the realistic goal is to formulate a practical calculation scheme for the kinetic evolution of a multigluon system. In order to make progress, one needs to make reasonable approximations that are consistent with the specific physical problem under study, and truncate the infinite hierarchy of Green functions.

In this section a closed set of approximate equations is derived that are in principle solvable, given a suitable physics scenario. The basic idea is to describe an evolving gluon system in terms of two distinct components, namely, *hard, short-range quantum fluctuations* and *soft, long-wavelength collective excitations*, which I assume to be separable by a characteristic space-time distance. It is clear that the relative proportions and interactions of hard and soft degrees of freedom must be calculated self-consistently from the equations of motion.

Starting from the in-in amplitude (1), the strategy of the procedure is the following.

(1) The exact expression of the in-in amplitude $Z_P\equiv\exp(iW_P)$ is rewritten in terms of soft and hard field modes by splitting the gauge field $\mathcal{A}_\mu=A_\mu+a_\mu$. Therefrom, one obtains an infinite set of coupled equations for the Green functions. In order to reduce this to a finite system, I make the following approximation.

Approximation 1. The functional $W_P=-i\ln Z_P$ is expressed in terms of connected one- and two-point functions $\mathcal{G}^{(1)},\mathcal{G}^{(2)}$ alone by eliminating $\mathcal{G}^{(n)}$ for $n\geqslant3$ as dynamical variables. Then the expectation values of $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ describe the induced soft mean field $\overline{A}_\mu$ and the hard (soft) correlation functions $\hat{\Delta}_{\mu\nu}$ $(\hat{D}_{\mu\nu})$.

(2) From the truncated functional W_P the corresponding effective action Γ_P is obtained, which generates the desired self-consistent equations of motion for $\overline{A}_\mu$, $\hat{\Delta}_{\mu\nu}$, and $\hat{D}_{\mu\nu}$. Here I make the following approximation.

Approximation 2. It is assumed that the soft field dynamics can be treated classically by the nonpropagating average field $\overline{A}_\mu$, and that the long-range propagation of soft modes, described by $\hat{D}_{\mu\nu}$ may be ignored at this level, i.e., $\hat{D}_{\mu\nu}\ll\overline{A}_\mu\overline{A}_\nu$. This assumption is motivated by the widely studied [24,13] observation that a classical treatment of the long-distance dynamics of bosonic quantum fields at high density, obeying the classical field equations, should provide a good approximation, if the soft modes are sufficiently occupied.

The original infinite equation system can then be reduced to a Yang-Mills equation for the classical, soft field $\overline{A}_\mu$, as it is induced by the current of hard quanta, and a Dyson-

Schwinger equation for the hard propagator $\hat{\Delta}_{\mu\nu}$ subject to the presence of the soft mean field and to quantum fluctuations. These field equations of motion are still of very intractable nonlinear character. They are further simplified to quantum-kinetic equations in Sec. III.

B. Separating soft and hard dynamics

The first step in the strategy is the separation of soft and hard physics in the path-integral formalism with Green functions of both the soft and hard quanta in the presence of the soft classical field that is induced by and feeding back to the quantum dynamics. A frequently used method for separate treatment of quantum and classical dynamics in field theory is the so-called ''background field method'' [29] which has been studied, e.g., in the context of dynamical symmetry breaking, vacuum structure, confinement and gravity, or for hot plasmas in finite temperature QCD. Within the background field method, one would split up the gauge field appearing in the classical action into an external classical background field and a quantum field which remains the sole dynamical variable in the path integral. I will, however, not follow this path, and rather prefer to *treat soft and hard physics on equal footing*, that is, to separate the gauge field into a soft classical field plus its soft quantum excitations, and a hard quantum field. Then both soft and hard fields can be quantized and remain as dynamical variables *a priori*.

The gauge field $\mathcal{A}_\mu$ appearing in the classical action $I_{\rm YM}[\mathcal{A}]$ is split up into a soft (long-range) part A_μ, and a hard (short-range) quantum field a_μ:

$$\mathcal{A}^a_\mu(x)=\int\frac{d^4k}{(2\pi)^4}e^{+ik\cdot x}\mathcal{A}^a_\mu(k)\,\theta(\mu^2-k_\perp^2)+\int\frac{d^4k}{(2\pi)^4}e^{+ik\cdot x}\mathcal{A}^a_\mu(k)\,\theta(k_\perp^2-\mu^2)\equiv A^a_\mu(x)+a^a_\mu(x). \tag{6}$$

This is the formal definition of the terms ''soft'' and ''hard,'' as used in this paper. The soft and hard physics are separated by the momentum scale μ which is at this point arbitrary. However, this arbitrariness can in principle be overcome by considering $\mu(x)$ as a dynamical *variable* depending on the space-time point x, rather than a fixed parameter, and determining it self-consistently from the local stability condition $d\mathcal{A}_\nu(x)/d\mu^2(x)=0$. From Eq. (6) it is obvious that the corresponding scale in space-time $\lambda\equiv 1/\mu$ divides soft and hard regimes in terms of the transverse wavelength of field modes, so that one may associate the soft field A_μ being responsible for long-range color collective effects, and the hard field a_μ embodying the short-range quantum dynamics. Consequently, the field strength tensor receives a soft part, a hard part, and a mixed contribution:

$$\mathcal{F}^a_{\mu\nu}(x)\equiv(F^a_{\mu\nu}[A]+f^a_{\mu\nu}[a]+\phi^a_{\mu\nu}[A,a])(x). \tag{7}$$

When quantizing this decomposed theory by writing down the appropriate in-in amplitude Z_P, one must be consistent with the gauge field decomposition (6) into soft and hard components and with the classical character of the former. Substituting the soft-hard mode decomposition (6) into Eq. (1), the functional integral of the in-in amplitude (C11) becomes

$$Z_P[\mathcal{K}]=\int\mathcal{D}A\mathcal{D}a\exp\{i(I[A]+I[a]+I[A,a])\}, \tag{8}$$

with the soft, hard, and mixed contribution, respectively,

$$I[A]=\int d^4x\left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a}-\frac{1}{2\alpha}(n\cdot A^a)^2\right)+\int d^4x\mathcal{K}^{(1)a}_\mu A^{\mu,a}+\int d^4xd^4yA^{\mu,a}\mathcal{K}^{(2)ab}_{\mu\nu}A^{\nu,b}+\cdots, \tag{9}$$

$$I[a]=\int d^4x\left(-\frac{1}{4}f^a_{\mu\nu}f^{\mu\nu,a}-\frac{1}{2\alpha}(n\cdot a^a)^2\right)+\int d^4x\mathcal{K}^{(1)a}_\mu a^{\mu,a}+\int d^4xd^4ya^{\mu,a}\mathcal{K}^{(2)ab}_{\mu\nu}a^{\nu,b}+\ \dots, \tag{10}$$

$$\begin{aligned}I[A,a]&=\int d^4x\left(-\frac{1}{4}\phi^a_{\mu\nu}\phi^{\mu\nu,a}-\frac{1}{2}\{\phi^a_{\mu\nu}F^{\mu\nu,a}+\phi^a_{\mu\nu}f^{\mu\nu,a}+F^a_{\mu\nu}f^{\mu\nu,a}\}\right)\\&=\int d^4x\{-gf^{abc}[(\partial^x_\mu A^a_\nu)(a^{\mu,b}a^{\nu,c}+A^{\mu,b}a^{\nu,c}+a^{\mu,b}A^{\nu,c})+(\partial^x_\mu a^a_\nu)(A^{\mu,b}A^{\nu,c}+a^{\mu,b}A^{\nu,c}+A^{\mu,b}a^{\nu,c})]-g^2f^{ace}f^{bde}[2A^a_\mu A^b_\nu a^{\mu,c}a^{\nu,d}+A^a_\mu A^b_\nu A^{\mu,c}a^{\nu,d}+a^a_\mu a^b_\nu a^{\mu,c}A^{\nu,d}]\}.\end{aligned} \tag{11}$$

Note that in Eqs. (10) and (11) terms involving two-products $\propto a_\mu\, A_\nu$ do not contribute to Z_P, because their expectation value vanishes due to the soft-hard separation (6) which defines a_μ and A_ν as complimentary.

At this point I make approximation 1 from above. It is assumed that initial state can be represented as an ensemble of incoherent hard gluons, each of which has very small spatial extent $\Delta r_\perp\ll\lambda=1/\mu$, corresponding to transverse momenta $k_\perp^2\gg\mu^2$. By definition of μ, the short-range character of these quantum fluctuations implies that the expectation value $\langle a_\mu\rangle$ vanishes at all times. However, the long-range correlations of the eventually populated soft modes with small momenta $k_\perp^2\ll\mu^2$ may lead to a collective mean field with nonvanishing $\langle A_\mu\rangle$. Accordingly, I impose the following condition on the expectation values of the fields:

$$\langle A^a_\mu(x)\rangle\begin{cases}=0 & \text{for}\ \ t\leqslant t_0,\\ \geqslant 0 & \text{for}\ \ t>t_0.\end{cases}\qquad \langle a^a_\mu(x)\rangle\overset{!}{=}0\qquad\text{for all } t. \tag{12}$$

Now I make approximation 2, that is, the quantum fluctuations of the soft field are ignored, assuming any multipoint correlations of soft fields to be small:

$$\langle A^{a_1}_{\mu_1}(x_1)\cdots A^{a_n}_{\mu_n}(x_n)\rangle \ll \langle A^{a_1}_{\mu_1}(x_1)\rangle\cdots\langle A^{a_n}_{\mu_1}(x_n)\rangle \quad \text{for all } n\geq 2; \tag{13}$$

i.e., take A_μ as a nonpropagating and nonfluctuating, classical field. In particular,

$$iD^{ab}_{\mu\nu}(x,y)\equiv\langle A^a_\mu(x)A^b_\nu(y)\rangle \ll \langle A^a_\mu(x)\rangle\langle A^b_\nu(y)\rangle, \tag{14}$$

so that the limit ''$D_{\mu\nu}\to 0$'' can be considered.

As explained in more detail in Appendix E, the generating functional for the *connected* Green functions

$$W_P[\mathcal{K}]=-i\ln Z_P[\mathcal{K}], \tag{15}$$

which generate the infinite set of connected n-point Green functions $\mathcal{G}^{(n)}$ via

$$(-i)\mathcal{G}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) \equiv \frac{\delta}{i\,\delta\mathcal{K}^{(n)}}W_P[\mathcal{K}]|_{\mathcal{K}=0} = \langle a^{a_1}_{\mu_1}(x_1)\cdots a^{a_k}_{\mu_k}(x_k)A^{a_{k+1}}_{\mu_{k+1}}(x_{k+1})\cdots A^{a_n}_{\mu_n}(x_n)\rangle^{(c)}_P, \tag{16}$$

is truncated at level $n\geq 3$ on the basis of approximations (12) and (14). As a result, W_P becomes a functional of the one-point function (soft mean field $\overline{A}$) and the two-point function (hard propagator $\hat{\Delta}$) *only*:

$$\mathcal{G}^{(1)a}_\mu(x)=\langle A^a_\mu(x)\rangle^{(c)}_P\equiv\overline{A^a_\mu}(x),$$

$$\mathcal{G}^{(2)ab}_{\mu\nu}(x,y)=\langle a^a_\mu(x)a^b_\nu(y)\rangle^{(c)}_P\equiv i\hat{\Delta}^{ab}_{\mu\nu}(x,y). \tag{17}$$

These relations define the soft, classical mean field $\overline{A}$, and the hard quantum propagator $\hat{\Delta}$ in terms of expectation values of soft and hard field operators A_μ and a_μ, respectively. One now readily obtains the *effective action* Γ_P (or proper vertex functional) via Legendre transformation (cf. Appendix E):

$$\Gamma_P[\mathcal{G}]\approx\Gamma_P[\overline{A},\hat{\Delta}]=W_P[\mathcal{K}^{(1)},\mathcal{K}^{(2)}]-\mathcal{K}^{(1)}\circ\overline{A} - \frac{1}{2}\mathcal{K}^{(2)}\circ(i\hat{\Delta}+\overline{AA}), \tag{18}$$

which is a functional of *only* the soft field $\overline{A}$ and the hard propagator $\hat{\Delta}$ as independent dynamical degrees of freedom.

The *equations of motion* for the mean field $\overline{A}$ and for the hard propagator $\hat{\Delta}$ in the presence of sources, follow now by differentiation of Eq. (18) with respect to $\overline{A}$ and $\hat{\Delta}$ (cf. Appendix E)

$$\frac{\delta\Gamma_P}{\delta\overline{A^a_\mu}(x)}=-\mathcal{K}^{(1)\mu,a}(x)-\int_P d^4y\; K^{(2)\mu\nu,ab}(x,y)\overline{A^{\nu,\,b}}(y), \tag{19}$$

$$\frac{\delta\Gamma_P}{\delta\hat{\Delta}^{ab}_{\mu\nu}(x,y)}=\frac{1}{2i}\mathcal{K}^{(2)\mu\nu,ab}(x,y). \tag{20}$$

The self-consistent equations of motion of the dynamically evolving system are then obtained from Eqs. (19) and (20) by (i) imposing initial conditions in terms of the $\mathcal{K}$ kernels at $t=t_0$ and (ii) by obtaining an explicit formula for Γ_P in terms of $\overline{A_\mu}$ and $\hat{\Delta}_{\mu\nu}$. Concerning the initial conditions, I confine myself to nonequilibrium initial states of Gaussian form (i.e., quadratic in the hard modes) and do not consider possible linear force terms. That is, I set

$$\mathcal{K}^{(1)}(x)|_{x^0=t_0}=0, \quad \mathcal{K}^{(2)}(x,y)|_{x^0=y^0=t_0}\geq 0. \tag{21}$$

To obtain an explicit expression for Γ_P, the formal loop expansion of Eq. (18) results in the well-known Cornwall-Jackiw-Tomboulis formula [30]

$$\Gamma_P[\overline{A},\hat{\Delta}]=\overline{I}_{\text{eff}}[\overline{A},a]-\frac{i}{2}\,\text{Sp}[\ln(\Delta_0^{-1}\hat{\Delta})-\overline{\Delta_0}^{-1}\hat{\Delta}+1] + \Gamma^{(2)}_P[\overline{A},\hat{\Delta}], \tag{22}$$

where $\text{Sp}[AB\ldots]\equiv \text{Tr}\int_P d^4x_1d^4x_2\cdots A(x_1)B(x_2)\cdots$ stands for both the trace over color and Lorentz indices, as well as the integration over all space-time positions, hence giving the expectation value $\langle AB\cdots\rangle_P$ as defined in Appendix C. The *physical interpretation* of the various terms in this expression for Γ_P is the following [11].

(i) The first term is of order $\hbar^0$ and is given by the classical action (9),(10) at $A=\overline{A}$ and switched-off sources $\mathcal{K}$:

$$\overline{I}_{\text{eff}}[\overline{A},a]\equiv[I[A]+I[a]+I[A,a]]_{A=\overline{A},\mathcal{K}=0}. \tag{23}$$

Notice that in the limit $a=0$, this reduces to the classical action for the soft mean field, $\overline{I}_{\text{eff}}[\overline{A},0]=I_{\text{YM}}[\overline{A}]+I_{\text{GF}}[n\cdot\overline{A}]$.

(ii) The second term in Eq. (22) is of order $\hbar^1$ and contains the contributions of the coupling between the soft mean field $\overline{A}$ and the hard quantum propagator $\hat{\Delta}$. The *free* propagator [see Fig. 3(a)] is given by $[\delta^2\overline{I}_{\text{eff}}[\overline{A},a]/\delta a(x)\delta a(y)]_{A=0;\,a=0}$ with $\overline{A}$ switched off, which yields

$$(\Delta_0^{-1})^{ab}_{\mu\nu}(x,y)=-\delta_{ab}\delta^4_P(x,y)d_{\mu\nu}(\partial_x)\partial_x^2, \tag{24}$$

where it is understood that the space-time arguments x and y in Δ_0 satisfy $(x-y)^2_\perp<1/\mu^2$, and

$$d_{\mu\nu}(\partial_x)\equiv g^{\mu\nu}-\frac{n^\mu\partial_x^\nu+n^\nu\partial_x^\mu}{n\cdot\partial_x}+(n^2+\alpha^{-1}\partial_x^2)\frac{\partial_x^\mu\partial_x^\nu}{(n\cdot\partial_x)^2}. \tag{25}$$

Even in the absence of quantum fluctuations, these contributions amount to a modification of the free propagator, such that the free propagator Δ_0 becomes an effective propagator $\overline{\Delta}$ in the mean field, *dressed* up by the presence of $\overline{A}$. This *mean field propagator* [see Fig. 3(b)], denoted by $\overline{\Delta}$, is obtained from $[\delta^2\overline{I}_{\text{eff}}[\overline{A},a]/\delta a(x)\delta a(y)]_{A=\overline{A};a=0}$ with finite $\overline{A}\neq 0$, which results in

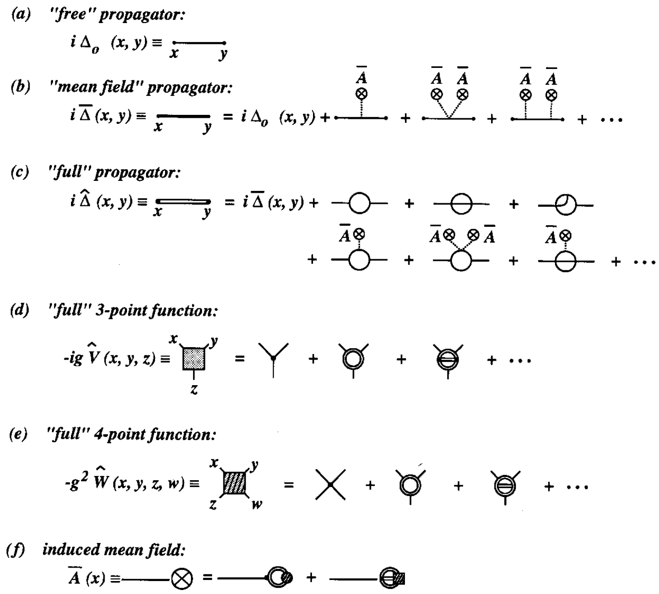

FIG. 3. Diagrammatics of the various terms used for the n-point functions appearing in the text. The two-point function $\mathcal{G}^{(2)}=i\Delta$ is the hard gluon propagator with the *free-field* propagator Δ_0 (no interactions), the *mean-field* propagator $\overline{\Delta}$ (including the interactions with the classical soft field $\overline{A}$), and the *full* propagator $\hat{\Delta}$ [including both mean-field and quantum (loop) interactions]. Similarly, the connected three-point function $\mathcal{G}^{(3)}=-ig\hat{V}$ and the four-point function $\mathcal{G}^{(4)}=-g^2\hat{W}$ contain soft mean-field plus hard quantum contributions with internal full propagator $\hat{\Delta}$. Finally, the one-point function is the soft mean field $\overline{A}$ that is generated by the hard gluons through the coupling to the full three-point and four-point functions $\hat{V}$ and $\hat{W}$.

$$(\overline{\Delta}^{-1})^{ab}_{\mu\nu}(x,y)=(\Delta_0^{-1})^{ab}_{\mu\nu}(x,y)-\overline{\Pi}^{ab}_{\mu\nu}(x,y), \tag{26}$$

where $\overline{\Pi}$ denotes the self-energy contribution associated with the presence of the mean field $\overline{A}\neq 0$. Its explicit expression is given below in Eq. (48). In other words, the effect of the mean field is to shift the pole in the free propagator Δ_0 of Eq. (24) by a dynamically induced ''mass'' term $\propto\overline{\Pi}$, which can produce screening and damping effects. Note that $\Delta_0^{-1}=\overline{\Delta}^{-1}|_{\overline{A}=0}$. It is important to realize that this mean field effect is still on the classical tree-level, and does not involve quantum fluctuations associated with radiative self-interactions among the hard gluons.

(iii) The last term $\Gamma_P^{(2)}$ in Eq. (22) represents the sum of all two-particle irreducible graphs of order $\hbar^2,\hbar^3,\ldots$ [30], with the *full* propagator $\hat{\Delta}$, dressed by both the soft mean field and the quantum self-interactions [see Fig. 3(c)]

$$\hat{\Delta}^{ab}_{\mu\nu}(x,y)\equiv\hat{\Delta}^{ab}_{[\overline{0}]\mu\nu}(x,y)+\delta\hat{\Delta}^{ab}_{[\overline{A}]\mu\nu}(x,y), \tag{27}$$

where the dependence of the full propagator on the soft mean field $\overline{A}$ is indicated by an explicit subscript, and

$$\hat{\Delta}^{ab}_{[\overline{0}]\mu\nu}=\hat{\Delta}^{ab}_{[\overline{A}]\mu\nu}|_{\overline{A}=0},\qquad \delta\hat{\Delta}^{ab}_{[\overline{A}]\mu\nu}|_{\overline{A}=0}=0. \tag{28}$$

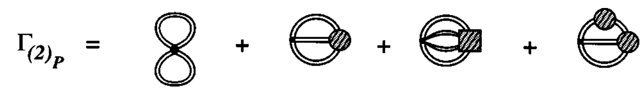

FIG. 4. The two-loop contribution $\Gamma_P^{(2)}$, Eqs. (29)–(33), to the effective action Γ_P of Eq. (22), in the diagrammatic representation of Fig. 3. Formally, $\Gamma_P^{(2)}$ is the sum of all two-particle irreducible graphs with internal lines representing the full gluon propagators $\hat{\Delta}$ and full three- and four-gluon vertices $\hat{V}$ and $\hat{W}$.

Note that $\hat{\Delta}_{[\overline{0}]}\neq\Delta_0$, that is, $\hat{\Delta}_{[\overline{0}]}$ denotes the *full* propagator for $\overline{A}=0$, whereas Δ_0 is the *free* propagator (24). The real (dispersive) part of $\Gamma_P^{(2)}$ contains the virtual loop corrections associated with the gluon self-interactions, whereas the imaginary (dissipative) part contains the emission, absorption, and scattering processes of hard gluons. In other words, $\Gamma_P^{(2)}$ embodies all the interesting quantum dynamics that is connected with renormalization group, entropy generation, dissipation, etc. The explicit form of $\Gamma_P^{(2)}$ is diagrammatically shown in Fig. 4, with the vertices and lines defined by Fig. 3. Suppressing color and Lorentz indices and employing a condensed notation, e.g., $\hat{\Delta}(x_1,x_2)\equiv\hat{\Delta}^{ab}_{\mu\nu}(x_1,x_2)$, the corresponding formula is

$$\Gamma_P^{(2)}[\overline{A},\hat{\Delta}]=\Gamma_{(1)}+\Gamma_{(2)}+\Gamma_{(3)}+\Gamma_{(4)}, \tag{29}$$

with the contributions

$$\Gamma_{(1)}=\frac{1}{8}g^2\int_P d^4x d^4y\int_P d^4x_1 d^4y_1 W_0(x,y,x_1,y_1)\times\hat{\Delta}(y_1,x_1)\hat{\Delta}(y,x), \tag{30}$$

$$\Gamma_{(2)}=\frac{i}{12}g^2\int_P d^4x d^4y\int_P\prod_{i=1}^{2}d^4x_i d^4y_i V_0(x,x_1,x_2)\times\hat{\Delta}(x_1,y_1)\hat{\Delta}(x_2,y_2)\hat{V}(y_2,y_1,y)\hat{\Delta}(y,x), \tag{31}$$

$$\Gamma_{(3)}=\frac{1}{48}g^4\int_P d^4x d^4y\int_P\prod_{i=1}^{3}d^4x_i d^4y_i W_0(x,x_1,x_2,x_3)\times\hat{\Delta}(x_1,y_1)\hat{\Delta}(x_2,y_2)\hat{\Delta}(x_3,y_3)\times\hat{W}(y_3,y_2,y_1,y)\hat{\Delta}(y,x), \tag{32}$$

$$\Gamma_{(4)}=\frac{i}{96}g^4\int_P d^4x d^4y\int_P\prod_{i=1}^{2}d^4x_i d^4y_i d^4z_i W_0(x,x_1,x_2,x_3)\times\hat{\Delta}(x_2,z_2)\hat{\Delta}(x_3,z_3)\hat{V}(z_3,z_2,z_1)\hat{\Delta}(z_1,y_1)\times\hat{\Delta}(x_1,y_2)\hat{V}(y_1,y_2,y)\hat{\Delta}(y,x). \tag{33}$$

The functions $\hat{V}$ and $\hat{W}$ are the *full* proper vertex functions for the three-gluon and four-gluon coupling, respectively. Their diagrammatic representation is shown in Figs. 3(d) and 3(e), and formally they are given by the functional derivatives of Γ_P at $\overline{A}\neq 0$, namely, $[\delta^n\Gamma_P/\delta a(x_1)\cdots\delta a(x_i)\delta A(x_{i+1})\cdots\delta A(x_n)]_{A=\overline{A};\,a=0}$ for $n=3$ and $n=4$, respectively,

$$-ig\hat{V}^{abc}_{\lambda\mu\nu}(x,y,z)=-igV^{abc}_{0\lambda\mu\nu}(x,y,z)+O(g^3),$$

$$-g^2\hat{W}^{abcd}_{\lambda\mu\nu\sigma}(x,y,z,w)=-g^2W^{abcd}_{0\lambda\mu\nu\sigma}(x,y,z,w)+O(g^4), \tag{34}$$

which, to lowest order in the coupling constant, reduce to the *bare* three- and four-gluon vertices V_0 and W_0, respectively:

$$\begin{aligned}V^{abc}_{0\lambda\mu\nu}(x,y,z)=f^{abc}\{&g_{\lambda\mu}(\partial_y-\partial_x)_\nu\delta^4_P(x,z)\,\delta^4_P(y,z)\\&+g_{\mu\nu}(\partial_z-\partial_y)_\lambda\delta^4_P(y,x)\,\delta^4_P(z,x)\\&+g_{\nu\lambda}(\partial_x-\partial_z)_\mu\delta^4_P(x,y)\,\delta^4_P(z,y)\}\end{aligned} \tag{35}$$

$$\begin{aligned}&W^{abcd}_{0\lambda\mu\nu\sigma}(x,y,z,w)\\&\quad=-\{(f^{ace}f^{bde}-f^{ade}f^{cbe})\ g_{\lambda\mu}g_{\nu\sigma}\\&\qquad+(f^{abe}f^{cde}-f^{ade}f^{bce})\ g_{\lambda\nu}g_{\mu\sigma}\\&\qquad+(f^{ace}f^{dbe}-f^{abe}f^{cde})\ g_{\lambda\sigma}g_{\nu\mu}\}\\&\qquad\times\delta^4_P(x,y)\,\delta^4_P(z,w)\,\delta^4_P(y,z).\end{aligned} \tag{36}$$

C. Equations of motion

As sketched above and discussed in more detail in Appendix E, the equations of motion (19) and (20) result from approximating the exact theory by truncation of the infinite hierarchy of equations for the n-point Green functions to the one-point function [the soft mean field $\overline{A(x)}$] and the two-point function [the hard propagator $i\hat{\Delta}(x,y)$], with all higher-point functions being combinations of these and connected by the three-gluon and four-gluon vertices $-ig\hat{V}(x,y,z)$ and $-g^2\hat{W}(x,y,z,w)$, respectively. Before writing down the explicit form of the resulting equations of motion, it is useful to summarize the terminology introduced in the course of the above discussion.

Mean field $\overline{A}$ [Fig. 3(f)]. Denotes the classical soft field as the expectation value of the gauge field A, which is induced by the abundance of emitted hard gluons and their collective motion. *Free* propagator $i\Delta_0$ [Fig. 3(a)]. Refers to the free propagation of hard gluons in the absence of interactions, i.e., vanishing coupling $g=0$. *Mean-field* propagator $i\overline{\Delta}$ [Fig. 3(b)]. Denotes the tree-level propagator without quantum corrections, i.e., the free propagator with an arbitrary number $\geqslant 0$ of attached external legs coupling to the soft mean field, but without closed loops that correspond to quantum self-interactions. *Full* propagator $i\hat{\Delta}$ [Fig. 3(c)]. Terms the dressed propagator of the hard quanta, that is, renormalized by both the interactions with the soft mean field *and* the self-interactions among the hard quanta. *Full* vertex functions $-ig\hat{V},-g^2\hat{W}$ [Figs. 3(d), 3(e)]. Represent the three-gluon and four-gluon vertices with the internal lines being the full hard propagator including mean-field and quantum interactions.

1. Yang-Mills equation for the soft mean field

The equation of motion for the soft field $\overline{A^a_\mu}(x)$, is given by Eq. (19), i.e., $\delta\Gamma_P/\delta\overline{A}=-\mathcal{K}^{(1)}-\mathcal{K}^{(2)}\circ\overline{A}$, from which one obtains, upon taking into account the initial condition (21), $\mathcal{K}^{(1)}=0$, the *Yang-Mills equation for* $\overline{A}$

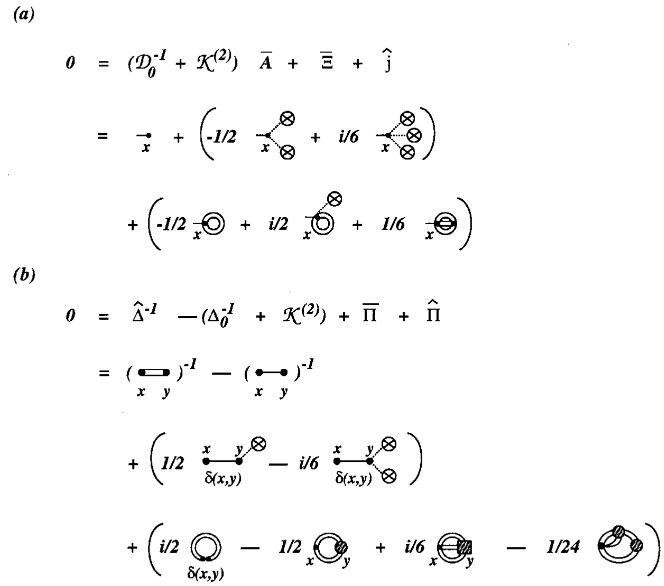

FIG. 5. Diagrammatic representation in terms of the rules of Fig. 3, of the equations of motion. (a) The Yang-Mills equations (37), (38) for the soft field $\overline{A}$ with the self-coupling contribution $\overline{\Xi}$, Eqs. (40)–(42), and the generating hard gluon current $\hat{j}$, Eqs. (200)–(204). (b) The Dyson-Schwinger equation (47) for the hard propagator $\hat{\Delta}$ with the mean-field polarization tensor $\overline{\Pi}$, Eqs. (48)–(50), and the quantum contribution $\hat{\Pi}$, Eqs. (51)–(55).

$$[\overline{D^{\lambda,ab}},\overline{F^b_{\lambda\mu}}](x)=-\hat{j}^a_\mu(x)-(\mathcal{K}^{(2)\ ab}_{\mu\lambda}\circ\overline{A^{\lambda,\ b}})(x), \tag{37}$$

where $[\overline{D},\overline{F}]=\overline{DF}-\overline{FD}$ with the covariant derivative defined as $\overline{D^\lambda}\equiv D^\lambda[\overline{A}]=\partial^\lambda_x-ig\overline{A^\lambda}(x)$ and $\overline{F_{\lambda\mu}}\equiv F_{\lambda\mu}[\overline{A}]=[\overline{D_\lambda},\overline{D_\mu}]/(-ig)$. The second term on the right side is the initial state contribution to the current, according to the condition (21), $\mathcal{K}^{(2)}\circ\overline{A}=\int_P d^4y\ \mathcal{K}^{(2)\ ab}_{\mu\lambda}(x,y)\overline{A^{\lambda,b}}(y)$.

Rewriting the left-hand side of Eq. (37) as

$$[\overline{D^{\lambda,ab}},\overline{F^b_{\lambda\mu}}](x)=\mathcal{D}^{-1ab}_{0\mu\lambda}\overline{A^{\lambda,\ b}}(x)+\overline{\Xi^a_\mu}(x),$$

$$\mathcal{D}^{-1ab}_{0\mu\lambda}\equiv\delta^{ab}(g_{\mu\lambda}\partial^2_x-\partial^x_\mu\partial^x_\lambda-n_\mu n_\lambda), \tag{38}$$

where, upon taking into account the gauge constraint (2), the $-n_\mu n_\lambda\overline{A^\lambda}$ in $\mathcal{D}^{-1}_{0\mu\lambda}\overline{A^\lambda}$ does not contribute, because $0=\langle n\cdot A\rangle=n^\nu\overline{A_\nu}$ and Eq. (37) may be expressed in the alternative form [see Fig. 5(a)]

$$\{(\mathcal{D}^{-1}_0+\mathcal{K}^{(2)})\overline{A}\}^a_\mu(x)+\overline{\Xi^a_\mu}(x)+\hat{j}^a_\mu(x)=0. \tag{39}$$

Here the function $\overline{\Xi}$ contains the soft-field self-coupling

$$\overline{\Xi^a_\mu}(x)=\overline{\Xi^a_{(1)\mu}}(x)+\overline{\Xi^a_{(2)\mu}}(x), \tag{40}$$

$$\overline{\Xi^a_{(1)\mu}}(x)=-\frac{g}{2}\int_P\prod_{i=1}^2 d^4x_iV^{abc}_{0\mu\nu\lambda}(x,x_1,x_2)\overline{A^{\nu,b}}(x_1)\times\overline{A^{\lambda,c}}(x_2), \tag{41}$$

$$\overline{\Xi}^{a}_{(2)\mu}(x) = + \frac{ig^2}{6} \int_P \prod_{i=1}^{3} d^4x_i W^{abcd}_{0\mu\nu\lambda\sigma}(x,x_1,x_2,x_3) \overline{A}^{\nu,b}(x_1) \times \overline{A}^{\lambda,c}(x_2) \overline{A}^{\sigma,d}(x_3), \tag{42}$$

and the current $\hat{j}$ is the *induced current* due to the hard quantum dynamics in the presence of the soft field $\overline{A}$:

$$\hat{j}^a_\mu(x) = \hat{j}^a_{(1)\mu}(x) + \hat{j}^a_{(2)\mu}(x) + \hat{j}^a_{(3)\mu}(x), \tag{43}$$

$$\hat{j}^a_{(1)\mu}(x) = -\frac{i\, g}{2} \int_P \prod_{i=1}^{2} d^4x_i V^{abc}_{0\mu\nu\lambda}(x,x_1,x_2) \hat{\Delta}^{\nu\lambda,\, bc}(x_1,x_2), \tag{44}$$

$$\hat{j}^a_{(2)\mu}(x) = -\frac{g^2}{2} \int_P \prod_{i=1}^{3} d^4x_i W^{abcd}_{0\mu\nu\lambda\sigma} \times (x,x_1,x_2,x_3) \overline{A}^{\nu,\, b}(x_1) \hat{\Delta}^{\lambda\sigma,\, cd}(x_2,x_3), \tag{45}$$

$$\hat{j}^a_{(3)\mu}(x) = -\frac{ig^3}{6} \int_P \prod_{i=1}^{3} d^4x_i d^4y_i W^{abcd}_{0\mu\nu\lambda\sigma} \times (x,x_1,x_2,x_3) \hat{\Delta}^{\nu\nu',\, bb'}(x_1,y_1) \hat{\Delta}^{\lambda\lambda',\, cc'} \times (x_2,y_2) \hat{\Delta}^{\sigma\sigma',\, dd'}(x_3,y_3) V^{abcd}_{0\mu'\nu'\lambda'\sigma'}(y_1,y_2,y_3). \tag{46}$$

It should be remarked that the function $\overline{\Xi}$ on the left-hand side of Eq. (37) contains the nonlinear self-coupling of the soft field $\overline{A}$ alone, whereas the induced current $\hat{j}$ on the right-hand side is determined by the hard propagator $\hat{\Delta}$, thereby generating the soft field.

2. *Dyson-Schwinger equation for the hard gluon propagator*

From the equation of motion (20) for the hard propagator $\hat{\Delta}^{ab}_{\mu\nu}(x,y)$, that is, $\delta\Gamma_P/\delta\hat{\Delta} = \mathcal{K}^{(2)}/(2i)$, one finds after incorporating condition (21), $\mathcal{K}^{(1)}=0$, the *Dyson-Schwinger equation for* $\hat{\Delta}$ [see Fig. 5(b)]:

$$\{(\hat{\Delta})^{-1} - (\Delta_0)^{-1} - \mathcal{K}^{(2)} + \overline{\Pi} + \hat{\Pi}\}^{ab}_{\mu\nu}(x,y) = 0, \tag{47}$$

where $\hat{\Delta}$ is the *fully dressed propagator* of the hard quantum fluctuations in the presence of the soft mean field, defined by Eq. (27), whereas Δ_0 is the *free propagator*, given by Eq. (24). The polarization tensor Π has been decomposed into two parts, a mean-field part $\overline{\Pi}$ and a quantum fluctuation part $\hat{\Pi}$. The *mean-field polarization tensor* $\overline{\Pi}$ incorporates the local interaction between the hard quanta and the soft mean field:

$$\overline{\Pi}^{ab}_{\mu\nu}(x,y) = \overline{\Pi}^{ab}_{(1)\mu\nu}(x,y) + \overline{\Pi}^{ab}_{(2)\mu\nu}(x,y), \tag{48}$$

$$\overline{\Pi}^{ab}_{(1)\mu\nu}(x,y) = \frac{ig}{2} \delta^4_P(x,y) \int_P d^4z V^{abc}_{0\mu\nu\lambda}(x,y,z) \overline{A}^{\lambda,c}(z), \tag{49}$$

$$\overline{\Pi}^{ab}_{(2)\mu\nu}(x,y) = \frac{g^2}{6} \delta^4_P(x,y) \int_P d^4z d^4w W^{abcd}_{0\mu\nu\lambda\sigma}(x,y,z,w) \times \overline{A}^{\lambda,c}(z) \overline{A}^{\sigma,d}(w), \tag{50}$$

plus terms of order $g^3\overline{A}^3$ which one may safely ignore within the present approximation scheme. The *fluctuation polarization tensor* $\hat{\Pi}$ contains the quantum self-interaction among the hard quanta in the presence of $\overline{A}$. It is given by the variation $2i\,\delta\Gamma^{(2)}_P/\delta\hat{\Delta}$ of the two-loop part $\Gamma^{(2)}_P$, Eq. (29), of the effective action Γ_P:

$$\hat{\Pi}^{ab}_{\mu\nu}(x,y) = \hat{\Pi}^{ab}_{(1)\mu\nu}(x,y) + \hat{\Pi}^{ab}_{(2)\mu\nu}(x,y) + \hat{\Pi}^{ab}_{(3)\mu\nu}(x,y) + \hat{\Pi}^{ab}_{(4)\mu\nu}(x,y), \tag{51}$$

$$\hat{\Pi}^{ab}_{(1)\mu\nu}(x,y) = -\frac{g^2}{2} \int_P d^4x_1 d^4y_1 W^{abcd}_{0\mu\nu\lambda\sigma}(x,y,x_1,y_1) \times \hat{\Delta}^{\lambda\sigma\,,cd}(y_1,x_1) \tag{52}$$

$$\hat{\Pi}^{ab}_{(2)\mu\nu}(x,y) = -\frac{i\, g^2}{2} \int_P \prod_{i=1}^{2} d^4x_i d^4y_i V^{acd}_{0\mu\lambda\sigma}(x,x_1,x_2) \times \hat{\Delta}^{\lambda\lambda',\, cc'}(x_1,y_1) \hat{\Delta}^{\sigma\sigma',\, dd'}(x_2,y_2) \times \hat{V}^{d'c'b}_{\sigma'\lambda'\nu}(y_2,y_1,y), \tag{53}$$

$$\hat{\Pi}^{ab}_{(3)\mu\nu}(x,y) = -\frac{g^4}{6} \int_P \prod_{i=1}^{3} d^4x_i d^4y_i W^{acde}_{0\mu\lambda\sigma\tau}(x,x_1,x_2,x_3) \times \hat{\Delta}^{\lambda\lambda',\, cc'}(x_1,y_1) \hat{\Delta}^{\sigma\sigma',dd'}(x_2,y_2) \times \hat{\Delta}^{\tau\tau',ee'}(x_3,y_3) \hat{W}^{e'd'c'b}_{\tau'\sigma'\lambda'\nu}(y_3,y_2,y_1,y), \tag{54}$$

$$\hat{\Pi}^{ab}_{(4)\mu\nu}(x,y) = -\frac{ig^4}{24} \int_P \prod_{i=1}^{2} d^4x_i d^4y_i \times d^4z_i W^{acde}_{0\mu\lambda\sigma\tau}(x,x_1,x_2,x_3) \hat{\Delta}^{\sigma\rho',df'}(x_2,z_2) \times \hat{\Delta}^{\tau\rho'',ef''}(x_3,z_3) \hat{V}^{f''f'f}_{\rho''\rho'\rho}(z_3,z_2,z_1) \times \hat{\Delta}^{\rho\lambda',fc'}(z_1,y_1) \hat{\Delta}^{\lambda\sigma',cd'}(x_1,y_2) \times \hat{V}^{c'd'}_{\lambda'\sigma'}(y_1,y_2,y). \tag{55}$$

Note that the usual Dyson-Schwinger equation in *vacuum* is contained in Eqs. (47)–(55) as the special case when the mean field vanishes, $\overline{A}(x)=0$, and initial state correlations are absent, $\mathcal{K}^{(2)}(x,y)=0$. In this case, the propagator becomes the usual vacuum propagator, since the mean-field contribution $\overline{\Pi}$ is identically zero, and the quantum part $\hat{\Pi}$ reduces to the vacuum contribution.

III. TRANSITION TO QUANTUM KINETICS

The equations of motion (37) or (39) for $\overline{F_{\mu\nu}}$ or $\overline{A_\mu}$ and Eq. (47) for $\hat{\Delta}_{\mu\nu}$, are nonlinear integrodifferential equations and clearly not solvable in all their generality. However, the field equations of motion (37) or (47) can be cast into much simpler quantum-kinetic equations with the help of the Wigner-function technique and gradient expansion, and the assumption of two-scale separation. As a result one obtains finally the three master equations mentioned in Sec. I: a simplified Yang-Mills equation describing the space-time change of $\overline{A}$ and two equations for the gluon propagator $\hat{\Delta}$, namely, first, an *evolution equation* for the QCD evolution in momentum space and, second, a *transport equation* for the space-time development in the presence of $\overline{A}$. In order to achieve this result, one needs to make a third key approximation (in addition to the two approximations of Sec. II A).

Approximation 3. It is assumed that the induced soft field $\overline{A_\mu}$ is slowly varying on the scale of the short-range, hard quantum fluctuations, that is, the gradient of the soft field is small compared to the Compton wavelength of the hard quanta. Then one can treat the quantum fluctuations of $\hat{\Delta}(r,k)$ at short distances separately from the collective effects represented by to the soft field $\overline{A(r)}$ with long wavelength.

A. Quantum and kinetic space-time regimes

The key to derive from (37) or (47) the corresponding approximate quantum-kinetic equations is the separability of hard and soft dynamics in terms of the space-time scale $\lambda \equiv 1/\mu$, where μ is the parametric momentum scale introduced in Eq. (6). This implies that one may characterize the dynamical evolution of the gluon system by a short-range *quantum scale* $r_{\rm qua} \ll \lambda$, and a comparably long-range *kinetic scale* $r_{\rm kin} \gtrsim \lambda$. Low-momentum collective excitations that may develop at the particular momentum scale $g\mu$ are thus well separated from the typical hard gluon momenta $k_\perp \geq \mu$, if $g \ll 1$. Therefore, collectivity can arise, because the wavelength of the soft oscillations $\sim 1/g\mu$ is much larger than the typical extension of the hard quantum fluctuations $\sim 1/\mu$. I emphasize that this notion of two characteristic scales is not just an academic construction, but rather is a typical property of quantum field theory. A simple example is a freely propagating electron. In this case, the quantum scale is given the Compton wavelength $\sim 1/m_e$ in the restframe of the charge, and measures the size of the radiative vacuum polarization cloud around the bare charge. The kinetic scale, on the other hand, is determined by the mean-free-path of the charge, which is infinite in vacuum, and in medium is inversely proportional to the local charge density times the interaction cross section $\sim 1/(n_e \sigma_{\rm int})$. Adopting this notion to the present case of gluon dynamics, I define $r_{\rm qua}$ and $r_{\rm kin}$ as follows.

The quantum scale $r_{\rm qua}$ measures the spatial extension of quantum fluctuations associated with virtual and real radiative emission and reabsorption off a given hard gluon, described by the hard propagator $\hat{\Delta}$. It can thus be interpreted as its Compton wavelength, corresponding to the typical transverse extension of the fluctuations and thus inversely proportional to the average transverse momentum

$$r_{\rm qua} \equiv \hat{\lambda} \simeq \frac{1}{\langle k_\perp \rangle}, \quad \langle k_\perp \rangle \geq \mu, \tag{56}$$

where the second relation is imposed by means of the definition (6) of hard and soft modes. In general, $\hat{\lambda}$ can be a space-time-dependent quantity, because the magnitude of $\langle k_\perp \rangle$ is determined by both the radiative self-interactions of the hard gluons and the interactions with the soft field.

The kinetic scale $r_{\rm kin}$ measures the range of the long-wavelength correlations, described by the soft mean-field $\overline{A}$, and may be parametrized in terms of the average transverse wavelength of soft modes $\langle q_\perp \rangle$, such that

$$r_{\rm kin} \equiv \overline{\lambda} \simeq \frac{1}{\langle q_\perp \rangle}, \quad \langle q_\perp \rangle \lesssim g\mu, \tag{57}$$

where $\overline{\lambda}$ may vary from one space-time point to another, because the population of soft modes $\overline{A(q)}$ is determined locally by the hard current $\hat{j}$ with dominant contribution from gluons with transverse momentum $\simeq \mu$.

The above classification of quantum-(kinetic-)scales specifies in space-time the relevant regime for the hard (soft) dynamics, so that the separability of the two scales $r_{\rm qua}$ and $r_{\rm kin}$ imposes the following condition on the relation between space-time and momentum:

$$\hat{\lambda} \ll \overline{\lambda} \quad \text{or} \quad \langle k_\perp \rangle \geq \mu \gg g\mu \approx \langle q_\perp \rangle. \tag{58}$$

The physical interpretation of Eq. (58) is simple: At short distances $r_{\rm qua} \ll 1/(g\mu)$ a hard gluon can be considered as an *incoherent quantum* which emits and partly reabsorbs daughter gluons, corresponding to the combination of real bremsstrahlung and virtual radiative fluctuations. Only a hard probe with a short wavelength $\hat{\lambda} \leq r_{\rm qua}$ can resolve this quantum dynamics. On the other hand, at larger distances $r_{\rm kin} \approx 1/(g\mu)$, a gluon appears as a *coherent quasiparticle*, that is, as an extended object with a changing transverse size corresponding to the extent of its intrinsic quantum fluctuations. This dynamical substructure is, however, not resolvable by long-wavelength modes $\overline{\lambda} \geq r_{\rm kin}$ of the soft field $\overline{A}$.

Accordingly, one may classify the quantum and kinetic regimes, respectively, by associating with two distinct space-time points x^μ and y^μ the following characteristic scales:

$$s^\mu \equiv x^\mu - y^\mu \sim \hat{\lambda} = \frac{1}{g\mu}, \quad \partial_s^\mu = \frac{1}{2}(\partial_x^\mu - \partial_y^\mu) \sim g\mu,$$

$$r^\mu \equiv \frac{1}{2}(x^\mu + y^\mu) \sim \overline{\lambda} = \frac{1}{\mu}, \quad \partial_r^\mu = \partial_x^\mu + \partial_y^\mu \sim \mu. \tag{59}$$

On the *kinetic scale* the effect of the soft field modes of $\overline{A}$ on the hard quanta involves the coupling $g\overline{A}$ to the hard propagator and is of the order of the soft wavelength $\overline{\lambda} = 1/(g\mu)$, so that one may characterize the soft field strength by

$$g\overline{A_\mu}(r) \sim g\mu, \quad g\overline{F_{\mu\nu}}(r) \sim g^2\mu^2, \tag{60}$$

plus corrections of order $g^2\mu^2$ and $g^3\mu^3$, respectively, which are assumed to be small.

On the *quantum scale*, on the other hand,

$$\hat{\Delta}^{-1}_{\mu\nu} \sim k_\perp^2 \gtrsim \mu^2 \gg g^2\mu^2 \sim g\overline{F_{\mu\nu}}, \tag{61}$$

and one expects that the short-distance fluctuations corresponding to emission and reabsorption of gluons with momenta $k_\perp \geqslant \mu$, are little affected by the long-range, soft mean field, because the color force $\sim g\overline{F}$ acting on a gluon with momentum $k_\perp \sim \mu$ produces only a very small change in its momentum.

B. The kinetic approximation

The realization of the two space-time scales, short-distance quantum and quasiclassical kinetic, allows us to reformulate the quantum field-theoretical problem as a relativistic many-body problem within kinetic theory. The key element is to establish the connection between the preceding description in terms of Green functions and a probabilistic kinetic description in terms of so-called Wigner functions [31]. Whereas the two-point functions, such as the propagator or the polarization tensor, depend on two separate space-time points x and y, their Wigner transforms utilizes a mixed space-time/momentum representation, which is particularly convenient for implementing the assumption of separated quantum and kinetic scales, i.e., that the long-wavelength field $\overline{A}$ is slowly varying in space-time on the scale of short-range quantum fluctuations. Moreover, the trace of the Wigner-transformed propagator is the quantum analogue of the single particle phase-space distribution of gluons, and therefore provides the basic quantity to make contact with kinetic theory of multiparticle dynamics [18].

In terms of the center-of-mass coordinate $r=\frac{1}{2}(x+y)$ and relative coordinate $s=x-y$ of two space-time points x and y, Eq. (59), one can express any two-point function $\mathcal{G}(x,y)$, such as $\hat{\Delta},\Pi$, in terms of the coordinates

$$\mathcal{G}^{ab}_{\mu\nu}(x,y)=\mathcal{G}^{ab}_{\mu\nu}\left(r+\frac{s}{2},r-\frac{s}{2}\right)\equiv\mathcal{G}^{ab}_{\mu\nu}(r,s). \tag{62}$$

The *Wigner transform* $\mathcal{G}(r,k)$ is then defined as the Fourier transform with respect to the relative coordinate s, being the canonical conjugate to the momentum k. In general, the necessary preservation of local gauge symmetry requires a careful definition that obeys the gauge transformation properties [7], but for the specific choice of gauge (2), the Wigner transform is simply [32,46].

$$\mathcal{G}(r,s)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot s}\mathcal{G}(r,k),$$

$$\mathcal{G}(r,k)=\int d^4s\,e^{ik\cdot s}\mathcal{G}(r,s). \tag{63}$$

The Wigner representation (63) will facilitate a systematic identification of the dominant contributions of the soft field $\overline{A}$ to the hard propagator $\hat{\Delta}$, a concept that was developed by Blaizot and Iancu [7]: First, one expands both $\overline{A}$ and $\hat{\Delta}=\hat{\Delta}_{[\overline{0}]}+\delta\hat{\Delta}_{[\overline{A}]}$ in terms of gradients of the long-range variation with the kinetic scale r and, second, one makes an additional expansion in powers of the soft field $\overline{A}$ and of the induced perturbation $\delta\hat{\Delta}_{[\overline{A}]}\sim g\hat{\Delta}_{[\overline{0}]}$.

1. Gradient expansion

To proceed, recall that the coordinate r^μ describes the kinetic space-time dependence $\sim r_{\rm kin}$, whereas s measures the quantum space-time distance $\sim r_{\rm qua}$. In translational invariant situations, e.g., in vacuum or thermal equilibrium, $\mathcal{G}(r,s)$ in Eq. (63) is independent of r^μ and sharply peaked about $s^\mu=0$. Here the range of the variation is fixed by $\lambda=1/\mu$, Eq. (56), corresponding to the confinement length const$\times1/\Lambda$ in the case of vacuum, or to the thermal wavelength const$\times1/T$ in equilibrium. On the other hand, in the presence of a slowly varying soft field $\overline{A}$ with a wavelength $\overline{\lambda}=1/(g\mu)$, Eq. (57), the s^μ dependence is little affected, while the acquired r^μ dependence will have a long-wavelength variation. In view of the estimates (59), one may therefore neglect the derivatives of $\mathcal{G}(r,k)$ with respect to r^μ which are of order $g\mu$, relative to those with respect to s^μ which are of order μ.

Hence one can perform the so-called *gradient expansion* of the soft field and the hard propagator and polarization tensor in terms of gradients $(s\cdot\partial_r)^n$, and keep only terms up to first order $n=1$, i.e.,

$$\overline{A_\mu}(x)=\overline{A_\mu}\left(r+\frac{s}{2}\right)\simeq\overline{A_\mu}(r)+\frac{s}{2}\cdot\partial_r\overline{A_\mu}(r), \tag{64}$$

and similarly for $\overline{A_\mu}(y)=\overline{A_\mu}(r-s/2)$, as well as

$$\hat{\Delta}_{\mu\nu}(x,y)=\hat{\Delta}_{\mu\nu}(r,s)\simeq\hat{\Delta}_{\mu\nu}(0,s)+s\cdot\partial_r\hat{\Delta}_{\mu\nu}(r,s) \tag{65}$$

$$\hat{\Pi}_{\mu\nu}(x,y)=\hat{\Pi}_{\mu\nu}(r,s)\simeq\hat{\Pi}_{\mu\nu}(0,s)+s\cdot\partial_r\hat{\Pi}_{\mu\nu}(r,s). \tag{66}$$

Then, by using the following conversion rules [11,23] to carry out the Wigner transformations:

$$\int d^4x'\,f(x,x')g(x',y)\Rightarrow f(r,k)g(r,k)+\frac{i}{2}[(\partial_k f)\cdot(\partial_r g)-(\partial_r f)\cdot(\partial_k g)] \tag{67}$$

$$h(x)g(x,y)\Rightarrow h(r)g(r,k)-\frac{i}{2}(\partial_r h)\cdot(\partial_k g),$$

$$h(y)\;g(x,y)\Rightarrow h(r)g(r,k)+\frac{i}{2}(\partial_r h)\cdot(\partial_k g), \tag{68}$$

$$\partial_x^\mu f(x,y)\Rightarrow\left(-ik^\mu+\frac{1}{2}\partial_r^\mu\right)f(r,k),$$

$$\partial_y^\mu f(x,y)\Rightarrow\left(+ik^\mu+\frac{1}{2}\partial_r^\mu\right)f(r,k), \tag{69}$$

the transformed polarization tensor $\Pi(r,k)$ is obtained from $\Pi(x,y)$, Eqs. (48) and (51), with

$$\Pi_{\mu\nu}(r,k)=\overline{\Pi}_{\mu\nu}(r,k)+\hat{\Pi}_{\mu\nu}(r,k), \tag{70}$$

where the soft *mean-field contribution* [cf. Eqs. (49), (50)] is

$$\overline{\Pi}^{ab}_{\mu\nu}(r,k)=(\overline{\Pi}_{(1)}+\overline{\Pi}_{(2)})^{ab}_{\mu\nu}(r,k), \tag{71}$$

$$\overline{\Pi}^{ab}_{(1)\mu\nu}(r,k)=\frac{ig}{2}V^{abc}_{0\mu\nu\lambda}(k,0,-k)\overline{A}^{\lambda,c}(r), \tag{72}$$

$$\overline{\Pi}^{ab}_{(2)\mu\nu}(r,k)=-\frac{ig}{6}W^{abcd}_{0\mu\nu\lambda\sigma}(k,0,0,-k)\overline{A}^{\lambda,c}(r)\overline{A}^{\sigma,d}(r), \tag{73}$$

and the *quantum contribution* [cf. Eqs. (51)–(55)] is

$$\hat{\Pi}^{ab}_{\mu\nu}(r,k)=(\hat{\Pi}_{(1)}+\hat{\Pi}_{(2)}+\hat{\Pi}_{(3)}+\hat{\Pi}_{(4)})^{ab}_{\mu\nu}(r,k), \tag{74}$$

$$\hat{\Pi}^{ab}_{(1)\mu\nu}(r,k)=+\frac{ig^2}{2}\int\frac{d^4q}{(2\pi)^4i}W^{abcd}_{0\mu\nu\lambda\sigma}(k,q,-q,-k)\times\hat{\Delta}^{\lambda\sigma,cd}(r,k), \tag{75}$$

$$\hat{\Pi}^{ab}_{(2)\mu\nu}(r,k)=+\frac{g^2}{2}\int\frac{d^4q}{(2\pi)^4i}V^{acd}_{0\mu\lambda\sigma}(k,-q,-q')\times\hat{\Delta}^{\lambda\lambda',cc'}(r,q)\hat{\Delta}^{\sigma\sigma',dd'}(r,q')\times\hat{V}^{d'c'b}_{\sigma'\lambda'\nu}(r;q',q,-k), \tag{76}$$

$$\hat{\Pi}^{ab}_{(3)\mu\nu}(r,k)=-\frac{g^4}{6}\int\frac{d^4q}{(2\pi)^4i}\frac{d^4p}{(2\pi)^4i}\times W^{acde}_{0\mu\lambda\sigma\tau}(k,-q,-q',-p)\hat{\Delta}^{\lambda\lambda',cc'}(r,q)\times\hat{\Delta}^{\sigma\sigma',dd'}(r,q')\hat{\Delta}^{\tau\tau',ee'}(r,p)\times\hat{W}^{e'd'c'b}_{\tau'\sigma'\lambda'\nu}(r;q,q',p,-k), \tag{77}$$

$$\hat{\Pi}^{ab}_{(4)\mu\nu}(r,k)=-\frac{g^4}{24}\int\frac{d^4q}{(2\pi)^4i}\frac{d^4p}{(2\pi)^4i}\times W^{acde}_{0\mu\lambda\sigma\tau}(k,-q,-q',-p')\hat{\Delta}^{\sigma\rho',df'}(r,q)\times\hat{\Delta}^{\tau\rho'',ef''}(r,q')\hat{V}^{f''f'f}_{\rho''\rho'\rho}(r;q,q',-p)\times\hat{\Delta}^{\rho\lambda',fc'}(r,p)\hat{\Delta}^{\lambda\sigma',cd'}(r,p')\times\hat{V}^{c'd'}_{\lambda'\sigma'}(r;p,p',-k). \tag{78}$$

Here the three- and four-gluon vertex functions from Eqs. (34), (35), and (36) depend explicitly on r:

$$\hat{V}(r;k_i)=V_0(k_i)+O[g^4f(r,k_i)],$$

$$\hat{W}(r;k_i)=W_0(k_i)+O[g^4f(r,k_i)], \tag{79}$$

with the *bare* pointlike vertices V_0,W_0 being r independent and given by

$$V^{abc}_{0\lambda\mu\nu}(k_1,k_2,k_3)=-if^{abc}\{g_{\lambda\mu}(k_1-k_2)_\nu+g_{\mu\nu}(k_2-k_3)_\lambda+g_{\nu\lambda}(k_3-k_2)_\mu\}. \tag{80}$$

$$W^{abcd}_{0\lambda\mu\nu\sigma}(k_1,k_2,k_3,k_4)=-\{(f^{ace}f^{bde}-f^{ade}f^{cbe})g_{\lambda\mu}g_{\nu\sigma}+(f^{abe}f^{cde}-f^{ade}f^{bce})g_{\lambda\nu}g_{\mu\sigma}+(f^{ace}f^{dbe}-f^{abe}f^{cde})g_{\lambda\sigma}g_{\nu\mu}\}. \tag{81}$$

With the above formulas, one can now convert both the Yang-Mills equation (37) and the Dyson-Schwinger equation (47) into a set of much simpler equations. For the Dyson-Schwinger equation, the Wigner transformation together with the gradient expansion yields *two* distinct equations for the hard propagator $\hat{\Delta}^{\mu\nu}(r,k)$, namely, (i) an *evolution equation*[2] and (ii) a *transport equation*. They are obtained [11,23] by taking the sum and difference of Wigner-transform of Eq. (47) and its adjoint, using the rules (67)–(69),

(i) *Evolution equation*:

$$\left(k^2-\frac{1}{4}\partial_r^2\right)\hat{\Delta}^{\mu\nu}(r,k)-\frac{1}{2}\{\overline{\Pi}^{\mu}_{\sigma},\hat{\Delta}^{\sigma\nu}\}(r,k)+\frac{i}{4}[\partial_r^\lambda\overline{\Pi}^{\mu}_{\sigma},\partial^k_\lambda\hat{\Delta}^{\sigma\nu}](r,k)=d^{\mu\nu}(k)\quad\hat{1}_P+\frac{1}{2}\{\hat{\Pi}^{\mu}_{\sigma},\hat{\Delta}^{\sigma\nu}\}(r,k)+\frac{i}{4}[\partial_k^\lambda\hat{\Pi}^{\mu}_{\sigma},\partial^r_\lambda\hat{\Delta}^{\sigma\nu}](r,k)-\frac{i}{4}[\partial_r^\lambda\hat{\Pi}^{\mu}_{\sigma},\partial^k_\lambda\hat{\Delta}^{\sigma\nu}](r,k). \tag{82}$$

(ii) *Transport equation*:

$$(k\cdot\partial_r)\hat{\Delta}^{\mu\nu}(r,k)+\frac{i}{2}[\overline{\Pi}^{\mu}_{\sigma},\hat{\Delta}^{\sigma\nu}](r,k)-\frac{1}{4}\{\partial_r^\lambda\overline{\Pi}^{\mu}_{\sigma},\partial^k_\lambda\hat{\Delta}^{\sigma\nu}\}(r,k)=-\frac{i}{2}[\hat{\Pi}^{\mu}_{\sigma},\hat{\Delta}^{\sigma\nu}](r,k)+\frac{1}{4}\{\partial_k^\lambda\hat{\Pi}^{\mu}_{\sigma},\partial^r_\lambda\hat{\Delta}^{\sigma\nu}\}(r,k)-\frac{1}{4}\{\partial_r^\lambda\hat{\Pi}^{\mu}_{\sigma},\partial^k_\lambda\hat{\Delta}^{\sigma\nu}\}(r,k), \tag{83}$$

where $\partial_r^2\equiv\partial_r\cdot\partial_r$, $[A,B]\equiv AB-BA$, $\{A,B\}\equiv AB+BA$. In Eqs. (82) and (83), $\hat{1}_P=1$ (0) for $\hat{\Delta}^F,\hat{\Delta}^{\overline{F}}$ ($\hat{\Delta}^>,\hat{\Delta}^<$) arises as the transform of $\delta^4_P(x,y)$. The function $d_{\mu\nu}(k)$ is the sum

[2]In Ref. [11] the ''evolution'' equation was called the ''renormalization'' equation, a term that may be misleading. In order to avoid confusion with the ''renormalization group'' equation, the name evolution equation appears more suitable.

over the gluon polarizations s [emerging from the Fourier transform of the operator (25)];

$$d_{\mu\nu}(k)=\sum_{s=1,2}\varepsilon_\mu(k,s)\cdot\varepsilon^*_\nu(k,s)$$

$$=g_{\mu\nu}-\frac{n_\mu k_\nu+n_\nu k_\mu}{n\cdot k}+(n^2+\alpha k^2)\frac{k_\mu k_\nu}{(n\cdot k)^2}. \quad (84)$$

with the properties[3] $d^\mu_\mu(k)=2$, $k_\mu d^{\mu\nu}(k)\xrightarrow{k^2\to 0}0$ and $n_\mu d^{\mu\nu}=0=d^{\mu\nu}n_\nu$. Furthermore, the initial state contribution $\mathcal{K}^{(2)}$ appearing in Eqs. (37) and (47), which contributes only at $r^0=t_0$, has been absorbed into the hard propagator

$$\hat{\Delta}^{-1}_{\mu\nu}(r,k)\equiv\hat{\Delta}^{-1}_{\mu\nu}(r,k)-\mathcal{K}^{(2)}_{\mu\nu}(r,k)\,\delta(r^0-t_0). \quad (85)$$

For the Yang-Mills equation (37) determining $\overline{F^{\mu\nu}}(r)$, one obtains on the same level of approximation a compact expression in terms of the hard current[4] $\hat{j}$:

$$[\overline{D^{\lambda,ab}},\overline{F^b_{\lambda\mu}}](r)=-\hat{j}^a_\mu(r)=-g\,\gamma^{\mu\nu\lambda\sigma}\int\frac{d^4k}{(2\pi)^2}\times\mathrm{Tr}\left\{T^a\left(k_\lambda\hat{\Delta}_{\nu\sigma}(r,k)+\frac{i}{2}[\overline{D^r_\lambda},\hat{\Delta}_{\nu\sigma}(r,k)]\right)\right\}, \quad (86)$$

where $\gamma_{\mu\nu\lambda\sigma}=2g_{\mu\nu}g_{\lambda\sigma}-g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\lambda}$ and $\overline{D^r_\lambda}=\partial^r_\lambda-ig\overline{A_\lambda}(r)$.

2. Expansion in powers of $g\overline{A}$

In order to isolate the leading effects of the soft mean field $\overline{A}$ on the hard quantum propagator $\hat{\Delta}$, I follow Ref. [7] to separate the quantum contribution from the mean field contribution on the basis of the assumption that the $\overline{A}$ field is slowly varying on the short-range scale of the quantum fluctuations. To do so, recall Eq. (27),

$$\hat{\Delta}(r,k)\equiv\hat{\Delta}_{[\overline{0}]}(r,k)+\delta\hat{\Delta}_{[\overline{A}]}(r,k), \quad (87)$$

with the quantum piece $\hat{\Delta}_{[\overline{0}]}$ and the mean-field part $\delta\hat{\Delta}_{[\overline{A}]}$ defined by

$$\hat{\Delta}^{-1}_{[\overline{0}]}=\hat{\Delta}^{-1}|_{\overline{A}=0}=\Delta_0^{-1}-\hat{\Pi}|_{\overline{A}=0},$$

$$\delta\hat{\Delta}^{-1}_{[\overline{A}]}=\overline{\Delta}^{-1}-\Delta_0^{-1}=-\overline{\Pi}, \quad (88)$$

where the free-field propagator Δ_0 and the mean-field propagator $\overline{\Delta}$ are given by Eqs. (24) and (26), respectively. Given the ansatz (87), with the feedback of the induced soft field to the hard propagator being contained in $\delta\hat{\Delta}_{[\overline{A}]}$, the latter is now expanded in powers of the soft field coupling $g\overline{A}$, and it is anticipated that the mean-field induced part $\delta\hat{\Delta}_{[\overline{A}]}$ is a correction being *at most g times* the quantum piece $\hat{\Delta}_{[\overline{0}]}$; that is,

$$\delta\hat{\Delta}_{[\overline{A}]}(r,k)=\sum_{n=1,\infty}\frac{1}{n!}(g\overline{A(r)}\cdot\partial_k)^n\hat{\Delta}_{[\overline{0}]}(k)$$

$$\simeq g\overline{A(r)}\cdot\partial_k\hat{\Delta}_{[\overline{0}]}(r,k) \quad (89)$$

and, to the same order of approximation,

$$\partial^\mu_r\,\delta\hat{\Delta}_{[\overline{A}]\,\mu\nu}(r,k)\simeq g(\partial^\mu_r\overline{A^\lambda})\,\partial^\lambda_k\hat{\Delta}_{[\overline{0}]\mu\nu}(r,k), \quad (90)$$

where, on the right side, the space-time derivative acts only on $\overline{A}$. Now the decomposition (87) with the approximation (89) is inserted into Eqs. (82), (83), (86), and all terms up to order $g^2\mu^2\hat{\Delta}_{[\overline{0}]}$ are kept. The resulting equations can be compactly expressed in terms of the *kinetic* momentum K_μ rather than the *canonical* momentum k_μ (as always in the context of interactions with a gauge field [33]), which for the class of axial-type gauges (2) amounts to the replacements

$$k_\mu\to K_\mu=k_\mu-g\overline{A_\mu}(r),\qquad \partial^r_\mu\to\overline{D^r_\mu}=\partial^r_\mu-g\,\partial^r_\mu\overline{A^\nu}(r)\,\partial^k_\nu. \quad (91)$$

Taking into account approximation 3 of Sec. III A implying $K^2\hat{\Delta}\gg\overline{D^2_r}\hat{\Delta}$, one finds for the evolution, transport, and Yang-Mills equation, Eqs. (82), (83), and (86), respectively,

$$\{K^2,\hat{\Delta}^{\mu\nu}_{[\overline{0}]}\}(r,K)=d^{\mu\nu}(K)+\frac{1}{2}\{\hat{\Pi}^\mu_\sigma,\hat{\Delta}^{\sigma\nu}_{[\overline{0}]}\}(r,K), \quad (92)$$

$$[K\cdot\overline{D_r},\hat{\Delta}^{\mu\nu}](r,K)=-\frac{i}{2}[\overline{\Pi^\mu_\sigma},\hat{\Delta}^{\sigma\nu}_{[\overline{0}]}](r,K)-\frac{i}{2}[\hat{\Pi}^\mu_\sigma,\hat{\Delta}^{\sigma\nu}_{[\overline{0}]}]\times(r,K), \quad (93)$$

$$[\overline{D^\lambda_r},\overline{F_{\lambda\mu}}](r)=-\hat{j}_\mu(r)=-g\int\frac{d^4K}{(2\pi)^2}\mathrm{Tr}\{-K_\mu\hat{\Delta}^\nu_\nu(r,K)+\hat{\Delta}^\nu_\mu(r,K)K_\nu\}, \quad (94)$$

where the color indices are suppressed, noting that $\hat{\Delta}^{ab}_{\mu\nu}=\hat{\Delta}_{\mu\nu}$, $\overline{F_{\lambda\mu}}=T^a\overline{F^a_{\lambda\mu}}$, $\hat{j}_\mu=T^a\hat{j}^a_\mu$, and $\mathrm{Tr}[\cdots]=\mathrm{Tr}\,T^a[\cdots]$.

3. The physical representation

One sees that the original Dyson-Schwinger equation (47) reduces in the kinetic approximation to the set of algebraic equations (92) and (93). Now recall (cf., Appendix B 3) that in the CTP framework these equations are still 2×2 matrix equations which mix the four different components of $\hat{\Delta}=(\hat{\Delta}^F,\hat{\Delta}^>,\hat{\Delta}^<,\hat{\Delta}^{\overline{F}})$ and of $\hat{\Pi}=(\hat{\Pi}^F,\hat{\Pi}^>,\hat{\Pi}^<,\hat{\Pi}^{\overline{F}})$. For the following it is more convenient to employ instead an equiva-

[3] This property reflects that in the noncovariant gauges (C6) only the two physical polarization states propagate, i.e., those with $\varepsilon_\mu k^\mu=0$. For comparison, in the covariant Feynman gauge, $d^{\mu\nu}=g^{\mu\nu}$, $d^\mu_\mu=4$, and $k_\mu d^{\mu\nu}=k^\nu\neq 0$.

[4] Note that in the kinetic approximation, the piece $\hat{j}_{(3)}$, Eq. (46), does not contribute, because it has two additional $\hat{\Delta}$ insertions and is down by a factor g/μ^4 as compared to $\hat{j}_{(1)}$ and $\hat{j}_{(2)}$.

lent set of independent functions, namely, the *retarded* and *advanced functions* $\hat{\Delta}^{\rm ret}$, $\hat{\Delta}^{\rm adv}$, plus the *correlation function* $\hat{\Delta}^{\rm cor}$, and analogously for $\hat{\Pi}$. This latter set is more directly connected with physical, observable quantities, and is commonly referred to as *physical representation* [19]

$$\hat{\Delta}^{\rm ret}=\hat{\Delta}^{F}-\hat{\Delta}^{<},\quad \hat{\Delta}^{\rm adv}=\hat{\Delta}^{F}-\hat{\Delta}^{>},\quad \hat{\Delta}^{\rm cor}=\hat{\Delta}^{<}+\hat{\Delta}^{>}. \tag{95}$$

Similarly, for the polarization tensor the retarded, advanced, and correlation functions are defined as [note the subtle difference to Eq. (95)]

$$\hat{\Pi}^{\rm ret}=\hat{\Pi}^{F}+\hat{\Pi}^{<},\quad \hat{\Pi}^{\rm adv}=\hat{\Pi}^{F}+\hat{\Pi}^{>},\quad \hat{\Pi}^{\rm cor}=-(\hat{\Pi}^{>}-\hat{\Pi}^{<}). \tag{96}$$

Loosely speaking, the retarded and advanced functions characterize the intrinsic quantum nature of a ''dressed'' gluon, describing its substructural state of emitted and reabsorbed gluons, whereas the correlation function describes the kinetic correlations among different such ''dressed'' gluons. The great advantage [19,22] of this physical representation is that, in general, the dependence on the phase-space occupation of gluon states (the local density) is essentially carried by the correlation functions $\hat{\Delta}^{>},\hat{\Delta}^{<}$, whereas the dependence of the retarded and advanced functions $\hat{\Delta}^{\rm ret},\hat{\Delta}^{\rm adv}$ on the local density is weak. More precisely, the retarded and advanced propagators and the imaginary parts of the self-energies embody the renormalization effects and dissipative quantum dynamics that is associated with short-distance emission and absorption of quantum fluctuations, whereas the correlation function contains both the effect of interactions with the soft mean field and of statistical binary scatterings among the hard gluons.

In going over to the physical representation, one finds then that Eqs. (92) and (93) give a set of ''self-contained'' equations for the retarded and advanced functions alone:

$$\{K^2,\hat{\Delta}^{\rm ret}_{\rm adv}\}_{\mu\nu}=d^{\mu\nu}+\frac{1}{2}(\Pi^{\rm ret}_{\rm adv}\hat{\Delta}^{\rm ret}_{\rm adv}+\hat{\Delta}^{\rm ret}_{\rm adv}\Pi^{\rm ret}_{\rm adv})_{\mu\nu}, \tag{97}$$

$$[K\cdot\overline{D_r},\hat{\Delta}^{\rm ret}_{\rm adv}]_{\mu\nu}=-\frac{i}{2}(\Pi^{\rm ret}_{\rm adv}\hat{\Delta}^{\rm ret}_{\rm adv}-\hat{\Delta}^{\rm ret}_{\rm adv}\Pi^{\rm ret}_{\rm adv})_{\mu\nu}, \tag{98}$$

plus a set of ''mixed'' equations for the correlation functions:

$$\{K^2,\hat{\Delta}^{\rm ret}_{\rm adv}\}_{\mu\nu}=-\frac{1}{2}(\Pi^{\rm ret}_{\rm adv}\hat{\Delta}^{\rm adv}+\Pi^{\rm ret}\hat{\Delta}^{\rm ret}_{\rm adv}+\hat{\Delta}^{\rm ret}_{\rm adv}\Pi^{\rm adv}+\hat{\Delta}^{\rm ret}\Pi^{\gtrless})_{\mu\nu}, \tag{99}$$

$$[K\cdot\overline{D_r},\hat{\Delta}^{\gtrless}]_{\mu\nu}=-\frac{i}{2}(\Pi^{\rm ret}_{\rm adv}\hat{\Delta}^{\rm adv}+\Pi^{\rm ret}\hat{\Delta}^{\rm ret}_{\rm adv}-\hat{\Delta}^{\rm ret}_{\rm adv}\Pi^{\rm adv}-\hat{\Delta}^{\rm ret}\Pi^{\rm ret}_{\rm adv})_{\mu\nu}. \tag{100}$$

Equations (97)–(100) may be further manipulated by the following trick. Let the imaginary and real components of the retarded and advanced propagators be denoted by

$$\hat{\rho}_{\mu\nu}\equiv 2\,{\rm Im}\hat{\Delta}_{\mu\nu}=i(\hat{\Delta}^{\rm ret}-\hat{\Delta}^{\rm adv})_{\mu\nu},$$

$${\rm Re}\hat{\Delta}_{\mu\nu}=\frac{1}{2}(\hat{\Delta}^{\rm ret}+\hat{\Delta}^{\rm adv})_{\mu\nu}, \tag{101}$$

with $\hat{\Delta}^{\rm ret}=(\hat{\Delta}^{\rm adv})^*$ and $\theta(K^0)\hat{\Delta}^{\rm ret}=\theta(-k^0)\hat{\Delta}^{\rm adv}$. The analogous decomposition of the polarization tensor in terms of its real and imaginary components defines the quantum part $\hat{\Pi}$ as the sum and difference of the retarded and advanced contributions, respectively:

$$\hat{\Gamma}_{\mu\nu}\equiv 2\,{\rm Im}\hat{\Pi}_{\mu\nu}=i(\hat{\Pi}^{\rm ret}-\hat{\Pi}^{\rm adv})_{\mu\nu},$$

$${\rm Re}\hat{\Pi}_{\mu\nu}=\frac{1}{2}(\hat{\Pi}^{\rm ret}+\hat{\Pi}^{\rm adv})_{\mu\nu}, \tag{102}$$

and similarly for the mean-field part $\overline{\Pi}$, associated with the presence of a soft field. The imaginary parts $\hat{\rho}$ and $\hat{\Gamma}$ are the *spectral density* and *spectral width*, respectively, of the hard gluons.

In terms of this representation one obtains from Eqs. (93), (94), and (97)–(100) the following final set of *master equations*:

$$\{K^2,\hat{\rho}\}_{\mu\nu}=\{{\rm Re}\hat{\Pi},\hat{\rho}\}_{\mu\nu}+\{\hat{\Gamma},{\rm Re}\hat{\Delta}_{[\bar{0}]}\}_{\mu\nu}+g(\overline{F^{\lambda}_{\mu}}\hat{\rho}_{\lambda\nu}+\hat{\rho}^{\lambda}_{\mu}\overline{F_{\lambda\nu}}), \tag{103}$$

$$\begin{aligned}[K\cdot\overline{D_r},\hat{\Delta}^{\rm cor}]_{\mu\nu}=&+i[\hat{\Pi}^{\rm cor},{\rm Re}\hat{\Delta}_{[\bar{0}]}]_{\mu\nu}+i[{\rm Re}\hat{\Pi},\hat{\Delta}^{\rm cor}_{[\bar{0}]}]_{\mu\nu}\\&-\frac{1}{2}\{\hat{\Pi}^{\rm cor},\hat{\rho}\}_{\mu\nu}-\frac{1}{2}\{\hat{\Gamma},\hat{\Delta}^{\rm cor}_{[\bar{0}]}\}_{\mu\nu}\\&-gK_{\lambda}\overline{F^{\lambda\sigma}}\partial^{K}_{\sigma}\hat{\Delta}^{\rm cor}_{[\bar{0}]\mu\nu}\\&-g(\overline{F^{\lambda}_{\mu}}\hat{\Delta}^{\rm cor}_{[\bar{0}]\lambda\nu}-\hat{\Delta}^{\rm cor\lambda}_{[\bar{0}]\mu}\overline{F_{\lambda\nu}}),\end{aligned} \tag{104}$$

$$[\overline{D^{\lambda}_r},\overline{F_{\lambda\mu}}]=-\hat{j}_{\mu}=-g\int\frac{d^4k}{(2\pi)^2}\times{\rm Tr}\{(-K_{\mu}\hat{\Delta}^{\rm cor\nu}_{\nu}+\hat{\Delta}^{\rm cor\nu}_{\mu}K_{\nu})\}. \tag{105}$$

The *physical significance* of Eqs. (103) and (104) is [11] that Eq. (103) determines, in terms of the spectral density $\hat{\rho}$, the state of a single gluon with respect to its virtual fluctuations and real emission (absorption) processes, corresponding to the real and imaginary parts of the retarded and advanced polarization tensor in the presence of the soft field $\overline{F}$. Equation (104), on the other hand, characterizes, in terms of the correlation function $\hat{\Delta}^{\rm cor}$, the correlations among different such gluon states. The polarization tensor appears here in distinct ways. The first two terms on the right-hand side account for scatterings between the single-gluon states. The next two terms incorporate the renormalization effects which result from the fact that the gluons between collisions do not behave as free particles, but change their dynamical structure due to virtual fluctuations, as well as real emission and absorption of quanta. The last two terms account for the soft interaction with the mean field $\overline{F}$. Equation (105) finally determines the rate of change of he soft field $\overline{F}$ by the hard gluon current, which involves the full correlation function $\hat{\Delta}^{cor}$.

The interlinked structure of Eqs. (103)–(105) is very convenient for explicit calculations (demonstrated in Sec. IV). It provides a systematic solution scheme, as discussed below, to solve for the three quantities of interest, namely, the spectral density $\hat{\rho}$, the correlation function $\hat{\Delta}^{cor}$, and the mean field $\overline{A}$. In view of Eqs. (103)–(105) the natural logic is a stepwise determination of $\hat{\rho}\rightarrow\hat{\Delta}^{cor}_{[\overline{0}]}\rightarrow\delta\hat{\Delta}^{cor}_{[\overline{A}]}\rightarrow\hat{\Delta}^{cor}\rightarrow\hat{j}\rightarrow\overline{F}$.

C. General solution scheme

Let me exemplify the above interpretation of Eqs. (103) and (104) in more quantitative detail (see also Refs. [19,22]). The *formal solution* of Eq. (103) for the retarded and advanced functions is [19]

$$\hat{\Delta}^{ret}_{\mu\nu}=\Delta^{ret}_{0\mu\nu}+(\Delta^{ret}_0\Pi^{ret}\hat{\Delta}^{ret})_{\mu\nu},$$

$$\hat{\Delta}^{adv}_{\mu\nu}=\Delta^{adv}_{0\mu\nu}+(\Delta^{adv}_0\Pi^{adv}\hat{\Delta}^{adv})_{\mu\nu}, \qquad (106)$$

where $\Pi^{ret}_{adv}=\hat{\Pi}^{ret}_{adv}+\overline{\Pi}^{ret}_{adv}$. This determines $\hat{\rho}_{\mu\nu}$ via Eq. (101). Once $\hat{\Delta}^{ret}_{adv}$ is known, the solution of Eq. (104) for the correlation function is given by [19]

$$\hat{\Delta}^{cor}_{\mu\nu}=-(\hat{\Delta}^{ret}\Delta_0^{cor-1}\hat{\Delta}^{adv})_{\mu\nu}+(\hat{\Delta}^{ret}\hat{\Pi}^{cor}\hat{\Delta}^{adv})_{\mu\nu}, \qquad (107)$$

with $\Pi^{cor}=\hat{\Pi}^{cor}+\overline{\Pi}^{cor}$. It has the general form [22]

$$\hat{\Delta}^{cor}_{\mu\nu}(r,K)=-i\hat{\rho}_{\mu\nu}(r,K)\mathrm{G}(r,K), \qquad (108)$$

i.e., the convolution of the spectral density $\hat{\rho}_{\mu\nu}$ with the phase-space density of hard gluons G:

$$\mathrm{G}(r,K)=1+2\mathrm{g}(r,K), \qquad (109)$$

where the 1 comes from the vacuum contribution of a single gluon state, and the $2\mathrm{g}$ represents the correlations with other hard gluons that are close by in phase-space. Note that the function g is constrained to be a real and even function in K (cf., Appendix F). From Eq. (108) it follows that the total number of gluons N in a space-time element d^4r is

$$\mathrm{N}(r)\equiv\frac{d\mathrm{N}}{d^4r}=\int\frac{d^4K}{(2\pi)^4}\mathrm{Tr}[d^{-1}_{\mu\nu}(K)i\hat{\Delta}^{cor}_{\mu\nu}(r,K)]$$

$$=\int\frac{d^4K}{(2\pi)^4}\hat{\rho}(r,K)\mathrm{G}(r,K), \qquad (110)$$

where $d_{\mu\nu}(K)$ is the polarization sum given by Eq. (84), $d^{-1}_{\mu\nu}=2d_{\mu\nu}$ and $\hat{\rho}=\frac{1}{2}d_{\mu\nu}\hat{\rho}_{\mu\nu}$, and an averaging over the transverse polarizations and the color degrees of freedom is understood.

The above formulas become immediately familiar when considering for illustration the simplest case of a noninteracting system of gluons, the *free-field* case. In this case, $\Pi=0$ and one finds, utilizing the formulas of Appendix F, for the free retarded and advanced functions

$$\Delta^{ret}_{0\mu\nu}(K)=\frac{d_{\mu\nu}(K)}{K^2+i\epsilon}, \quad \Delta^{adv}_{0\mu\nu}(K)=\frac{d_{\mu\nu}(K)}{K^2-i\epsilon}. \qquad (111)$$

Hence, the free-field spectral density ρ_0 which is the difference between Δ^{ret}_0 and Δ^{adv}_0, is on-shell:

$$-i\rho_{0\mu\nu}=\Delta^{ret}_{0\mu\nu}-\Delta^{adv}_{0\mu\nu}=2\pi\delta(K^2)d_{\mu\nu}(K), \qquad (112)$$

by means of the principal-value (PV) formula $(K^2\pm i\epsilon)^{-1}=\mathrm{PV}(1/K^2)\mp i\pi\delta(K^2)$. The free-field correlation function Δ^{cor}_0 is then readily determined via Eq. (108):

$$\Delta^{cor}_{0\mu\nu}(r,K)=-2\pi i\delta(K^2)\mathrm{G}_0(r,K)d_{\mu\nu}(K), \qquad (113)$$

and so, with $\mathrm{G}_0=1+2\mathrm{g}_0$, the number of on-shell gluons per d^4r is

$$\mathrm{N}_0(r)=\int\frac{d^3K}{(2\pi)^3 2K^0}\mathrm{G}_0(r,\vec{K}),$$

$$\mathrm{G}_0(r,\vec{K})=\mathrm{G}_0(r,K)|_{K^0=|\vec{K}|}=(2\pi)^3 2K^0\frac{d\mathrm{N}_0}{d^3K}. \qquad (114)$$

The free-field exercise, Eqs. (111)–(114), illustrates the two main properties, which hold also for the general interacting case, Eqs. (106)–(110).

(i) The spectral density $\hat{\rho}_{\mu\nu}(r,K)$ describes the ''dressing'' of a *single* gluon state with momentum K with respect to its radiative quantum fluctuations, i.e., its fluctuating coat of emitted and reabsorbed gluons. The function $\mathrm{Tr}[d^{-1}_{\mu\nu}\hat{\rho}^{\mu\nu}]$ is the *intrinsic* gluon distribution, that is, the number of gluons inside this gluon state. The spectral density is a property of the state itself and therefore is nonvanishing even in vacuum, in the absence of a medium. For on-shell particles $\hat{\rho}_{\mu\nu}\propto\delta(K^2)$, and therefore there are no intrinsic gluons present.

(ii) The correlation function $\hat{\Delta}^{cor}_{\mu\nu}(r,K)$ describes an interacting *ensemble* of such fluctuating gluon states, and is given by the number density $\mathrm{G}(r,K)$ of those gluons weighted with their spectral density $\hat{\rho}_{\mu\nu}$, containing the intrinsic gluon density of each of them. For the noninteracting case, it obviously reduces to an ensemble of on-shell particles with $K^0=|\vec{K}|$.

In closure of this section, a generic *solution scheme* may be the following iteration recipe (which is exemplified in the next section).

(1) Solve the evolution equation (103) for $\hat{\Delta}^{ret}_{adv}$ and the associated spectral density $\hat{\rho}$ at starting point $t=t_0$ with specified initial condition $\hat{\rho}(t_0)=\rho_0$ at a large initial momentum or energy scale Q. This can be done just as in free space, except that the kinetic momentum $K=k-g\overline{A}$ carries now an implicit dependence on the soft field $\overline{A}$ with specified initial value $\overline{A(t_0)}$.

(2) Solve the transport equation (104) for the correlation function $\hat{\Delta}^{cor}=\hat{\Delta}^{cor}_{[\overline{0}]}+\delta\hat{\Delta}^{cor}_{[\overline{A}]}$. This involves (a) the construction of $\hat{\Delta}^{cor}_{[\overline{0}]}$ with the help of $\hat{\rho}$ and $\hat{\Delta}^{ret}_{adv}$ from step (1) and (b)

the calculation of the mean-field-induced correction $\delta\hat{\Delta}^{\rm cor}_{[\bar{0}]}$ from the right side of Eq. (104). The resulting space-time evolution of $\hat{\Delta}^{\rm cor}$ describes then the evolution of the gluon density G within a time interval between t_0 and $t_1 \sim 1/\langle K_{\perp 1}\rangle$, corresponding to the evolution from Q down to a mean $K_{\perp 1}$ at t_1.

(3) Insert the solution for the full correlation function $\hat{\Delta}^{\rm cor}$ into the current $\hat{j}_\mu$ on the left side of Eq. (105) and integrate over all momenta K from the initial momentum scale Q down to the hard-soft scale μ. This gives the current induced by the motion of the total aggregate of hard gluons during the evolution between t_0 and t_1. Then solve the Yang-Mills equation (105) to determine the soft field $\overline{A}$ (equivalently $\overline{F}$) that is generated at t_1 as a result of the hard gluon evolution.

(4) Return to step (1) and proceed with second iteration, replacing $\overline{A(t_0)}$ by $\overline{A(t_1)}$, and so forth.

IV. SAMPLE CALCULATION: HARD GLUON EVOLUTION WITH SELF-GENERATED SOFT FIELD

This section is devoted to exemplifying the practical applicability of the developed formalism by following the solution scheme of Sec. III C for the specific physics scenario advocated in the introduction and schematically illustrated in Fig. 1. I consider a high-energy beam current of hard gluons as it evolves in space-time and momentum space, and eventually induces its soft mean field.

A. The physics scenario

(i) The initial state is modeled as an ensemble of a number $\mathcal{N}_0$ of uncorrelated hard gluons. The Lorentz-frame of reference is the one where the gluons move with the speed of light in the $+z$ direction. The initial gluon beam is prepared at

$$r_0^\mu = (t_0, \vec{r}_{\perp 0}, z_0), \quad t_0 = z_0 = 0, \quad 0 \leqslant r_{\perp 0} \leqslant R, \quad (115)$$

corresponding at t_0 to a sheet located with longitudinal position z_0 with transverse extent up to a maximum R, specified later.

(ii) The initial hard gluons are imagined to be produced at some very large momentum scale $Q^2 \gg \Lambda^2$, with their energies and longitudinal momentum along the z axis being $\simeq Q$. These gluons are therefore strongly concentrated around the light cone with momenta

$$k^0 \simeq k^3 \approx Q, \quad \frac{k_\perp^2}{Q^2} \approx 0, \quad (116)$$

and hence have very small spatial extent $\Delta r \sim 1/Q$. That is, the initial state gluons are taken as *bare* quanta without any radiation field around them.

(iii) The subsequent timelike evolution of these bare gluons proceeds then by two competing processes: (a) the regeneration of the radiation field by emission and reabsorption of virtual quanta and (b) the bremsstrahlung emission of real gluonic offspring. As a consequence, phase space will be populated with progressing time by more and more gluons. The typical energies decrease, whereas the average transverse momentum increases [cf. Fig. 1(a)], but yet within the hard momentum range

$$Q^2 \gg k_\perp^2 \geqslant \mu^2 \gg \Lambda^2. \quad (117)$$

Eventually, the evolving gluon system reaches the point at which the transverse momenta become of the order of the energies. This point is defined to be characterized by the scale μ—the transition from hard, perturbative to soft, nonperturbative regimes. When $K_\perp^2 \lesssim \mu^2$, the individual gluons cannot be resolved anymore, and their coherent color current acts as the source of the soft mean field.

(iv) Because of the restricted kinematic region (117) of the hard gluon dynamics, the coupling $\alpha_s = g^2/4\pi$ satisfies

$$\alpha_s(k_\perp^2) \ll 1, \quad \alpha_s(k_\perp^2)\ln(Q^2/k_\perp^2) \simeq 1, \quad \text{for all } k_\perp^2 \geqslant \mu^2, \quad (118)$$

so that a perturbative evaluation of the hard gluon interactions is applicable, provided $\mu \gtrsim 1$ GeV. The perturbative analysis in the following subsections will be restricted to leading order: the hard gluon interactions then include *only radiative self-interactions* $\sim g^2$, but *no gluon-gluon scatterings*[5] $\sim g^4$, or other higher-loop contributions. Hence, for the hard gluon propagator $\hat{\Delta} = \hat{\Delta}_{[\bar{0}]} + \delta\hat{\Delta}_{[\bar{A}]}$ of Eqs. (87), (88), the required accuracy for the quantum contribution $\hat{\Delta}_{[\bar{0}]}$ is

$$\hat{\Delta}_{[\bar{0}]} = \Delta_0 + \hat{C}(g^2;\Delta_0) + O(g^4), \quad (119)$$

where Δ_0 is the free-field solution. On top of this the interaction of the hard gluons with the soft field is treated as a correction $\delta\hat{\Delta}_{[\bar{A}]}$ as in Eq. (89), to leading order $\sim g\overline{A}$ to the solution $\hat{\Delta}_{[\bar{0}]}$ of Eq. (119):

$$\delta\hat{\Delta}_{[\bar{A}]} = \overline{C(g\overline{A};\Delta_{[\bar{0}]})} + O(g^2\overline{A^2}). \quad (120)$$

Although this so defined physics scenario, with an initial state of bare gluons, being only statistically correlated and incoherent, may appear to be rather academic, it has in fact valuable physical relevance. For example, it may be viewed as the idealized version of the initial density of materialized gluons in the very early stage of a high-energy collision of two heavy nuclei. In this example, one expects the materialization of a large number $\mathcal{N}_0$ of virtual gluons in the wave functions of the colliding nuclei, to occur very shortly after the nuclear overlap by means of hard scatterings. If one imagines the time of nuclear overlap equal to $t_0 = 0$, and

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[5]Aside from $g^4 \ll g^2$, the neglect of scatterings is reasonable here, because for a beam of almost collinearly moving gluons, the *collision rate*, i.e., the number of collisions per unit time and unit volume, $R_{\rm coll} = dN_{\rm coll}/dt d^3r \propto \int d^3k_1 d^3k_2 f_{g_1}(k_1) f_{g_2}(k_2) \overline{\Sigma}|M_{g_1 g_2}|^2$, vanishes if the relative velocity $v_{12} = |v_1 - v_2| = |k_1/\omega_1 - k_2/\omega_2|$ of any two gluons tends to zero, and hence their total invariant mass $\hat{s} = (k_1 + k_2)^2$. The suppression of gluon scattering arises from $|M|^2$, being a function of the Mandelstam variables $\hat{s}, \hat{t}, \hat{u}$ only, vanishes if the gluons move parallel, because then $\hat{s} \simeq \hat{t} \simeq \hat{u} \approx 0$ and the scattering matrix element tends to zero.

assume the average momentum transfer of initial hard scatterings $\approx Q^2$, then the above idealistic scenario acquires a more realistic meaning.

B. Choice of light cone gauge and kinematics

For the purpose of calculational convenience, I will henceforth work in the *light cone gauge* which is a special case of the axial-type gauges (2). It is defined by Eqs. (C3)–(C6) of Appendix C, that is,

$$n\cdot\mathcal{A}^a=0,\quad n^2=0\quad(\mathcal{A}^a_\mu=\overline{A^a_\mu},a^a_\mu),\tag{121}$$

corresponding to the gauge fixing term in Eq. (3)

$$I_{\rm GF}[n\cdot\mathcal{A}]=\int_P d^4x\left(-\frac{1}{2\alpha}[n\cdot\mathcal{A}^a(x)]^2\right),\quad\text{with }\alpha\to 0.\tag{122}$$

I choose the lightlike vector n^μ parallel to the direction of motion of the gluon beam along the forward light cone:

$$n^\mu=(n^0,\vec{n}_\perp,n^3)=(1,\vec{0}_\perp,-1)\tag{123}$$

and employ *light cone variables*, i.e., for any four-vector v^μ,

$$v^\mu=(v^+,v^-,\vec{v}_\perp),\quad v^2=v^+v^- - v_\perp^2,\tag{124}$$

$$v^\pm=v_\mp=v^0\pm v^3,\quad \vec{v}_\perp=(v^1,v^2),\quad v_\perp=\sqrt{\vec{v}_\perp^{\,2}},\tag{125}$$

$$v_\mu w^\mu=\frac{1}{2}(v^+w^-+v^-w^+)-\vec{v}_\perp\cdot\vec{w}_\perp.\tag{126}$$

Then $n^\mu=(n^+,n^-,\vec{n}_\perp)=(0,1,\vec{0}_\perp)$, so that the gauge constraint (121) reads

$$n\cdot\mathcal{A}=\mathcal{A}^+=\mathcal{A}_-=0,\tag{127}$$

and the nonvanishing components of the gauge-field tensor $\overline{F^{\mu\nu}}=-\overline{F^{\nu\mu}}$ are

$$\overline{F^{+-}}=-\partial^+\overline{A^-},\quad \overline{F^{+\,i}}=\partial^+\overline{A^i},$$
$$\overline{F^{-\,i}}=\partial^-\overline{A^i}-\partial^i\overline{A^-}-ig[\overline{A^-},\overline{A^i}],$$
$$\overline{F^{ij}}=\partial^i\overline{A^j}-\partial^j\overline{A^i}-ig[\overline{A^i},\overline{A^j}],\tag{128}$$

where $\partial^\pm=\partial/\partial r^\pm$ and the index $i=1,2$ labels the transverse components.

Finally, the kinematic imposition (116) reads in terms of light cone variables

$$K^+K^-=K^2+K_\perp^2\ll(K^+)^2,\tag{129}$$

$$K^+\simeq 2K^0\simeq 2K^3,\quad K^-\simeq 0,\quad K_\perp^2\gg K^2.\tag{130}$$

Physically this implies that the hard gluons are effectively on mass shell, i.e., their actual virtuality (degree of off-shellness) K^2 is small compared to $K_\perp^2$, the transverse momentum squared, and negligibly small compared to the scale $(K^+)^2$. Within this kinematic regime, I henceforth consider $K^2/(K^+)^2\to 0$.

C. Properties of $\hat{\Delta}_{\mu\nu}$ and $\Pi_{\mu\nu}$ in the light cone representation

The most general Lorentz decomposition of the polarization tensor $\Pi=\hat{\Pi}+\overline{\Pi}$ in light cone gauge can be written as $\Pi^{ab}_{\mu\nu}(r,K)=\delta^{ab}\Pi_{\mu\nu}(r,K)$, with

$$\Pi_{\mu\nu}(r,K)=\left(g_{\mu\nu}-\frac{K_\mu K_\nu}{K^2}\right)\Pi_\perp+\left(\frac{K_\mu K_\nu}{K^2}\right)\Pi_\parallel+\left(\frac{n_\mu K_\nu+K_\mu n_\nu}{n\cdot K}\right)\Pi_1+\left(\frac{K^2 n_\mu n_\nu}{(n\cdot K)^2}\right)\Pi_2,\tag{131}$$

where $\Pi_\perp,\Pi_\parallel,\Pi_1,\Pi_2$ are scalar functions of dimension mass squared and depend on the four-vectors r^μ and $K^\mu=k^\mu-g\overline{A^\mu}$. In light cone gauge, the Ward identity for the gluon propagator [27]

$$\lim_{\alpha\to 0}\left\{\frac{1}{\alpha}(n\cdot K)n^\mu\hat{\Delta}_{\mu\nu}+\frac{1}{(2\pi)^4}K_\nu\right\}\overset{!}{=}0\tag{132}$$

enforces $\Pi_{\mu\nu}$ to be transverse with respect to n_μ and symmetric in its arguments and indices:

$$n^\mu\Pi^{ab}_{\mu\nu}=0=\Pi^{ab}_{\mu\nu}n^\nu,\quad \Pi^{ab}_{\mu\nu}=\Pi^{ba}_{\nu\mu},\tag{133}$$

which implies that

$$\Pi_\parallel=-\Pi_1=+\Pi_2.\tag{134}$$

Therefore, with $n\cdot K=K^+$,

$$\Pi^{ab}_{\mu\nu}(r,K)=\delta^{ab}\left(g_{\mu\nu}-\frac{K_\mu K_\nu}{K^2}\right)\Pi_\perp+\delta^{ab}\left(\frac{K_\mu K_\nu}{K^2}+\frac{n_\mu K_\nu+K_\mu n_\nu}{K^+}+\frac{K^2 n_\mu n_\nu}{(K^+)^2}\right)\Pi_\parallel,\tag{135}$$

$$\Pi_\perp=\frac{1}{2}\left(g_{\mu\nu}+\frac{K^2}{(K^+)^2}n_\mu n_\nu\right)\Pi^{\mu\nu},\quad \Pi_\parallel=3\Pi_\perp-\Pi^\mu_\mu.\tag{136}$$

The corresponding full gluon propagator is given by the inverse of $(\hat{\Delta})^{-1}=(\Delta_0)^{-1}-\Pi$. Using the free-field form

$$\Delta^{ab}_{0\,\mu\nu}(r,K)=\delta^{ab}d_{\mu\nu}(K)\Delta_0(r,K),$$

$$d_{\mu\nu}(K)=g_{\mu\nu}-\frac{n_\mu K_\nu+K_\mu n_\nu}{K^+}+\frac{K^2 n_\mu n_\nu}{(K^+)^2},\tag{137}$$

with the scalar functions $\Delta_0\equiv(\Delta_0^{\rm ret},\Delta_0^{\rm adv},\Delta_0^{\rm cor})$ [cf., Eqs. (111)–(113)],

$$\Delta_0^{\substack{\mathrm{ret}\\ \mathrm{adv}}}(K)=\frac{1}{K^2\pm i\epsilon},\quad \Delta_0^{\mathrm{cor}}(r,K)=-2\pi i\,\delta(K^2)\mathrm{G}_0(r,K), \tag{138}$$

one finds

$$\hat{\Delta}^{ab}_{\mu\nu}(r,K)=\delta^{ab}\Delta_0(r,K)\left(\frac{1}{1-\Pi_\perp/K^2}\right)\left\{g_{\mu\nu}-\frac{n_\mu K_\nu+K_\mu n_\nu}{K^+}+\frac{K^2 n_\mu n_\nu}{(K^+)^2}\left(\frac{\Pi_\parallel/K^2}{1-(\Pi_\perp-\Pi_\parallel)/K^2}\right)\right\}. \tag{139}$$

Now, because of Eq. (130), the last term in Eq. (139) vanishes for $K^2/(K^+)^2\rightarrow 0$, and the full propagator $\hat{\Delta}$ can be expressed as the free-field counterparts Δ_0 times a scalar *form-factor function* $\mathcal{Z}$ whose momentum dependence contains only the Lorentz invariants $n\cdot K=K^+$ and $K_\perp^2$:

$$\hat{\Delta}^{ab}_{\mu\nu}(r,K)=\delta^{ab}d_{\mu\nu}(K)\Delta_0(r,K)\mathcal{Z}(r,K^+,K_\perp^2), \tag{140}$$

where, because of $K^2/(K^+)^2\rightarrow 0$, the function $d_{\mu\nu}$ reduces now to

$$d_{\mu\nu}(K)=g_{\mu\nu}-\frac{n_\mu K_\nu+K_\mu n_\nu}{K^+} \tag{141}$$

and the form factor $\mathcal{Z}$ is related to the polarization tensor by

$$\mathcal{Z}(r,K^+,K_\perp^2)=\frac{1}{1-\Pi_\perp/K^2}, \tag{142}$$

with boundary condition

$$\mathcal{Z}(0,K^+,K_\perp^2)|_{K=Q}=1. \tag{143}$$

Here Q is the renormalization point, determined by the momentum scale of the initial state hard gluons (which is specified in the next subsection).

The great advantage of the light cone gauge becomes evident now: the solution of the full retarded, advanced, and correlation functions (106)–(108) boils down to calculating a single scalar function for each of them, namely, the form factor $\mathcal{Z}$, which is simply multiplied to the free-field forms (111)–(113). For the retarded and advanced functions, with

$$\mathcal{Z}^{\substack{\mathrm{ret}\\ \mathrm{adv}}}=\frac{1}{1-\Pi_\perp^{\substack{\mathrm{ret}\\ \mathrm{adv}}}/K^2} \tag{144}$$

one has

$$\hat{\Delta}^{\mathrm{ret}}_{\mu\nu}(K)=\frac{d_{\mu\nu}(K)}{K^2+i\epsilon}\mathcal{Z}^{\mathrm{ret}}(r,K^+,K_\perp^2),$$

$$\hat{\Delta}^{\mathrm{adv}}_{0\mu\nu}(K)=\frac{d_{\mu\nu}(K)}{K^2-i\epsilon}\mathcal{Z}^{\mathrm{adv}}(r,K^+,K_\perp^2), \tag{145}$$

which satisfy the useful relations $\hat{\Delta}^{\mathrm{ret}}_{\mu\nu}=(\hat{\Delta}^{\mathrm{adv}}_{\mu\nu})^*$ and $\hat{\Delta}^{\mathrm{ret}}_{\mu\nu}(K^0,\vec{K})=\hat{\Delta}^{\mathrm{adv}}_{\mu\nu}(-K^0,\vec{K})$. Defining

$$\mathcal{Z}^\rho\equiv\mathcal{Z}^{\mathrm{ret}}-\mathcal{Z}^{\mathrm{adv}}, \tag{146}$$

the spectral density follows immediately as

$$\hat{\rho}_{\mu\nu}(r,K)=i(\hat{\Delta}^{\mathrm{ret}}-\hat{\Delta}^{\mathrm{adv}})_{\mu\nu}(r,K)=d_{\mu\nu}(K)(-2\pi i)\mathcal{Z}^\rho(r,K^+,K_\perp^2), \tag{147}$$

and the correlation function is obtained as

$$\hat{\Delta}^{\mathrm{cor}}_{\mu\nu}(r,K)=d_{\mu\nu}(K)(-2\pi i)\mathcal{Z}^\rho(r,K^+,K_\perp^2)[1+2\mathrm{g}(r,K)]. \tag{148}$$

D. Specifying the initial state

To fix the initial conditions for the scenario described in Sec. VI A, both $\hat{\Delta}$ and $\overline{A}$ have to be provided with initial values at $r^0\equiv t_0=0$. The initial condition for the hard propagator is chosen as

$$\hat{\Delta}_{\mu\nu}(r,K)|_{r^0=r^3=0}=\Delta_{0\mu\nu}(0,\vec{r}_\perp,K), \tag{149}$$

referring to a statistical ensemble of bare gluon states at time $r^0=0$, which can be characterized by a single-particle density matrix of the Gaussian form as given by Eq. (B14) of Appendix B. This ansatz corresponds to an initial state source term in Eq. (47) of the form

$$\Delta^{\mathrm{cor}}_{0\mu\nu}(r,K)|_{r=(0,\vec{r}_\perp,0)}=\mathcal{K}^{(2)}_{\mu\nu}(r,K)\,\delta(r^0)\,\delta(r^3)=\rho_{0\mu\nu}(K)\mathrm{G}_0(r,K). \tag{150}$$

As assumed in Sec. IV A, the initial ensemble consists of a total number $\mathcal{N}_0$ of bare gluons with total invariant mass $\mathcal{Q}^2$, all moving with equal fractions of the total momentum $\mathcal{Q}^\mu=Q^\mu/\mathcal{N}_0$. That is, each gluon moves initially with momentum $Q^\mu=(Q^+,0,0_\perp)$ collinearly to the others along the light cone. Throughout the ultrarelativistic limit it is understood that $Q^2\rightarrow\infty$, i.e., $Q^+\gg\Lambda$, where $\Lambda\approx 0.2-0.3$ GeV. The spatial distribution of these $\mathcal{N}_0$ initial gluons at $r^0=0$ is taken as a δ distribution along the light cone at $r^3\equiv z_0=0$, and a random distribution transverse to the light cone motion. That is, the initial multigluon ensemble is prepared at light cone position $r^+=t_0+z_0=0$ and light cone time $r^-=t_0-z_0$ with a transverse smearing $0\leqslant r_\perp\leqslant\mathcal{N}_0/\sqrt{Q^2}$, where the typical transverse extent of each gluon is $\delta r_\perp\approx 1/Q^2\ll 1$ fm. Accordingly, the initial state spectral density ρ_0 in Eq. (150) is taken as

$$\rho_0(K)=\frac{(2\pi)^4}{K^+}\delta(K^+-Q^+)\,\delta\left(K^--\frac{K_\perp^2+Q^2}{K^+}\right)\delta^2(\vec{K}_\perp),$$

$$\int\frac{dK^+dK^-d^2K_\perp}{(2\pi)^4}\rho_0(K)=1. \tag{151}$$

The corresponding retarded and advanced functions $\Delta_0^{\substack{\mathrm{rel}\\ \mathrm{adv}}}$ are of the form (111)

$$\Delta_0^{\substack{\mathrm{ret}\\ \mathrm{adv}}}(K)=\mathrm{PV}\left(\frac{1}{K^2}\right)\mp\frac{i}{2}\rho_0(K). \tag{152}$$

Finally, the initial state correlation function $\Delta_0^{\rm cor}$ is the convolution of ρ_0 with the density of bare gluons at the scale Q and light cone time (position) $r^-=r^+=0$:

$$\Delta_{0\mu\nu}^{\rm cor}(r,K)=\int d^4K'(2\pi)^4 d_{\mu\nu}(K')\rho_0(K')\mathrm{G}_0(r,K), \tag{153}$$

where $\mathrm{G}_0(r,K)\equiv\mathrm{G}_0(r)\mathrm{G}_0(K)$ with

$$\mathrm{G}_0(r)=\frac{\mathcal{N}_0}{\pi}\,\delta(r^-)\,\delta(r^+)\,\theta(1-\mathcal{N}_0 r_\perp^2 Q^2),$$

$$\mathrm{G}_0(K)=\frac{(2\pi)^4}{K^+}\,\delta(K^+-Q^+)\,\delta\!\left(K^- - \frac{K_\perp^2+Q^2}{K^+}\right)\delta^2(\vec{K}_\perp). \tag{154}$$

The visualization of the initial gluon density G_0 in Eq. (154) is a *two-dimensional color-charge density*: It is spread out in the two transverse directions $\vec{r}_\perp$ in a disc with radius $R=1/\sqrt{\mathcal{N}_0 Q^2}=1/\mathcal{Q}$, and a δ function in longitudinal direction at $r^+=0$ at time $r^-=0$. The normalization is such that the total number $\mathcal{N}_0$ of initial bare gluons is given by

$$\int dr^- dr^+ d^2r_\perp \mathrm{G}_0(r,K)\equiv\mathcal{N}_0\mathrm{G}_0(K). \tag{155}$$

Finally, because of this statistical ensemble of almost pointlike, bare gluons, one does not expect any collective mean-field behavior at initial time $r^0=t_0=0$ and at large Q^2, so that the magnitude of the soft field is initially equal to zero which is consistent with Eq. (12):

$$\overline{A_\mu}(r^+,r^-,\vec{r}_\perp)|_{r^-=r^+=0}=0. \tag{156}$$

This completes the construction of the initial state, starting from which I now address the solution of the set of equations (103)–(105).

E. Solving for the spectral density $\hat{\rho}_{\mu\nu}$

To find the spectral density $\hat{\rho}_{\mu\nu}$, the solution of $\hat{\Delta}^{\rm ret}_{[\bar{0}]}$ and $\hat{\Delta}^{\rm adv}_{[\bar{0}]}$ is needed. The first correction to the free-field solution (112) arises from two contributions: (a) from the one-loop hard gluon self-interaction of order g^2 that is contained in the hard polarization tensor $\hat{\Pi}$ and (b) from the coupling of the hard gluon propagator to the soft field $\overline{A}$ in $\overline{\Pi}$ which is of order $g\overline{A}$. Within the perturbative scheme (119) and (120) the retarded and advanced propagators are to be evaluated to order g^2 from Eq. (106) with the internal propagators in $\Pi^{\rm ret}_{\rm adv}$ taken as the free-field solutions

$$\hat{\Delta}^{\rm ret}_{\substack{\rm adv\\ [\bar{0}]}}=\Delta^{\rm ret}_{\substack{\rm adv\\ 0}}+\Delta^{\rm ret}_{\substack{\rm adv\\ 0}}\hat{\Pi}^{\rm ret}_{\rm adv}[g^2;\Delta_0]\Delta^{\rm ret}_{\substack{\rm adv\\ 0}}, \tag{157}$$

with the subsidiary condition $[K\cdot\overline{D_r},\hat{\Delta}^{\rm ret}_{\substack{\rm adv\\ [\bar{0}]}}]=0+O(g^4)$. To order g^2, the gluon polarization tensor $\hat{\Pi}$ as given by Eqs. (74)–(78), reduces to the one-loop term $\hat{\Pi}^{(2)}$, because the tadpole term $\hat{\Pi}^{(1)}$ vanishes as usual in the context of dimensional regularization [27], and the two-loop terms $\hat{\Pi}^{(3)},\hat{\Pi}^{(4)}$ are of order g^4. Hence, $(\hat{\Pi}^{\rm ret}-\hat{\Pi}^{\rm adv})[g^2;\Delta_0]$ in Eq. (157) reduces to

$$\begin{aligned}(\hat{\Pi}^{\rm ret}-\hat{\Pi}^{\rm adv})^{ab}_{\mu\nu}(r,K) &= -\frac{ig^2}{2}\int\frac{d^4q}{(2\pi)^4}V^{acd}_{0\mu\lambda\sigma}(K,-q,-K+q)\hat{V}^{d'c'b}_{0\sigma'\lambda'\nu}(r;K\\ &\quad -q,q,-K)\,\delta^{cc'}\,\delta^{dd'}d^{\lambda\lambda'}(q)d^{\sigma\sigma'}(K-q)\\ &\quad\times\{\Delta_0^{\rm adv}(r,q)\Delta_0^{\rm cor}(r,K-q)-\Delta_0^{\rm cor}(r,q)\Delta_0^{\rm ret}(r,K-q)\},\end{aligned} \tag{158}$$

where $\Delta^{ab}_{0\mu\nu}(r,K)=\Delta_0(r,K)d^{\mu\nu}(K)\delta^{ab}$ are the zeroth order solutions (111) and (113).

The mean-field contribution (71)–(73) to the retarded and advanced components of $\overline{\Pi}$, on the other hand, vanishes, because $\overline{F_{\mu\nu}}=T^a\overline{F^a_{\mu\nu}}$ is antisymmetric and traceless[6]:

$$\begin{aligned}(\overline{\Pi}^{\rm ret}_{\rm adv}\hat{\Delta}^{\rm ret}_{\rm adv})^{ab}_{\mu\nu}(r,K) &= -2g\,\delta^{ab}d_{\mu\nu}\Delta^{\rm ret}_{\substack{\rm adv\\ 0}}(r,K)\left(\frac{1}{3}g^{\rho\lambda}\overline{F_{\rho\lambda}}(r)\right)\\ &=0.\end{aligned} \tag{159}$$

Hence, the dependence on the soft field $\overline{F_{\mu\nu}}$ or $\overline{A_\mu}$ is resident only implicitly in the kinetic momentum $K_\mu=k_\mu-g\overline{A_\mu}$, so that Eq. (103) becomes formally identical to the case of $\overline{A}=0$, in which $K_\mu=k_\mu$. Exploiting this formal analogy, one can evaluate explicitly $\hat{\Pi}^{\rm ret}_{\mu\nu}-\hat{\Pi}^{\rm adv}_{\mu\nu}$ in the kinematic range $Q^2\gtrsim(K^+)^2\gg K_\perp^2\geq\mu^2$ by using standard techniques of QCD evolution calculus [11,35]. Inserting into Eq. (158) the free-field expressions for $\Delta_0^{\rm ret}$, $\Delta_0^{\rm adv}$, and $\Delta_0^{\rm cor}$, from Eqs. (111), (113), one finds that to $O(g^2)$ the polarization tensor $\hat{\Pi}^{\rm ret}_{\rm adv}$ does not depend on r; hence one may write

$$\hat{\Pi}^{\rm ret}_{\rm adv}(r,K)\equiv\hat{\Pi}^{\rm ret}_{\rm adv}(K)\qquad\mathcal{Z}^{\rm ret}_{\rm adv}(r,K)\equiv\mathcal{Z}^{\rm ret}_{\rm adv}(K). \tag{160}$$

Using the light cone variables (126), for the momenta, together with the light cone phase-space element

$$d^4q\overset{``q^2\to0''}{=}\frac{1}{2}dq^+dq^-d^2q_\perp\,\delta(q^+q^--q_\perp^2)=\frac{\pi}{2}\frac{dq^+}{q^+}dq_\perp^2, \tag{161}$$

and using Eqs. (136) and (142), $\hat{\Pi}^{\rm ret}_\perp-\hat{\Pi}^{\rm adv}_\perp=\frac{1}{2}(\hat{\Pi}^{\rm ret}_{\rm adv})^\mu_\mu$ one finds the form factors $\mathcal{Z}^{\rm ret}_{\rm adv}=(1-\Pi^{\rm ret}_{\substack{\rm adv\\ \perp}}/K^2)^{-1}$ to leading-log accuracy:

[6]Note, however, that this cancellation occurs only in the light cone gauge (122) with gauge parameter $\alpha=0$. In a general noncovariant gauge with $\alpha\neq0$, one encounters on the right-hand side of Eq. (159) a finite term $\alpha\,(n\cdot\partial_r)[n_\nu\overline{A_\mu}(r)+n_\mu\overline{A_\nu}(r)]$.

$$\mathcal{Z}^{\rm ret}(K^+,K_\perp^2) = \exp\left\{-\frac{1}{2}\int_{K_\perp^2}^{K^{+2}=Q^2} dq_\perp^2 \int_0^{K^+}\frac{dq^+}{q^+}\,\frac{\alpha_s(q_\perp^2)}{2\pi q_\perp^2}\,\gamma\left(\frac{q^+}{K^+}\right)\right\}, \tag{162}$$

$$\mathcal{Z}^{\rm adv}(K^+,K_\perp^2)\,\theta(K^+) = -\mathcal{Z}^{\rm ret}(-K^+,K_\perp^2)\,\theta(-K^+), \tag{163}$$

where

$$\alpha_s(q_\perp^2) = \frac{4\pi}{11\ln(q_\perp^2/\Lambda^2)},$$

$$\gamma(z) = 2N_c\left(z(1-z)+\frac{1-z}{z}+\frac{z}{1-z}\right), \tag{164}$$

and $z=q^+/K^+,\ 1-z=q'^+/K^+,\ q'=K-q$. The effective form-factor function $\mathcal{Z}^\rho$ can be approximately evaluated:

$$\mathcal{Z}^\rho(K^+,K_\perp^2) = \mathcal{Z}^{\rm ret}-\mathcal{Z}^{\rm adv} \approx \begin{cases} \exp\left\{-\dfrac{3\alpha_s}{2\pi}\ln^2\left(\dfrac{Q^2}{K_\perp^2}\right)\right\} & \text{for } K_\perp^2 \geqslant \mu Q, \\ \exp\left\{-\dfrac{3\alpha_s}{2\pi}\left[\dfrac{1}{2}\ln^2\left(\dfrac{Q^2}{\mu^2}\right)-\ln^2\left(\dfrac{K_\perp^2}{\mu^2}\right)\right]\right\} & \text{for } K_\perp^2 < \mu\, Q. \end{cases} \tag{165}$$

Substituting $\mathcal{Z}^\rho$ into Eq. (145) for $\hat{\Delta}^{\rm ret}_{\mu\nu}$ and $\hat{\Delta}^{\rm adv}_{\mu\nu}$, one obtains for the spectral density $\hat{\rho} = i d^{-1}_{\mu\nu}(\hat{\Delta}^{\rm ret}_{\mu\nu}-\hat{\Delta}^{\rm adv}_{\mu\nu})$,

$$\hat{\rho}(K^+,K_\perp^2) = \mathcal{Z}^\rho(K^+,K_\perp^2)\frac{(2\pi)^4}{K^+}\Bigg[\delta(K^+-Q^+)\,\delta(K_\perp^2) + \int_{K_\perp^2}^{Q^2}\frac{dq_\perp^2}{q_\perp^2}\,\frac{\alpha_s(q_\perp^2)}{2\pi} \times\int_0^1 dz\,\gamma(z)\,\hat{\rho}\left(\frac{K^+}{z},q_\perp^2\right)\mathcal{Z}^{\rho-1}\left(K^+,\frac{q_\perp^2}{z}\right)\Bigg]. \tag{166}$$

The previously advocated interpretation of the spectral density $\hat{\rho}$ of an initial state gluon as the density of its ‘‘intrinsic’’ gluon fluctuations becomes clearer now: $\hat{\rho}$ represents the structure function of a gluon that was initialized as a bare state at Q^2. Looking at this gluon state with a resolution scale $K_\perp^2$, one sees at $K_\perp^2 = Q^2$ only the initial bare gluon itself, because $\mathcal{Z}^\rho(Q^+,Q^2)=1$, Eq. (143), and the integral term in Eq. (166) vanishes. For $K_\perp^2 < Q^2$, the form factor $\mathcal{Z}^\rho(K^+,K_\perp^2)$ decreases [cf. Eqs. (162), (165)], and so the first term, which is the probability that the gluon remains in its bare initial state, is suppressed by $\mathcal{Z}^\rho$, whereas the integral term, which is the adjoint probability that the gluon contains a distribution of intrinsic gluons, increases with weight $\mathcal{Z}^\rho(K^+,K_\perp^2)/\mathcal{Z}^\rho(K^+,q_\perp^2/z)$. Hence the evolution of the spectral density $\hat{\rho}$ describes the change of structure of the initially bare gluon state due to real and virtual emission and absorption of daughter gluons, corresponding to the generation of virtual Coulomb field coat and real bremsstrahlung, respectively.

Equation (166) can be solved in closed form by using the following trick to effectively eliminate $\mathcal{Z}^\rho$. First, note that $\hat{\rho}$ satisfies the momentum sum rule [19]

$$\int_0^{K^+} dq^+ q^+ \hat{\rho}(q^+,q_\perp^2) = (K^+)^2,$$

$$\int_0^{K^+} dq^+ q^+ \frac{\partial}{\partial q_\perp^2}\hat{\rho}(q^+,q_\perp^2) = 0, \tag{167}$$

for any value of $q_\perp^2$. Equation (167) is nothing but a manifestation of light cone momentum conservation, meaning that the aggregate of q^+ momentum from intrinsic gluons must add up to the total K^+ of the gluon state composed of those. This is a general property, which is immediately evident in the free-field case. Next, multiply Eq. (166) by q^+/K^+ and integrate over q^+ from 0 to K^+, which yields on account of the sum rule (167)

$$1 = \mathcal{Z}^\rho(K^+,K_\perp^2)\left[1+\int_{K_\perp^2}^{Q^2}\frac{dq_\perp^2}{q_\perp^2}\,\frac{\alpha_s(q_\perp^2)}{2\pi} \times\int_0^1 dz\,\gamma(z)\,\mathcal{Z}^{\rho-1}\left(K^+,\frac{q_\perp^2}{z}\right)\right], \tag{168}$$

which does not contain $\hat{\rho}$. Next, multiply this formula with $\mathcal{Z}^{\rho-1}(K^+,K_\perp^2)$ from the left, and then differentiate with respect to $K_\perp^2$ by applying $K_\perp^2\,\partial/\partial K_\perp^2$:

$$\left(K_\perp^2\frac{\partial}{\partial K_\perp^2}\mathcal{Z}^{\rho-1}(K^+,K_\perp^2)\right)\hat{\rho}(K^+,K_\perp^2) + \mathcal{Z}^{\rho-1}(K^+,K_\perp^2) \times\left(K_\perp^2\frac{\partial}{\partial K_\perp^2}\hat{\rho}(K^+,K_\perp^2)\right) = -\mathcal{Z}^{\rho-1}(K^+,K_\perp^2)\frac{\alpha_s(K_\perp^2)}{2\pi}\int_0^1 dz\,\gamma(z)\frac{1}{z}\hat{\rho}\left(\frac{K^+}{z},zK_\perp^2\right). \tag{169}$$

Using Eq. (162), the derivative $\partial \mathcal{Z}^{\rho}/\partial K_\perp^2$ on the left-hand side can be rewritten as

$$K_\perp^2 \frac{\partial}{\partial K_\perp^2} \mathcal{Z}^{\rho-1}(K^+,K_\perp^2) = -\mathcal{Z}^{\rho-1}(K^+,K_\perp^2)\frac{1}{2}\frac{\alpha_s(K_\perp^2)}{2\pi} \times \int_0^1 dz\,\gamma(z). \tag{170}$$

Substituting this into Eq. (166) and multiplying by $\mathcal{Z}^{\rho}$, one obtains a differential evolution equation in the manner of Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [34,35] that involves only $\hat{\rho}$, but not $\mathcal{Z}^{\rho}$ anymore[7]:

$$K_\perp^2 \frac{\partial}{\partial K_\perp^2} \hat{\rho}(K^+,K_\perp^2) = \frac{\alpha_s(K_\perp^2)}{2\pi}\int_0^1 \frac{dz}{z}\,\gamma(z)\left[\hat{\rho}\left(\frac{K^+}{z},K_\perp^2\right) - \frac{z}{2}\hat{\rho}(K^+,K_\perp^2)\right]. \tag{171}$$

The explicit solution of this equation is well known [36,37]:

$$\hat{\rho}(K^+,K_\perp^2) = \rho_0(K^+,K_\perp^2) + \rho_1(K^+,K_\perp^2)\exp\left[-\frac{N_c}{12\pi}g(K_\perp^2)\right] \times \exp\left[\sqrt{\frac{4N_c}{11\pi}g(K_\perp^2)h(K^+)}\right], \tag{172}$$

where

$$\rho_0(K^+,K_\perp^2) = \frac{(2\pi)^4}{K^+}\,\delta(K^+-Q^+)\,\delta(K_\perp^2-Q^2),$$

$$\rho_1(K^+,K_\perp^2) = \frac{(2\pi)^4}{K^+}\frac{1}{\sqrt{4\pi}}\left[\frac{N_c}{11\pi}\,g(K_\perp^2)\right]^{1/4}[h(K^+)]^{-3/4},$$

$$g(K_\perp^2) = \ln\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(K_\perp^2/\Lambda^2)}\right], \quad h(K^+) = \ln\left(\frac{Q^+}{K^+}\right). \tag{173}$$

F. Solving for the correlation function $\hat{\Delta}^{\rm cor}_{\mu\nu}$

Within the perturbative scheme (119) and (120), the calculation of $\hat{\Delta}^{\rm cor} = \hat{\Delta}^{\rm cor}_{[\bar{0}]} + \delta\hat{\Delta}^{\rm cor}_{[\bar{A}]}$ is most conveniently split into two steps.

(1) The quantum contribution $\hat{\Delta}^{\rm cor}_{[\bar{0}]}$ is evaluated to order g^2 from Eq. (107), i.e., the hard polarization tensor $\hat{\Pi}^{\rm cor}[g^2,\Delta_0]$ is to be calculated again in one-loop approximation with free-field internal propagators. The mean-field part polarization tensor, on the other hand, is set to zero in this first step: $\overline{\Pi}^{\rm cor}=0$.

(2) The mean-field-induced correction $\delta\hat{\Delta}^{\rm cor}_{[\bar{A}]}$ in leading order $g\bar{A}$ is then added by calculating $\overline{\Pi}^{\rm cor}[g;\Delta_0]$. The quantum part now is set to zero in this second step: $\hat{\Pi}^{\rm cor}=0$ [as it is already contained in $\hat{\Delta}^{\rm cor}_{[\bar{0}]}$ from step (1)].

1. The contribution $\hat{\Delta}^{\rm cor}_{[\bar{0}]}$

Since to order g^2 only the radiative self-interaction contributes to the hard propagator $\hat{\Delta}_{[\bar{0}]}$, and scattering processes that could alter the gluon trajectories are absent, the transport equation for the part $\hat{\Delta}_{[\bar{0}]}$ simplifies to

$$[K\cdot\overline{D_r},\hat{\Delta}^{\rm cor}_{[\bar{0}]}] = 0 + O(g^4). \tag{174}$$

Therefore, with respect to the space-time variable r, Eq. (174) implies a free-streaming behavior in the presence of the soft mean field, as implicitly contained in $K=k-g\bar{A}$, that is, $\hat{\Delta}^{\rm cor}_{[\bar{0}]}(r,K) = \hat{\Delta}^{\rm cor}_{[\bar{0}]}(r'-Vr^-)$ with $V_\mu = K_\mu/K^+$ and $r'<r$. Hence, Eq. (107) with $\hat{\Delta}^{\rm cor}_{[\bar{0}]}\to\Delta^{\rm cor}_0$ remains to be considered:

$$\hat{\Delta}^{\rm cor}_{[\bar{0}]} = -\Delta_0^{\rm ret}(\Delta_0^{\rm cor-1} - \hat{\Pi}^{\rm cor}[g^2,\Delta_0])\Delta_0^{\rm adv}. \tag{175}$$

The easiest way to obtain $\hat{\Delta}^{\rm cor}_{[\bar{0}]}$ is to use the formula (108) and simply convolute the total number density of gluons G $=1+2$g with the spectral density $\hat{\rho}$ obtained in the preceding subsection. To prove that the relation (108) is indeed consistent, one calculates instead $\hat{\Delta}^{\rm cor}_{[\bar{0}]}$ from Eq. (175) directly. The procedure is fully analogous to the previous subsection except that, instead of $\Pi^{\rm ret}-\Pi^{\rm adv}$, one needs to evaluate $\Pi^{>}+\Pi^{<}$. The resulting form of $\hat{\Pi}^{\rm cor}[g^2,\Delta_0]$ in Eq. (175) is

$$\begin{aligned}(\hat{\Pi}^{\rm cor})^{ab}_{\mu\nu}(r,K) &= -(\hat{\Pi}^{>}+\hat{\Pi}^{<})^{ab}_{\mu\nu}(r,K)\\ &= \frac{ig^2}{2}\int\frac{d^4q}{(2\pi)^4}V^{acd}_{0\mu\lambda\sigma}(K,-q,-K+q)\\ &\quad\times\hat{V}^{d'c'b}_{0\sigma'\lambda'\nu}(r;K-q,q,-K)\\ &\quad\times\delta^{cc'}\delta^{dd'}d^{\lambda\lambda'}(q)d^{\sigma\sigma'}(K-q)\\ &\quad\times\{\Delta_0^{>}(r,q)\Delta_0^{<}(r,K-q)\\ &\quad+\Delta_0^{<}(r,q)\Delta_0^{>}(r,K-q)\},\end{aligned} \tag{176}$$

where in the integral, the free-field forms $\Delta^{\gtrless}_{0\mu\nu}$ are given by (cf. Appendix F):

$$\Delta^{\gtrless ab}_{0\mu\nu}(r,K) = \Delta_0^{\gtrless}(r,K)d^{\mu\nu}(K)\delta^{ab},$$

$$\Delta_0^{\gtrless}(r,K) = (-2\pi i)\,\delta(K^2)[\,\theta(\pm K^+) + {\rm g}_0(r,\pm K)]. \tag{177}$$

Inserting into Eq. (176) the expressions (177), and observing that in Eq. (107) $\Pi^{\rm cor}$ is sandwiched between $\hat{\Delta}^{\rm ret}$ and $\hat{\Delta}^{\rm adv}$, i.e., appears only in the combination

$$\begin{aligned}\hat{\Delta}^{\rm ret}_{\mu\lambda}\hat{\Pi}^{\rm cor}_{\lambda\rho}\hat{\Delta}^{\rm adv}_{\rho\nu} &\propto d_{\mu\lambda}(K)d_{\lambda\tau}(q)d_{\tau\rho}(K-q)d_{\rho\nu}(K)\\ &\propto d_{\mu\nu}(K)\hat{\Pi}^{\rm cor}_{\perp}(K),\end{aligned} \tag{178}$$

[7]It should be noted that in obtaining (171), the fact that $\hat{\rho}(K^+/z,zK_\perp^2)\simeq\hat{\rho}(K^+/z,K_\perp^2)$ was used—a property that is due to the very weak z dependence of the $K_\perp^2$ argument of $\hat{\rho}$ [35].

where $\hat{\Pi}_\perp^{\rm cor}=g_{\rho\tau}\hat{\Pi}_{\rho\tau}^{\rm cor}$, and $d_{\mu\lambda}d_{\lambda\nu}=d_{\mu\nu}$, one finds after a calculation analogous as in the preceding subsection the following result for the $\hat{\Delta}_{[\bar{0}]}$ part of the correlation function:

$$\hat{\Delta}_{[\bar{0}]}^{\rm cor}(r,K^+,K_\perp^2)={\rm Tr}[d_{\mu\nu}^{-1}(K)\hat{\Delta}_{[\bar{0}]\mu\nu}^{\rm cor}(r,K)]$$
$$=\int_{K_\perp^2}^{Q^2}\frac{dK_\perp'^2}{K_\perp'^2}\frac{\alpha_s(K_\perp'^2)}{2\pi}\int_0^1\frac{dz}{z}\gamma(z)$$
$$\times\left[\{\hat{\rho}(1+2{\rm g})\}(r,K^+/z,K_\perp'^2)\right.$$
$$\left.-\frac{z}{2}\{\hat{\rho}(1+2{\rm g})\}(r,K^+,K_\perp'^2)\right]. \tag{179}$$

Comparison with Eq. (171) reveals that $\hat{\Delta}_{[\bar{0}]}^{\rm cor}$ is indeed the convolution of the spectral density $\hat{\rho}$ with the total gluon density ${\rm G}=1+2{\rm g}$, as advocated by Eq. (108). Hence,

$${\rm G}_{[\bar{0}]}(r,K^+,K_\perp^2)={\rm G}_0(r,Q^+,0_\perp)$$
$$+\int_{K_\perp^2}^{Q^2}\frac{dK_\perp'^2}{K_\perp'^2}\frac{\alpha_s(K_\perp'^2)}{2\pi}\int_0^1 dz\,\gamma(z)$$
$$\times\left[\frac{1}{z}{\rm G}_{[\bar{0}]}\left(r,\frac{K^+}{z},K_\perp'^2\right)\right.$$
$$\left.-\frac{1}{2}{\rm G}_{[\bar{0}]}(r,K^+,K_\perp'^2)\right], \tag{180}$$

where ${\rm G}_0(r,Q^+,0_\perp)$ is the initial gluon density (154). In the limit $z\ll 1$ the integral (180) can be approximately evaluated analytically [38,39]. This gives an estimate of the gluon multiplicity [40,41] as a function of $K_\perp^2$ at fixed space-time point r:

$${\rm G}_{[\bar{0}]}(r,K_\perp^2)=i\int_0^{Q^+}dK^+{\rm G}_{[\bar{0}]}(r,K^+,K_\perp^2)$$
$$={\rm G}_0(r,Q^2)\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(K_\perp^2/\Lambda^2)}\right)^{-1/4}$$
$$\times\exp\left\{2\sqrt{\frac{N_c}{11\pi}}\left[\sqrt{\ln\left(\frac{Q^2}{\Lambda^2}\right)}\right.\right.$$
$$\left.\left.-\sqrt{\ln\left(\frac{K_\perp^2}{\Lambda^2}\right)}\right]\right\}, \tag{181}$$

where ${\rm G}_0(r,Q^2)$ is given by Eq. (154). It is evident that in the kinematic regime $Q^2\geqslant K_\perp^2\geqslant\mu^2$, the hard gluon multiplicity is characterized by a rapid growth as the gap between the initial scale Q^2 and $K_\perp^2$ increases.

2. The contribution $\delta\hat{\Delta}_{[A]}^{\rm cor}$

The leading-order mean-field contribution $\delta\hat{\Delta}_{[A]}^{\rm cor}$ is now to be added to the result for $\hat{\Delta}_{[\bar{0}]}^{\rm cor}$, Eq. (179). To do so, one needs to evaluate $\overline{\Pi}^{\rm cor}$ to order $g\overline{A}$, using the free-field solutions (111)–(113) and set $\hat{\Pi}^{\rm cor}=0$. The analogon of Eq. (175) for $\delta\hat{\Delta}_{[A]}^{\rm cor}$ is

$$\delta\hat{\Delta}_{[A]}^{\rm cor}=-\Delta_{[\bar{0}]}^{\rm ret}\overline{\Pi}^{\rm cor}[g,\Delta_0]\Delta_{[\bar{0}]}^{\rm adv}, \tag{182}$$

and $\overline{\Pi}^{\rm cor}[g,\Delta_0]$ can be read off from Eq. (104), giving the contribution

$$gK_\lambda\overline{F^{\lambda}}^{\sigma}(r)\partial_\sigma^K\Delta_{[\bar{0}]\mu\nu}^{{\rm cor}ab}(K)+g[\overline{F_\mu^{\lambda}}(r)\Delta_{[\bar{0}]\lambda\nu}^{\rm cor}(K)$$
$$-\Delta_{[\bar{0}]\mu}^{{\rm cor}\lambda}(K)\overline{F_{\lambda\nu}}(r)]^{ab}$$
$$=g\delta^{ab}d_{\mu\nu}(K)K_\lambda\overline{F^{\lambda}}^{\sigma}(r)\partial_\sigma^K\Delta_{[\bar{0}]}^{{\rm cor}ab}(K). \tag{183}$$

The second term on the left side cancels, because $\overline{F_\mu^{\lambda}}\hat{\Delta}_{[\bar{0}]\lambda\nu}^{\rm cor}\propto\overline{F_\mu^{\lambda}}g_{\lambda\nu}-\overline{F_\mu^{\lambda}}(n_\lambda K_\nu+K_\lambda n_\nu)/K^+$, and $n^+=-n^-=1,n_\perp=0$, $K^-=0$. Notice that this is a specific feature of the employed light cone representation, and does not hold in a general noncovariant gauge. With Eq. (183), the function $\delta\hat{\Delta}_{[A]}^{\rm cor}$ satisfies the transport equation

$$[K\cdot\overline{D_r},\delta\hat{\Delta}_{[A]}^{\rm cor}]=-gd_{\mu\nu}K_\lambda\overline{F^{\lambda}}^{\sigma}\partial_\sigma^K\Delta_{[\bar{0}]}^{\rm cor}. \tag{184}$$

To solve Eq. (184), it is convenient to express $\delta\hat{\Delta}_{[A]\mu\nu}^{\rm cor}$ in terms of a new function $\Phi_\mu=T^a\Phi_\mu^a$ [7], defined by

$$\delta\hat{\Delta}_{[A]\mu\nu}^{\rm cor}(r,K)=d_{\mu\nu}(K)g\Phi^\lambda(r,K)\partial_\lambda^K\hat{\Delta}_{[\bar{0}]}^{\rm cor}(r,K), \tag{185}$$

where $\hat{\Delta}_{[\bar{0}]}^{\rm cor}$ is the solution (179). In terms of the function Φ^μ, the transport equation (184) becomes now

$$[K\cdot\overline{D_r},\Phi_\mu(r,K)]=\overline{F_{\mu\nu}}(r)K^\nu. \tag{186}$$

The function Φ^μ evidently satisfies

$$K_\mu\Phi^\mu(r,K)=0\Rightarrow\Phi^-=\frac{2}{K^+}\vec{K}_\perp\cdot\vec{\Phi}_\perp,\quad\Phi^+=0, \tag{187}$$

i.e., Φ^- is not an independent variable, but is expressible in terms of the transverse components $\vec{\Phi}_\perp$, and Φ^+ is suppressed by K^-/K^+ and therefore may be set to zero. The interpretation of the function Φ_μ, as was pointed out by Blaizot and Iancu [7], is that the component $g\Phi_\mu$ corresponds to the kinetic momentum $K_\mu=k_\mu-g\overline{A}_\mu$ that is acquired by a gluon propagating in the presence of the soft field $\overline{A}_\mu$ or $\overline{F}_{\mu\nu}$. The condition (187) reflects then the fact that the light cone energy transferred by the soft field, namely, $g\Phi^-$, equals the mechanical work done by the Lorentz force $g\vec{V}_\perp\cdot\Delta\vec{K}_\perp=g\vec{V}_\perp\cdot\vec{\Phi}_\perp$, where $V^\mu=K^\mu/K^+=(1,0,\vec{V}_\perp)$ is the velocity.

The transport equation (186) for Φ_μ can be readily solved [7] with the help of the retarded and advanced functions $\Delta_{0\mu\nu}^{\rm ret\atop adv}=d_{\mu\nu}\,\Delta_0^{\rm ret\atop adv}$,

$$\Phi_\mu(r,K)=i\int d^4r'\,\Delta_0^{\rm ret}(r-r',K)\overline{F_{\mu\nu}}(r')K^\nu$$
$$-i\int d^4r'\,\Delta_0^{adv}(r-r',K)\overline{F_{\mu\nu}}(r')K^\nu. \qquad (188)$$

The free-field retarded and advanced functions admit the space-time representation [7]

$$\Delta_0^{\rm ret}(r-r',K)=-i\,\theta(r^- - r'^-)\,\delta[r^+ - r'^+ -(r^- - r'^-)]$$
$$\times\delta^2\left(\vec{r}_\perp-\vec{r}'_\perp-\frac{\vec{K}_\perp}{K^+}(r^- - r'^-)\right),$$
$$\Delta_0^{\rm adv}(r-r',K)=+i\,\theta(r'^- - r)\,\delta[r'^+ - r^+ -(r'^- - r^-)]$$
$$\times\delta^2\left(\vec{r}'_\perp-\vec{r}_\perp-\frac{\vec{K}_\perp}{K^+}(r'^- - r^-)\right), \qquad (189)$$

and therefore $i(\Delta_0^{\rm ret}-\Delta_0^{\rm adv})(r-r',K)=2\,\delta[r^+-r'^+-(r^- - r'^-)]\,\delta^2[\vec{r}_\perp-\vec{r}'_\perp-(\vec{K}_\perp/K^+)(r^- - r'^-)]$. Insertion into Eq. (188) then yields

$$\Phi_\mu(r,K)=2K^\nu\int_0^{r^-}dr'^-\,\overline{F_{\mu\nu}}\left(r-\frac{K}{K^+}r'^-\right)\equiv 2K^\nu\overline{F}_{\mu\nu}(r). \qquad (190)$$

Substituting this result into Eq. (185) and using the light cone components of $\overline{F_{\mu\nu}}$, Eq. (128), the result for the mean-field induced correction $\delta\hat{\Delta}^{\rm cor}_{[\overline{A}]}$ is

$$\delta\hat{\Delta}^{\rm cor}_{[\overline{A}]}(r,K)={\rm Tr}[d^{-1}_{\mu\nu}(K)\,\delta\hat{\Delta}^{\rm cor}_{[\overline{A}]\mu\nu}(r,K)]$$
$$=-g\overline{F_{\perp+}}(r)\left(K_\perp\frac{\partial}{\partial K^+}-K^+\frac{\partial}{\partial K_\perp}\right)$$
$$\times\hat{\Delta}^{\rm cor}_{[\overline{0}]}(r,K), \qquad (191)$$

where $\perp$ denotes the transverse vector components $i=1,2$, and $\Phi_\perp=\frac{1}{2}(\Phi^1+\Phi^2)$, $K_\perp=1/\sqrt{2}(K^1+K^2)$.

With Eq. (190), the addition $\delta{\rm G}_{[\overline{A}]}(r,K^+,K_\perp^2)$ to the gluon density ${\rm G}_{[\overline{0}]}(r,K^+,K_\perp^2)$ of Eq. (180) is

$$\delta{\rm G}_{[\overline{A}]}(r,K^+,K_\perp^2)=-2g\frac{K^+}{K_\perp}\overline{F_{\perp+}}(r)$$
$$\times\left(K_\perp^2\frac{\partial}{\partial K_\perp^2}{\rm G}_{[\overline{0}]}(r,K^+,K_\perp^2)\right)$$
$$+O[K_\perp^2/(K^+)^2], \qquad (192)$$

where the explicit form of derivative term in brackets can be easily read off the right-hand side of Eq. (180).

G. Expansion in space-time of the hard gluon ensemble

As argued before in Eq. (174), the evolution of the hard gluon density G, described by $\hat{\Delta}^{\rm cor}$, can in the present context be viewed as a purely multiplicative cascade of gluon emissions, since to order g^2 and due to the quasicollinear motion of the gluons, statistical scatterings between them do not contribute. Therefore the space-time development of ${\rm G}(r,K^+,K_\perp^2)$ with respect to $r=(r^-,r^+,\vec{r}_\perp)$ is of ‘‘free-streaming nature.’’ That is, the expansion with time of the ensemble of gluons as a whole proceeds through a deterministic diffusion in momentum and space-time, as qualitatively sketched in Fig. 1(a).

To quantify this heuristic picture, one needs to invoke the uncertainty principle to relate the development in space and time to the evolution in momentum space, i.e., with respect to K^+ and $K_\perp^2$. Specifically, what is the characteristic time r^- in the chosen Lorentz frame that it takes to build up the density ${\rm G}(r,K^+,K_\perp^2)$ from the initial form ${\rm G}_0(r_0,Q^+,0_\perp)$ at time $r_0^-=0$. Viewing the gluon evolution as a cascade of successive branchings $K_{n-1}\to K_n+K'_n$, where n labels the generation in the cascade tree, the lifetime of gluon K_{n-1} is given by the time span Δr_n^- that it takes to emit and form the daughters K_n and K'_n as individual offspring, that is, by the formation time

$$\Delta r_n^-=\frac{1}{2}\left(\frac{K_n^+}{K_{\perp n}^2}-\frac{K_n^{+\prime}}{K_{\perp n}^{\prime 2}}\right)=\frac{K_{n-1}^+}{K_{\perp n}^2}\equiv\tau_n\gamma_n, \qquad (193)$$

with $K_n^{+\prime}=K_{n-1}^+-K_n^+$, $\vec{K}_{\perp n}=-\vec{K}'_{\perp n}$ and $K_{\perp n}\equiv\sqrt{\vec{K}^2_{\perp n}}$. Here $\tau_n=1/K_{\perp n}$ and $\gamma_n=K_{n-1}^+/K_{\perp n}$ play the role of the proper time and the Lorentz γ factor, respectively , in agreement with the uncertainty principle. Similarly, the average longitudinal and transverse distances traveled by the gluons K_n and K'_n during the time span Δr_n^- are

$$\Delta r_n^+=\frac{1}{2}(V_n^+ + V_n^{+\prime})\Delta r_n^-\simeq\frac{K_{n-1}^+}{K_{\perp n}^2}, \qquad (194)$$

$$\Delta r_{\perp n}=|\vec{V}_{\perp n}-\vec{V}'_{\perp n}|\Delta r_n^-\simeq\frac{2}{K_{\perp n}}, \qquad (195)$$

where $V^\mu=K^\mu/K^+$ and $K_\perp\ll K^+$ is assumed as before. The average total time $\langle r^-\rangle$ elapsed up to the nth cascade generation with mean gluon momentum K^+ and $K_\perp^2$, and the associated spatial spread $\langle r^+\rangle$, $\langle r_\perp\rangle$, of the diffusing gluon ensemble, is then obtained by weighting the evolution of the gluon density G, Eq. (180), with $\Delta r(K^+,K_\perp^2)\equiv(\Delta r^-,\Delta r^+,\Delta r_\perp)$ from Eqs. (193)–(195). Taking the real emission part of Eq. (180), differentiating it with respect to $K_\perp^2$, convoluting it with the weight $\Delta r(K^+,K_\perp^2)$, integrating over all possible branchings, and normalizing it to the density ${\rm G}(r,K^+,K_\perp^2)$ itself, the desired average is

$$\langle r(K^+,K_\perp^2)\rangle=\frac{1}{{\rm G}(r,K^+,K_\perp^2)}\int_{K_\perp^2}^{Q^2}\frac{dK_\perp'^2}{K_\perp'^2}\frac{\alpha_s(K_\perp'^2)}{2\pi}$$
$$\times\left[\int_{K^+}^{Q^+}\frac{dK^{+\prime}}{K^{+\prime}}{\rm G}(r,K^+,K_\perp'^2)\Delta r(K^{+\prime},K_\perp'^2)\right]$$
$$\times\int_0^1\frac{dz}{z}\gamma(z){\rm G}(r,K^+/z,K_\perp'^2). \qquad (196)$$

This complicated formula can be approximately evaluated from the known behavior of G, as has been worked out in detail in [42]. For $K^+ \ll Q^+$ the result is, up to powers of $\ln(Q^+/K^+)$, the following estimate:

$$\langle r^-(K^+,K_\perp^2)\rangle \simeq \langle r^+(K^+,K_\perp^2)\rangle = \frac{K^+}{K_\perp^2}\mathcal{T}(K^+,K_\perp^2),$$

$$\langle r_\perp(K^+,K_\perp^2)\rangle = \frac{2}{K_\perp}\mathcal{T}(K^+,K_\perp^2), \tag{197}$$

where

$$\mathcal{T}(K^+,K_\perp^2) = c_1(K_\perp^2)\exp\left[-c_2(K_\perp^2)\sqrt{\ln\left(\frac{Q^+}{K^+}\right)}\right], \tag{198}$$

with $c_1, c_2 > 0$ very slowly varying functions of $K_\perp^2$. This estimate shows that those gluons which are emitted either with large $K_\perp^2$ or with small K^+/Q^+, appear the earliest in time r^- and contribute the quickest to the diffusion in $r^+, r_\perp$.

H. Constructing the hard current $\hat{j}_\mu$ and the induced soft field $\overline{A_\mu}$

The final task of the solution scheme of Sec. III C is to solve for the soft field $\overline{A_\mu}$ or $\overline{F_{\mu\nu}}$, which is induced by the color current $\hat{j}_\mu$, being generated by the aggregate of initial plus emitted hard gluons from the evolution of the gluon density (180). In the equation of motion for $\overline{F_{\mu\nu}}$, recall Eq. (105),

$$[\overline{D_r^\lambda}, \overline{F_{\lambda\mu}}]^a(r) = -\hat{j}_\mu^a(r), \tag{199}$$

the current on the right-hand side is determined by the hard gluon correlation function $\hat{\Delta}_{\mu\nu}^{\rm cor} = \hat{\Delta}_{[\overline{0}]\mu\nu}^{\rm cor} + \delta\hat{\Delta}_{[\overline{A}]\mu\nu}^{\rm cor}$, and therefore by the gluon density $\mathrm{G} = \mathrm{G}_{[\overline{0}]} + \delta\mathrm{G}_{[\overline{A}]}$, as obtained in the previous subsection:

$$\hat{j}_\mu(r) = T^a\hat{j}_\mu^a(r) = -g\int\frac{d^4k}{(2\pi)^2}\,\mathrm{Tr}\{T^a[K_\mu\hat{\Delta}_\nu^{\mathrm{cor}\,\nu}(r,K) - K_\nu\hat{\Delta}_\mu^{\mathrm{cor}\,\nu}(r,K)]\}. \tag{200}$$

The first point to be made here is that, for the light cone gauge condition $A^+ = 0$, the gauge-field tensor $\overline{F_{\mu\nu}}$ has only the nonvanishing components (128), and if one requires in addition $A^- = 0$, then in Eq. (199), $\overline{D^\lambda}\,\overline{F_{\lambda\mu}} = \delta_\perp^\lambda\delta_{\mu+}\overline{D_\perp}\,\overline{F_{\perp+}}$. The second observation is that the left-hand side of Eq. (200) is essentially the density G of hard gluons weighted with their momentum K^μ. Because the gluons evolve with the velocity $V^\mu = K^\mu/K^+$ along the light cone, at a given light cone time r^- and corresponding coordinate $r^+ = r_0^+ + V^+ r^- = r^-$, these gluons appear as an extremely thin Lorentz-contracted sheet, but are spread out in transverse direction $r_\perp$ over a disc with radius $\sim 1/\langle K_\perp\rangle$. As a consequence, the gluon current $\hat{j}^\mu = (\hat{j}^+, \hat{j}^-, \vec{\hat{j}}_\perp)$ has only a component in the + direction, $\hat{j}^\mu = \delta^{\mu +}\hat{j}^+$ [24,43]. Denoting as before the two transverse vector components $i = 1,2$ by $\perp$ with summation convention $a_\perp b_\perp \equiv \Sigma_{i=1,2}a_i b_i$, Eq. (199) now becomes

$$[\overline{D_\perp}, \overline{F_{+\perp}}]^a(r) = [\delta^{ab}\partial_\perp - gf^{abc}\overline{A_\perp^c}(r), \overline{F_{+\perp}^b}](r) = -\hat{j}_+^a(r), \tag{201}$$

where $\hat{j}_+^a$ is the color-charge density at $r^- = r^+$:

$$\hat{j}_+^a(r) = gT^a\mathcal{J}(\vec{r}_\perp)\,\delta(r^- - r^+), \tag{202}$$

where

$$\mathcal{J}(\vec{r}_\perp) = 2\pi\int_0^{Q^+}\frac{dK^+}{(2\pi)^3 2K^+}\times\int_{\mu^2}^{Q^2}dK_\perp^2\ \mathrm{Tr}[T^aK^+\hat{\Delta}^{\rm cor}(r,K^+,K_\perp^2)] \tag{203}$$

and $\hat{\Delta}^{\rm cor} = d_{\mu\nu}^{-1}(\hat{\Delta}_{[\overline{0}]\mu\nu}^{\rm cor} + \delta\hat{\Delta}_{[\overline{A}]\mu\nu}^{\rm cor})$, using Eqs. (180), (192):

$$\hat{\Delta}^{\rm cor}(r,K^+,K_\perp^2) = 2\left(1 - 2g\frac{K^+}{K_\perp}\overline{\mathrm{F}_{\perp+}}(r)K_\perp^2\frac{\partial}{\partial K_\perp^2}\right)\times\mathrm{G}_{[\overline{0}]}(r,K^+,K_\perp^2). \tag{204}$$

Equations (203) and (204) follow from the fact that on the left-hand side of Eq. (200) the correlation function obeys the transversality condition $K^\mu\hat{\Delta}_{\mu\nu}^{\rm cor} = K^\mu d_{\mu\nu}\hat{\Delta}^{\rm cor} = 0$ and because $K^\mu\hat{\Delta}_\nu^{\mathrm{cor}\,\nu} = K^\mu d_\nu^\nu\hat{\Delta}^{\rm cor} = 2K^\mu\hat{\Delta}^{\rm cor}$. Notice that in Eq. (203) the limits of the integration over K^+ and $K_\perp^2$ correspond to the average time r^- and spatial extent $r_\perp$ of the gluon system, as estimated in Eq. (197) above, and hence, $\mathcal{J}$ accounts for the total gluon multiplicity accumulated by the evolution between Q and μ.

Integrating both sides of Eq. (201) over r^+, r^-, and using Eqs. (203) and (204), gives

$$\mathcal{J}(\vec{r}_\perp) = \int_0^{r^+}dr^+\int_0^{r^-}dr^-(\partial_\perp\overline{F_{\perp+}} - ig[\overline{A_\perp}, \partial_\perp\overline{F_{\perp+}}])\times(r^+,r^-,\vec{r}_\perp). \tag{205}$$

An approximate method to determine the soft field from Eq. (205) is to adopt the approach of Kovchegov [43], who recently calculated the light cone gauge field induced by an ultrarelativistic current of quarks with a uniform momentum distribution, using the known form of the light cone gauge potential of a single color charge [44]. Applying his concept to the present case of gluons with a nonuniform distribution $\mathrm{G}(r,K)$, the first step is to write the color-charge density $\mathcal{J}$ of Eq. (203) in a "discretized version" as a superposition of $\mathcal{N}$ individual gluon charges:

$$\hat{j}_+^a(r) = g\ \mathrm{Tr}\left[\sum_{i=1}^{\mathcal{N}}T_i^a\,\delta(r^+ - r_i^+)\,\delta(r^- - r_i^-)\times\delta^2(\vec{r}_\perp - \vec{r}_{\perp i})\mathcal{J}(\vec{r}_{\perp i})\right]\delta(r^- - r^+), \tag{206}$$

where $\mathcal{N}$ is the total number of gluons at a given $r^+=r^-$, $\mathcal{N}(r)=\pi\int_0^{Q^+}[dK^+/(2\pi)^3 2K^+]\int_{\mu^2}^{Q^2}dK_\perp^2 \mathrm{G}(r,K)$. Now the approximate solution to Eq. (205) for the light cone gauge potential $\overline{A_\mu}$ at space-time point r is obtained by the superposition of contributions that are induced by the hard gluons at points r_i. Following Kovchegov [43] in detail, the result is that $\overline{A_\mu}$ has only nonvanishing transverse components

$$\overline{A^+}(r)=\overline{A^-}(r)=0,$$

$$\overline{\vec{A}_\perp}(r)=2\pi g\sum_{i=1}^{\mathcal{N}} \theta(r^+-r_i^+)\ \theta(r^- -r_i^-)\ln\left(\frac{\vec{r}_\perp-\vec{r}_{\perp\ i}}{|\vec{r}_\perp-\vec{r}_{\perp i}|^2}\right) \times\mathcal{J}(\vec{r}_{\perp i})\ \mathrm{Tr}[T_i^a S(r) T^a S^{-1}(r)], \tag{207}$$

where

$$S(r)=\prod_{i=1}^{\mathcal{N}}\exp\left[2\pi i g^2 T^a T_i^a\,\theta(r^+-r_i^+)\,\theta(r^- -r_i^-)\times\ln\left(\frac{\vec{r}_\perp-\vec{r}_{\perp i}}{|\vec{r}_\perp-\vec{r}_{\perp i}|^2}\right)\right]. \tag{208}$$

It is important to note that Eq. (207) is only an approximate solution of Eq. (205) for the induced soft field. It is an *estimate* of the classical equation of motion for the soft mean field $\overline{\vec{A}_\perp}$ that is generated by the collective motion of a given configuration of hard gluons with a distribution $\mathrm{G}(r,K)$. In other words, Eq. (207) is the non-Abelian Weizsäcker-Williams field due to the hard gluons.

I. Conclusion

Let me summarize the input and results of the preceding sample calculation for the evolution of a high-energy gluon beam along the light cone. On the basis of the calculation scheme of Sec. III C, the logic of the application proceeded in the following steps.

(1) Choice of light cone gauge with gauge vector n along the gluon beam direction K^+/K and gauge constraint $A^+=0$.

(2) Specification of the initial bare gluon ensemble at time $r^-=0$ with a momentum distribution of equal momenta $K^+=Q^+$, $K_\perp=0$, and a spatial distribution being uniform $r_\perp\leqslant R$ in the transverse plane, but a δ-function sheet in longitudinal beam direction at $r^+=0$.

(3) Calculation of the retarded and advanced functions $\hat{\Delta}^{\mathrm{ret}}_{\mathrm{adv}}$ and the associated spectral density $\hat{\rho}$ to order g^2 from the initial values of the hard gluon propagators. The result is stated by Eqs. (172) and (173).

(4) Evaluation of the quantum part $\hat{\Delta}^{\mathrm{cor}}_{[\overline{0}]}$ of the correlation function, involving the result for $\hat{\Delta}^{\mathrm{ret}}_{\mathrm{adv}}$ of point (3). The solution for $\hat{\Delta}^{\mathrm{cor}}_{[\overline{0}]}$ and the corresponding gluon phase-space density $\mathrm{G}_{[\overline{0}]}$ is given by Eqs. (180) and (181), respectively.

(5) Evaluation of the mean-field part $\delta\hat{\Delta}^{\mathrm{cor}}_{[\overline{A}]}$, involving the solution for $\hat{\Delta}^{\mathrm{cor}}_{[\overline{0}]}$ of point (4). The result for $\delta\hat{\Delta}^{\mathrm{cor}}_{[\overline{A}]}$ and the correction to the gluon density $\delta\mathrm{G}_{[\overline{A}]}$ is given by Eqs. (191) and (192), respectively.

(6) Construction of the hard gluon current $\hat{j}$ from the solution $\hat{\Delta}^{\mathrm{cor}}=\hat{\Delta}^{\mathrm{cor}}_{[\overline{0}]}+\delta\hat{\Delta}^{\mathrm{cor}}_{[\overline{A}]}$ of points (4) and (5), with the explicit form given by the formulas (202)–(204). Approximate evaluation of the soft mean field $\overline{A}$ from the classical Yang-Mills equation (201) with resulting Weizsäcker-Williams form (207).

With this procedure, the original master equations (103)–(105) are solved in first iteration to order $g^2(1+g\overline{A})$. One could in principle now repeat this cycle, with the first-order solutions replacing the zeroth-order forms as input. This, however, is beyond the scope of the current paper.

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APPENDIX A: CONVENTIONS AND NOTATION

Throughout the paper pure $\mathrm{SU}(3)_c$ Yang-Mills theory for $N_c=3$ colors is considered, in the absence of quark degrees of freedom, with the *gauge field tensor*

$$\mathcal{F}^a_{\mu\nu}=\partial^x_\mu\mathcal{A}^a_\nu-\partial^x_\nu\mathcal{A}^a_\mu+g f^{abc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu \tag{A1}$$

and the classical *Yang-Mills Lagrangian*

$$\mathcal{L}_{\mathrm{YM}}(x)=-\frac{1}{4}\mathcal{F}^a_{\mu\nu}(x)\mathcal{F}^{\mu\nu,\ a}(x)=-\frac{1}{2}\{(\partial^x_\mu\mathcal{A}^a_\nu)^2-(\partial^x_\mu\mathcal{A}^a_\nu)\times(\partial^\nu_x\mathcal{A}^{\nu,\ a})\}(x)+g f_{abc}\{(\partial^x_\mu\mathcal{A}^a_\nu)\mathcal{A}^{\mu,b}\mathcal{A}^{\nu,c}\}(x)+g^2 f^{abc}f^{ab'c'}\{\mathcal{A}^b_\mu\mathcal{A}^c_\nu\mathcal{A}^{\mu,b'}\mathcal{A}^{\nu,c'}\}(x). \tag{A2}$$

Because only gluonic degrees of freedom are considered, only the *fundamental representation* of color space is relevant, with the color indices $a,b,\ldots,$ running from 1 to N_c. The generators of the SU(3) color group are the traceless Hermitian matrices T_a with the structure constants f^{abc}, as matrix elements, satisfying

$$\mathrm{Tr}(T^a,T^b)=N_c\delta^{ab},\quad [T^a,T^b]=+if^{abc}T_c,\quad -if^{abc}=(T^a)^{bc}. \tag{A3}$$

In compact notation,

$$\mathcal{A}_\mu\equiv T^a\mathcal{A}^a_\mu,\quad \mathcal{F}_{\mu\nu}\equiv T^a\mathcal{F}^a_{\mu\nu}=\partial^x_\mu\mathcal{A}_\nu-\partial^x_\nu\mathcal{A}_\mu-ig[\mathcal{A}_\mu,\mathcal{A}_\nu]=\frac{1}{(-ig)}[D_\mu,D_\nu], \tag{A4}$$

where $\partial_\mu^x \equiv \partial/\partial x^\mu$ acting on the space-time argument $x^\mu = (x^0, \vec{x})$. The *covariant derivative*, denoted by D_μ, is defined as

$$D_\mu(x) \equiv \partial_\mu^x - igT^a \mathcal{A}_\mu^a(x) = \partial_\mu^x - ig\mathcal{A}_\mu(x) \tag{A5}$$

and its adjoint is $D_\mu^\dagger(y) \equiv \partial_\mu^y + ig\mathcal{A}_\mu(y)$. In components, using Eq. (A3),

$$D_\mu^{ab}(x) = \delta^{ab}\partial_\mu^x - gf^{abc}\mathcal{A}_\mu^c(x), \tag{A6}$$

with the color coupling strength g being related to the strong coupling $\alpha_s = g^2/(4\pi)$. In general, for any color matrix O with matrix elements $O_{ab}(x)$, the action of the covariant derivative is

$$[D_\mu, O(x)] \equiv \partial_\mu^x O(x) - ig[\mathcal{A}_\mu(x), O(x)], \tag{A7}$$

and in particular, the covariant derivative of the field strength tensor reads $[D_\mu, \mathcal{F}_{\nu\lambda}] = \partial_\mu^x \mathcal{F}_{\nu\lambda} - ig[\mathcal{A}_\mu, \mathcal{F}_{\nu\lambda}]$.

The convention for placing indices and labels are such that *color indices* $a, b, \ldots$, are always written as superscripts, whereas all other labels may be subscripts or superscripts. In particular, the *Lorentz vector indices* $\mu, \nu, \ldots$, may be raised or lowered according to the Minkowski metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and the usual convention for summation over repeated indices is understood. Finally, some shorthand notation is employed: namely,

$$A \cdot B \equiv A_\mu g^{\mu\nu} B_\nu, \quad K \cdot (AB) \equiv K_{\mu\nu} A^\mu B^\nu, \tag{A8}$$

$$A \circ B \equiv \int_P d^4x A(x) \cdot B(x),$$

$$K \circ (AB) \equiv \int_P d^4x d^4y K(x,y) \cdot [A(x)B(y)], \tag{A9}$$

where the label P under the integral sign refers to the integration of the time components x^0 (y^0) along a closed path in the complex time plane.

APPENDIX B: BASICS OF THE CLOSED-TIME-PATH FORMALISM

1. The in-in amplitude Z_P

The key problem in this paper is describing the dynamical development of a multiparticle system (here gluons), that evolves from an initially prepared quantum state, e.g., produced by a high-energy particle collision. There is a crucial difference between the evolution of the system in *in vacuum* (which means, free space in the absence of surrounding matter) and *in medium* (which could be either an external matter distribution, or an internal particle density induced by the gluons themselves). As illustrated by Fig. 2 in the Introduction, this difference arises from the interactions, and hence, nontrivial statistical correlations between the gluons and the particles of the environment.

In the case of *vacuum*, the usual quantum field theory describes the time evolution of the system by the vacuum-vacuum transition amplitude, also called the in-out amplitude [see Fig. 2(a), left panel]. That is, one starts at time t_0 in the remote past with appropriate asymptotic in state and evolves it to t_∞ in the asymptotic future, by means of the time evolution operator $U(t_\infty, t_0)$. Multiplication with the Hermitian conjugate counterpart, which corresponds to a backward evolution from t_∞ to t_0 under the action of $U^\dagger(t_0, t_\infty)$. The resulting in-out amplitude may be interpreted as the sum over all n-point Green functions for space-time points along a path in the complex t plane, exclusively on the upper (lower) branch for the forward (backward) evolution. In vacuum there is no correlation between the two time branches and so, for instance, the two-point Green functions are the usual time-ordered Feynman Δ^F (anti-time-ordered $\overline{\Delta^F}$) propagator [see Fig. 2(a), right panel]. Because $U^\dagger(t_0, t_\infty) = U(t_\infty, t_0)$, one has $\Delta^F(t_1, t_2) = -\overline{\Delta^F}(t_2, t_1)$.

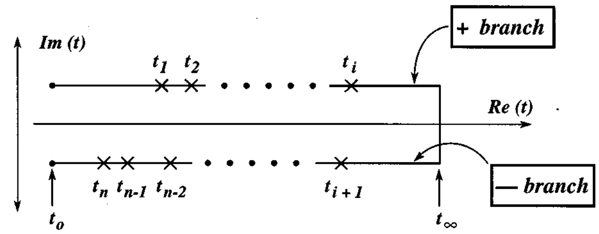

FIG. 6. The close-time path in the complex t plane for the evolution of operator expectation values in an arbitrary initial state. Any point on the forward, positive branch $t_0 \to t_\infty$ is understood at an earlier instant than any point on the backward, negative branch $t_\infty \to t_0$.

In the case of a *medium*, the above concept fails, because of the *a priori* presence of medium particles described by the density matrix $\hat{\rho}(t_0)$. Instead one has to construct a generalized transition amplitude, called the in-in amplitude, which accounts for the nontrivial initial state at t_0 embodied in the density matrix $\hat{\rho}(t_0)$, and evolves the system in the presence of the medium from t_0 to t_∞ in the future, by means of the time evolution operator $U(t_0, t_\infty)$ [see Fig. 2(b), left panel]. Because now $U^\dagger(t_0, t_\infty) \neq U(t_\infty, t_0)\hat{\rho}(t_0)$, forward and backward contributions are not merely conjugate to each other, but interfere, giving rise to statistical correlations between upper and lower time branch of the contour in the t plane. As a consequence the space of Green functions is enlarged by non-time-ordered correlation functions. For example, the two-point functions are now $\Delta^F, \overline{\Delta^F}$ plus the new functions $\Delta^<$ and $\Delta^>$ [see Fig. 2(b), right panel].

The fundamental quantity of interest is the in-in amplitude Z_P for the evolution of the initial quantum state $|\mathrm{in}\rangle$ forward in time into the remote future, starting from a specified initial state that could be either the vacuum or a medium. Within the CTP formalism the amplitude Z_P can be evaluated by time integration over the *closed-time path* P in the complex t plane. As illustrated in Fig. 6, this closed contour extends from $t = t_0$ to $t = t_\infty$ in the remote future along the positive $(+)$ branch and back to $t = t_0$ along the negative $(-)$ branch, where any point on the $+$ branch is understood at an earlier instant than any point on the $-$ branch:

$$Z_P[\mathcal{J}, \hat{\rho}] \equiv Z_P[\mathcal{J}^+, \mathcal{J}^-, \hat{\rho}] = \mathrm{Tr}\{U^\dagger_{\mathcal{J}^-}(t_0, t) U_{\mathcal{J}^+}(t, t_0)\hat{\rho}(t_0)\}, \tag{B1}$$

where the $\hat{\rho}(t_0)$ is the initial state density matrix, and U and $U^\dagger$ are the time evolution operator and its adjoint

$$U_{\mathcal{J}^+}(t,t_0)=T\exp\left\{-i\left[\int_{t_0}^{t}dt'\,d^3x'\,\mathcal{J}^+(x')\cdot\mathcal{A}^+(x')\right]\right\},$$

$$U^\dagger_{\mathcal{J}^+}(t_0,t)=T^\dagger\exp\left\{+i\left[\int_{t_0}^{t}dt'\,d^3x'\,\mathcal{J}^-(x')\cdot\mathcal{A}^-(x')\right]\right\},\tag{B2}$$

with T $(T^\dagger)$ denoting the time (antitime) ordering operator. Note that J_+ (J_-) is the source along the positive (negative) branch of the closed-time path of Fig. 6, and in general $J_+ \neq J_-$, so that Z_P depends on two different sources. If these are set equal, one has $Z_P(J,J,\rho)=\mathrm{Tr}\hat{\rho}$, which is equal to unity in the absence of initial correlations, being a statement of unitarity.

Z_P contains the full information about the development of the initial state via the creation, interaction, and destruction of quanta, through the agency of the sources: the quanta are initially created (e.g., by particle collision), they evolve by further creation and annihilation (real and virtual emission or absorption as well as scattering), and are finally destroyed (e.g., by detection in a calorimeter). Both the act of initial creation and final destruction represent the external sources $\mathcal{J}$ in the sense of a probing apparatus, whereas the intermediate dynamics is governed by the underlying quantum theory. Hence, in order to describe the time evolution of the initially prepared quantum system, to the final detected state, the knowledge of Z_P allows us to extract objectively the self-contained development of the system, when the external influence removed (i.e., the sources are switched off).

The functional Z_P can be represented as a path integral by imposing boundary conditions in terms of complete sets of eigenstates of the gauge fields $\mathcal{A}_\mu$ at initial time $t=t_0$:

$$\mathcal{A}(t_0,\vec{x})|\mathcal{A}^+(t_0)\rangle=\mathcal{A}^+(\vec{x})|\mathcal{A}^+(t_0)\rangle,$$

$$\mathcal{A}(t_0,\vec{x})|\mathcal{A}^-(t_0)\rangle=\mathcal{A}^-(\vec{x})|\mathcal{A}^-(t_0)\rangle,\tag{B3}$$

and in the remote future at $t=t_\infty$:

$$\mathcal{A}'(t_\infty,\vec{x})|\mathcal{A}'(t_\infty)\rangle=\mathcal{A}'(\vec{x})|\mathcal{A}'(t_\infty)\rangle.\tag{B4}$$

Then, making use of the completeness of the eigenstates, one obtains from Eq. (B1) the following functional integral representation for Z_P:

$$Z_P[\mathcal{J}^+,\mathcal{J}^-,\hat{\rho}]$$

$$=\int \mathcal{D}\mathcal{A}^+\mathcal{D}\mathcal{A}^-\mathcal{D}\mathcal{A}'\langle\mathcal{A}^-(t_0)|U^\dagger_{\mathcal{J}^-}(t_0,t_\infty)|$$

$$\times\mathcal{A}(t_\infty)\rangle\langle\mathcal{A}(t_\infty)|U_{\mathcal{J}^+}(t_\infty,t_0)|\mathcal{A}^+(t_0)\rangle$$

$$\times\langle\mathcal{A}^+(t_0)|\hat{\rho}|\mathcal{A}^+(t_0)\rangle.\tag{B5}$$

The first two amplitudes are the transition amplitudes in the presence of $\mathcal{J}^+$ and $\mathcal{J}^-$, whereas the density matrix element incorporates the initial state correlations at t_0 at the endpoints of the closed-time path P. Hence, one obtains the path integral representation for Z_P in analogy to usual field theory [20,21]

$$Z_P[\mathcal{J}^+,\mathcal{J}^-,\hat{\rho}]=\int \mathcal{D}\mathcal{A}^+\mathcal{D}\mathcal{A}^-\exp[i(I[\mathcal{A}^+]+\mathcal{J}^+\circ\mathcal{A}^+)$$

$$-i(I^*[\mathcal{A}^-]+\mathcal{J}^-\circ\mathcal{A}^-)]\mathcal{M}[\hat{\rho}],\tag{B6}$$

where

$$\mathcal{M}(\hat{\rho})=\langle\mathcal{A}^+(t_0)|\hat{\rho}|\mathcal{A}^-(t_0)\rangle.\tag{B7}$$

The generalized classical action $I[\mathcal{A}]$ accounts for all four field orderings on the closed-time path P:

$$I[\mathcal{A}]\equiv I[\mathcal{A}^+]-I^*[\mathcal{A}^-]=I^{(0)}[u_{\alpha\beta}\mathcal{A}^\alpha_\mu\mathcal{A}^\beta_\nu]$$

$$+I^{(1)}[gv_{\alpha\beta\gamma}(\partial_\mu\mathcal{A}^\alpha_\nu)\mathcal{A}^\beta_\mu\mathcal{A}^\gamma_\nu]$$

$$+I^{(2)}[g^2w_{\alpha\beta\gamma\delta}\mathcal{A}^\alpha_\mu\mathcal{A}^\beta_\nu\mathcal{A}^\gamma_\mu\mathcal{A}^\delta_\nu],\tag{B8}$$

where the correspondence with the terms of Eq. (A2) is obvious (the color indices are suppressed here), and where $\alpha,\beta,\gamma,\delta=+,-$:

$$u_{\alpha\beta}=u^{\alpha\beta}=\mathrm{diag}(1,-1),\quad v_{\alpha\beta\gamma}=\delta_{\alpha\beta}u_{\beta\gamma},$$

$$w_{\alpha\beta\gamma\delta}=\mathrm{sign}(\alpha)\,\delta_{\alpha\beta}\delta_{\beta\gamma}\delta_{\gamma\delta},\tag{B9}$$

with the usual summation convention over repeated Greek indices $\alpha,\beta,\ldots$. Equations (B6)–(B8) represent the detailed version of the compact form (C1) used in Sec. II as the starting point, except for the Faddeev-Popov determinant and the gauge fixing constraint, which is omitted here and addressed in Appendix C.

2. The density matrix and the initial state

Turning to the properties of the initial state incorporated in the functional $\mathcal{M}(\hat{\rho})$ with the density matrix $\hat{\rho}(t_0)$, we denote by t_0 the initial point of time from which on the evolution of the multigluon state is followed, and assume that all the dynamics prior to t_0 is contained in the form of the *initial state*:

$$|\mathrm{in}\rangle\equiv|\mathcal{A}(t_0)\rangle=\prod_{\mu,a}|\mathcal{A}^a_\mu(t_0)\rangle.\tag{B10}$$

The initial state at $t=t_0$ can be constructed by expanding the gauge field operator $\mathcal{A}$ in the Heisenberg representation in terms of a Fock basis of noninteracting single-gluon states, the in-basis,

$$\mathcal{A}^a_\mu(t_0,\vec{x})=\int\frac{d^3k}{(2\pi)^3}\,\theta(k^0)(2\pi)\,\delta(k^2)$$

$$\times\sum_s\,[e^{-ik\cdot x}\hat{c}^a_\mu(k,s)+e^{ik\cdot x}\hat{c}^{a\dagger}_\mu(k,s)]\tag{B11}$$

so that a particular Fock state is given by (suppressing color and Lorentz indices)

$$|n^{(1)},n^{(2)},\ldots,n^{(\infty)}\rangle=\prod_i \frac{1}{\sqrt{n^{(i)}!}}(\hat{c}^\dagger(k_i,s_i))^{n^{(i)}}|0\rangle. \tag{B12}$$

Here $\hat{c}^\dagger$ ($\hat{c}$) is the creation (annihilation) operator for a single-gluon state with definite four-momentum k and spin s, satisfying $\hat{c}^\dagger|0\rangle=1$ ($\hat{c}|0\rangle=0$) and the $n^{(i)}$ are the occupation numbers of the different gluon states, $n^{(i)}\equiv\langle n^{(i)}|\hat{c}(k_i,s_i)\hat{c}^\dagger(k_i,s_i)|n^{(i)}\rangle$. Finally, $|0\rangle$ denotes the vacuum state or a ground state different than the vacuum (e.g., a hadron). Thus, a general multigluon state $|\phi\rangle$ at time t_0 is given by a superposition of such states:

$$|\mathcal{A}(t_0)\rangle=\prod_{\mu,a}\sum_{n^{(i)}} C^a_\mu(n^{(1)},n^{(2)},\ldots,n^{(\infty)})\times|n^{(1)},n^{(2)},\ldots,n^{(\infty)}\rangle, \tag{B13}$$

with real-valued coefficients C. Alternatively, the initial state of the system at t_0 can be characterized by the *density matrix*

$$\hat{\rho}(t_0)\equiv|\mathcal{A}(t_0)\rangle\langle\mathcal{A}(t_0)|,\quad (\hat{\rho}_0)_{ij}\equiv\langle n^{(i)}|\hat{\rho}(t_0)|n^{(j)}\rangle. \tag{B14}$$

For instance, the case of empty vacuum corresponds to a diagonal density matrix $\hat{\rho}(t_0)=|0\rangle\langle 0|$ with $(\hat{\rho}_0)_{ij}\propto\delta_{ij}$, whereas a general density matrix that describes any form of a single-particle density distribution at t_0 is

$$\hat{\rho}(t_0)=N\exp\left[\sum_s\int_\Omega d^3x\int\frac{d^3k}{(2\pi)^3 2k_0}\times\theta(k_0)F(t_0,\vec{x},k)\hat{c}^{a\dagger}_\mu(k,s)\hat{c}^a_\mu(k,s)\right], \tag{B15}$$

where Ω denotes the hypersurface of the initial values and F is a c-number function related to the single-particle phase-space density of gluons around $\vec{x}+d\vec{x}$ with four-momentum within $k^\mu+dk^0d\vec{k}$, and N a normalization factor. The form (B15) describes a large class of interesting nonequilibrium systems [19], and contains as a special case the thermal equilibrium distribution, namely, when $t\to -i/T$ and $F(t_0,\vec{x},k)\to k_0\delta(k^2)T^{-1}$, so that $\hat{\rho}(T_0)\to N\exp[-\hat{\mathcal{H}}_{\rm YM}/T]$.

3. Perturbation theory and Feynman rules

The convenient feature of the CTP formalism is that it is formally completely analogous to standard quantum field theory, except for the fact that the fields have contributions from both time branches. In particular, one obtains as in usual field theory, from the path-integral representation (C11) the n-point Green functions $G^{(n)}(x_1,\ldots,x_n)$, which, however, now include all correlations between points on either positive and negative time branches

$$G^{(n)}_{\alpha_1\alpha_2\ldots\alpha_n}(x_1,\ldots,x_n)=\frac{1}{Z_P[0]}\frac{\delta}{i\,\delta\mathcal{K}^{(n)}}Z_P[\mathcal{K}]\Bigg|_{\mathcal{K}=0},$$

$$\alpha_i=\pm, \tag{B16}$$

depending on whether the space-time points x_i lie on the $+$ or $-$ time branch. One can then construct a perturbative expansion of the nonequilibrium Green functions in terms of modified Feynman rules (as compared to standard field theory) [18,19,22].

(i) All local one-point functions $G^{(1)}_\alpha(x)$, such as the gauge-field or the color current, are "vectors" with two components:

$$\mathcal{A}(x)\equiv\begin{pmatrix}\mathcal{A}^+\\ \mathcal{A}^-\end{pmatrix},\quad \mathcal{J}(x)\equiv\begin{pmatrix}\mathcal{J}^+\\ \mathcal{J}^-\end{pmatrix}. \tag{B17}$$

Similarly, all two-point functions $G^{(2)}_{\alpha\beta}(x,y)$, such as the gluon propagator $i\Delta_{\mu\nu}$ and the polarization tensor $\Pi_{\mu\nu}$, are 2×2 matrices with components,

$$\Delta(x_1,x_2)\equiv\begin{pmatrix}\Delta^{++} & \Delta^{+-}\\ \Delta^{-+} & \Delta^{--}\end{pmatrix},\quad \Pi(x_1,x_2)\equiv\begin{pmatrix}\Pi^{++} & \Pi^{+-}\\ \Pi^{-+} & \Pi^{--}\end{pmatrix}. \tag{B18}$$

Explicitly, the components of the propagator are

$$\Delta^F_{\mu\nu}(x,y)\equiv\Delta^{++}_{\mu\nu}(x,y)=-i\langle T\mathcal{A}^+_\mu(x)\mathcal{A}^+_\nu(y)\rangle,$$

$$\Delta_{\mu\nu}{<}(x,y)\equiv\Delta^{+-}_{\mu\nu}(x,y)=-i\langle\mathcal{A}^+_\nu(y)\mathcal{A}^-_\mu(x)\rangle,$$

$$\Delta_{\mu\nu}{>}(x,y)\equiv\Delta^{-+}_{\mu\nu}(x,y)=-i\langle\mathcal{A}^-_\mu(x)\mathcal{A}^+_\nu(y)\rangle,$$

$$\Delta^{\overline{F}}_{\mu\nu}(x,y)\equiv\Delta^{--}_{\mu\nu}(x,y)=-i\langle\overline{T}\mathcal{A}^-_\mu(x)\mathcal{A}^-_\nu(y)\rangle, \tag{B19}$$

where Δ^F is the usual time-ordered Feynman propagator, $\Delta^{\overline{F}}$ is the corresponding anti-time-ordered propagator, and $\Delta^>$ ($\Delta^<$) is the unordered correlation function for $x_0>y_0$ ($x_0<y_0$). In compact notation,

$$\Delta_{\mu\nu}(x,y)=-i\langle T_P\mathcal{A}(x)\mathcal{A}(y)(y)\rangle, \tag{B20}$$

where the generalized time-ordering operator T_P is defined as

$$T_PA(x)B(y):=\theta_P(x_0,y_0)A(x)B(y)+\theta_P(y_0,x_0)B(y)A(x), \tag{B21}$$

with the θ_P function defined as

$$\theta_P(x_0,y_0)=\begin{cases}1 & \text{if } x_0 \textit{ succeeds } y_0 \text{ on the contour } P,\\ 0 & \text{if } x_0 \textit{ precedes } y_0 \text{ on the contour } P.\end{cases} \tag{B22}$$

Higher order products $A(x)B(y)C(z)\cdots$ are ordered analogously. Finally, the generalized δ_P function on the closed-time path P is defined as

$$\delta^4_P(x,y):=\begin{cases} +\delta^4(x-y) & \text{if } x_0 \text{ and } y_0 \text{ from positive branch,} \\ -\delta^4(x-y) & \text{if } x_0 \text{ and } y_0 \text{ from negative branch,} \\ 0 & \text{otherwise.} \end{cases} \tag{B23}$$

(ii) The number of elementary vertices is doubled, because each propagator line of a Feynman diagram can be either of the four components of the Green functions. The interaction vertices in which all the fields are on the $+$ branch are the usual ones, while the vertices in which the fields are on the $-$ branch have the opposite sign. On the other hand, combinatoric factors, rules for loop integrals, etc., remain the same as in usual field theory. Specifically, the three-gluon and four-gluon vertices

$$G^{(3)}_{\alpha\beta\gamma}(x_1,x_2,x_3)\equiv\int_P d^4x G^{(2)}_{\alpha'\alpha}(x_1,x)G^{(2)}_{\beta'\beta}(x_2,x) \times G^{(2)}_{\gamma'\gamma}(x_3,x)\Gamma^{(3)}_{\alpha'\beta'\gamma'}(x),$$

$$G^{(4)}_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) \equiv\int_P d^4x G^{(2)}_{\alpha'\alpha}(x_1,x)G^{(2)}_{\beta'\beta}(x_2,x)G^{(2)}_{\gamma'\gamma}(x_3,x) \times G^{(2)}_{\delta'\delta}(x_4,x)\Gamma^{(4)}_{\alpha'\beta'\gamma'\delta'}(x),$$

with $\Gamma^{(3)}(x)$ and $\Gamma^{(4)}(x)$ denoting the elementary, amputated vertices (with the external legs removed), have, for fixed $\alpha,\beta,\gamma,\delta$, two components. For instance, as in Fig. 7, for the external points on the $+$-branch,

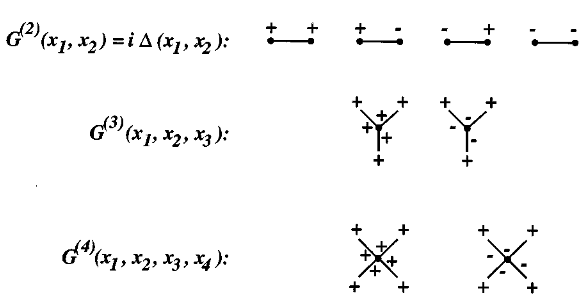

FIG. 7. Example for the appearance of additional contributions to the n-point functions $G^{(n)}$ for the propagator $G^{(2)}$, the three-vertex $G^{(3)}$, and the four-vertex $G^{(4)}$ In usual quantum field theory referring to free space or "vacuum," only the $+$ graphs are nonzero. In the CTP formalism, accounting for the presence of surrounding matter or "medium," new diagrams arise that correspond to statistical correlations between the field living on the $+$ and $-$ time branches of Fig. 6.

for $\alpha=\beta=\gamma=+$: $\quad \Gamma^{(3)}_{\alpha'\beta'\gamma'}(x)=(\Gamma^{(3)}_{+++},-\Gamma^{(3)}_{---})$,

for $\alpha=\beta=\gamma=\delta=+$:

$$\Gamma^{(4)}_{\alpha'\beta'\gamma'\delta'}(x)=(\Gamma^{(4)}_{++++},-\Gamma^{(4)}_{----}).$$

APPENDIX C: THE in-in AMPLITUDE Z_P FOR QCD IN NONCOVARIANT GAUGES

For the case of QCD, a path-integral representation of the in-in amplitude Z_P is obtained along the lines of Appendix B 1, except that one has to extend the generic formula (B6) to account for eliminating the spurious gauge degrees of freedom by the usual Faddeev-Popov procedure [45]. The gauge theory version of Eq. (B6) for the class of noncovariant gauges (2) reads, therefore,

$$Z_P[\mathcal{J},\hat\rho]=\mathcal{N}\int \mathcal{D}\mathcal{A}\ \det\mathcal{F}\delta(f[\mathcal{A}])\exp\{i(I[\mathcal{A},\mathcal{J}])\}\mathcal{M}(\hat\rho), \tag{C1}$$

where $\mathcal{A}=(\mathcal{A}^+,\mathcal{A}^-)$ and $\mathcal{J}=(\mathcal{J}^+,\mathcal{J}^-)$ have two components, living on the $+$ and $-$ time branches of Fig. 2. Physical expectation values are defined as functional averages over T_P-ordered products of n field operators ($n\geqslant1$), weighted by Z_P:

$$\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_P\equiv\frac{1}{Z_P[0,\hat\rho]}\int\mathcal{D}\mathcal{A}\ \det\mathcal{F}\delta(f[\mathcal{A}]) \times\exp\{i(I[\mathcal{A},\mathcal{J}])\} \times\mathcal{M}(\hat\rho)T_P[\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)]. \tag{C2}$$

The structure of the functional Z_P in Eqs. (C1) and (C2) is the following.

(i) The functional integral (with normalization $\mathcal{N}$) is over all gauge field configurations with measure $\mathcal{D}\mathcal{A}\equiv\Pi_{\mu,a}\mathcal{D}\mathcal{A}^a_\mu$, subject to the condition of gauge fixing, here for the *class of noncovariant gauges* defined by

$$f^a[\mathcal{A}]:=n\cdot\mathcal{A}^a(x)-B^a(x)\Rightarrow\langle n^\mu\mathcal{A}^a_\mu(x)\rangle=0,\quad n^\mu\equiv\frac{n^\mu}{\sqrt{|n^2|}}, \tag{C3}$$

where n^μ is a constant four-vector, being either spacelike ($n^2<0$), timelike ($n^2>0$), or lightlike ($n^2=0$). With this choice of gauge class the *local gauge constraint* on the fields $\mathcal{A}^a_\mu(x)$ in the path-integral (C1) becomes

$$\det\mathcal{F}\delta(n\cdot\mathcal{A}^a-B^a)=\text{const}\times\exp\{iI_{\text{GF}}[n\cdot\mathcal{A}]\}, \tag{C4}$$

$$I_{\mathrm{GF}}[n\cdot\mathcal{A}]=-\frac{1}{2\alpha}\int_P d^4x[n\cdot\mathcal{A}^a(x)]^2, \tag{C5}$$

where $\det\mathcal{F}$ is the Faddeev-Popov determinant (which in the case of the noncovariant gauges turns out to be a constant factor, cf., Appendix C), and where $\delta(n\cdot\mathcal{A})\equiv\Pi_a\delta(n\cdot\mathcal{A}^a)$. The right side translates this constraint into a *gauge fixing* functional I_{GF}. The particular choice of the vector n^μ and of the real-valued parameter α is dictated by the physics or computational convenience, and is distinguished further within the class of noncovariant gauges [27,28]:[8]

$$\text{homogeneous axial gauge:}\quad n^2<0,\quad \alpha=0,$$

$$\text{inhomogenous axial gauge:}\quad n^2<0,\quad \alpha=1,$$

$$\text{temporal axial gauge:}\quad n^2>0,\quad \alpha=0,$$

$$\text{light cone gauge:}\quad n^2=0,\quad \alpha=0. \tag{C6}$$

(ii) The exponential I is the *effective classical action* with respect to both the $+$ and the $-$ time contour, $I[\mathcal{A},\mathcal{J}]\equiv I[\mathcal{A}^+,\mathcal{J}^+]-I^*[\mathcal{A}^-,\mathcal{J}^-]$, including the usual Yang-Mills action $I_{\mathrm{YM}}=\int d^4x\mathcal{L}_{\mathrm{YM}}$, plus the source $\mathcal{J}$ coupled to the gauge field $\mathcal{A}$:

$$I[\mathcal{A},\mathcal{J}]=-\frac{1}{4}\int_P d^4x\mathcal{F}^a_{\mu\nu}(x)\mathcal{F}^{\mu\nu,a}(x)+\int_P d^4x\mathcal{J}^a_\mu(x)\mathcal{A}^{\mu,a}(x)\equiv I_{\mathrm{YM}}[\mathcal{A}]+\mathcal{J}\circ\mathcal{A}. \tag{C7}$$

(iii) The form of the initial state at $t=t_0$ as described by the density matrix $\hat\rho$ [an example is given in Appendix B 2, Eq. (B14)] is embodied in the function $\mathcal{M}(\hat\rho)$ which is the density-matrix element of the gauge fields at initial time t_0

$$\mathcal{M}(\hat\rho)=\langle\mathcal{A}^+(t_0)|\hat\rho|\mathcal{A}^-(t_0)\rangle\equiv\exp(i\mathcal{K}[\mathcal{A}]), \tag{C8}$$

where $\mathcal{A}^\pm$ refers to the $+$ and $-$ time branch *at* t_0, respectively (cf., Fig. 2). The functional $\mathcal{K}$ may be expanded in a series of nonlocal kernels corresponding to multipoint correlations concentrated at $t=t_0$:

$$\mathcal{K}[\mathcal{A}]=\mathcal{K}^{(0)}+\int_P d^4x\mathcal{K}^{(1)a}_\mu(x)\mathcal{A}^{\mu,a}(x)+\frac{1}{2}\int_P d^4xd^4y\mathcal{K}^{(2)ab}_{\mu\nu}(x,y)\mathcal{A}^{\mu,a}(x)\mathcal{A}^{\nu,b}(y)\cdots\equiv\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{K}^{(n)}\circ[\mathcal{A}(1)\mathcal{A}(2)\cdots\mathcal{A}(n)]. \tag{C9}$$

Clearly, the sequence of kernels $\mathcal{K}^{(n)}$ contains as much information as the original density matrix. In the special case that $\hat\rho$ is diagonal, the kernels $\mathcal{K}^{(n)}=0$ for all n, and the usual ''vacuum field theory'' is recovered.

The path-integral representation (C1) can be rewritten in a more convenient form. First, the gauge-fixing functional $I_{\mathrm{GF}}[n\cdot\mathcal{A}]$ is implemented, using Eq. (C5). Second, the series representation (C9) is inserted into the initial state functional $\mathcal{M}(\hat\rho)$. Third, $\mathcal{K}^{(0)}$ is absorbed in the overall normalization $\mathcal{N}$ of Z_P (henceforth set to unity), and the external source $\mathcal{J}$ in the one-point kernel $\mathcal{K}^{(1)}$:

$$\mathcal{K}^{(0)}:=i\ln\mathcal{N},\quad \mathcal{K}^{(1)}:=\mathcal{K}^{(1)}+\mathcal{J}. \tag{C10}$$

Then Eq. (C1) becomes

$$Z_P[\mathcal{J},\hat\rho]\Rightarrow Z_P[\mathcal{K}]=\int\mathcal{D}\mathcal{A}\exp\{i(I[\mathcal{A},\mathcal{K}])\}, \tag{C11}$$

where, instead of Eq. (C7),

$$I[\mathcal{A},\mathcal{K}]\equiv I_{\mathrm{YM}}[\mathcal{A}]+I_{\mathrm{GF}}[n\cdot\mathcal{A}]+\mathcal{K}^{(1)}\circ\mathcal{A}+\frac{1}{2}\mathcal{K}^{(2)}\circ(\mathcal{A}\mathcal{A})+\frac{1}{6}\mathcal{K}^{(3)}\circ(\mathcal{A}\mathcal{A}\mathcal{A})+\cdots. \tag{C12}$$

The objects of physical relevance are the n-point *Green functions* $G^{(n)}$, defined as the coefficients in a functional expansion of Z_P

$$Z_P[\mathcal{K}]=Z_P[0]\sum_{n=1}^{\infty}\frac{i^n}{n!}\int\prod_{i=1}^{n}d^4x_iG^{(n)}(x_1,\ldots,x_n)\mathcal{K}^{(n)}\times(x_1,\ldots,x_n), \tag{C13}$$

that is, the $G^{(n)}$ are functional averages in the sense of Eq. (C2):

$$G^{(n)a_1\ldots a_n}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n)\equiv\langle\mathcal{A}^{a_1}_{\mu_1}(x_1)\cdots\mathcal{A}^{a_n}_{\mu_n}(x_n)\rangle_P=\frac{1}{Z_P[0]}\frac{\delta}{i\,\delta\mathcal{K}^{(n)}}Z_P[\mathcal{K}]|_{\mathcal{K}=0}. \tag{C14}$$

The practical evaluation of Z_P amounts therefore to calculating the $G^{(n)}$ in the expansion (C13) up to the order of desired accuracy. For instance, the one-, two-, and three-point Green functions according to Eq. (C14) are

$$G^{(1)a}_\mu(x)=\langle\mathcal{A}^a_\mu(x)\rangle_P, \tag{C15}$$

[8]The analogy with the class of covariant gauges defined by $f^a[\mathcal{A}]:=\partial_x\cdot\mathcal{A}^a-B^a$, instead of Eq. (C3), is evident: in place of Eq. (C5), it results in the familiar gauge-fixing functional $\exp\{-i/2\alpha\int_P d^4x(\partial\cdot\mathcal{A}^a)^2\}$, where $\alpha=1$ gives the *Feynman gauge* and $\alpha=0$ the *Landau gauge*.

$$G^{(2)ab}_{\mu\nu}(x,y)=\langle\mathcal{A}^a_\mu(x)\mathcal{A}^b_\nu(y)\rangle_P=\frac{\delta}{i\,\delta\mathcal{K}^{(1)a}_\mu(x)}\langle\mathcal{A}^b_\nu(y)\rangle_P+\langle\mathcal{A}^a_\mu(x)\rangle_P\langle\mathcal{A}^b_\nu(y)\rangle_P,$$

$$G^{(3)abc}_{\mu\nu\lambda}(x,y,z)=\langle\mathcal{A}^a_\mu(x)\mathcal{A}^b_\nu(y)\mathcal{A}^c_\lambda(z)\rangle_P=\frac{1}{2}\left(\frac{\delta}{i\,\delta\mathcal{K}^{(2)ab}_{\mu\nu}(x,y)}+\frac{\delta}{i\,\delta\mathcal{K}^{(1)a}_\mu(x)}\,\frac{\delta}{i\,\delta\mathcal{K}^{(1)b}_\nu(y)}\right)\langle\mathcal{A}^c_\lambda(z)\rangle_P+\frac{\delta}{i\,\delta\mathcal{K}^{(1)a}_\mu(x)}[\langle\mathcal{A}^b_\nu(y)\rangle_P\langle\mathcal{A}^c_\lambda(z)\rangle_P]+\langle\mathcal{A}^a_\mu(x)\rangle_P\langle\mathcal{A}^b_\nu(y)\rangle_P\langle\mathcal{A}^c_\lambda(z)\rangle_P.$$

Higher order Green functions are generated in a way similar to Eq. (C14).

APPENDIX D: NONCOVARIANT GAUGES AND THE ABSENCE OF GHOSTS

In this appendix the standard procedure of gauge field quantization is applied to the class of noncovariant gauges (C6), and it is shown that ghost degrees of freedom are indeed absent, reducing the general nonlinear dynamics of QCD essentially to a linear QED-type dynamics. For an excellent review and bibliography, see Ref. [27]. Recall that under local gauge transformations

$$g[\theta^a]\equiv\exp[-i\,\theta^a(x)T^a],\tag{D1}$$

the gauge fields transform as

$$\mathcal{A}^a_\mu\rightarrow\mathcal{A}^{(\theta)a}_\mu=g[\theta^a]\mathcal{A}^a_\mu g^{-1}[\theta^a],\tag{D2}$$

implying that $\mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu}=\mathcal{F}^{(\theta)a}_{\mu\nu}\mathcal{F}^{(\theta)a}_{\mu\nu}$, that is the gauge invariance of the Yang-Mills action $I_{\rm YM}[\mathcal{A}]$. However, the source term $\mathcal{J}\circ\mathcal{A}$ in the generating functional Z_P of Eq. (C1) is not gauge invariant under the transformations (D1). Consequently, the *naive* functional

$$Z^{(\rm naive)}_P=\int\mathcal{D}\mathcal{A}\exp\{i(I_{\rm YM}[\mathcal{A}]+\mathcal{J}\circ\mathcal{A})\}\times\mathcal{M}(\hat{\rho})\tag{D3}$$

is also not a gauge invariant quantity. As is well known, this can be remedied by applying the formal Faddeev-Popov [45] procedure and integrating the path-integral Z_P over all possible gauge transformations $g(\theta^a)$ subject to the linear subsidiary condition

$$\phi^a[\mathcal{A}^{(\theta)}_\mu]\equiv n^\mu\mathcal{A}^{(\theta)a}_\mu(x)-\beta^a(x)\stackrel{!}{=}0,\tag{D4}$$

with normalized spacelike vector n^μ and $\beta^a(x)$ an arbitrary weight function. The Faddeev-Popov trick to implement the constraint (D4) in the noninvariant functional $Z^{(\rm naive)}_P$ by multiplying with

$$1=\int\mathcal{D}\theta\prod_a\,\delta(\phi^a[\mathcal{A}^{(\theta)}_\mu])\,\det\mathcal{F},\tag{D5}$$

where the determinant is Jacobian for the change of variables $\phi^a\rightarrow\theta^a$,

$$(\det\mathcal{F})^{ab}=\det\left(\frac{\delta\phi^a[\mathcal{A}^{(\theta)}_\mu]}{\delta\theta^b}\right)_{\phi^a[\mathcal{A}^{(\theta)}_\mu]=0}=\left\{\int\mathcal{D}\theta\prod_a\,\delta(\phi^a[\mathcal{A}^{(\theta)}_\mu])\right\}^{-1}.\tag{D6}$$

Following this procedure one arrives at

$$Z^{(\rm naive)}_P\rightarrow Z_P=\int\mathcal{D}\mathcal{A}\,\det\mathcal{F}\prod_a\,\delta(\phi^a[\mathcal{A}_\mu])\times\exp\{i(I_{\rm YM}[\mathcal{A}]+\mathcal{J}\circ\mathcal{A})\}\times\mathcal{M}(\hat{\rho}),\tag{D7}$$

which is now a gauge invariant expression due to the proper account of the subsidiary condition (D4) that guarantees the correct transformation properties of the gauge fields in the presence of the sources $\mathcal{J}$.

To obtain the final form of Z_P as quoted in Eq. (C1), one integrates functionally over the arbitrary functions $\beta^a(x)$ introduced in Eq. (D4), by choosing, e.g., a Gaussian weight functional

$$w[\beta^a]=\exp\left\{-\frac{i}{2\alpha}\int_P d^4x[\beta^a(x)]^2\right\},\tag{D8}$$

with the real valued parameter $0\leqslant\alpha\leqslant1$, upon which the Faddeev-Popov determinant $\det\mathcal{F}$ can be rewritten in a more suitable way:

$$\det\mathcal{F}=\int\mathcal{D}\beta\prod_a\,\exp\left\{-\frac{i}{2\alpha}\int_P d^4x[\beta^a(x)]^2\right\}\times\delta[n^\mu\mathcal{A}^{(\theta)a}_\mu(x)-\beta^a(x)].\tag{D9}$$

In order to calculate the determinant, it is sufficient to integrate over θ^a in a small vicinity where the argument of the δ function passes through zero at given $\mathcal{A}^{(\theta)a}$ and β^a. For *infinitesimal gauge transformations*

$$g[\theta^a]\rightarrow\delta g[\theta^a]=1-i\,\theta^a(x)T^a,\tag{D10}$$

the gauge fields transform as

$$\mathcal{A}^a_\mu\rightarrow\mathcal{A}^a_\mu+\delta\mathcal{A}^a_\mu,\qquad\delta\mathcal{A}^a_\mu=f^{abc}\theta^b\mathcal{A}^c_\mu-\frac{1}{g}\partial^x_\mu\theta^a,\tag{D11}$$

so that one obtains

$$\delta[n^\mu\mathcal{A}^{(\theta)a}_\mu(x)-\beta^a(x)]=\delta\left(n^\mu\mathcal{A}^{(\theta)a}_\mu(x)+f^{abc}\theta^b n^\mu\mathcal{A}^{(\theta)c}_\mu-\frac{1}{g}n^\mu\partial^x_\mu\theta^a-\beta^a\right)=\delta\left(f^{abc}\theta^b\beta^c-\frac{1}{g}n^\mu\partial^x_\mu\theta^a\right),\tag{D12}$$

because $n^\mu \mathcal{A}_\mu^{(\theta)a} = \beta^a$. This latter expression is evidently independent of $\mathcal{A}_\mu^a$. Therefore, when substituted into Eq. (D9), det$\mathcal{F}$ is also explicitly independent of the gauge fields, and hence can be pulled out of the path integral Z_P and absorbed in an (irrelevant) normalization, which may be set equal to unity. The final result is then

$$Z_P[\mathcal{J},\hat{\rho}] = \int \mathcal{D}\mathcal{A} \exp\{i(I_{\rm YM}[\mathcal{A}] + I_{\rm GF}[n\cdot\mathcal{A}] + \mathcal{J}\circ\mathcal{A})\} \times \mathcal{M}(\hat{\rho}), \tag{D13}$$

where, from Eq. (D9),

$$I_{\rm GF}[n\cdot\mathcal{A}] \equiv \exp\left\{-\frac{i}{2\alpha}\int_P d^4x [n\cdot\mathcal{A}^a(x)]^2\right\}. \tag{D14}$$

In conclusion, the property of gauge field independence of the Faddeev-Popov determinant proves that there are indeed no ghost fields coupling to the gluon fields, hence the formulation is *ghost-free*.

APPENDIX E: THE TRUNCATED EFFECTIVE ACTION $\Gamma_P[\bar{A},\hat{\Delta}]$

The generating functional for the *connected* Green functions, denoted by $\mathcal{G}^{(n)}$, is defined as usual:

$$W_P[\mathcal{K}] = -i \ln Z_P[\mathcal{K}]. \tag{E1}$$

From W_P one obtains the *connected* Green functions $\mathcal{G}^{(n)}$ by functional differentiation analogous to Eq. (C14) in terms of mixed products of a_μ and A_μ fields:

$$(-i)\mathcal{G}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) \equiv \frac{\delta}{i\,\delta\mathcal{K}^{(n)}} W_P[\mathcal{K}]\Big|_{\mathcal{K}=0} = \langle a^{a_1}_{\mu_1}(x_1)\cdots a^{a_k}_{\mu_k}(x_k) A^{a_{k+1}}_{\mu_{k+1}}(x_{k+1})\cdots A^{a_n}_{\mu_n}(x_n)\rangle_P^{(c)}, \tag{E2}$$

where the superscript (c) indicates the ''connected parts.'' It follows then that

$$\frac{\delta W_P}{\delta\mathcal{K}^{(1)\mu,a}(x)} = \mathcal{G}^{(1)a}_\mu(x),$$

$$\frac{\delta W_P}{\delta\mathcal{K}^{(2)\mu\nu,ab}(x,y)} = \frac{1}{2}[\mathcal{G}^{(2)ab}_{\mu\nu}(x,y) + \mathcal{G}^{(1)a}_\mu(x)\mathcal{G}^{(1)b}_\nu(y)],$$

$$\frac{\delta W_P}{\delta\mathcal{K}^{(3)\mu\nu\lambda,abc}(x,y,z)} = \frac{1}{6}[\mathcal{G}^{(3)abc}_{\mu\nu\lambda}(x,y,z) + 3\mathcal{G}^{(2)ab}_{\mu\nu}(x,y)\ \mathcal{G}^{(1)c}_\lambda(z) + \mathcal{G}^{(1)a}_\mu(x)\mathcal{G}^{(1)b}_\nu(y)\mathcal{G}^{(1)c}_\lambda(z)], \tag{E3}$$

where, for example,

$$\mathcal{G}^{(1)a}_\mu(x) = \langle A^a_\mu(x)\rangle_P^{(c)} + \langle a^a_\mu(x)\rangle_P^{(c)},$$

$$\mathcal{G}^{(2)ab}_{\mu\nu}(x,y) = \langle A^a_\mu(x)A^b_\nu(y)\rangle_P^{(c)} + \langle a^a_\mu(x)a^b_\nu(y)\rangle_P^{(c)},$$

$$\mathcal{G}^{(3)abc}_{\mu\nu\lambda}(x,y,z) = \langle A^a_\mu(x)A^b_\nu(y)A^c_\lambda(z)\rangle_P^{(c)} + \langle a^a_\mu(x)a^b_\nu(y)a^c_\lambda(z)\rangle_P^{(c)}, \tag{E4}$$

and similarly expresses higher order Green functions which involve 4,5, . . . , space-time points.

W_P of Eq. (E1) involves the sources $\mathcal{K}$ that do not have any immediate physical interpretation; it is more convenient to work with the corresponding effective action Γ_P, the generating functional for the proper vertex functions, which determines the equations of motion for the physically relevant Green functions. The *effective action* Γ_P is defined as the multiple Legendre transform, and is obtained by eliminating the source variables $\mathcal{K}$ in favor of the connected Green functions $\mathcal{G}$:

$$\Gamma_P[\mathcal{G}] = W_P[\mathcal{K}] - \mathcal{K}^{(1)}\circ\mathcal{G}^{(1)} - \frac{1}{2}\mathcal{K}^{(2)}\circ(\mathcal{G}^{(2)} + \mathcal{G}^{(1)}\mathcal{G}^{(1)}) - \frac{1}{6}\mathcal{K}^{(3)}\circ(\mathcal{G}^{(3)} + 3\mathcal{G}^{(2)}\mathcal{G}^{(1)} + \mathcal{G}^{(1)}\mathcal{G}^{(1)}\mathcal{G}^{(1)}) - \cdots. \tag{E5}$$

So far no approximations have been made. The variation of Γ_P with respect to the Green functions $\mathcal{G}^{(n)}$ would yield an infinite set of coupled equations, the analogue of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [14]

$$\frac{\delta\Gamma_P}{\mathcal{G}^{(1)}} = -\mathcal{K}^{(1)} - \mathcal{K}^{(2)}\circ\mathcal{G}^{(1)} - \frac{1}{2}\mathcal{K}^{(3)}\circ(\mathcal{G}^{(2)} + \mathcal{G}^{(1)}\mathcal{G}^{(1)}) - \cdots,$$

$$\frac{\delta\Gamma_P}{\mathcal{G}^{(2)}} = -\frac{1}{2}\mathcal{K}^{(2)} - \frac{1}{2}\mathcal{K}^{(3)}\circ\mathcal{G}^{(1)} - \cdots,$$

$$\frac{\delta\Gamma_P}{\mathcal{G}^{(3)}} = -\frac{1}{6}\mathcal{K}^{(3)} - \cdots, \quad \text{etc.} \tag{E6}$$

At this point approximations 1 and 2 of Sec. II A are invoked. It is assumed that the initial state is a (dilute) ensemble of hard gluons of very small spatial extent $\ll\lambda$, corresponding to transverse momenta $k_\perp^2 \gg \mu^2$. By definition of λ, or μ, the short-range character of these quantum fluctuations implies that the expectation value $\langle a_\mu\rangle$ vanishes at all times. However, the long-range correlations of the eventually populated soft modes with very small momenta $k_\perp^2 \ll \mu^2$ may lead to a collective mean field with nonvanishing $\langle A\rangle$. Accordingly, the following condition is imposed on the expectation values of the fields:

$$\langle A^a_\mu(x)\rangle \begin{cases} = 0 & \text{for } t \le t_0, \\ \ge 0 & \text{for } t > t_0, \end{cases} \quad \langle a^a_\mu(x)\rangle \stackrel{!}{=} 0 \quad \text{for all } t. \tag{E7}$$

Furthermore, the quantum fluctuations of the soft field are ignored, assuming any multipoint correlations of soft fields to be small,

$$\langle A^{a_1}_{\mu_1}(x_1)\cdots A^{a_n}_{\mu_n}(x_n)\rangle \ll \langle A^{a_1}_{\mu_1}(x_1)\rangle\cdots\langle A^{a_n}_{\mu_1}(x_n)\rangle \quad \text{for all } n\geqslant 2; \tag{E8}$$

i.e., A_μ is treated as a nonpropagating and nonfluctuating, classical field. Hence, the set of Green functions (E4) reduces to

$$\mathcal{G}^{(1)a}_{\mu}(x)=\langle A^a_\mu(x)\rangle^{(c)}_P \equiv \overline{A^a_\mu}(x),$$

$$\mathcal{G}^{(2)ab}_{\mu\nu}(x,y)=\langle a^a_\mu(x)a^b_\nu(y)\rangle^{(c)}_P \equiv i\hat{\Delta}^{ab}_{\mu\nu}(x,y). \tag{E9}$$

These relations define the soft, classical mean field $\overline{A}$, and the hard quantum propagators $\hat{\Delta}$.

Now the hierarchy is truncated for $n\geqslant 3$. However, to perform this truncation properly, one must eliminate all the $\mathcal{G}^{(3)}_\mu$, $\mathcal{G}^{(4)}_\mu$, etc., as dynamical variables by introducing [20]

$$\Gamma_P[\mathcal{G}^{(1)},\mathcal{G}^{(2)}]\equiv\Gamma_P[\mathcal{G}^{(1)},\mathcal{G}^{(2)},\tilde{\mathcal{G}}^{(3)},\tilde{\mathcal{G}}^{(4)},\cdots], \tag{E10}$$

where $\tilde{\mathcal{G}}^{(n)}$ for all $n\geqslant 3$ are functionals of the one- and two-point functions alone, and are determined by the implicit equations

$$\tilde{\mathcal{G}}^{(n)}:=\tilde{\mathcal{G}}^{(n)}[\mathcal{G}^{(1)},\mathcal{G}^{(2)}],$$

$$\frac{\delta\Gamma_P[\mathcal{G}^{(1)},\mathcal{G}^{(2)}]}{\delta\tilde{\mathcal{G}}^{(n)}[\mathcal{G}^{(1)},\mathcal{G}^{(2)}]}=0 \qquad \text{for all } n\geqslant 3. \tag{E11}$$

From Eqs. (E3) and (E4) one sees that then the infinite set of Green functions reduces to involve only $\mathcal{G}^{(1)}_\mu=\overline{A_\mu}$ and $\mathcal{G}^{(2)}_{\mu\nu}=i\hat{\Delta}_{\mu\nu}$, so that Γ_P becomes a functional of only the soft mean field $\overline{A_\mu}$ and the hard propagator $\hat{\Delta}_{\mu\nu}$:

$$\Gamma_P[\mathcal{G}]\approx\Gamma_P[\overline{A},\hat{\Delta}]=W_P[\mathcal{K}^{(1)},\mathcal{K}^{(2)}]-\mathcal{K}^{(1)}\circ\overline{A} - \frac{1}{2}\mathcal{K}^{(2)}\circ(i\hat{\Delta}+\overline{AA}). \tag{E12}$$

The *equations of motion* for the mean field $\overline{A}$ and for the hard propagator $\hat{\Delta}$ in the presence of external sources, follow now from Eqs. (E4), (E6), and (E12):

$$\frac{\delta\Gamma_P}{\delta\overline{A^a_\mu}(x)}=-\mathcal{K}^{(1)\mu,a}(x)-\int_P d^4y K^{(2)\mu\nu,ab}(x,y)\overline{A^{\nu,b}}(y), \tag{E13}$$

$$\frac{\delta\Gamma_P}{\delta\hat{\Delta}^{ab}_{\mu\nu}(x,y)}=\frac{1}{2i}\mathcal{K}^{(2)\mu\nu,ab}(x,y). \tag{E14}$$

APPENDIX F: ANALYTIC PROPERTIES OF THE FREE-FIELD PROPAGATORS

The components of the free-field propagator $\Delta^{ab}_{0\mu\nu}$ are defined as in Eq. (B19), i.e.,

$$\Delta^{Fab}_{0\mu\nu}(x,y)=-i\langle Ta^a_\mu(x)a^b_\nu(y)\rangle,$$

$$\Delta^{\overline{F}ab}_{0\mu\nu}(x,y)=-i\langle \overline{T}a^a_\mu(x)a^b_\nu(y)\rangle,$$

$$\Delta^{>ab}_{0\mu\nu}(x,y)=-i\langle a^a_\mu(x)a^b_\nu(y)\rangle,$$

$$\Delta^{<ab}_{0\mu\nu}(x,y)=-i\langle a^b_\nu(y)a^a_\mu(x)\rangle. \tag{F1}$$

For free fields, one may write

$$\Delta^{ab}_{0\mu\nu}(x,y)=\delta^{ab}d_{\mu\nu}(\partial_x)\Delta_0(x,y) \quad (\Delta\equiv\Delta^F,\Delta^{\overline{F}},\Delta^>,\Delta^<), \tag{F2}$$

where $d_{\mu\nu}(\partial_x)$ is defined by Eq. (25), and the functions Δ_0 on the right side are the *scalar* parts of the propagators. The $F,\overline{F},>,<$ components of the latter obey the following free-field equations with different boundary conditions:

$$\vec{\partial}^2_x\Delta^F_0(x,y)=\Delta^F_0(x,y)\overleftarrow{\partial}^2_y=+\delta^4(x,y),$$

$$\vec{\partial}^2_x\Delta^{\overline{F}}_0(x,y)=\Delta^{\overline{F}}_0(x,y)\overleftarrow{\partial}^2_y=-\delta^4(x,y),$$

$$\vec{\partial}^2_x\Delta^>_0(x,y)=\Delta^>_0(x,y)\overleftarrow{\partial}^2_y=0,$$

$$\vec{\partial}^2_x\Delta^<_0(x,y)=\Delta^<_0(x,y)\overleftarrow{\partial}^2_y=0, \tag{F3}$$

and the identities

$$\Delta^F_0(x,y)=\theta(x_0,y_0)\Delta^>_0(x,y)+\theta(y_0,x_0)\Delta^<_0(x,y),$$

$$\Delta^{\overline{F}}_0(x,y)=\theta(x_0,y_0)\Delta^<_0(x,y)+\theta(y_0,x_0)\Delta^>_0(x,y). \tag{F4}$$

Because of the relations (F4), the set of equations (F3) can be solved by only two independent functions, namely, (i) a purely imaginary and odd function $i\Delta^-$ and (ii) a purely real and even function Δ^+:

$$i\Delta^-(x,y)\equiv\delta^{ab}d_{\mu\nu}(\partial_x)\langle[a_\mu(x),a^b_\nu(y)]\rangle,$$
$$=i(\Delta^>_0-\Delta^<_0)(x,y),$$

$$\Delta^+(x,y)\equiv\delta^{ab}d_{\mu\nu}(\partial_x)\langle\{a_\mu(x),a^b_\nu(y)\}\rangle=i(\Delta^>_0+\Delta^<_0)(x,y). \tag{F5}$$

From Eq. (F3) it follows that these functions obey

$$\vec{\partial}^2_x\Delta^-(x,y)=\Delta^-(x,y)\overleftarrow{\partial}^2_y=0, \quad \Delta^-(x,y)=-\Delta^-(y,x),$$

$$\vec{\partial}^2_x\Delta^+(x,y)=\Delta^+(x,y)\overleftarrow{\partial}^2_y=0, \quad \Delta^+(x,y)=+\Delta^+(y,x), \tag{F6}$$

with the general solutions

$$\Delta^{-}(x,y)=-i\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}2\pi\delta(k^2)\times[g_1(k)-g_2(-k)],$$

$$\Delta^{+}(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}2\pi\delta(k^2)[g_1(k)+g_2(-k)], \tag{F7}$$

where the functions

$$g_1(k)\equiv\theta(k_0)+f_1(k),\quad g_2(-k)\equiv\theta(-k_0)+f_2(-k), \tag{F8}$$

contain the positive and negative frequency modes, respectively. Here $\theta(\pm k_0)$ is the vacuum contribution, while $f_{1,2}(\pm k)$ are the additional contributions from a medium.

From Eqs. (F5)–(F7), one can now infer immediately the analytic properties of f_1, f_2, corresponding to those of g_1, g_2.

(1) One observes, because Δ^- is purely imaginary and Δ^+ is purely real, that

$$f_1(k),f_2(k)=\text{real}$$

must hold.

(2) Because the commutator of free fields, i.e., the imaginary function Δ^-, must be independent of the state of the medium

$$\Delta^{-}(x,y)\overset{!}{=}-i\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}2\pi\delta(k^2)\times[\theta(k_0)-\theta(-k_0)],$$

it follows that

$$f_1(k)=f_2(-k)\equiv f(k).$$

(3) Because the anticommutator, i.e., the real function Δ^+ must satisfy

$$\int d^4x d^4y h^*(x)\Delta^{+}(x,y)h(y)\overset{!}{\geqslant}0$$

for any smooth, but in general complex-valued function h, it follows that

$$f(\vec{k})\equiv\int dk^0 f(k^0,\vec{k})\geqslant 0\qquad\text{for all }\vec{k},$$

and so the ''on-shell'' function $f(\vec{k})$ is positive definite may indeed be identified with the positive definite phase-space density dN/d^3k.

The free-field solutions of $\Delta^F,\Delta^{\overline{F}},\Delta^>,\Delta^<$ can now be easily reconstructed using the following identities implied by Eqs. (F1) and (F4):

$$2\Delta^F(x,y)=-i\Delta^+(x,y)+[2\theta(x^0,y^0)-1]\Delta^-(x,y),$$

$$2\Delta^{\overline{F}}(x,y)=-i\Delta^+(x,y)+[2\theta(y^0,x^0)-1]\Delta^-(x,y), \tag{F9}$$

$$2\Delta^>(x,y)=-i\Delta^+(x,y)+\Delta^-(x,y),$$

$$2\Delta^<(x,y)=-i\Delta^+(x,y)-\Delta^-(x,y), \tag{F10}$$

from which, upon Fourier transformation, one obtains

$$\Delta_0^F(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left(\frac{+1}{k^2+i\epsilon}-i2\pi\delta(k^2)f(k)\right),$$

$$\Delta_0^{\overline{F}}(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left(\frac{-1}{k^2-i\epsilon}-i2\pi\delta(k^2)f(k)\right),$$

$$\Delta_0^>(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\{-i2\pi\delta(k^2)[\theta(k_0)+f(k)]\},$$

$$\Delta_0^<(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\{-i2\pi\delta(k^2)[\theta(-k_0)+f(k)]\}. \tag{F11}$$

Finally, inferring the corresponding free-field forms of the retarded, advanced, and correlation functions is straightforward:

$$\Delta_0^{\text{ret}}(x,y)=+\theta(x_0,y_0)\Delta^-(x,y)=(\Delta_0^F-\Delta_0^<)(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left(\frac{1}{k^2+2i\epsilon}\right),$$

$$\Delta_0^{\text{adv}}(x,y)=-\theta(y_0,x_0)\Delta^-(x,y)=(\Delta_0^F-\Delta_0^>)(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left(\frac{1}{k^2-2i\epsilon}\right),$$

$$\Delta_0^{\text{cor}}(x,y)=-i\Delta^+(x,y)=(\Delta_0^>+\Delta_0^<)(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\{-i2\pi\delta(k^2)[1+2f(k)]\}. \tag{F12}$$

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[32] It should be remarked that the Wigner function can be transformed from the here employed class of noncovariant gauges to any other gauge by utilizing the general form of Eq. (63) with

$$\mathcal{G}^{ab}_{\mu\nu}(r,s) \equiv U^{aa'}\left(r, r+\frac{s}{2}\right)\mathcal{G}^{a'b'}_{\mu\nu}\left(r+\frac{s}{2}, r-\frac{s}{2}\right)U^{bb'}\left(r-\frac{s}{2}, r\right),$$

and $U(x,y)$ is the ''parallel transporter'' or ''link operator'' [7,46] that guarantees that the Wigner function transforms covariantly under local gauge transformations, when moving on an oriented straight line $z(\xi) := x + \xi(y-x)$, $0 \leqslant \xi \leqslant 1$ from space-time point x to point y:

$$U^{ab}(x,y) = \delta^{ab} P \exp\left\{ ig \int_0^\infty dz^\mu \mathcal{A}_\mu(z) \right\}$$
$$= \delta^{ab}\left(1 + P \sum_{n=1}^{\infty} \prod_{i=1}^{n} ig\ z_i \cdot \mathcal{A}(z_i)\right).$$

The symbol P denotes the path ordering of the color matrices in the exponential and $\mathcal{A} = T^a \mathcal{A}^a_\mu$, where $\mathcal{A}$ stands for A or a. Choosing the gauge vector n^μ such that it is parallel to the direction of propagation $n \| z$ the gauge constraint $n \cdot \mathcal{A} = 0$ implies $z \cdot \mathcal{A} = 0$, and as a consequence one has then $U^{ab}(x,y) \overset{n\cdot\mathcal{A}=0}{=} \delta^{ab}$, so that

$$\hat{\mathcal{G}}^{ab}_{\mu\nu}(r,k) \overset{n\cdot\mathcal{A}=0}{=} \int d^4 s e^{ik\cdot s} \mathcal{G}^{ab}_{\mu\nu}\left(r+\frac{s}{2}, r-\frac{s}{2}\right).$$

This property is another very advantageous feature of the noncovariant gauges: the link operator becomes a unit matrix in color space, and so the complicated nonlinear dependence (in the gauge fields) of the Wigner functions reduces to a form identical to those in classical field theory. This is of course not possible in a general gauge, in particular, in covariant gauges one has to account for the infinite number of nonlinear terms in the series representation of the path-ordered exponential.

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