Ferromagnetic domain wall and a primordial magnetic field

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We argue that a coherent magnetic field is generated spontaneously when a large domain wall is created in the early Universe. It is caused by two-dimensional massless fermions bounded to the domain wall soliton. We point out that the magnetic field is a candidate for a primordial magnetic field. [S0556-2821(97)01716-5]

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Domain walls [1] arise in any unification model with a discrete symmetry in Higgs potentials in which the symmetry is often used to reduce complication in the potentials. Unfortunately, the domain walls are unfavorable from a cosmological point of view [2]; the energy density of the domain wall dominates in the Universe and invalidates the scenario of the history of the early Universe. To cure this problem several ideas [2–5] have been proposed, but there is not yet a definite solution. Here we do not address this problem.

In this paper we analyze magnetic properties of the domain wall (hereafter, we simply call it the wall). We show spontaneous generation of a magnetic field B in the presence of the wall. The size of the coherence of the magnetic field is comparable to that of the wall. We assume that the size L of a domain wall is the same as the distance to the horizon, as indicated by numerical simulation [6] of the creation of the walls in the early Universe. Thus the field is a candidate of the primordial magnetic field [7] in the Universe. The result of the spontaneous generation of the field depends on the presence of fermion zero modes [8] on the wall. The zero modes exist in general when a fermion couples with the domain wall soliton. It implies the existence of massless fermions on the two-dimensional wall, although the fermions are massive when they are unbounded to the wall. As we will show later, the free energy of such a fermion gas under the magnetic field B is proportional to B. The sign of the coefficient of this term is negative. Thus the amount of the decrease of the energy is proportional to B. On the other hand, the field energy is proportional to B^2. Therefore, spontaneous generation of a magnetic field arises since the total energy of the system decreases with the sufficiently weak magnetic field. As the size of the wall is the same as the distance to the horizon, the coherent length of such a magnetic field is so large that it may be a candidate for a primordial magnetic field leading to an intergalactic or galactic magnetic field [9]. The strength of the field depends on the period when the wall is created and when it disappears. For instance, if it is created in the electroweak phase transition at about a temperature of 1 TeV, the strength of the field is about 10^9 G. It evolves to the field with its strength 10^{-15} G at recombination of photons and electrons.

We now present detailed calculations. Hereafter, we consider a large flat domain wall with its size L assumed to be the same as the distance to the horizon. First, we discuss fermionic zero modes [8]. The modes exist in general when the fermion ψ couples with a Higgs field ϕ; strictly speaking, the existence of the modes depends on how the field ϕ couples with the fermion. Here we assume for simplicity the real Higgs field ϕ whose potential has a discrete symmetry, $\phi\to-\phi$ and the following Yukawa coupling, $g\bar{\psi}\psi\phi+\text{H.c.}$ The classical field ϕ behaves near the wall located at $x_3=0$ such as $\phi(x_3)\to\pm v$ as $x_3\to\pm\infty$; $\pm v$ are vacuum expectation values of the Higgs field ϕ. The field has no dependence of coordinates x_1 and x_2. Then, it is easy to find the zero modes by solving a Dirac equation with energy E equal to zero, $(E\gamma_0+i\vec{\gamma}\vec{\partial}+i\gamma_3\partial_3+g\phi)\psi=0$ where $\vec{\gamma}(\vec{\partial})$ is a two-dimensional vector tangent to the wall. Adopting the representation of γ metrics,

$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad\text{and}\quad\gamma^i=\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix},\tag{1}$$

we find the solutions with zero energy,

$$\psi=\begin{pmatrix}u\\i\sigma_3u\end{pmatrix},\quad u=\begin{pmatrix}a\\b\end{pmatrix}\exp[-z(x_3)],\tag{2}$$

where $z(x_3)$ is defined as $\partial z(x_3)/\partial x_3=\phi(x_3)$and a and b are constants. Thus there are two zero energy modes. These may be viewed as zero energy bound states of massive fermion ψ with its mass $m=\sqrt{g}v$. It turns out that when the fermions move on the wall (in this case a and b are functions of x_1 and x_2), their energies are proportional to the absolute value of their momenta. Thus they are massless fermions in two-dimensional space of the wall. Note that such states bounded to the wall loose two dynamical degrees of freedom; components u_1 and u_2 of $\psi=\binom{u_1}{u_2}$ are not independent of each other. Namely, spin degrees of freedom are lost so that the two-dimensional massless fermion (antifermion) has only one dynamical degree of freedom [10].

In order to show spontaneous generation of magnetic field we need to calculate both a thermodynamical potential and a vacuum energy of these fermions under a magnetic field perpendicular to the wall. For this purpose we need to find energy spectrum under a constant magnetic field B; $\vec{B}=(0,0,B)$. Choosing a gauge such as $\vec{A}=(-x_2,x_1,0)B/2$, we solve the above Dirac equation with gauge potential $\vec{A}$. Then, relevant solutions take the form of Eq. (2) with both of a and b being functions of x_1 and x_2; these are solutions representing the cyclotron motions of the fermions bounded to the wall. Their spectra can be obtained easily, $E_n=\pm\sqrt{2eBn}(n\geqslant 0)$ with degeneracy per unit area being

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given by $eB/2\pi$; e is the charge of the fermion. Eigenstates are characterized with integer n and orbital angular momentum m $(n,m\geq 0)$:

$$\psi=\begin{pmatrix} u \\ i\sigma_3 u \end{pmatrix}, \quad u=c\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\exp[-z(x_3)], \tag{3}$$

with

$$v_1=\rho^m e^{im\theta}L_m{}^{(n)}e^{-eB\rho^2/4} \quad \text{and}$$
$$v_2=\rho^m e^{i(m+1)\theta}\partial_\rho L_m{}^{(n)}e^{-eB\rho^2/4}/E_n, \tag{4}$$

where $L_m{}^{(n)}(eB\rho^2/2)$ is the Laguerre function and c is a normalization constant $(\rho^2=x_1^2+x_2^2)$. These solutions represent states localized around the domain wall. In addition to the solutions, there are solutions representing scattering states and cyclotron motions in the x_1-x_2 plane.

It is assumed that the states representing fermions bounded to the wall need to satisfy a energy condition such as $E_n<m$. Otherwise, the states with $E_n\geq m$ energetically favor jumping out of the wall and hence would not be relevant to the property of the wall. Indeed, these states are expected to become unstable once we include even small couplings with the scattering states. They also become unstable when finite curvature or irregularity of the wall is present. It is shown in Ref. [11] that fermion zero modes on a string soliton become unstable due to an effect of the string curvature when they have energies higher than a certain critical energy. Similar effect on the fermion zero modes is expected in the case of the wall. Thus in a realistic circumstance they decay and corresponding fermions go away from the wall, unless the energy condition is satisfied. Therefore, it is conceived that they are irrelevant for the property of the wall. Hereafter, we assume this energy condition.

Now, let us calculate the thermodynamical potential Ω under the condition that the temperature β^{-1} is much smaller than the mass of the fermion. Then it is given for free gas of the fermions such that

$$\Omega=\Omega_+ + \Omega_- + \Omega_0,$$

$$\Omega_{\pm,}=-\beta^{-1}N_d\sum_{n=1}^{\infty}\ \ln(1+e^{-E_n\beta}),$$

$$\Omega_0=-\beta^{-1}N_d\ \ln 2, \tag{5}$$

where $N_d=eBL^2/2\pi$ is degeneracy of a Landau level (L^2 is surface area of the wall). Summation $\Sigma_{E_n\leq m}$ over the states satisfying the energy condition has been replaced by the summation $\Sigma_{n=1}^{\infty}$ over all the states since the condition $\beta^{-1}\ll m$ is assumed. The indices $(\pm,0)$ denote quantities of fermions, antifermions, and zero energy states $(E_{n=0}=0)$, respectively. For simplicity, we have assumed a vanishing chemical potential associated with the fermion number, namely, the fermion number in the Universe is assumed to be negligibly small just as baryon number in the Universe.

As we are interested in the behavior of the system at a small magnetic field, the sum over integer n can be evaluated by using the Poisson resummation formula,

$$\sum_{n=1}^{\infty} f(n\hat\beta^2)=\sum_{m=-\infty}^{\infty}\int_0^{\infty}dx f(x\hat\beta^2)e^{2\pi imx}$$
$$=\int_0^{\infty}dx f(x\hat\beta^2)$$
$$+2\sum_{m=1}^{\infty}\int_0^{\infty}dx f(x\hat\beta^2)\cos(2\pi mx), \tag{6}$$

with $\hat\beta^2\equiv 2eB\beta^2$, where $f(x)=\ln(1+e^{-\sqrt{x}})$. Expanding the second term in small $\hat\beta$, we obtain the thermodynamical potential for $\hat\beta\rightarrow 0$ such that

$$\Omega=-\frac{1}{2\pi}\ \beta^{-3}L^2\int_0^{\infty}\frac{y^2dy}{1+e^y}+\frac{\sqrt{2eB}eB\zeta(-1/2)L^2}{2\pi}$$
$$+O(\beta(eB)^2), \tag{7}$$

where the first term represents a contribution of free massless fermions without the magnetic field and the second one represents a correction by the weak magnetic field $eB\ll\beta^{-2}$. This first term can be obtained by using the spectrum of the energy, $E_k=|k|$ (k is two-dimensional momentum) and by taking account of the fact that both the fermion and antifermion possess only one dynamical degree of freedom.

Similarly, we calculate a vacuum energy of the two-dimensional massless fermion on the wall. We also assume the energy condition; zero point oscillations of fermions with energies higher than m are associated not with the property of the wall, but with the property of the outside of the wall. Therefore, the vacuum energy of the wall is given by

$$E_{\text{vac}}=-\sum_{n=1}^{F}\ \sqrt{2eBn}N_d$$
$$\approx-\frac{m^3L^2}{6\pi}-\frac{eBmL^2}{4\pi}-\frac{\sqrt{2eB}eB\zeta(-1/2)L^2}{2\pi}, \tag{8}$$

where $F=m^2/2eB$ is the number of Landau levels, energies of whose states are less than the mass m of the fermion; $\zeta(z)$ is the ζ function. We have taken a limit of small magnetic field $(eB\ll m^2)$ in the calculation. The first term represents the vacuum energy of the fermion without magnetic field. The term can be expressed as $-L^2\int_{|k|<m}|k|dk^2/(2\pi)^2$. This quantity should be renormalized to the surface energy of the wall. The third term has already been obtained [12] as a vacuum energy of two-dimensional massless fermion without the energy condition mentioned above.

A comment is in order. In the summation of the zero point energy we have assumed the existence of the physical cutoff F; the states localized on the wall with energies higher than m do not exist in physical circumstances. Such states exist only when they do not interact with any other modes, e.g., the scattering states, oscillation modes of the wall, etc. The states, however, are quite unstable against any small couplings existing in realistic circumstances so that these localized states would decay and disappear. Hence, it is reasonable to conceive that the only states which really exist when we include all the couplings, are scattering states whose energies are higher than m and bound states whose energies are less than m. Thus, the physical cutoff F may exist. This assumption is essential for deriving a result of ferromagnetism of the wall, although we have not yet succeeded to demonstrate it. We elaborate on this point in a future publication.

We can see that the magnetic field reduces both the thermal energy and the vacuum energy of the fermion; the amounts of the reduction are proportional to $B\sqrt{B}L^2$ and BL^2, respectively. Namely, the whole free energy E_B associated with the magnetic field turns out to be

$$F_B = -\frac{eBmL^2}{4\pi} + \frac{B^2L^3}{2\mu}, \tag{9}$$

where the second term represents the energy of the magnetic field inside the horizon with its volume L^3. μ is the permeability of the Universe. [The thermal effect of the order of $B^{3/2}$ in Eq. (7) cancels with the corresponding effect of the vacuum energy [13]. But this cancellation does not necessarily hold in any case: For instance, when the chemical potential of the fermion in the Universe is nonvanishing, the cancellation does not hold. The thermal effect dominates over the effect of the vacuum energy.] Hence, the magnetic field is obtained by minimizing this energy,

$$B_r = \frac{em\mu}{4\pi L}. \tag{10}$$

This is the magnetic field associated with the domain wall soliton in the Universe. The strength of the field becomes small with the expansion of the Universe. In the radiation-dominated universe the distance L to the horizon behaves with temperature β^{-1} such as $M_{\rm PL}\beta^2/\sqrt{f}$ ($M_{\rm PL}$ is the Planck mass and f is the total number of massless degrees of freedom at temperature β^{-1}) so that B_r decreases in such a way as $B_r \sim \beta^{-2}\sqrt{f}/M_{\rm PL}$. Numerically, when the domain wall arises at the electroweak phase transition, e.g., $\beta^{-1} \sim 100$ GeV ($f \sim 100$), the magnitude of the magnetic field is of order of 10^6 G with the use of the top quark mass $m = 175$ GeV.

We have so far assumed in the evaluation of the thermodynamical potential that the temperature is much smaller than the mass of the fermion, i.e., $\beta^{-1} \ll m$. The fermions bounded to the wall never escape even if they are excited thermally: So we have performed infinite sum in the previous calculation. On the contrary, when the temperature is much larger than the mass of the fermion, they can escape from the wall. So in the evaluation of the thermodynamical potential we should take account of the energy condition; only the fermions with their energies less than the mass m contribute to the potential,

$$\Omega_{\pm,} = -\beta^{-1}N_d\sum_{n=1}^{F}\ln(1+e^{-E_n\beta}) \cong -\beta^{-1}N_dF\ln 2,$$

$$\Omega_0 = -\beta^{-1}N_d\ln 2, \tag{11}$$

where we have taken the temperature β^{-1} being much larger than the mass m of the fermion. Therefore, the whole free energy depending on the magnetic field is

$$F_B = -\frac{eBL^2\ln 2}{2\pi\beta} + \frac{B^2L^3}{2\mu}, \tag{12}$$

where the vacuum energy has been neglected because β^{-1} is much larger than m so that the energy is much smaller than the thermal energy in Eq. (12). The thermal energy dominates over the vacuum energy. We see that imposition of the magnetic field reduces the free energy of the fermion gas. Then the field generated spontaneously is given by

$$B_r = \frac{e\mu\ln 2}{2\pi\beta L}. \tag{13}$$

Numerically, $B_r \cong 10^6$ G at $\beta^{-1} = 100$ GeV, or $B_r \cong 1$ G at $\beta^{-1} = 1$ GeV.

Comparing this result with the previous one in Eq. (10), we understand that generation of the magnetic field B_r originates in the vacuum energy of the fermion when the temperature is less than the mass of the fermion, while it originates in the thermal energy of the fermion when the temperature is higher than the mass. In the latter case the magnetic field increases the number N_B of the states with their energies less than the mass m:

$$N_B - N_{B=0} = \frac{eBL^2}{2\pi}\sum_{n=0}^{F-1} - \frac{L^2}{(2\pi)^2}\int_{|k|\leq m} dk^2$$
$$= \frac{eBL^2(F - m^2/2eB)}{2\pi} > 0, \tag{14}$$

where F is such that $\sqrt{2eBF} > m > \sqrt{2eB(F-1)}$. Thus it leads to increasing entropy S of the gas and hence decreasing free energy ($= E - S\beta^{-1} \approx -S\beta^{-1}$). This is the reason for the spontaneous generation of the magnetic field when the temperature is higher than the mass of the fermion.

As we have shown, the spontaneous generation of magnetic field arises in the presence of the domain wall soliton. The generation of the magnetic field is a natural consequence of fermion dynamics on the domain wall soliton. Hereafter, we wish to discuss briefly phenomenological application of the result.

Let us consider a realistic model where the wall is created at the electroweak phase transition and the fermions are quarks or leptons: Such models exist, e.g., a next-to-minimal supersymmetric standard model [14]. The transition temperature may be about 1 TeV and it is much larger than any masses of quarks and leptons. Thus the above formula in Eq. (13) is applicable for this case. Then, the magnetic field is roughly 10^9 G at the temperature of 1 TeV with its coherent length $L = 10^{-2}$ cm. This magnetic field is naively expected to evolve to the field with its strength 10^{-15} G and coherent length 10^{10} cm around the period of the recombination. Note that the evolution of the field is such that $Ba^2 = $ const because its flux is conserved owing to very large electric conductivity of the Universe [9]; a is the cosmic scale factor in Robertson-Walker metric. As the Universe expands, the size of the wall becomes large and the magnetic field becomes weak. These walls are expected to decay [2–5] eventually before they dominate the energy of the Universe. They, however, leave magnetic fields with various strengths and coherent lengths. To determine the strength and the coherent length of the field more precisely, we have to take into account dissipation of magnetic field and the number of the fermion species of the zero modes.

In summary, we have shown assuming the energy condition that the magnetic field is generated spontaneously when the domain wall is present in the early Universe. This magnetic field is a candidate for a primodial magnetic field leading to intergalactic or galactic magnetic field in the present Universe. We have only discussed the flat domain wall in this paper, but similar results can be also obtained in the case of spherical domain walls. We will present these results in future publications.

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