

# Generalized symmetries and invariant matter couplings in two-dimensional dilaton gravity

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New features of the generalized symmetries of generic two-dimensional dilaton models of gravity are presented and invariant gravity-matter couplings are introduced. We show that there is a continuum set of Noether symmetries, which contains half a DeWitt algebra. Two of these symmetries are area-preserving transformations. We show that gravity-matter couplings which are invariant under area preserving transformations only contribute to the dynamics of the dilaton-gravity sector with a reshaping of the dilaton potential. The interaction with matter by means of invariant metrics is also considered. We show in a constructive way that there are metrics which are invariant under two of the symmetries. The most general metrics and minimal couplings that satisfy this condition are found. [S0556-2821(97)01416-1]

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## I. INTRODUCTION

Currently, one of the main objectives of theoretical physics is to devise a quantum theory of gravity. The four-dimensional Einstein-Hilbert gravity theory and, in general, four- and higher-dimensional models of gravity is unfortunately very complex to handle. Toy models which share their most relevant features with Einstein's gravity should, therefore, play an important role here. Two-dimensional (2D) dilaton models of gravity are general covariant models which in addition to the two-dimensional metric also involve a scalar (dilaton) field (for a review see Ref. [1]). When coupled to matter, these models have solutions describing the formation of two-dimensional black holes and Hawking radiation [2,3]. Moreover, and unlike their four-dimensional counterparts, these models are renormalizable. Thus they may be a useful tool to explore the final fate of black holes and solve the information puzzle.

Unfortunately, in spite of their being much simpler than their higher-dimensional cousins, only few of them, notably the Callan-Giddings-Harvey-Strominger (CGHS) model, have been shown to be solvable when interacting with matter. This represents a serious drawback, as it is difficult to distinguish in the dynamics of these models which is due to a particular feature of the CGHS model and which corresponds to general properties of gravity. These theories would, then, be far more useful if the developments which have been made with the CGHS model could be extended to more general ones, especially spherically symmetric gravity. If this were the case, we could be more confident that the experience gained from these two-dimensional toy models would actually be useful in the four-dimensional case. Moreover, the quantum nature of the CGHS model remains elusive (see, for instance, Ref. [4]). Other solvable models might not face the difficulties which have been found when trying to quantize this model.

As is well known, solvability is usually related to invariances, this being the reason that classical solvability usually implies quantum solvability. New generalized symmetries (we use the adjective "generalized" because they involve derivatives of the fields) have been recently uncovered for generic dilaton gravity which generalize those of the CGHS model [5,6]. Therefore, it is natural to study whether or not these symmetries can be used to find invariant gravity-matter interactions, thereby providing solvable models.

In the present paper, we shall consider two different approaches to introduce symmetric gravity-matter couplings. In the first approach (Sec. IV), we consider couplings which are invariant under area-preserving transformations of the metric. In the second one (Sec. V), we consider couplings which are constructed by means of an invariant metric. For the first approach, we show that, with regard to the gravity sector, interaction with general area-preserving couplings simply amounts to a reshaping of the dilaton potential. For the second approach, we show that metrics and conformal couplings can be constructed which are invariant under two of the symmetries. The most general invariant metrics and conformal couplings are constructed. In the last section, we briefly discuss several natural continuations of the present developments.

First, the CGHS model and the generalized symmetries of the generic 2D dilaton models are briefly reviewed and some features are presented.

## II. TWO-DIMENSIONAL DILATON GRAVITIES AND THE CGHS MODEL

The generic models of two-dimensional dilaton gravity are defined by means of the action

$$S_{\text{GDG}}(\tilde{g}, \tilde{\phi}) = \frac{1}{2\pi} \int d^2x \sqrt{-\tilde{g}} [D(\tilde{\phi})\tilde{R} + H(\tilde{\phi})(\nabla\tilde{\phi})^2 + F(\tilde{\phi})] - S_M, \quad (1)$$

where  $D$ ,  $H$ , and  $F$  are arbitrary functions, and  $S_M$  is the gravity-matter interaction term.

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A result which is particularly useful is that after suitable redefinitions of the two-dimensional metric  $\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu}$  and the dilaton field  $\tilde{\phi} \rightarrow \phi$ , any action can be brought to the form [7,8]

$$\tilde{S}_{\text{GDG}} = S_V - S_M, \quad (2)$$

where

$$S_V = \frac{1}{2\pi} \int d^2x \sqrt{-g} [R\phi + V(\phi)]. \quad (3)$$

The CGHS or string-inspired model of two-dimensional dilaton gravity [9,2], with the action

$$S_{\text{CGHS}} = \frac{1}{2\pi} \int d^2x \sqrt{-g} [(R\phi + 4\lambda^2) - \frac{1}{2}(\nabla f)^2] \quad (4)$$

has attracted particular attention because it is exactly solvable at the classical as well as the semiclassical levels [10,11]. Solvability of this model is due to the existence of a sufficient number of free-field and Liouville equations, and this, in turn, is related to the existence of symmetries. Let us for the moment restrict our attention to the model without cosmological constant, in which case the symmetries are particularly simple. It is easy to see that, in addition to the familiar transformation of the matter sector,

$$\delta_f \phi = 0, \quad \delta_f g_{\mu\nu} = 0, \quad \delta_f f = \epsilon, \quad (5)$$

the following two transformations are also symmetries of the string-inspired model (4) with  $\lambda = 0$ :

$$\delta_R \phi = \epsilon, \quad \delta_R g_{\mu\nu} = 0, \quad \delta_R f = 0, \quad (6)$$

$$\delta_\phi \phi = 0, \quad \delta_\phi g_{\mu\nu} = \epsilon g_{\mu\nu}, \quad \delta_\phi f = 0. \quad (7)$$

These symmetries correspond to the free-field equations

$$\square f = 0, \quad R = 0, \quad \square \phi = 0. \quad (8)$$

In turn, these free-field equations imply classical solvability for the model. This can be easily seen by choosing the conformal gauge for the metric:

$$ds^2 = 2e^\rho dx^+ dx^- \quad (9)$$

in terms of which we have  $R = -2e^{-\rho} \partial_+ \partial_- \rho$  and  $\square = 2e^{-\rho} \partial_+ \partial_-$ .

### III. NEW SYMMETRIES IN GENERIC 2D DILATON GRAVITY

Now, let us go back to the general Lagrangian in Eq. (2). It can be shown that generic dilaton gravity without matter is highly symmetric, also [5,6]. This provides a symmetry-based explanation for the solvability of these models.

For the Lagrangian in Eq. (2) we have

$$\begin{aligned} \delta \mathcal{L} = & \frac{\sqrt{-g}}{2\pi} \{ [R + V'(\phi) - T_\phi] \delta \phi + [g_{\mu\nu} \square \phi - \frac{1}{2} g_{\mu\nu} V(\phi) - \nabla_\mu \nabla_\nu \phi - T_{\mu\nu}] \delta g^{\mu\nu} + (E - L)_A \delta f^A \\ & + \nabla_\alpha [ -\phi (g^{\mu\nu} \nabla^\alpha \delta g_{\mu\nu} - g^{\alpha\mu} \nabla^\nu \delta g_{\mu\nu}) - \nabla^\alpha \phi g_{\mu\nu} \delta g^{\mu\nu} + \nabla_\nu \phi \delta g^{\nu\alpha} + j_M^\alpha ] \}. \end{aligned} \quad (10)$$

Here,  $f^A$  are the matter fields,  $(E - L)_A = 0$  are the Euler-Lagrange equations of motions for these fields, and  $j_M^\mu$  is the matter contribution to the symplectic potential current of the model.

The equations of motions are, therefore,

$$\begin{aligned} R + V'(\phi) &= T_\phi, \\ g_{\mu\nu} \square \phi - \frac{1}{2} g_{\mu\nu} V(\phi) - \nabla_\mu \nabla_\nu \phi &= T_{\mu\nu}, \\ (E - L)_A &= 0. \end{aligned} \quad (11)$$

In absence of matter, it is easy to show that the general action for the 2D dilaton models (2) is invariant under the following symmetries (note the change in notation with respect to Refs. [5, 6]):

$$\delta_a \phi = 0, \quad \delta_a g_{\mu\nu} = g_{\mu\nu} a_\sigma \nabla^\sigma \phi - \frac{1}{2} (a_\mu \nabla_\nu \phi + a_\nu \nabla_\mu \phi),$$

$$\delta_1 \phi = 0, \quad \delta_1 g_{\mu\nu} = \epsilon_1 \left( \frac{g_{\mu\nu}}{(\nabla \phi)^2} - 2 \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^4} \right),$$

$$\delta_2 \phi = \epsilon_2, \quad \delta_2 g_{\mu\nu} = \epsilon_2 V \left( \frac{g_{\mu\nu}}{(\nabla \phi)^2} - 2 \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^4} \right), \quad (12)$$

where  $a_\mu$  is any arbitrary constant bivector.

The Noether currents are, respectively,

$$J^{\mu\nu} = g^{\mu\nu} E, \quad j_1^\mu = \frac{\nabla^\mu \phi}{(\nabla \phi)^2}, \quad j_2^\mu = j_R^\mu + V \frac{\nabla^\mu \phi}{(\nabla \phi)^2}, \quad (13)$$

where  $E = \frac{1}{2} [(\nabla \phi)^2 - J(\phi)]$  with  $J(\phi)$  a primitive of  $V(\phi)$ :  $J'(\phi) = V(\phi)$ .

Now,  $j_1^\mu$  and  $j_2^\mu - j_R^\mu$  satisfy the integrability condition

$$\epsilon^{\mu\nu} \nabla_\mu (j_1)_\nu = 0 = \epsilon^{\mu\nu} \nabla_\mu (j_2 - j_R)_\nu. \quad (14)$$

Thus, the conservation law for the currents  $j_1^\mu, j_2^\mu$  turns out to imply the existence of two free fields. The free-field equations are, respectively,

$$\begin{aligned} \square j_1 &= 0, \\ R + \square j_2 &= 0, \end{aligned} \quad (15)$$

where

$$j_1 = \int^\phi \frac{d\tau}{2E+J(\tau)} \quad (16)$$

and

$$j_2 = \ln(2E+J). \quad (17)$$

Now, Eqs. (15) lead directly to the general solution to the equations of motion (11). To show this, let us choose light-cone coordinates and fix the residual conformal gauge as follows:

$$j_1 = \frac{1}{2}(x^+ - x^-). \quad (18)$$

Then, the classical general solution of the theory can be (implicitly) given as

$$\int^\phi \frac{d\tau}{2E+J(\tau)} = \frac{1}{2}(x^+ - x^-),$$

$$e^\rho = -\frac{1}{2}[2E+J(\phi)]. \quad (19)$$

Moreover, because of the peculiar conservation law for  $E$ ,

$$\partial_\mu E = 0, \quad \mu = 1, 2, \quad (20)$$

if  $j^\mu$  is a conserved current, so is  $f(E)j^\mu$ , for arbitrary function  $f$ . In the particular case of  $j_1^\mu$ , these conserved currents turn out to be the Noether current of local (generalized) symmetries:

$$\delta_f \phi = 0,$$

$$\delta_f g_{\mu\nu} = -\epsilon f'(E) \left( g_{\mu\nu} - \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^2} \right)$$

$$+ \epsilon f(E) \left( \frac{g_{\mu\nu}}{(\nabla \phi)^2} - 2 \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^4} \right).$$

In particular,  $\delta_E$  is given by:

$$\delta_E \phi = 0, \quad \delta_E g_{\mu\nu} = -\frac{\epsilon_3}{2} \left[ g_{\mu\nu} + J \left( \frac{g_{\mu\nu}}{(\nabla \phi)^2} - 2 \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^4} \right) \right]. \quad (21)$$

These symmetries close the algebra

$$[\delta_f, \delta_g] = \frac{1}{2} \delta_{(f'g - g'f)}. \quad (22)$$

If restricted to analytic functions, they define half a DeWitt algebra

$$[\delta_n, \delta_m] = \frac{1}{2} \delta_{(n-m)}. \quad (23)$$

In particular,  $\delta_1$ ,  $\delta_E$ , and  $\delta_{E^2}$ , close a  $\mathfrak{sl}(2, R)$  algebra.

In the limiting case  $V=0$ , the symmetry  $\delta_2$  is the symmetry  $\delta_R$  of the string-inspired model with no cosmological constant. Moreover,  $-2\delta_E$  coincides with  $\delta_\phi$ . It is apparent, therefore, that we have generalized the symmetries of the CGHS model to an arbitrary 2D dilaton gravity model.

We should emphasize that the transformations just described are symmetries for generic dilaton models. For particular potentials, these symmetries present special features. Notably, it turns out that, although for a generic potential none of these symmetries is conformal, for  $V=4\lambda^2$  (the string-inspired model) or  $V=4\lambda^2 e^{\beta\phi}$  (the so-called exponential model [5]), a linear combination of these symmetries is conformal. These conformal symmetries are  $\delta_2 - 4\lambda^2 \delta_1$  and  $\delta_2 + 2\beta \delta_E$ , respectively. The coupling to conformal matter, therefore, preserves this symmetry. However, these two models are the only ones for which a combination of the above symmetries is conformal [5]. Moreover, except for  $V=0$ , the interaction with conformal matter destroys the invariance under the other symmetries.

#### IV. AREA-PRESERVING COUPLINGS

Conformal invariance provides little room to move in: it does not serve for generic potentials but only for very particular ones. Therefore, to find solvable models for arbitrary potentials, we should go beyond conformal symmetry and consider interactions which are invariant under (some of) the generalized symmetries,  $\delta_f$  and  $\delta_2$ , which have been described above.

An important feature shared by  $\delta_1$  and  $\delta_2$  (but none of the other symmetries) is that they are area-preserving transformations, that is,  $g^{\mu\nu} \delta_{1,2} g_{\mu\nu} = 0$ . Therefore, if  $S_M$  is invariant under area-preserving transformations, the whole action  $S_{\text{GDG}}$  will be invariant under  $\delta_1$  and  $\delta_2$ .

Invariance under area-preserving transformations (APTs) requires that the traceless part of the energy-momentum tensor vanishes. Thus,

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} T^\alpha{}_\alpha \equiv \frac{1}{2} g_{\mu\nu} T. \quad (24)$$

Hence, when the coupling is to area-preserving matter, the equations of motion (11) take the form

$$R + V'(\phi) - T_\phi = 0,$$

$$g_{\mu\nu} \square \phi - \frac{1}{2} g_{\mu\nu} V(\phi) - \nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} T = 0,$$

$$(E-L)_A = 0. \quad (25)$$

We can consider  $(1/2\pi) \int d^2x \sqrt{-g} V(\phi)$  to be part of the gravity-matter interaction term  $S_M$ . Therefore, without loss of generality and after a bit of algebra, we can write the equations of motion as

$$R = T_\phi,$$

$$\square \phi = T,$$

$$\nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \square \phi = 0,$$

$$(E-L)_A = 0. \quad (26)$$

The next-to-last equation implies that the vector

$$k^\mu = \frac{\epsilon^{\mu\nu}}{\sqrt{-g}} \nabla_\nu \phi \quad (27)$$

satisfies the Killing equation  $\nabla_{(\mu} k_{\nu)} = 0$  on shell.

Now, invariance under diffeomorphisms of  $S_M$  implies that on solutions of the equations of motion for the matter fields we have

$$0 = T_\phi \nabla_\mu \phi + 2 \nabla^\nu T_{\mu\nu} = T_\phi \nabla_\mu \phi + \nabla_\mu T. \quad (28)$$

Hence, we also have

$$k^\mu \nabla_\mu T = 0. \quad (29)$$

Therefore, on solutions of the equations of motion for the matter fields,  $T$  can be written as a function of the dilaton field:

$$T = T(\phi). \quad (30)$$

Moreover, Eq. (28) implies that  $T_\phi \equiv T_\phi(\phi)$  and

$$T_\phi(\phi) = -T'(\phi). \quad (31)$$

Therefore, we have shown that, with respect to the dilaton-gravity sector, the interaction with area-preserving matter simply amounts to a modification of the dilaton potential  $V(\phi) \rightarrow \tilde{V}(\phi) = V(\phi) - T$ .

#### A. Particular cases

For a theory to be invariant under area-preserving symmetries, it is a necessary and sufficient condition that its action depends on  $g_{\mu\nu}$  exclusively through the measure  $\sqrt{-g}$  [12].

Invariance under all area-preserving transformations imposes a severe restriction on the construction of interaction terms. However, in two dimensions, there is a large class of interactions, notably Yang-Mills and generalized gauge theories, which satisfies this requirement [12]. In fact, in two dimensions any interaction term where the metric only raises antisymmetrized indices is invariant under area-preserving transformations. For instance, for a Yang-Mills theory in two dimensions, we have  $F_{\mu\nu} = \epsilon_{\mu\nu} \tilde{F}$ . Therefore, the action has the form

$$S_{\text{YM}} = \int d^2x \sqrt{-g} \text{Tr} F^{\mu\nu} F_{\mu\nu} = -2 \int d^2x \frac{\text{Tr} \tilde{F}^2}{\sqrt{-g}}. \quad (32)$$

Minimally coupling these gauge fields to matter fields destroys these symmetries. Nevertheless, this does not rule out that more general couplings might be found which are APT invariant, or at least invariant under some of these symmetries (see below, Sec. V).

First of all, let us consider an interaction term which is invariant under area-preserving transformations of the metric and which does not depend on the dilaton field  $\phi$ . Then, we have

$$\nabla_\nu T = 0 \Rightarrow T = \text{const}. \quad (33)$$

In this case, therefore, in as far as the gravity sector is concerned, the interaction with matter simply amounts to a constant shift of the potential

$$V \rightarrow \tilde{V} = V - Q, \quad (34)$$

where  $Q = T^\alpha_\alpha = \text{const}$ .

Consider now a coupling

$$S_M = \frac{1}{2\pi} \int d^2x \sqrt{-g} \mathcal{L}_M, \quad (35)$$

with

$$\mathcal{L}_M = W(\phi) (\sqrt{-g})^s \tilde{\mathcal{L}}, \quad (36)$$

where  $\tilde{\mathcal{L}}$  depends only on the matter fields. Most of the couplings with gauge fields which have been considered in the literature are of this form. For instance, in Ref. [13], a coupling of this form with an Abelian gauge field is considered and in Ref. [14] a similar coupling with a Yang-Mills field is studied.

In this case, invariance under diffeomorphisms implies

$$W(\phi)^{s/(1+s)} (\sqrt{-g})^s \tilde{\mathcal{L}} = Q = \text{const}. \quad (37)$$

Therefore, the shift in the potential is nonconstant now but given by

$$V \rightarrow \tilde{V} = V - (1+s) Q W(\phi)^{1/(1+s)}. \quad (38)$$

As a matter of fact, as precise a result can also be obtained with any interaction term in which the measure  $\sqrt{-g}$  and the dilaton field  $\phi$  appear only through a product  $U(\phi) \sqrt{-g}$ . Namely, let  $S_M$  be a gravity-matter interaction term such that the dependence of its scalar density  $\sqrt{-g} \mathcal{L}_M$  with respect to the measure  $\sqrt{-g}$  and the dilaton field  $\phi$  is of the form

$$\sqrt{-g} \mathcal{L}_M \equiv (\sqrt{-g} \mathcal{L}_M)(y, A), \quad (39)$$

with  $y = \sqrt{-g} U(\phi)$ . Then, on matter fields which obey the equations of motions, we have

$$\frac{\delta(\sqrt{-g} \mathcal{L}_M)}{\delta y} = Q = \text{const}. \quad (40)$$

Therefore, the dynamics of the dilaton-gravity sector is the same as that without matter fields but with a different potential  $\tilde{V}$ , with

$$\tilde{V}(\phi) = V(\phi) - Q U(\phi). \quad (41)$$

#### V. COUPLING BY MEANS OF AN INVARIANT METRIC

Another way of producing symmetric matter-gravity interactions is by considering couplings which involve only an invariant metric  $\bar{g}_{\mu\nu}$ . That is, let  $\bar{g}_{\mu\nu}$  be a metric which is invariant under a transformation  $\delta$  which, in turn, is a symmetry of  $S_V$ . Then,  $\delta$  is also a symmetry of the action

$$S = S_V + \int d^2x \sqrt{-g} \mathcal{L}(\bar{g}_{\mu\nu}, f^A). \quad (42)$$

However, to demand strict invariance of the metric is in fact too restrictive a requirement. A more relaxed but sufficient condition is to require  $\sqrt{-g} \mathcal{L}(\bar{g}, f^A)$  to be invariant. Consider, for instance, a minimal coupling

$$\sqrt{-g} \mathcal{L} = \sqrt{-g} \bar{g}^{\mu\nu} \bar{\nabla}_\mu f \bar{\nabla}_\nu f, \quad (43)$$

where  $f$  is a scalar field. This interaction term is invariant under  $\delta$  if  $\bar{g}_{\mu\nu}$  is conformally invariant:

$$\delta\bar{g}_{\mu\nu} = K\bar{g}_{\mu\nu}, \quad (44)$$

with arbitrary scalar quantity  $K$ .

We shall restrict ourselves to metrics of the form

$$\bar{g}_{\mu\nu} = A g_{\mu\nu} + B \nabla_\mu \phi \nabla_\nu \phi, \quad (45)$$

with  $A = A((\nabla\phi)^2, \phi)$  and  $B = B((\nabla\phi)^2, \phi)$ . We have

$$\det g = A^2 \det g \left( 1 + \frac{B}{A} (\nabla\phi)^2 \right),$$

$$\bar{g}^{\mu\nu} = \frac{1}{A} \left( g^{\mu\nu} - \frac{\nabla^\mu \phi \nabla^\nu \phi}{\frac{A}{B} + (\nabla\phi)^2} \right),$$

$$\sqrt{-g} \bar{g}^{\mu\nu} = \sqrt{-g} \left( 1 + \frac{B}{A} (\nabla\phi)^2 \right)^{1/2} \left( g^{\mu\nu} - \frac{\nabla^\mu \phi \nabla^\nu \phi}{\frac{A}{B} + (\nabla\phi)^2} \right). \quad (46)$$

Because of the transformation properties

$$\delta_f E = \frac{\epsilon}{2} f(E), \quad \delta_f \phi = 0, \quad (47)$$

$$\delta_2 E = 0, \quad \delta_2 \phi = \epsilon, \quad (48)$$

it is best to consider  $A = A(E, \phi)$  and  $B = B(E, \phi)$ .

Invariance under  $\delta_f$  and  $\delta_2$  requires, respectively,

$$\frac{1}{2} f(E) \frac{\delta A}{\delta E} + \frac{f(E)A}{2E+J} + A f'(E) = K_f A,$$

$$\frac{1}{2} f(E) \frac{\delta B}{\delta E} - 2 \frac{f(E)A}{(2E+J)^2} - \frac{f'(E)A}{2E+J} = K_f B,$$

$$\frac{\delta A}{\delta \phi} + \frac{AV}{2E+J} = K_2 A,$$

$$\frac{\delta B}{\delta \phi} - 2 \frac{AV}{(2E+J)^2} = K_2 B.$$

With these premises, it is not difficult to show that the most general conformally invariant or strictly invariant metrics  $\bar{g}_{\mu\nu}$  (and invariant minimal couplings  $\sqrt{-g} \bar{g}^{\mu\nu}$ ) are of the following form ( $A_f, A_2, A, \lambda_f$ , and  $\lambda_2$  are functions of their arguments and  $\lambda$  is a constant).

Conformally invariant under  $\delta_f$ :

$$\bar{g}_{\mu\nu} = A_f(E, \phi) \left( g_{\mu\nu} + \frac{\lambda_f^2(\phi)(\nabla\phi)^4 f^2(E) - 1}{(\nabla\phi)^2} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (49)$$

Strictly invariant under  $\delta_f$ :

$$\begin{aligned} \bar{g}_{\mu\nu} &= \frac{A_f(\phi)}{(\nabla\phi)^2 f^2(E)} \\ &\times \left( g_{\mu\nu} + \frac{\lambda_f^2(\phi)(\nabla\phi)^4 f^2(E) - 1}{(\nabla\phi)^2} \nabla_\mu \phi \nabla_\nu \phi \right). \end{aligned} \quad (50)$$

Conformally invariant under  $\delta_2$ :

$$\bar{g}_{\mu\nu} = A_2(E, \phi) \left( g_{\mu\nu} + \frac{\lambda_2^2(E)(\nabla\phi)^4 - 1}{(\nabla\phi)^2} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (51)$$

Strictly invariant under  $\delta_2$ :

$$\bar{g}_{\mu\nu} = \frac{A_2(E)}{(\nabla\phi)^2} \left( g_{\mu\nu} + \frac{\lambda_2^2(E)(\nabla\phi)^4 - 1}{(\nabla\phi)^2} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (52)$$

Conformally invariant under  $\delta_f$  and  $\delta_2$ :

$$\bar{g}_{\mu\nu} = A(E, \phi) \left( g_{\mu\nu} + \frac{\lambda^2(\nabla\phi)^4 f^2(E) - 1}{(\nabla\phi)^2} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (53)$$

Strictly invariant under  $\delta_f$  and  $\delta_2$ :

$$\bar{g}_{\mu\nu} = \frac{1}{(\nabla\phi)^2 f^2(E)} \left( g_{\mu\nu} + \frac{\lambda^2(\nabla\phi)^4 f^2(E) - 1}{(\nabla\phi)^2} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (54)$$

The most general minimal couplings which are invariant under  $\delta_f$  are of the form

$$\begin{aligned} \sqrt{-g} \bar{g}^{\mu\nu} &= \sqrt{-g} \lambda_f(\phi) (\nabla\phi)^2 f(E) \\ &\times \left( g^{\mu\nu} - \frac{\lambda_f^2(\phi)(\nabla\phi)^4 f^2(E) - 1}{\lambda_f^2(\phi)(\nabla\phi)^6 f^2(E)} \nabla^\mu \phi \nabla^\nu \phi \right). \end{aligned} \quad (55)$$

The most general minimal couplings which are invariant under  $\delta_2$  are of the form

$$\begin{aligned} \sqrt{-g} \bar{g}^{\mu\nu} &= \sqrt{-g} \lambda_2(E) (\nabla\phi)^2 \\ &\times \left( g^{\mu\nu} - \frac{\lambda_2^2(E)(\nabla\phi)^4 - 1}{\lambda_2^2(E)(\nabla\phi)^6 f^2(E)} \nabla^\mu \phi \nabla^\nu \phi \right). \end{aligned} \quad (56)$$

The most general minimal couplings which are invariant under  $\delta_f$  and  $\delta_2$  are of the form

$$\begin{aligned} \sqrt{-g} \bar{g}^{\mu\nu} &= \sqrt{-g} (\nabla\phi)^2 f(E) \\ &\times \left( g^{\mu\nu} - \frac{\lambda^2(\nabla\phi)^4 - 1}{\lambda^2(\nabla\phi)^6 f^2(E)} \nabla^\mu \phi \nabla^\nu \phi \right). \end{aligned} \quad (57)$$

Therefore, metrics and conformal couplings exist which are invariant under both  $\delta_f$  and  $\delta_2$ , for any function  $f = f(E)$ . However, it also follows that no metric or conformal coupling can be found which is invariant under  $\delta_f$  and  $\delta_g$  unless  $f$  and  $g$  are proportional to one another.

The metrics which are conformally invariant under  $\delta_1$  and  $\delta_2$  turn out to be of the form

$$\bar{g}_{\mu\nu} = A(E, \phi) \left( g_{\mu\nu} + \frac{\lambda^2 (\nabla \phi)^4 - 1}{(\nabla \phi)^2} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (58)$$

Strict invariance under  $\delta_1$  and  $\delta_2$  requires  $A \propto 1/(\nabla \phi)^2$ .

Moreover, the only minimal couplings to a scalar field which are invariant under  $\delta_1$  and  $\delta_2$  are proportional to

$$\begin{aligned} \sqrt{-\bar{g}} (\bar{\nabla} f)^2 &= \sqrt{-g} (\nabla \phi)^2 \\ &\times \left( g^{\mu\nu} - \frac{\lambda^2 (\nabla \phi)^4 - 1}{\lambda^2 (\nabla \phi)^6} \nabla^\mu \phi \nabla^\nu \phi \right) \nabla_\mu f \nabla_\nu f. \end{aligned} \quad (59)$$

Thus, save for a constant parameter, there is only one minimal coupling which is invariant under  $\delta_1$  and  $\delta_2$ .

## VI. DISCUSSION

Consider again the CGHS and the exponential models coupled to conformal matter. Both these models are invariant under a conformal transformation but not the same transformation. The CGHS model coupled to conformal matter is invariant under  $\delta_2 - 4\lambda^2 \delta_1$ , whereas the exponential model is invariant with respect to  $\delta_2 + 2\beta \delta_E$ . Moreover, the coupling to matter does not preserve any of the other symmetries that have been discussed in the present paper. As has been shown in Ref. [15], invariance of  $\sigma$  models, as the CGHS or exponential models minimally coupled to matter, implies that these models contain a free field as well as a field which obeys a Liouville equation. However, as the Liouville equation does not appear as the conservation equation of a Noether current, this equation is not directly related to an invari-

ance of the theory. Therefore, as there is not enough symmetry, no quantum solvability should be expected. In fact, the quantum nature of Liouville theory has remained elusive (see, for instance, Ref. [16]) and much the same can be said about the quantum nature of the CGHS model (see, for instance, Ref. [4]).

Unlike the CGHS and the exponential models, which are invariant with respect to one symmetry only (in this respect, the model with  $V=0$  is somewhat special as it is invariant under two symmetries), we have constructed ‘‘minimal’’ couplings which are invariant under two symmetries. Therefore, we expect that this additional invariance of the models will imply quantum as well as classical solvability. The analysis in Ref. [15] does not apply to our models and a different analysis should be made. A detailed discussion of this question deserves a separate study.

Finally, we would like to mention that, for more than two decades now, it has been known that Einstein’s gravity, when restricted to metrics with two *commuting* Killing vector fields, acquires a large number of non-Abelian symmetries, the so-called Geroch group [17]. It is clear that our results may have some relationship with, or be a generalization of, Geroch’s group. We hope to establish that relationship and communicate it in a future publication.

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