Effective field theories in the large-N limit

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Various effective field theories in four dimensions are shown to have exact nontrivial solutions in the limit as the number N of fields of some type becomes large. These include extended versions of the U(N) Gross-Neveu model, the nonlinear O(N) σ model, and the CP^{N-1} model. Although these models are not renormalizable in the usual sense, the infinite number of coupling types allows a complete cancellation of infinities. These models provide qualitative predictions of the form of scattering amplitudes for arbitrary momenta, but because of the infinite number of free parameters, it is possible to derive quantitative predictions only in the limit of small momenta. For small momenta the large-N limit provides only a modest simplification, removing at most a finite number of diagrams to each order in momenta, except near phase transitions, where it reduces the infinite number of diagrams that contribute for low momenta to a finite number. [S0556-2821(97)05016-9]

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I. INTRODUCTION

There are a number of instructive models that can be exactly solved in the limit where the number N of fields becomes very large [1]. Well-known examples include the linear and nonlinear σ models [2], the Gross-Neveu model [3], and the CP^{N-1} model [4]. In four dimensions none of these models except the linear σ model is conventionally renormalizable, so their large-N limit has usually been studied either by introducing an ultraviolet cutoff, or by working in two dimensions, where the simpler versions of these models are renormalizable.

There is an alternative approach to infinities. In effective field theories that are not renormalizable in the usual ''power-counting'' sense, infinities are canceled by renormalization of coupling constants provided we include in the Lagrangian every interaction allowed by symmetry principles. Even though this means that the Lagrangian contains an infinite number of interaction terms, it is often possible to derive useful results in such theories by expanding in power of energy rather than coupling constants [5].

In this paper I will show how nontrivial finite results can be obtained by passing to the limit of large N in various four-dimensional effective field theories that are not renormalizable in the conventional sense. In this task we will encounter problems both of combinatorics and of renormalization.

The combinatoric problems here can be illustrated by recalling the Gross-Neveu model in its original form. The action is

$$I[\psi]=\int d^2x[-\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r-(g/N)(\overline{\psi_r}\psi_r)^2], \tag{1}$$

where ψ_r is a set of N fermion fields in two spacetime dimensions, forming the defining representation of a U(N) symmetry, and g is a constant that is held fixed as $N\to\infty$. As an aid to counting factors of $1/N$, one cancels the quartic term in Eq. (1) by adding an expression that is quadratic in an auxiliary field σ, and that vanishes when σ is integrated out. This results in the replacement of Eq. (1) with the equivalent action

$$\begin{aligned}I[\psi,\sigma]&=I[\psi]+(N/4g)\int d^2x[\sigma+(2g/N)\overline{\psi_r}\psi_r]^2\\&=\int d^2x[-\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r+\sigma\overline{\psi_r}\psi_r+(N/4g)\sigma^2].\end{aligned} \tag{2}$$

Since the fermion field appears quadratically in Eq. (2), it may be integrated out, yielding an effective action for σ:

$$\Gamma[\sigma]=(N/4g)\int d^2x\,\sigma^2-iN\mathrm{Tr}\ln(\gamma^\mu\partial_\mu-\sigma). \tag{3}$$

The whole action for σ is proportional to N, so the contribution of graphs with L σ loops to the effective action for σ is suppressed by a factor N^{1-L}. Because this method uses the special properties of integrals over Gaussians, it is often said that this method is limited to models in which the interaction is a product of just two bilinear currents [6], as in Eq. (1).

There are also special problems with infinities when an auxiliary field is introduced in order to impose some constraint on the N-component field, as in the nonlinear σ model in four dimensions. In the original form of this model the Lagrangian is

$$\mathcal{L}=-\tfrac{1}{2}f^2\partial_\mu\pi_r\partial^\mu\pi_r, \tag{4}$$

where f is an N-independent constant with the dimensions of mass, and the scalar fields π_r form an O(N) N vector, constrained by

$$\pi_r\pi_r=N. \tag{5}$$

The counting of powers of N becomes much easier if one replaces this constraint with a Lagrange multiplier term, so that the Lagrangian becomes

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$$\mathcal{L}=-\tfrac{1}{2}f^2\partial_\mu\pi_r\partial^\mu\pi_r-\tfrac{1}{2}f^2\lambda(\pi_r\pi_r-N), \tag{6}$$

with π_r now unconstrained. Integrating out the auxiliary field $\lambda(x)$ yields the Lagrangian (4), with the π_r constrained by Eq. (5). If instead we integrate over the π_r we find an effective action for the auxiliary field:

$$\Gamma[\lambda]=\frac{iN}{2}\mathrm{Tr}\ \ln(\Box-\lambda)+\frac{Nf^2}{2}\int d^4x\lambda. \tag{7}$$

Because both terms are proportional to N, the Green's functions for $\lambda(x)$ are given by using the effective action (7) in the tree approximation. As well known, this theory is nonrenormalizable. We can see this in Eq. (7). The field $\lambda(x)$ here has dimensionality $+2$, so the trace term may be written (aside from an inconsequential constant term) as

$$\tfrac{1}{2}i\mathrm{Tr}\ \ln(\Box-\lambda)=\int d^4x[\mathcal{I}_1\lambda+\mathcal{I}_2\lambda^2]+T_f[\lambda], \tag{8}$$

where the $\mathcal{I}_a$ are divergent constants, and $T_f[\lambda]$ is finite. The infinite term $\mathcal{I}_1$ can be canceled by an infinity in the parameter f^2, leaving a finite remainder $\tfrac{1}{2}Nf_R^2\int d^4x\lambda$ in $\Gamma(\lambda)$, with $f_R^2=f^2+2\mathcal{I}_1$. But the term $\mathcal{I}_2$ cannot be canceled in this way. We could, of course, add a term proportional to λ^2 to the Lagrangian (6), with a coefficient whose infinite part cancels the infinite constant $\mathcal{I}_2$ in Eq. (8), but then we would lose the constraint (5), and this would be the linear σ model rather than the nonlinear σ model. If we view this as an effective field theory then the Lagrangian (4) is just the first term in an infinite series involving higher powers of the currents $\partial_\mu\phi_r\partial_\nu\phi_r$ and higher derivatives, but it is not immediately obvious how these higher terms will allow us to cancel the infinity in $\mathcal{I}_2$.

We run into a similar problem when an auxiliary field is introduced to impose a condition of gauge invariance. The classic example here is the CP^{N-1} model in four dimensions. This model contains a set of N complex scalar fields u_r, subject to the constraint that

$$u_r^\dagger(x)u_r(x)=N. \tag{9}$$

In order that the $u_r(x)$ at each x should form a CP^{N-1} manifold, we must require the action to be invariant under ''gauge'' transformations

$$\delta u_r(x)=i\epsilon(x)u_r(x), \tag{10}$$

with $\epsilon(x)$ an arbitrary real infinitesimal function. In the original CP^{N-1} model, this is accomplished by taking the action as

$$I=-f^2\int d^4x(\partial^\mu u_r-ia^\mu u_r)^\dagger(\partial_\mu u_r-ia_\mu u_r), \tag{11}$$

where f is an N-independent constant with the dimensions of mass, and $a_\mu(x)$ may be defined as the bilinear

$$a_\mu\equiv-(i/2N)[u_r^\dagger\partial_\mu u_r-(\partial_\mu u_r^\dagger)u_r], \tag{12}$$

which under the gauge transformation (10) changes by

$$\delta a_\mu=\partial_\mu\epsilon. \tag{13}$$

Equivalently, we can replace $a_\mu(x)$ in Eq. (11) with an independent auxiliary field $A_\mu(x)$, so that the action is

$$I=-f^2\int d^4x(\partial^\mu u_r-iA^\mu u_r)^\dagger(\partial_\mu u_r-iA_\mu u_r). \tag{14}$$

Since $A_\mu(x)$ enters quadratically in Eq. (14), the path integral over $A_\mu(x)$ is done by giving it a value at which the action (14) is stationary with respect to $A_\mu(x)$, which turns out to give $A_\mu(x)=a_\mu(x)$. To enforce the constraint (9) we can add a Lagrange multiplier term $-f^2\int d^4x\lambda(u_r^\dagger u_r-N)$, with $\lambda(x)$ another auxiliary field, and $u_r(x)$ now unconstrained. Since $u_r(x)$ enters quadratically in the action it can be integrated out, yielding an effective action for the auxiliary fields

$$\Gamma[A,\lambda]=iN\mathrm{Tr}\ \ln[D_\mu D^\mu-\lambda]+Nf^2\int d^4x\lambda, \tag{15}$$

where here $D_\mu\equiv\partial_\mu-iA_\mu$. Because each term is proportional to N, the contribution of graphs with L loops to the Green's functions for λ and a_μ is suppressed by a factor N^{1-L}. Equation (15) displays the problem with renormalizability [7] in four dimensions: Dimensional analysis and gauge invariance show that the infinite part of the trace is a linear combination of $\int d^4x\lambda$, $\int d^4x\lambda^2$, and $\int d^4x(\partial_\mu A_\nu-\partial_\nu A_\mu)^2$ with divergent coefficients, and although the infinite part of the coefficient of $\int d^4x\lambda$ can be canceled by an infinite term in f^2, there is nothing here that can cancel the infinite coefficients of $\int d^4x\lambda^2$ or $\int d^4x(\partial_\mu A_\nu-\partial_\nu A_\mu)^2$. Treating this as an effective field theory, we would certainly have to add terms to the action involving $\int d^4x(\partial_\mu a_\nu-\partial_\nu a_\mu)^2$, where a_μ is defined by Eq. (12), but this would not cancel the infinity in $\Gamma[\lambda,A]$ proportional to $\int d^4x(\partial_\mu A_\nu-\partial_\nu A_\mu)^2$. We cannot add a term proportional to $\int d^4x(\partial_\mu A_\nu-\partial_\nu A_\mu)^2$ without making $A_\mu(x)$ an independent dynamical field, thus removing the most interesting aspect of the theory, the appearance of long-range forces in a theory without an elementary gauge field.

Finally, there is a problem that always confronts us in dealing with effective field theories: how to use a theory with an infinite number of free parameters to derive physical predictions. As we shall see, the large-N limit can give qualitative information about the *form* of S-matrix elements, but in effective field theories it is not possible actually to calculate the functions that appear in S-matrix elements except in the low-energy limit. In the extended Gross-Neveu model considered in Sec. II it turns out that there are usually only a finite number of graphs that contribute to each order in energy, whatever the value of N, so for low energy the large-N limit leads only to modest simplifications. As shown in Secs. III and IV, the same is true in the extended nonlinear σ model and the extended CP^{N-1} model, with one interesting exception: Near the phase transitions at which the broken symmetries of these models are restored, there is an infinite number of graphs of the same order in energy, which can be summed only in the large-N limit.

The original motivation of this work was to decide whether the appearance of a spin-one ''photon'' in the two-dimensional CP^{N-1} model occurs also in four-dimensional versions of this model, when the problem of infinities is handled by treating the model as an effective quantum field theory. Section IV shows that the answer to this question is yes, but as discussed in Sec. V, this result is less surprising than might be supposed.

II. THE EXTENDED GROSS-NEVEU MODEL

To illustrate the use of the large-N limit in effective field theories, let us consider the general class of models with a set of N massless fermion fields in d spacetime dimensions, transforming according to the defining representation of a global U(N) symmetry. Any U(N)-invariant action will be a functional of a set of bilinear currents $j_\ell(x)$ that are invariant under U(N), such as the currents $j_0=\overline{\psi_r}\psi_r$, $j_{1\mu}=\overline{\psi_r}\partial_\mu\psi_r$, etc. We will consider a class of extended four-dimensional Gross-Neveu models, with an action of the form[1]

$$I[\psi]=-\int d^dx\,\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r+NF[j/N], \tag{16}$$

where $F[\tau]$ is some N-independent functional. The original Gross-Neveu action (1) is a special case of Eq. (16), with $F[j/N]$ quadratic in the particular current $j_0=\overline{\psi_r}\psi_r$. As we shall see, the N dependence given the second term in the action (16) makes the theory nontrivial but soluble [8], as in the original Gross-Neveu model.

The action (16) may be replaced with an equivalent action

$$I[\psi,\sigma]=\int d^4x[-\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r+\sigma_\ell(x)j_\ell(x)]+NG[\sigma,N], \tag{17}$$

where $\exp\{iNG[\sigma,N]\}$ is the functional Fourier transform with respect to $N\tau$ of $\exp\{iNF[\tau]\}$:

$$\exp\{iNG[\sigma,N]\}$$
$$\equiv\int\prod_{\ell,x}d\tau_\ell(x)\exp\left\{-iN\int d^4x\,\sigma_\ell(x)\tau_\ell(x)+iNF[\tau]\right\}. \tag{18}$$

Of course, $\exp\{iNF[\tau]\}$ is then (up to an unimportant constant factor) the Fourier transform of $\exp\{iNG[\sigma,N]\}$:

$$\exp\{iNF[\tau]\}\propto\int\prod_{\ell,x}d\sigma_\ell(x)\exp\left\{iN\int d^4x\,\sigma_\ell(x)\tau_\ell(x)+iNG[\sigma,N]\right\}. \tag{19}$$

The integral over the $\sigma_\ell(x)$ in the functional integral of $\exp\{iI[\psi,\sigma]\}$ thus yields

$$\int\prod_{\ell,x}d\sigma_\ell(x)\exp\{iI[\psi,\sigma]\}\propto\exp\{iI[\psi]\}, \tag{20}$$

so the action $I[\psi,\sigma]$ given by Eq. (17) is equivalent to the original action (16).

In the limit of large N the Fourier integral (18) may be done by setting $\tau_\ell(x)$ at the stationary "point" $\tau^\sigma(x)$ of the integrand, at which

$$\left.\frac{\delta F[\tau]}{\delta\tau_\ell(x)}\right|_{\tau=\tau^\sigma}\equiv\sigma_\ell(x), \tag{21}$$

so that $G[\sigma,N]$ approaches an N-independent functional, the Legendre transform of $F[\tau]$:

$$G[\sigma,N]\to G[\sigma]\equiv F[\tau^\sigma]-\int d^dx\,\sigma_\ell(x)\tau^\sigma_\ell(x). \tag{22}$$

That is, for $N\to\infty$, the action may be taken as

$$I[\psi,\sigma]=\int d^4x[-\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r+\sigma_\ell(x)j_\ell(x)]+NG[\sigma]. \tag{23}$$

We could just as well have taken the action (23) as our starting point, with $G[\sigma]$ an arbitrary N-independent functional; the only difference would be that then the theory would be equivalent to one with an action of the original form (16), but with $F[\tau]$ independent of N only in the limit $N\to\infty$.

Now, we want to calculate the quantum effective action $\Gamma[\psi,\sigma]$, which by definition in the tree approximation gives the same result as the sum of all loop and tree graphs calculated using the action (23). According to the usual prescription, we must replace $\psi_r(x)$ and $\sigma_\ell(x)$ in Eq. (23) with sums $\psi_r(x)+\psi'_r(x)$ and $\sigma_\ell(x)+\sigma'_\ell(x)$ and integrate over the quantum perturbations $\psi'_r(x)$ and $\sigma'_\ell(x)$, including only one-particle-irreducible graphs. A standard power-counting argument gives, to leading order in $1/N$,

$$\Gamma[\psi,\sigma]\to\int d^4x[-\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r+\sigma_\ell(x)j_\ell(x)]+NT[\sigma]+NG[\sigma], \tag{24}$$

where $T[\sigma]$ is an N-independent functional of σ, such as the functional Tr $\ln(\gamma^\mu\partial_\mu-\sigma)$ in Eq. (3), defined in general by the integral of a Gaussian

$$\exp\{iNT[\sigma]\}\equiv\int\left[\prod_{n,x}d\psi'_r(x)\right]\exp\left\{i\int d^dx[-\overline{\psi'_r}\gamma^\mu\partial_\mu\psi'_r+\sigma_\ell(x)j'_\ell(x)]\right\}. \tag{25}$$

[To obtain Eq. (24), note first that purely fermionic loops yield a term $NT[\sigma]$ in the effective action, which just makes an additive contribution to $NG[\sigma]$. The σ_ℓ propagators are then given by the inverse of the coefficient of the quadratic term in $NT[\sigma]+NG[\sigma]$, and are hence proportional to $1/N$, while the purely bosonic vertices (including those derived from $T[\sigma]$) make contributions proportional to N. Thus a graph with V_σ purely bosonic vertices, V_ψ fermion-fermion-boson vertices, I_ψ internal fermion lines (excluding those in purely fermionic loops), and I_σ internal boson lines

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[1]The free-field action could itself be regarded as a linear term in $NF[j/N]$, but it is convenient to treat it separately.

makes a contribution of order $N^{V_\sigma - I_\sigma}$. But $2V_\psi = 2I_\psi + F$, where F is the number of external fermion lines (i.e., factors of the classical ψ field), and the number L of loops (with all purely fermionic loops counted as bosonic vertices) is $L = I_\psi + I_\sigma - V_\psi - V_\sigma + 1$, so the number of powers of N is $V_\sigma - I_\sigma = 1 - L - F/2$. The leading graphs are, therefore, those with no loops, but the only one-particle-irreducible graphs with no loops are those consisting of just a single vertex, which yield the result (24).]

For instance, the amplitude for fermion-fermion scattering is given to leading order in $1/N$ by the tree graphs in which a single σ line is exchanged between the external fermions, with the vertices given by the term $\int d^4x \sigma_{\not\ell}(x) j_{\not\ell}(x)$ in Eq. (24), and the σ propagator given by the inverse of the coefficient of the quadratic term in $NG[\sigma] + NT[\sigma]$. Unlike the case of the original Gross-Neveu model, the terms of higher order in $1/N$ arise not only from loop graphs that correct the effective action (24), but also from higher-order terms in $G[\sigma, N]$.

Now, let us take up the problem of renormalization in the large-N limit. Although the one-loop functional $T[\sigma]$ is highly nonlocal, its infinite part is ''perturbatively local,'' that is, it is the integral of a series (in general infinite) of products of fields and their derivatives with divergent coefficients, of which only a finite number of terms contribute to the tree amplitude for any given process to any finite order in momentum. In order to cancel these infinities, it is necessary that $G[\sigma]$ be a general perturbatively local functional, subject to no constraints other than symmetry properties that also constrain $T[\sigma]$. But any perturbatively local $G[\sigma]$ can be obtained as the Legendre transform of some perturbatively local $F[\tau]$, provided only that

$$\mathrm{Det}\mathcal{M} \neq 0, \quad \mathcal{M}_{\not\ell x, my} \equiv \frac{\delta^2 G[\sigma]}{\delta\sigma_{\not\ell}(x)\,\delta\sigma_m(y)}. \tag{26}$$

[To see this, it is only necessary to note that the functional $G[\sigma]$ is the Legendre transform of a functional $F[\tau]$ given by the inverse Legendre transform of $G[\sigma]$:

$$F[\tau] = G[\sigma^\tau] + \int d^d x \sigma^\tau_{\not\ell}(x) \tau_{\not\ell}(x), \tag{27}$$

with σ^τ the stationary ''point'' of the expression on the right-hand side:

$$\left.\frac{\delta G[\sigma]}{\delta\sigma_{\not\ell}(x)}\right|_{\sigma=\sigma^\tau} = -\tau_{\not\ell}(x). \tag{28}$$

In terms of Feynman graphs, this just says that $F[\tau]$ is given a sum of tree Feynman graphs calculated from the action $G[\sigma] + \int d^d x \sigma_{\not\ell}(x)\tau_{\not\ell}(x)$, for which the propagator is $\mathcal{M}^{-1}_{\not\ell x, my}$. Therefore, $F[\tau]$ is perturbatively local if $G[\sigma]$ is, and if $\mathrm{Det}\mathcal{M} \neq 0$.] Furthermore, the condition that $\mathrm{Det}\mathcal{M} \neq 0$ can always be satisfied by adding a finite quadratic term to $G[\sigma]$, and subtracting the same term from $T[\sigma]$. Apart from symmetries, the only constraint on $F[\tau]$ is that it be perturbatively local, so it is always possible to choose an $F[\tau]$ that gives whatever $G[\sigma]$ is needed to cancel the infinities in $T[\sigma]$.

This works out in a particularly simple way if the currents $j_{\not\ell}(x)$ have dimensionality (in powers of mass, with $\hbar = c = 1$) less than the spacetime dimensionality, so that the $\sigma_{\not\ell}(x)$ have positive dimensionality. In this case the infinite part $T_\infty[\sigma]$ of $T[\sigma]$ is the integral of a *polynomial* in the $\sigma_{\not\ell}(x)$ and their derivatives, with infinite constant coefficients. For instance, consider an extended Gross-Neveu model in four dimensions, with an action of the form

$$I[\psi] = -\int d^4x\, \overline{\psi_r} \gamma^\mu \partial_\mu \psi_r + NF[j_0/N], \tag{29}$$

where $F[j_0/N]$ is an arbitrary even local functional of the single current:

$$j_0 = \overline{\psi_r}\psi_r. \tag{30}$$

We take $F[j/N]$ even so that the action will be invariant under a discrete chiral symmetry transformation $\psi_r \to \gamma_5 \psi_r$, which, if unbroken, keeps the fermions massless. Here, $\sigma(x)$ has dimensionality $+1$, and chiral symmetry tells us that the functional $T[\sigma]$ defined by Eq. (25) is even in σ, so this functional takes the form

$$T[\sigma] = \int d^4x [\mathcal{I}_0 + \mathcal{I}_1\sigma^2 + \mathcal{I}_2 \sigma\Box\sigma + \mathcal{I}_3\sigma^4] + T_f[\sigma], \tag{31}$$

where the $\mathcal{I}_a$ are infinite constants, and T_f is a finite functional, which for constant σ takes the well-known form [9]

$$T_f[\sigma] = -\frac{1}{32\pi^2}\int d^4x \sigma^4 \ln\sigma^2.$$

The functional $F[\tau]$ may be expanded in a series of even local operators of increasing dimensionality

$$F[\tau] = \int d^4x [A_0 + A_1\tau^2 + A_2\tau\Box\tau + A_3\tau^4 + A_4\tau\Box\Box\tau + \cdots], \tag{32}$$

in which case Eqs. (21) and (22) give

$$G[\sigma] = \int d^4x \left[A_0 - \frac{1}{4A_1}\sigma^2 + \frac{A_2}{4A_1^2}\sigma\Box\sigma + \frac{A_3}{16A_1^4}\sigma^4 + \left(\frac{A_4}{4A_1^2} - \frac{A_2^2}{4A_1^3}\right)\sigma\Box\Box\sigma + \cdots\right]. \tag{33}$$

In order to cancel infinities, we must take the bare parameters as

$$A_0 = C_0 - \mathcal{I}_0, \quad A_1 = -\tfrac{1}{4}[C_1 - \mathcal{I}_1]^{-1}, \quad A_2 = \frac{C_2 - \mathcal{I}_2}{4[C_1 - \mathcal{I}_1]^2},$$

$$A_3 = \frac{C_3 - \mathcal{I}_3}{16[C_1 - \mathcal{I}_1]^4},$$

$$A_4=\frac{C_4}{4[C_1-\mathcal{I}_1]^2}-\frac{[C_2-\mathcal{I}_2]^2}{4[C_1-\mathcal{I}_1]^3},\quad \ldots, \tag{34}$$

where the C_a are the finite renormalized coupling parameters that appear in the final result

$$G[\sigma]+T[\sigma]=\int d^4x[C_0+C_1\sigma^2+C_2\sigma\Box\sigma+C_3\sigma^4+C_4\sigma\Box\Box\sigma+\cdots]+T_f[\sigma]. \tag{35}$$

It is important but not surprising that the infinite number of unrenormalized constants A_a can be chosen to give finite results for $\Gamma[\sigma]$, despite the fact that the Gross-Neveu model is not conventionally renormalizable in four dimensions. What is somewhat surprising is that this is possible with an interaction given by a power series in the single current $j_0=\overline{\psi_r}\psi_r$ and its derivatives, without needing to include additional currents, such as $j_{1\mu}\equiv\overline{\psi_r}\partial_\mu\psi_r$. Although it is not necessary to include additional currents in order to cancel infinities, in the spirit of effective field theory we really should include all U(N)-invariant currents in the action. This complicates the cancellation of infinities through renormalization, but as we have shown earlier, it does not make it impossible.

What good is a theory like this, that has an infinite number of arbitrary parameters? For an action of the form (29), the effective action (24) takes the form

$$\Gamma[\psi,\sigma]=-\int d^4x\,\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r+\int d^4x\,\sigma\overline{\psi_r}\psi_r+NG[\sigma]+NT[\sigma], \tag{36}$$

with $G[\sigma]+T[\sigma]$ given by Eq. (35). To calculate scattering amplitudes we must use this as the action in the tree approximation. For instance, the invariant amplitude for a fermion-fermion scattering process $A+B\to C+D$ takes the form

$$M(A+B\to C+D)=\delta_{r_Ar_C}\delta_{r_Br_D}(\overline{u_C}u_A)(\overline{u_D}u_B)\Delta(t)-\delta_{r_Ar_D}\delta_{r_Br_C}(\overline{u_D}u_A)(\overline{u_C}u_D)\Delta(u), \tag{37}$$

where t and u are the Mandelstam variables $t=-(p_A-p_C)^2$ and $u=-(p_A-p_D)^2$, and $\Delta(t)$ is the σ propagator. This particular form of the scattering amplitude is a consequence of the assumption that the action has the form (29), with only the one current j_0, and it is valid in the large-N limit for arbitrary values of the fermion momenta.

Unfortunately, to go further and actually calculate the propagator $\Delta(s)$ without knowing the infinite number of free parameters in $F[\tau]$ or $G[\sigma]$, it is necessary to take the limit of small momenta. But in this limit, little is gained by also letting N become large.

To compare the consequences of the large N and low momentum approximations, let us consider an amplitude calculated using the action (29) in the equivalent form

$$I[\psi,\sigma]=-\int d^4x\,\overline{\psi_r}\gamma^\mu\partial_\mu\psi_r+\int d^4x\,\sigma\overline{\psi_r}\psi_r+NG[\sigma], \tag{38}$$

with the N dependence of $G[\sigma]$ left unspecified. Take all incoming and outgoing momenta of the order of some small momentum Q, and suppose that infinities are canceled by renormalizing at momenta also of order Q. Fermion propagators go as Q^{-1}, while σ propagators go as constants (any quadratic terms in $G[\sigma]$ involving derivatives being treated as interactions) and each loop introduces four factors of Q, so an amplitude with I_ψ internal fermion lines and L loops goes as Q^ν, where

$$\nu=4L-I_\psi+\sum_i d_iV_i, \tag{39}$$

where V_i is the number of purely bosonic interactions of type i, and d_i is the number of derivatives in such an interaction. This may be rewritten by using the familiar formulas

$$2I_\psi+E_\psi=2V_\psi, \tag{40}$$

$$2I_\sigma+E_\sigma=\sum_i V_in_i+V_\psi, \tag{41}$$

and

$$L=I_\psi+I_\sigma-\sum_i V_i-V_\psi+1, \tag{42}$$

where I_σ is the number of internal σ lines, E_ψ and E_σ are the numbers of external fermion and σ lines, V_ψ is the number of fermion-fermion-σ vertices, and n_i is the number of σ fields in the purely bosonic interaction of type i. Equation (39) then may be put in the form

$$\nu=2L+\sum_i \Delta_iV_i+2-\frac{E_\psi}{2}-E_\sigma+2, \tag{43}$$

where

$$\Delta_i=n_i+d_i-2. \tag{44}$$

Now, all σ interactions have either $n_i\geqslant 4$ or $n_i=2$ and $d_i\geqslant 2$ (the term proportional to $\int\sigma^2d^4x$ in $G[\sigma]$ being treated as the kinematic action for the σ field), so for all purely bosonic interactions $\Delta_i\geqslant 2$. A scattering amplitude for a fixed process (i.e., E_ψ and E_σ fixed) is, therefore, dominated for $Q\to 0$ by *tree* graphs with any number of fermion-fermion-σ vertices but no loops and no pure σ interactions. These graphs are a subset of those we encountered in the limit of large N, so the large-N limit would introduce no further simplification here.

The large-N limit becomes relevant in the next order in Q^2, which according to Eqs. (43) and (44), is given by graphs with any number of fermion-fermion-σ vertices and either one loop or with no loops and one pure σ interaction with $\Delta_i=2$. The graphs with a single interaction from $G[\sigma]$ or a single purely fermionic loop reproduce what we would find by using the large-N effective action (36) to this order in Q^2. But here there is also a class of graphs that does not appear in the large-N limit: the graphs constructed out of only fermion-fermion-σ vertices and containing a single loop, which is not purely fermionic. (For instance, in fermion-fermion scattering these are the graphs in which a pair of σ lines are exchanged between the fermions.) The large-N limit is, therefore, useful here, but not very useful, because even without it there are only a finite number of graphs to each order in Q^2. This is just a consequence of the fact that this is a theory where all interactions are nonrenormalizable. In the next section we will see an example where the large-N limit is much more important, because there is an infinite number of graphs to each order in small momenta, which cannot be summed except in the limit of large N.

III. THE NONLINEAR σ MODEL: INTEGRATING IN AN ORDER PARAMETER

Auxiliary fields are sometimes introduced to enforce constraints on the other fields, as well as to help in counting factors of $1/N$. The classic example is the nonlinear O (N) σ model, which in its usual form has Lagrangian (4). Here, we will consider a class of extended nonlinear σ models, with Lagrangians of the form

$$I[\pi]=-\frac{f^2}{2}\int d^4x\,\partial_\mu\pi_r\partial^\mu\pi_r+Nf^2F[j/N], \tag{45}$$

where π_r is a set of N scalar fields, satisfying the constraint

$$\pi_r\pi_r=N; \tag{46}$$

$j(x)$ is the O(N)-invariant scalar current with the minimum number of derivatives:

$$j\equiv\tfrac{1}{2}\partial_\mu\pi_r\partial^\mu\pi_r; \tag{47}$$

f^2 is an arbitrary positive constant; and $F[\tau]$ is a functional that, apart from being perturbatively local and N independent, can be chosen as we like. The N dependence in Eq. (45) has been chosen so that this model will be soluble but nontrivial in the limit $N\to\infty$.

As in the case of the extended Gross-Neveu model, we shall introduce an auxiliary field $\sigma(x)$, and replace Eq. (45) with the equivalent action

$$I[\pi,\sigma]=Nf^2G[\sigma,N]-f^2\int d^4x(1+\sigma)j, \tag{48}$$

where $\exp\{iNf^2G[\sigma,N]\}$ is the functional Fourier transform with respect to $f^2N\tau$ of $\exp\{iNf^2F[\tau]\}$:

$$\exp\{iNf^2G[\sigma,N]\}\equiv\int\prod_x d\tau(x)\exp\left\{iNf^2\int d^4x\,\sigma(x)\tau(x)+iNf^2F[\tau]\right\}. \tag{49}$$

It is easy to see that we get back to Eq. (45) when we integrate out $\sigma(x)$, but it will be convenient instead to use the action in the form (48).

In the limit of large N the functional $G[\sigma,N]$ approaches an N-independent functional given by the Legendre transform of $F[\tau]$,

$$G[\sigma,\tau]\to G[\sigma]=\int d^4x\,\sigma\tau^\sigma+F[\tau^\sigma], \tag{50}$$

with τ^σ defined by

$$\left.\frac{\delta F[\tau]}{\delta\tau(x)}\right|_{\tau=\tau^\sigma}=-\sigma(x). \tag{51}$$

At this point, it is not so obvious how infinite counterterms in the functional $G[\sigma]$ can be used to cancel the ultraviolet divergence proportional to $\int d^4x\lambda^2$ that is encountered when we introduce a Lagrange multiplier $\lambda(x)$ and integrate over the π_r. Yet we know that this is possible, because the cancellation of infinities is obvious in an extended linear σ model, in which the action is an arbitrary perturbatively local functional of an unconstrained N-vector field ϕ_r, and from such a model it is always possible to construct a nonlinear σ model by integrating out the massive order parameter represented by the O(N) scalar $\sqrt{\phi_r\phi_r}$. This suggests that we should show the cancellation of infinities in the extended nonlinear σ model by using the ingredients appearing in Eq. (48) to construct something like a linear σ model.

For this purpose, let us define new fields

$$\phi_r\equiv f\sqrt{1+\sigma}\,\pi_r. \tag{52}$$

Using the constraint (46), the current (47) may be written as

$$j=\frac{1}{2f^2(1+\sigma)}\partial_\mu\phi_r\partial^\mu\phi_r-\frac{N}{8(1+\sigma)^2}\partial_\mu\sigma\partial^\mu\sigma. \tag{53}$$

Also, the constraint now reads

$$\phi_r\phi_r=Nf^2(1+\sigma) \tag{54}$$

and will again be imposed by introducing a Lagrange multiplier $\lambda(x)$. The action (48) is thereby replaced with the equivalent action

$$I[\phi,\sigma,\lambda]=Nf^2G'[\sigma]-\int d^4x\{\tfrac{1}{2}\lambda[\phi_r\phi_r-Nf^2(1+\sigma)]+\tfrac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r\}, \tag{55}$$

where

$$G'[\sigma]\equiv G[\sigma]+\frac{1}{8}\int d^4x\frac{\partial_\mu\sigma\partial^\mu\sigma}{1+\sigma}. \tag{56}$$

Now, it is $\lambda(x)$ rather than $\sigma(x)$ that interacts with the N vector of scalar fields, so the ϕ_r scattering amplitude may be calculated in terms of the effective action for $\lambda(x)$ and $\phi_r(x)$, which is obtained by integrating out $\sigma(x)$. The part of the action involving $\sigma(x)$ is proportional to N, and does not involve the $\phi_r(x)$, so we can integrate out $\sigma(x)$ to leading order in $1/N$ by setting $\sigma(x)$ equal to the value $\sigma^\lambda(x)$ where Eq. (55) is stationary with respect to $\sigma(x)$:

$$\left.\frac{\delta G'[\sigma]}{\delta\sigma(x)}\right|_{\sigma=\sigma^\lambda}=-\tfrac{1}{2}\lambda(x). \tag{57}$$

This gives an action for $\phi_r(x)$ and $\lambda(x)$:

$$I[\phi,\lambda]=NH[\lambda]-\int d^4x[\tfrac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r+\tfrac{1}{2}\lambda\phi_r\phi_r], \tag{58}$$

where $H[\lambda]$ is another Legendre transform

$$H[\lambda]\equiv f^2G'[\sigma^\lambda]+\tfrac{1}{2}f^2\int d^4x\lambda(1+\sigma^\lambda). \tag{59}$$

The same reasoning as in the previous section (with λ and π_r replacing σ and ψ_r) shows that to leading order in $1/N$, the quantum effective action, which in tree approximation gives the complete scattering amplitude, is here

$$\Gamma[\phi,\lambda]=\Gamma[\lambda]-\int d^4x[\tfrac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r+\tfrac{1}{2}\lambda\phi_r\phi_r], \tag{60}$$

where

$$\Gamma[\lambda]=NH[\lambda]+\tfrac{1}{2}iN\mathrm{Tr}\ \ln(\Box-\lambda). \tag{61}$$

Following the same argument as in the previous section, by choosing $F[\tau]$ we can make $G'[\sigma]$ and $H[\lambda]$ any perturbatively local functionals we like, so we can adjust $H[\lambda]$ to cancel the infinite terms in the one-loop trace Tr $\ln(\Box-\lambda)$ proportional to $\int d^4x\lambda$ and $\int d^4x\lambda^2$. For λ spacetime independent, this gives

$$V(\lambda)=\frac{N\lambda^2\ln(\lambda/M^2)}{64\pi^2}+N\sum_{n=0}^{\infty}c_n\lambda^n, \tag{62}$$

where the c_n are model-dependent, N-independent finite constants; M is a constant that can be chosen, for instance, so that $c_2=1$; and $V[\lambda]$ is the ''effective potential,'' defined so that for a spacetime-independent λ,

$$\Gamma[\lambda]=-\mathcal{V}_4V(\lambda), \tag{63}$$

with $\mathcal{V}_4$ the spacetime volume.

To identify the possible phases of this model, we must examine the possible spacetime-independent vacuum expectation values of the fields ϕ_r and λ, at which the effective action (60) is stationary. These phases are of two different types.

A. Broken symmetry phase

In this phase the vacuum expectation value of $\lambda(x)$ vanishes, while $\phi_r(x)$ has a vacuum expectation $\sqrt{N}v_r$, given by the solution of the equation

$$\left.\frac{\delta\Gamma[\lambda]}{\delta\lambda(x)}\right|_{\lambda(x)=0}=-\left.\frac{\partial V(\lambda)}{\partial\lambda}\right|_{\lambda=0}=\tfrac{1}{2}Nv_rv_r. \tag{64}$$

From Eqs. (62) and (64) we see the system will be in this phase if $c_1<0$, and that in this case v_r is N independent and given by

$$v_rv_r=-2c_1. \tag{65}$$

To see the particle content of the theory in this phase, we note that the terms in the effective action (60) of second order in the displacement of the fields from their equilibrium value have a coefficient matrix given in momentum space by $-\tfrac{1}{2}\mathcal{D}(k)$, where $\mathcal{D}(k)$ is the matrix

$$\mathcal{D}_{rs}(k)=k^2\delta_{rs},\quad \mathcal{D}_{\lambda\lambda}(k)=NA(k^2),$$
$$\mathcal{D}_{r\lambda}(k)=\mathcal{D}_{\lambda r}(k)=\sqrt{N}v_r. \tag{66}$$

Here, $A(k^2)$ is an N-independent, one-loop amplitude derived from the part of $\Gamma[\lambda]$ quadratic in λ, which is easily calculated to be

$$A(k^2)=\frac{\ln(k^2/\mathcal{M}^2)}{32\pi^2}+\sum_{n=1}^{\infty}d_n(k^2)^n, \tag{67}$$

with d_n another set of model-dependent finite constants, and $\mathcal{M}$ is a constant chosen so that the constant term d_0 in the sum is absent. The scalar propagator $\Delta(k)=\mathcal{D}^{-1}(k)$, hence, has elements

$$\Delta_{rs}(k)=\frac{1}{k^2}\left(\delta_{rs}-\frac{v_rv_s}{v^2-k^2A(k^2)}\right),$$

$$\Delta_{\lambda\lambda}(k)=-\frac{k^2}{N[v^2-k^2A(k^2)]},$$

$$\Delta_{r\lambda}(k)=\Delta_{\lambda r}(k)=\frac{v_r}{\sqrt{N}[v^2-k^2A(k^2)]}, \tag{68}$$

where $v^2\equiv v_rv_r$. The pole in Δ_{rs} at $k^2=0$ clearly arises from the Goldstone bosons of $O(N)/O(N-1)$. This pole occurs only in the propagators of the components of ϕ_r in directions perpendicular to v_r. There may be other particles associated with poles at model-dependent nonzero masses arising from the vanishing of the denominators $v^2-k^2A(k^2)$ in the propagators of the fields $v_r\phi_r$ and λ, but without special assumptions about $H[\lambda]$ we can say nothing about them, except that they do not mix with the Goldstone bosons. The invariant amplitude for Goldstone boson-Goldstone boson scattering is given by the λ-λ element of the propagator, as

$$M(ap,bq\to a'p',b'q')=\frac{1}{N}\left[\frac{\delta_{ab}\delta_{a'b'}s}{v^2+sA(-s)}+\frac{\delta_{aa'}\delta_{bb'}t}{v^2+tA(-t)}+\frac{\delta_{ab'}\delta_{a'b}u}{v^2+uA(-u)}\right], \tag{69}$$

where a, b, a', b' run over the Goldstone directions, from 1 to $N-1$ (with v_r taken in the N direction), and s, t, u are the usual Mandelstam variables: $s=-(p+q)^2$, $t=-(p-p')^2$, and $u=-(p-q')^2$.

Even with the function $A(s)$ unknown, this specific form of the scattering amplitude is a nontrivial consequence of the action (45) in the limit of large N. But to go further and calculate the actual value of the scattering amplitude we need to restrict ourselves to low energies.

In the extreme low-energy limit, Eq. (69) reduces to the usual low-energy Goldstone boson scattering amplitude [10],

$$M(ap,bq\rightarrow a'p',b'q')$$

$$\rightarrow \frac{4}{F_\pi^2}[\delta_{ab}\delta_{a'b'}s+\delta_{aa'}\delta_{bb'}t+\delta_{ab'}\delta_{a'b}u], \tag{70}$$

provided we identify the Goldstone boson decay amplitude F_π (equal to ≈ 184 MeV for pions) as

$$F_\pi = 2v\sqrt{N}. \tag{71}$$

In this low-energy limit, nothing is gained by also taking N large.

The large-N limit does produce some simplification in the terms of higher order in energy. According to Eqs. (67), (69), and (71), the term in the Goldstone boson scattering amplitude of fourth order in momenta is

$$M^{(4)}(ap,bq\rightarrow a'p',b'q') = -\frac{N\delta_{ab}\delta_{a'b'}}{2\pi^2F_\pi^4}s^2\ln(-s/\mathcal{M}^2)$$

$$+ \quad \text{crossed terms}, \tag{72}$$

which may be compared with the exact formula for the terms in the amplitude of fourth order in momenta[2]

$$M^{(4)}_{aba'b'} = \frac{\delta_{ab}\delta_{a'b'}}{F_\pi^4}\Bigg[-\frac{N-3}{2\pi^2}s^2\ln(-s)$$

$$-\frac{1}{12\pi^2}(u^2-s^2+3t^2)\ln(-t)$$

$$-\frac{1}{12\pi^2}\ (t^2-s^2+3u^2)\ln(-u)-cs^2$$

$$-c'(t^2+u^2)\Bigg]+\ \text{crossed terms}, \tag{73}$$

where c and c' are unknown constants. We see that the effect of taking the large-N limit here is just to eliminate a few of the terms in Eq. (73).

Inspection of Eqs. (67) and (69) shows that not only the terms in the scattering amplitude of second and fourth order in a generic momentum k, but all the "leading log" terms of order $k^{2(n+1)}(\ln k^2)^n$ for $n\geqslant 0$, are uniquely determined by the first, model-independent term in Eq. (67), with no dependence on the coefficients d_n or the model-dependent functional $H[\lambda]$. These are just the model-independent consequences of unitarity and the broken O(N) symmetry alone, specialized to the case of large N. It is far easier to calculate the leading logarithms by using a large-N model and then passing to the low-energy limit where the results become model independent, as we have done here, than by evaluating the model-independent leading log terms for general N and then passing to the limit of large N, but it is still true that to each order in energy there are only a finite number of diagrams, whether or not we invoke the large-N limit.

B. Unbroken symmetry phase

In this phase the vacuum expectation value of $\phi_r(x)$ vanishes, while $\lambda(x)$ has a vacuum expectation value λ_0, given by the solution of the equation

$$\frac{\delta\Gamma[\lambda]}{\delta\lambda(x)}\bigg|_{\lambda(x)=\lambda_0} = -\frac{\partial V(\lambda)}{\partial\lambda}\bigg|_{\lambda=\lambda_0} = 0. \tag{74}$$

Here, the O(N) symmetry is unbroken, and in the large-N limit we have a degenerate multiplet of scalars with squared mass λ_0, so the system will be in this phase only if the stationary point λ_0 of $V(\lambda)$ is *positive*. The S-matrix elements for these degenerate scalars are given in the limit of large N by using the effective action (60) in the tree approximation; for instance, the Feynman scattering amplitude is

$$M(rp,sq\rightarrow r'p',s'q') = [\delta_{rs}\delta_{r's'}\Delta(s)+\delta_{rr'}\delta_{ss'}\Delta(t)$$

$$+\delta_{rs'}\delta_{r's}\Delta(u)], \tag{75}$$

where Δ is the λ propagator, of order $1/N$.

Since λ_0 is generically of the same order as whatever characteristic squared mass scale appears in the functional $\Gamma[\lambda]$, there is no way to use the model to make useful quantitative predictions about masses and scattering amplitudes in this phase without making special assumptions about the constants appearing in $H[\lambda]$. From the large-N limit we can only infer conclusions such as Eq. (75) about the general form of scattering amplitudes.

C. Phase transition

As we have seen, in general, without knowing all the constants c_n in the potential $V(\lambda)$, we can say nothing about the masses in the unbroken symmetry phase except that they are O(N) degenerate, and without knowing all the constants d_n in $A(k^2)$ we can say nothing about the masses in the broken symmetry phase except for the existence of a massless multiplet of Goldstone bosons. We can do better in the case where the constants are tuned so that the system is near the transition between the two phases.

In the unbroken symmetry phase the system is near the phase transition if λ_0, although nonzero, is small. For small λ, Eq. (62) becomes

$$V(\lambda)\rightarrow\frac{N\lambda^2\ln(\lambda/M^2)}{64\pi^2}+c_1\lambda. \tag{76}$$

(Recall that M has been chosen to make c_2 vanish.) Condition (74) then becomes

$$c_1 = -\frac{\lambda_0\ln(e^{1/2}\lambda_0/M^2)}{32\pi^2}. \tag{77}$$

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[2]This agrees with the result of Ref. [5] for the physical case $N=4$. The term of form $\delta_{ab}\delta_{a'b'}s^2\ln(-s)$ is proportional to $N-3$ rather than $N-1$ because it receives contributions from graphs that do not have index loops as well as from those that do.

This has small positive solutions for λ_0 as long as c_1 is small and *positive*. On the other hand, in the broken symmetry phase the system is near the phase transition if v_r is small, which according to Eq. (65) requires that c_1 be small and *negative*. Thus there is a second-order phase transition between these two phases when $c_1=0$, regardless of the values of the other c_n.

Near this phase transition in the broken symmetry phase the large-N approximation allows us to sum amplitudes to all orders in the ratio of momenta to the small vacuum expectation value v, provided the momenta are small compared with all the other mass scales characteristic of $H[\lambda]$. Equation (68) shows that in this case the Goldstone boson scattering amplitude has poles at $s=-m^2$, $t=-m^2$, and $u=-m^2$, with m given in terms of v by

$$v^2=-m^2A(-m^2)\to-\frac{m^2\ln(-m^2/\mathcal{M}^2)}{32\pi^2}, \qquad (78)$$

indicating the presence of an unstable light O($N-1$)-singlet particle with complex mass m. This is not unexpected; continuity suggests that the $N-1$ Goldstone bosons should be joined near the phase transition by an additional scalar whose mass must vanish at the phase transition, in order to allow a smooth transition to the unbroken symmetry phase, where the N degenerate scalars become massless at the phase transition. The mass m given by Eq. (78) is complex because this scalar can decay into Goldstone bosons.

There are other possibilities: near the phase transition the unbroken phase could have a degenerate multiplet of light scalars belonging to any representation of O(N) that contains the ($N-1$)-vector representation of O($N-1$), not necessarily the defining representation. The Goldstone bosons of the broken symmetry phase would then be joined at the phase transition with those massless scalars that are needed to fill out this representation when O(N) is restored. Our transformation of the theory has emphasized the possibility that near the phase transition the light degenerate multiplet of the unbroken phase forms an N-vector, but of course we do not know that the c_1 parameter encountered in this transformed theory *is* small. To explore other possible types of phase transition, we would have to transform the theory in other ways, and then assume that the parameter corresponding to c_1 in those transformed theories is small.

The smallness of c_1 opens up a much more powerful role for the large-N approximation. The expectation value $\langle\lambda\rangle$ is then small or zero, and the propagator of ϕ goes as k^{-2} for a four-momentum k which though small is larger than $\langle\lambda\rangle$. On the other hand, the term in the action of second order in λ has a momentum-independent term which is not small near the phase transition, so the λ propagator must be regarded as of zeroth order in momenta. We can count powers of momentum and/or $\sqrt{c_1}$ in any diagram by dimensional analysis, with the fields ϕ_r and λ taken as having dimensions one and two (in powers of momentum), respectively.

With this understanding, the action (58) contains one superrenormalizable (‘‘relevant’’) term $c_{1\mathrm{B}}\int d^4x\lambda$; three renormalizable (‘‘marginal’’) terms

$$-c_{2\mathrm{B}}\int d^4x\lambda^2, \quad -\tfrac{1}{2}\int d^4x\,\partial_\mu\phi_r\partial^\mu\phi_r,$$

$$-\tfrac{1}{2}\int d^4x\lambda\,\phi_r\phi_r;$$

and an infinite number of nonrenormalizable (‘‘irrelevant’’) terms, including terms of second order in derivatives of λ that act as corrections to the momentum-independent zeroth-order λ propagator. [The subscript B on $c_{1\mathrm{B}}$ and $c_{2\mathrm{B}}$ indicates that these are bare couplings, chosen to give finite values to the c_1 and c_2 in Eq. (62). To leading order in $1/N$ there is no renormalization of the coefficients of $\int d^4x\,\partial_\mu\phi_r\partial^\mu\phi_r$ and $\int d^4x\lambda\,\phi_r\phi_r$; these coefficients are fixed to be $-1/2$ by a choice of normalization of λ and ϕ_r.] The presence of a superrenormalizable term prevents an expansion in powers of momenta alone, but near the phase transition with c_1 small, we can expand any scattering amplitude in powers of the overall scale of momenta and $\sqrt{c_1}$. The leading term in this expansion is given by Feynman diagrams involving only the renormalizable and superrenormalizable interactions listed above.

The presence of the renormalizable interaction $\int d^4x\lambda\,\phi_r\phi_r$ means that there is an infinite number of multiloop graphs of leading order in momenta and/or $\sqrt{c_1}$. Here, the large-N limit offers the huge simplification, of reducing the complete quantum effective action to the simple form (60), only now with $H[\lambda]$ containing only terms linear and quadratic in λ:

$$\Gamma[\phi,\lambda]=-\int d^4x[\tfrac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r+\tfrac{1}{2}\lambda\,\phi_r\phi_r] -Nc_{1\mathrm{B}}\int d^4x\lambda-Nc_{2\mathrm{B}}\int d^4x\lambda^2 +\tfrac{1}{2}iN\mathrm{Tr}\ln(\Box-\lambda). \qquad (79)$$

Using Eq. (79) in the tree approximation gives the terms in scattering amplitudes of leading order in $1/N$ and in small momenta and $\sqrt{c_1}$. For instance, in the broken symmetry phase the function $A(k^2)$ appearing in Eqs. (66)–(69) is here simply given by the first term in Eq. (67)

$$A(k^2)=\frac{\ln(k^2/\mathcal{M}^2)}{32\pi^2}. \qquad (80)$$

For $v\to0$ there is just one solution of the equation $v^2=k^2A(k^2)$ with $k^2\to0$ [the only case where Eq. (80) can be trusted]. This solution has Re $k^2<0$, and corresponds to an unstable scalar particle that can decay into pairs of Goldstone bosons.

IV. THE EXTENDED CP^{N-1} MODEL: INTEGRATING IN A GAUGE FIELD

In addition to helping us to count factors of $1/N$ and enforcing constraints on the fields, auxiliary fields are sometimes introduced in order to enforce a condition of gauge invariance. The leading example of this sort is the CP^{N-1} model [4] which in its original form has an action given by Eqs. (11) and (12). Here, we shall consider a class of extensions of the CP^{N-1} model in which nontrivial finite results may be obtained in the limit of large N in four spacetime dimensions. As we shall see, just as in its original two-dimensional version, this model has the remarkable property that a long-range Coulomb force arises even though no elementary gauge field is introduced into the action.

The extended CP^{N-1} models to be considered here contain a set of N complex scalar fields u_r, subject to the constraint that

$$u_r^\dagger(x)u_r(x)=N. \tag{81}$$

The action is invariant under ''gauge'' transformations

$$\delta u_r(x)=i\epsilon(x)u_r(x), \tag{82}$$

with $\epsilon(x)$ an arbitrary real infinitesimal function. For a soluble model which yields nontrivial finite results, it turns out to be sufficient to take the action in the form

$$I[u]=Nf^2\int d^4x(-b+a_\mu a^\mu)+Nf^2F[a,b], \tag{83}$$

where

$$a_\mu\equiv-(i/2N)[u_r^\dagger\partial_\mu u_r-(\partial_\mu u_r^\dagger)u_r], \tag{84}$$

$$b\equiv(1/N)(\partial_\mu u_r^\dagger)\partial^\mu u_r, \tag{85}$$

and $F[a,b]$ is an arbitrary N-independent Lorentz-invariant perturbatively local functional, invariant under the transformation induced by the gauge transformation (82):

$$\delta b(x)=2a^\mu(x)\partial_\mu\epsilon(x), \quad \delta a_\mu(x)=\partial_\mu\epsilon(x). \tag{86}$$

The first term in Eq. (83) is a rewritten version of the action (11), (12) of the original CP^{N-1} model; this term could have been included in $NF[a,b]$, but it is convenient to display the kinematic part of the action explicitly. In principle we should include all SU(N)-invariant bilinear currents in addition to a_μ and b, but the effects of other currents are suppressed at small momenta, and the addition of an arbitrary functional of a_μ and b is enough to allow the cancellation of infinities. Note that $a^\mu(x)$ is now given by Eq. (84), and is *not* taken as an independent field, because we will need to include terms in $F[a,b]$ involving $\partial_\mu a_\nu-\partial_\nu a_\mu$, and we are trying to see how a Maxwell field can arise without its being put in from the beginning. The photon will appear here in quite a different way.

For the sake of variety, we will take a different approach to counting powers of $1/N$ here, which gives the same result as the functional Fourier transform used in the previous sections. We introduce a *pair* of new auxiliary fields $\alpha_\mu(x)$, $\rho_\mu(x)$ and $\beta(x)$, $\sigma(x)$ for each of the bilinears $a_\mu(x)$ and $b(x)$ appearing in the action, writing Eq. (83) in the equivalent form

$$I=Nf^2\int d^4x[-b+\alpha_\mu\alpha^\mu+\sigma(\beta-b)$$
$$+\rho^\mu(\alpha_\mu-a_\mu)]+Nf^2F[\alpha,\beta]. \tag{87}$$

Integrating out the $\sigma(x)$ and $\rho_\mu(x)$ yields δ functions which set $\beta(x)=b(x)$ and $\alpha_\mu(x)=a_\mu(x)$, taking us back to Eq. (83). Instead, we shall first integrate out the $\alpha_\mu(x)$ and $\beta(x)$. Since the terms in Eq. (87) that depend on $\alpha_\mu(x)$ or $\beta(x)$ are simply proportional to N and do not depend on the u_r, in the limit of large N we can set these fields equal to the values at which Eq. (87) is stationary with respect to $\alpha_\mu(x)$ and $\beta(x)$, giving

$$I=-Nf^2\int d^4x[(1+\sigma)b+\rho^\mu a_\mu]+Nf^2G[\rho,\sigma], \tag{88}$$

where G is the Legendre transform of $F+\int d^4x\,\alpha_\mu\alpha^\mu$,

$$G[\rho,\sigma]=\Big[F[\alpha,\beta]+\int d^4x\,\alpha_\mu\alpha^\mu$$
$$+\int d^4x(\rho^\mu\alpha_\mu+\sigma\beta)\Big]_{\mathrm{staty}}, \tag{89}$$

with the subscript ''staty'' meaning that we set $\alpha_\mu(x)$ and $\beta(x)$ equal to values satisfying conditions that make the quantity in square brackets stationary:

$$\frac{\delta F}{\delta\alpha^\mu(x)}=-2\alpha_\mu(x)-\rho_\mu(x), \quad \frac{\delta F}{\delta\beta(x)}=-\sigma(x). \tag{90}$$

The Legendre transform of a general perturbatively local functional is just another general perturbatively local functional, so we can regard $G[\rho,\sigma]$ as arbitrary, except for a gauge invariance condition. Using Eq. (90) and the invariance of $F[a,b]$ under the transformation (86), we easily see that

$$0=\int d^4x\,\partial_\mu\epsilon\left(2\alpha^\mu\frac{\delta F[\alpha,\beta]}{\delta\beta}+\frac{\delta F[\alpha,\beta]}{\delta\alpha_\mu}\right)$$
$$=\int d^4x\,\partial_\mu\epsilon\left(-2(1+\sigma)\frac{\delta G[\rho,\sigma]}{\delta\rho_\mu}-\rho^\mu\right). \tag{91}$$

It follows that we can define a new functional

$$G'[\rho,\sigma]\equiv G[\rho,\sigma]+\frac{1}{4}\int d^4x\frac{\rho_\mu\rho^\mu}{1+\sigma} \tag{92}$$

that is invariant under the transformations

$$\delta\rho_\mu=-2(1+\sigma)\partial_\mu\epsilon, \quad \delta\sigma=0. \tag{93}$$

This suggests that we should define a gauge field

$$A_\mu\equiv-\frac{\rho_\mu}{2(1+\sigma)}, \tag{94}$$

which according to Eq. (93) has the gauge transformation property

$$\delta A_\mu=\partial_\mu\epsilon. \tag{95}$$

In the original version of the CP^{N-1} model, $F=0$, and then Eq. (90) shows that $\rho_\mu=-2\alpha_\mu=-2a_\mu$ and $\sigma=0$, so Eq. (94) gives $A_\mu=a_\mu$. But in the general case with $F\neq0$, it is incorrect to identify A_μ with a_μ.

We are not yet ready to add a Lagrange multiplier term $-\int d^4x\lambda(u_r^\dagger u_r-N)$ and integrate out the u_r fields, because then we would again encounter an infinite term proportional to $\int d^4x\lambda^2$, and it is not yet clear how this could be canceled. Instead, we will first redefine the fields to introduce an order parameter, as we did in the previous section for the nonlinear σ model. Define

$$z_r \equiv f\sqrt{1+\sigma}u_r, \tag{96}$$

subject to the constraint

$$z_r^\dagger z_r = Nf^2(1+\sigma). \tag{97}$$

The bilinears (84) and (85) then take the form

$$a_\mu = \frac{-i}{2f^2N(1+\sigma)}[z_r^\dagger \partial_\mu z_r - (\partial_\mu z_r)^\dagger z_r], \tag{98}$$

$$b = \frac{1}{f^2N(1+\sigma)}\partial_\mu z_r^\dagger \partial^\mu z_r - \frac{1}{4(1+\sigma)^2}\partial_\mu\sigma\partial^\mu\sigma. \tag{99}$$

The action given by Eq. (88) now may be written

$$I = -\int d^4x (D_\mu z_r)^\dagger D^\mu z_r + Nf^2 G''[A,\sigma], \tag{100}$$

where D_μ is the gauge-covariant derivative

$$D_\mu z_r \equiv \partial_\mu z_r - iA_\mu z_r, \tag{101}$$

and G'' is another arbitrary gauge-invariant perturbatively local functional

$$G''[A,\sigma] \equiv \frac{1}{4}\int d^4x \frac{\partial_\mu\sigma\partial^\mu\sigma}{1+\sigma} + G'[\rho,\sigma]. \tag{102}$$

Now, we may enforce the constraint (97) by introducing a Lagrange multiplier term

$$-\int d^4x \lambda[z_r^\dagger z_r - Nf^2(1+\sigma)], \tag{103}$$

which preserves gauge invariance if we define $\lambda(x)$ to be gauge invariant. After integrating out the σ field, the action becomes

$$I = -\int d^4x (D_\mu z_r)^\dagger D^\mu z_r - \int d^4x \lambda z_r^\dagger z_r + NH[A,\lambda], \tag{104}$$

where $H[A,\lambda]$ is yet another Legendre transform

$$H[A,\lambda] = f^2\left[G''[A,\sigma] + \int d^4x(1+\sigma)\lambda\right]_{\sigma=\sigma^\lambda}, \tag{105}$$

with $\sigma^\lambda(x)$ equal to the $\sigma(x)$ at which the quantity in square brackets on the right-hand side of Eq. (105) is stationary

$$\left.\frac{\delta G''[A,\sigma]}{\delta\sigma(x)}\right|_{\sigma=\sigma^\lambda} = -\lambda(x). \tag{106}$$

The z_r field is now unconstrained.

We want to calculate the effective action $\Gamma[z,A,\lambda]$, which in the tree approximation gives the same result as the sum of all loop and tree graphs calculated using the action (104). Following the same reasoning as in Sec. II (with z_r replacing ψ_r, and λ and A replacing σ), we find that this is given to leading order in $1/N$ by

$$\Gamma[z,A,\lambda] = -\int d^4x (D_\mu z_r)^\dagger D^\mu z_r - \int d^4x \lambda z_r^\dagger z_r + \Gamma[A,\lambda], \tag{107}$$

where

$$\Gamma[A,\lambda] = iN\mathrm{Tr}\,\ln[D_\mu D^\mu - \lambda] + NH[A,\lambda]. \tag{108}$$

Gauge invariance and dimensional analysis tell us that the infinite part of the first term in Eq. (108) is a linear combination of the gauge-invariant functionals $\int d^4x\lambda$, $\int d^4x\lambda^2$, and $\int d^4x(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$. But $H[A,\lambda]$ is an arbitrary perturbatively local functional, constrained only by invariance under the gauge transformation (95), so there is no problem in adjusting it to cancel these infinities.

Like the nonlinear sigma model, the CP^{N-1} model can exist in several phases, characterized here by different spacetime-independent vacuum expectation value of the scalar fields z_r and λ, with $A^\mu(x)=0$. In analyzing these phases, we shall make use of the fact that for $A^\mu(x)=0$ and $\lambda(x)$ constant, $\Gamma[A,\lambda]$ may be expressed as in Eq. (63) in terms of an effective potential $V(\lambda)$

$$\Gamma[0,\lambda] = -\mathcal{V}_4 V(\lambda), \tag{109}$$

with $\mathcal{V}_4$ the spacetime volume. The effective potential here is given by a formula such as Eq. (62),

$$V(\lambda) = \frac{N\lambda^2\ln(\lambda/M^2)}{32\pi^2} + N\sum_{n=1}^{\infty} c_n\lambda^n, \tag{110}$$

with c_n a set of new constant coefficients depending on $H[0,\lambda]$, and M a new constant that can be chosen to make $c_2=0$. (The coefficient in the first term is $1/32\pi^2$ instead of $1/64\pi^2$ because z_r is complex.) In analyzing the vector particle mass and the ''charge'' of the z_r particles, we will also need to study the term in $\Gamma[A,\lambda]$ of second order in the photon field for constant $\lambda(x)$, which gauge invariance requires must take the form

$$\Gamma^{(2)}[A,\lambda] = \frac{N}{2}\int d^4x A^\mu(x)(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)f(-\Box,\lambda)A^\nu(x). \tag{111}$$

Evaluating the trace in Eq. (108), gives

$$f(q^2,\lambda) = -\frac{1}{16\pi^2}\int_0^1 dx(1-2x)^2\ln\left(\frac{\lambda+q^2x(1-x)}{W^2}\right) + \sum_{n=0}^{\infty} f_n(q^2)\lambda^n, \tag{112}$$

where $f_n(q^2)$ are N-independent functions of q^2 analytic at $q^2=0$, arising from the unknown functional $H[A,\lambda]$, and W is another mass parameter, which can be chosen to make $f_0(0)=0$.

A. Broken symmetry phase

In this phase $z_r(x)$ has a nonvanishing vacuum expectation value $\sqrt{N}v_r$ while the vacuum expectation values of $\lambda(x)$ and $A^\mu(x)$ both vanish, which requires that

$$\left.\frac{\partial V(\lambda)}{\partial\lambda}\right|_{\lambda=0}=-Nv^2, \tag{113}$$

where $v^2\equiv v_r^* v_r$. Since $\Gamma\propto N$, v_r is N independent. For small constant λ, the effective potential (110) is

$$V(\lambda)\to\frac{N\lambda^2\ln(\lambda/M^2)}{32\pi^2}+Nc_1\lambda. \tag{114}$$

Hence condition (113) gives

$$c_1=-v^2. \tag{115}$$

In analyzing the degrees of freedom in this phase, it is very convenient to eliminate the scalar-vector mixing in Eq. (107) by adopting unitarity gauge, in which $\mathrm{Im}(v_r^* z_r)=0$. Taking $v_N=v$ real and $v_i=0$ for $i=1,\ldots,N-1$, this means that z_N is real, while the z_i are still complex. The z_i are massless Goldstone boson fields, while z_N has the same sort of mixing with λ that we saw in the previous section; the terms in the action of second order in λ and/or $z_N-\sqrt{N}v$ have a coefficient matrix given by

$$\mathcal{D}_{NN}(k^2)=k^2,\quad \mathcal{D}_{\lambda\lambda}(k^2)=NA(k^2),$$
$$\mathcal{D}_{N\lambda}(k)=\mathcal{D}_{\lambda N}(k)=\sqrt{N}v, \tag{116}$$

where now

$$A(k^2)=\frac{\ln(k^2/\mathcal{M}^2)}{16\pi^2}+\sum_{n=1}^{\infty} f_n(k^2)^n, \tag{117}$$

with f_n yet another set of model-dependent constants, and $\mathcal{M}$ is a constant chosen so that the term f_0 in the sum is absent. The scalar mass m is given by the condition that this has a zero determinant at $k^2=-m^2$:

$$-m^2A(-m^2)=v^2. \tag{118}$$

Without further information about the functional $H[0,\lambda]$, we cannot tell whether there actually is a massive scalar in the spectrum of z_N and λ.

To study the vector particles in this phase, we note that according to Eq. (107), the term in $\Gamma[v,A,0]$ of second order in the photon field is given in this phase by

$$\Gamma^{(2)}[\sqrt{N}v,A,0]=-Nv^2\int d^4x A_\mu(x)A^\mu(x)+\Gamma^{(2)}[A,0], \tag{119}$$

where $\Gamma^{(2)}[A,0]$ is defined by Eq. (111). There is a vector particle of mass $\mu\neq 0$ if

$$\mu^2 f(-\mu^2,0)=2v^2. \tag{120}$$

Without special assumptions about $H[A,0]$, it is not possible to tell this has a solution, much less to calculate the vector boson mass μ. But it is clear that any massive scalar or vector particles would have to be unstable, because they could decay into the Goldstone bosons z_i.

B. Unbroken symmetry phase

In this phase λ has a nonzero vacuum expectation value λ_0, satisfying the condition

$$\left.\frac{\partial V(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_0}=0, \tag{121}$$

which allows $z_r(x)$ as well as $A^\mu(x)$ to have vanishing expectation values. Equation (107) shows that in this phase the z_r have squared mass λ_0, so λ_0 must be positive. The photon propagator in momentum space equals

$$\Delta_{\mu\nu}(k)=\frac{\eta_{\mu\nu}}{k^2Nf(k^2,\lambda_0)}+\ \text{gauge terms},$$

where ''gauge terms'' denote gauge-dependent terms proportional to $k_\mu k_\nu$. Because the z_r for $\lambda_0\neq 0$ have a finite mass the function $f(k^2,\lambda_0)$ is analytic at $k^2=0$, so the photon here is massless. Also, the renormalized gauge field is $\sqrt{Nf(0,\lambda_0)}A^\mu$, so the z_r charge is $1/\sqrt{Nf(0,\lambda_0)}$, and is hence of order $1/\sqrt{N}$. Without making special assumptions about the functional $H[A,\lambda]$, it is impossible to say anything more about the values of the z_r squared mass λ_0 or the z_r charge.

C. Phase transition

As in the case of the nonlinear σ model, we can obtain more detailed results when the model is near a phase transition between the broken and unbroken symmetry phases. In the unbroken symmetry phase, the model is near this phase transition if the λ_0 satisfying Eq. (121) is positive and small. In this case Eqs. (110) and (121) give

$$c_1=-\frac{\lambda_0\ln(e^{1/2}\lambda_0/M^2)}{16\pi^2}, \tag{122}$$

which has positive small solutions for λ_0 as long as c_1 be small and positive. On the other hand, in the broken symmetry phase the model is near the phase transition if v is small, which according to Eq. (115) requires that c_1 is small and negative. Thus there is a second-order phase transition at $c_1=0$, irrespective of the values of other parameters.

To analyze the low-momentum limit near a phase transition, we note that the action (104) contains a single super-renormalizable term $-Nc_{1\mathrm{B}}\int d^4x\lambda$; four strictly renormalizable terms

$$-Nc_{2\mathrm{B}}\int d^4x\lambda^2;\quad -\tfrac{1}{4}NZ\int d^4x(\partial_\mu A_\nu-\partial_\nu A_\mu)^2;$$
$$-\int d^4x(D_\mu z_r)^\dagger D^\mu z_r;\quad -\int d^4x\lambda z_r^\dagger z_r;$$

and an infinite number of nonrenormalizable terms. [The subscript B again indicates bare values, adjusted to give finite values to c_1 and c_2 in Eq. (110).] In the limit where

c_1 and all momenta are small, we can ignore the nonrenormalizable interactions, and calculate scattering amplitudes by using the quantum effective interaction (107) in the tree approximation, now with

$$\Gamma[A,\lambda]=iN\text{Tr}\,\ln[D_\mu D^\mu-\lambda]-Nc_{1\text{B}}\int d^4x\lambda -Nc_{2\text{B}}\int d^4x\lambda^2-\tfrac{1}{4}NZ\int d^4x(\partial_\mu A_\nu-\partial_\nu A_\mu)^2. \tag{123}$$

This tells us that, for example, the potential $V(\lambda)$ is given by Eq. (114); that for constant $\lambda(x)$ the function $f(q^2,\lambda)$ appearing in the formula (111) for the term in $\Gamma[A,\lambda]$ of second order in the vector field is here

$$f(q^2,\lambda)=-\frac{1}{16\pi^2}\int_0^1 dx(1-2x)^2\ln\left(\frac{\lambda+q^2x(1-x)}{W^2}\right); \tag{124}$$

and that the function $A(k^2)$ appearing in the scalar two-point function (116) is

$$A(k^2)=\frac{\ln(k^2/\mathcal{M}^2)}{32\pi^2}. \tag{125}$$

As an example of the use of these results, let us look more closely at the properties of the particles near the phase transition. In the unbroken symmetry phase for small λ_0, Eq. (124) gives

$$f(0,\lambda_0)\rightarrow-\frac{N}{48\pi^2}\ln\left(\frac{\lambda_0}{W^2}\right). \tag{126}$$

This is positive but diverges for $\lambda_0\rightarrow 0$, so that the z_r charge $1/\sqrt{Nf(0,\lambda_0)}$ vanishes at the phase transition. In the broken symmetry phase the vector boson mass is determined by the function $f(q^2,0)$, which for $q^2\rightarrow 0$ is given by Eq. (124) as

$$f(q^2,0)\rightarrow-\frac{1}{48\pi^2}\left[\ln\left(\frac{q^2}{W^2}\right)-\frac{8}{3}\right]. \tag{127}$$

Equation (120) for the vector boson squared mass μ^2 has a single solution that vanishes as $v\rightarrow 0$, indicating the presence of a single light vector particle. Also, in the broken symmetry phase near the phase transition, Eq. (118) for the scalar mass m takes the form

$$\frac{-m^2\ln(-m^2/\mathcal{M}^2)}{32\pi^2}=v^2. \tag{128}$$

This has one solution for m^2 that vanishes as $v\rightarrow 0$, indicating the presence of single massive but light scalar particle. These solutions for μ^2 and m^2 both have positive real part but are complex, reflecting the fact that both of these particles are unstable, because they can decay into pairs of Goldstone bosons.

V. DYNAMICAL GAUGE BOSONS: A REMARK

The CP^{N-1} model has attracted much attention because of the appearance of a massless gauge boson in a theory involving only scalar fields. It is important to recognize that this phenomenon does not depend on the existence of the gauge symmetry (82), or indeed on any of the symmetry properties of the action.

This can be seen by a very general argument [11]. Consider a theory that is invariant under a gauge group G, with various matter multiplets forming various representations of G. Suppose that one of these multiplets consists of scalar fields, some of which have vacuum expectation values that completely break the gauge symmetry. Integrate out the massive gauge vector bosons in unitarity gauge. We then have a perturbatively local effective field theory, with no hint of the original gauge invariance. It seems pretty clear that if we allow arbitrary interactions in the original theory, then in this way we obtain a completely general effective field theory of the remaining fields. But this procedure can be reversed, so *out of any effective field theory with no gauge symmetry and possibly no global symmetry, we can obtain a theory with any broken gauge symmetry.*

The point is that a spontaneously broken gauge symmetry in itself has no predictive power [12]. Of course, it can have plenty of predictive power if the gauge coupling is *weak*, but for this we have to fine-tune the parameters in the action. In the CP^{N-1} model studied in the previous section, this fine-tuning is achieved by the condition that c_1 is small.

To illustrate the possibility of constructing a broken gauge symmetry in an effective field theory that has no symmetry to begin with, consider a theory of Dirac fields $\psi_i(x)$ with an action of form

$$I[\psi]=-\int d^4x\sum_i\overline{\psi_i}\gamma^\mu\partial_\mu\psi_i-G[\psi], \tag{129}$$

where $G[\psi]$ is an essentially arbitrary perturbatively local functional of the fermion fields. We can choose $G[\psi]$ so that this action has *no* internal symmetries, if we like, not even fermion conservation. This action can be obtained by integrating out a vector field $A_\mu(x)$ in the action

$$I[\psi,A]=-\int d^4x\sum_i\overline{\psi_i}\gamma^\mu\partial_\mu\psi_i-G'[\psi]-\frac{M^2}{2}\int d^4xA_\mu A^\mu +\int d^4xA_\mu j^\mu-\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}, \tag{130}$$

where M is an arbitrary mass parameter, $F_{\mu\nu}(x)\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$, and

$$G'[\psi]\equiv G[\psi]-\frac{1}{2}\int d^4xj_\mu(x)\frac{1}{M^2-\Box}j^\mu(x) -\frac{1}{2M^2}\int d^4x\partial_\nu j^\nu(x)\frac{1}{M^2-\Box}\partial_\mu j^\mu(x), \tag{131}$$

where $j^\mu(x)$ is the current

$$j^\mu \equiv \sum_i q_i \overline{\psi_i} \gamma^\mu \psi_i , \qquad (132)$$

with q_i an arbitrary set of real parameters. As long as $M \neq 0$, G' is still perturbatively local. The action (130) can be obtained from another action

$$I[\psi,A,u] = -\frac{M^2}{2}\int d^4x |\partial_\mu u - iA_\mu u|^2 - \int d^4x \sum_i \overline{\psi_i}\gamma^\mu(\partial_\mu - iq_i A_\mu)\psi_i - \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + G'[\psi'] \qquad (133)$$

with u a scalar field constrained by $|u|^2=1$, and

$$\psi_i' \equiv \psi_i u^{-q_i}. \qquad (134)$$

The action (133) is invariant under the gauge transformation

$$\psi_i(x) \to e^{iq_i\alpha(x)}\psi_i(x), \quad u(x) \to e^{i\alpha(x)}u(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x), \qquad (135)$$

so the action (130) can be obtained from Eq. (133) by adopting the unitarity gauge, in which $u=1$. [The action (133) is also perturbatively local, because in deriving perturbation theory, we expand around $u=1$ rather than $u=0$.] Yet, there is no trace of this gauge invariance or even global invariance in the action (129) with which we started.

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