Heavy quark effective theory and nonrelativistic QCD Lagrangian to order α_s/m^3

Aneesh V. Manohar
Department of Physics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0319
(Received 17 January 1997)

The HQET and NRQCD Lagrangian is computed to order α_s/m^3. The computation is performed using dimensional regularization to regulate the ultraviolet and infrared divergences. The results are consistent with reparametrization invariance to order $1/m^3$. Some subtleties in the matching conditions for NRQCD are discussed. [S0556-2821(97)00813-8]

PACS number(s): 12.39.Hg, 12.20.−m, 12.38.Bx

I. INTRODUCTION

Heavy quark effective theory (HQET) [1] and nonrelativistic QCD (NRQCD) [2,3] are two effective theories that describe the interactions of almost on-shell heavy quarks. HQET describes the interactions of quarks of mass m in which the momentum transfer p is much smaller than m. The HQET Lagrangian has an expansion in powers of p/m. HQET is typically applied to hadrons containing a single heavy quark, such as the B meson, in which $p\sim\Lambda_{\rm QCD}$, the scale of the strong interactions. The HQET expansion is thus an expansion in powers of $\Lambda_{\rm QCD}/m$. NRQCD describes the interactions of nonrelativistic quarks, and is typically applied to $\overline{Q}Q$ bound states such as the Υ. The NRQCD Lagrangian also has an expansion in powers of $1/m$. The momentum transfer in NRQCD is of order mv, so that the small expansion parameter in NRQCD is the velocity v. The size of a term in the NRQCD Lagrangian can be estimated using velocity counting rules [3]. The basic difference between HQET and NRQCD can be seen from the first two terms in the effective Lagrangian:

$$\mathcal{L}=Q^\dagger(iD^0)Q+Q^\dagger\frac{\mathbf{D}^2}{2m}Q. \tag{1}$$

In HQET, the first term is of order $\Lambda_{\rm QCD}$, and the second term is of order $\Lambda^2_{\rm QCD}/m$, whereas in NRQCD both terms are of order mv^2. As a result, the quark propagator in HQET is $i/(k^0+i\epsilon)$, and in NRQCD it is

$$\frac{i}{(k^0-\mathbf{k}^2/2m+i\epsilon)}. \tag{2}$$

The HQET and NRQCD Lagrangian is computed in this paper to one loop and order $1/m^3$. Only the terms bilinear in fermions are considered here. There are also four-quark operators in the effective Lagrangian. Their coefficients are order α_s, and can be obtained simply from tree-level matching.

II. MATCHING CONDITIONS AND POWER COUNTING

The HQET effective theory matching computation is a straightforward generalization of known results to order $1/m^2$ [4–8]. One can compute diagrams in the full and effective theories, and match to a given order in $1/m$. Since the HQET propagator is m independent, the HQET power counting is manifest — one counts powers of $1/m$ directly from the vertex factors. This means that graphs with a vertex of order $1/m^r$ do not make any contributions to terms of order $1/m^s$, with $s<r$ to any order in the loop expansion.

The use of NRQCD as an effective field theory is more subtle. NRQCD with the propagator, Eq. (2), cannot be used as an effective Lagrangian to compute matching corrections, since the velocity power counting breaks down.[1] The matching conditions for NRQCD should be computed using the HQET power counting, by expanding in p^μ/m. After the HQET Lagrangian has been computed, it can be used for computing bound state properties using the NRQCD velocity power counting rules. In other words, the NRQCD propagator Eq. (2) should be thought of as the infinite series

$$\frac{1}{k^0-\mathbf{k}^2/2m+i\epsilon}=\frac{1}{k^0}+\frac{\mathbf{k}^2}{2m(k^0)^2}+\cdots, \tag{3}$$

where one uses the right-hand side inside any ultraviolet divergent Feynman graph. This is necessary when a cutoff such as dimensional regularization is used to regulate the Feynman graphs. NRQED matching conditions have previously been computed using a momentum space cutoff [9]. In this case, there is no difference between using the left- or right-hand sides of Eq. (3). However, a momentum space cutoff cannot be used in NRQCD, since it breaks gauge invariance.

The difference between using the two forms of Eq. (3) in a loop graph can be illustrated by a simple example. Consider the integral

$$\int_0^\infty dk^2\,\frac{(k^2)^a}{(k^2+m_1^2)(k^2+m_2^2)}=\frac{\pi}{\sin\pi a}\,\frac{(m_1^2)^a-(m_2^2)^a}{m_1^2-m_2^2}. \tag{4}$$

The denominator of a typical NRQCD loop graph has poles at $k^2\sim p^2$, where p is the external momentum, *and* at $k^2\sim m^2$ where m is the quark mass. The graph can be written in the form Eq. (4), where the scales m_1 and m_2 in Eq. (4) can be taken to be of order p and m, respectively. The power a increases as one considers more and more divergent loop

[1]I would like to thank M. Luke for extensive discussions on this point. Some of these issues will be discussed in a future publication. See also [15].

graphs in the effective theory. One can see immediately that the NRQCD power counting breaks down. Loop graphs with insertions of higher dimension operators are divergent, and can be proportional to positive powers of m because of the $(m_2^2)^a$ term in Eq. (4). The positive powers of m from the loop integral can compensate for inverse powers of m in the coefficient, and the entire effective Lagrangian expansion breaks down. Now consider the same integral, but first expand

$$\frac{1}{k^2+m_2^2}=\frac{1}{k^2}-\frac{m_2^2}{k^4}+\cdots,$$

evaluate the integral, and then resum the series. The answer is

$$\int_0^\infty dk^2\,\frac{(k^2)^a}{(k^2+m_1^2)(k^2+m_2^2)}=\frac{\pi}{\sin\pi a}\,\frac{(m_1^2)^a}{m_1^2-m_2^2}. \tag{5}$$

The integral is missing the $(m_2^2)^a$ term since it is nonanalytic at the origin for $a\neq$integer, which is where the integral is evaluated in $4-\epsilon$ dimensions. Equation (5) only has inverse powers of the high momentum scale $m_2\sim m$, and leads to an acceptable effective field theory. Thus NRQCD and HQET matching conditions are computed in the same way, and the two Lagrangians are the same.

III. THE LAGRANGIAN

The continuum NRQED effective Lagrangian at one loop has previously been computed [9] using a photon mass to regulated the infrared divergences, and a momentum space cutoff. This procedure cannot be used in a non-abelian gauge theory such as QCD. The kinetic terms in the NRQCD Lagrangian at one loop have previously been computed by Morningstar [10] using a lattice regulator. The computations in this article will be done in the continuum using dimensional regularization for the infrared and ultraviolet divergences, and for on-shell external states. This has the advantage that one can freely use the equations of motion to reduce the number of operators in the effective Lagrangian [11]. The most general effective Lagrangian to order $1/m^3$ (up to field redefinitions) is

$$\begin{aligned}\mathcal{L}=Q^\dagger\Big\{ & iD^0+c_2\frac{\mathbf{D}^2}{2m}+c_4\frac{\mathbf{D}^4}{8m^3}+c_F g\frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m}+c_D g\frac{[\mathbf{D}\cdot\mathbf{E}]}{8m^2}+ic_S g\frac{\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D})}{8m^2}+c_{W1}g\frac{\{\mathbf{D}^2,\boldsymbol{\sigma}\cdot\mathbf{B}\}}{8m^3}-c_{W2}g\frac{\mathbf{D}^i\boldsymbol{\sigma}\cdot\mathbf{B}\mathbf{D}^i}{4m^3}\\ &+c_{p'p}g\frac{\boldsymbol{\sigma}\cdot\mathbf{D}\mathbf{B}\cdot\mathbf{D}+\mathbf{D}\cdot\mathbf{B}\boldsymbol{\sigma}\cdot\mathbf{D}}{8m^3}+ic_M g\frac{\mathbf{D}\cdot[\mathbf{D}\times\mathbf{B}]+[\mathbf{D}\times\mathbf{B}]\cdot\mathbf{D}}{8m^3}+c_{A1}g^2\frac{\mathbf{B}^2-\mathbf{E}^2}{8m^3}-c_{A2}g^2\frac{\mathbf{E}^2}{16m^3}+c_{A3}g^2\mathrm{Tr}\left(\frac{\mathbf{B}^2-\mathbf{E}^2}{8m^3}\right)\\ &-c_{A4}g^2\mathrm{Tr}\left(\frac{\mathbf{E}^2}{16m^3}\right)+ic_{B1}g^2\frac{\boldsymbol{\sigma}\cdot(\mathbf{B}\times\mathbf{B}-\mathbf{E}\times\mathbf{E})}{8m^3}-ic_{B2}g^2\frac{\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{E})}{8m^3}\Big\}Q,\end{aligned} \tag{6}$$

which is the HQET-NRQCD Lagrangian in the special frame $v=(1,0,0,0)$, and the notation of [9] has been used. The covariant derivative is $D^\mu=\partial^\mu+igA^{\mu a}T^a=(D^0,-\mathbf{D})$. Covariant derivatives in square brackets act only on the fields within the brackets. The other covariant derivatives act on all fields to the right. The subscripts F, S, and D stand for Fermi, spin-orbit, and Darwin, respectively. The last seven terms in Eq. (6) were not given in Ref. [9], since they were not required for the computation done there. The last four terms can be omitted for QED.

In an arbitrary frame, Eq. (6) can be written as

$$\begin{aligned}\mathcal{L}_v=\overline{Q}_v\Big\{ & iD\cdot v-c_2\frac{D_\perp^2}{2m}+c_4\frac{D_\perp^4}{8m^3}-c_F g\frac{\sigma_{\alpha\beta}G^{\alpha\beta}}{4m}-c_D g\frac{v^\alpha[D_\perp^\beta G_{\alpha\beta}]}{8m^2}+ic_S g\frac{v_\lambda\sigma_{\alpha\beta}\{D_\perp^\alpha,G^{\lambda\beta}\}}{8m^2}+c_{W1}g\frac{\{D_\perp^2,\sigma_{\alpha\beta}G^{\alpha\beta}\}}{16m^3}\\ &-c_{W2}g\frac{D_\perp^\lambda\sigma_{\alpha\beta}G^{\alpha\beta}D_{\perp\lambda}}{8m^3}+c_{p'p}g\frac{\sigma^{\alpha\beta}(D_\perp^\lambda G_{\lambda\alpha}D_{\perp\beta}+D_{\perp\beta}G_{\lambda\alpha}D_\perp^\lambda-D_\perp^\lambda G_{\alpha\beta}D_{\perp\lambda})}{8m^3}-ic_M g\frac{D_{\perp\alpha}[D_{\perp\beta}G^{\alpha\beta}]+[D_{\perp\beta}G^{\alpha\beta}]D_{\perp\alpha}}{8m^3}\\ &+c_{A1}g^2\frac{G_{\alpha\beta}G^{\alpha\beta}}{16m^3}+c_{A2}g^2\frac{G_{\mu\alpha}G^{\mu\beta}v_\alpha v_\beta}{16m^3}+c_{A3}g^2\mathrm{Tr}\left(\frac{G_{\alpha\beta}G^{\alpha\beta}}{16m^3}\right)+c_{A4}g^2\mathrm{Tr}\left(\frac{G_{\mu\alpha}G^{\mu\beta}v_\alpha v_\beta}{16m^3}\right)-ic_{B1}g^2\frac{\sigma_{\alpha\beta}[G^{\mu\alpha},G_\mu{}^\beta]}{16m^3}\\ &-ic_{B2}g^2\frac{\sigma_{\alpha\beta}[G^{\mu\alpha},G^{\nu\beta}]v_\mu v_\nu}{16m^3}\Big\}Q_v,\end{aligned} \tag{7}$$

where

$$D_\perp^\mu=D^\mu-v^\mu v\cdot D. \tag{8}$$

The tree-level matching conditions can be obtained by integrating out the antiquark components, and making a field redefinition to eliminate terms with $v\cdot D$ acting on the quark fields. The ''standard'' form of the HQET Lagrangian after integrating out the antiquark fields is

$$\mathcal{L}_v = \overline{Q_v}\left\{ iv\cdot D + i\not{D}_\perp \frac{1}{2m+iv\cdot D} i\not{D}_\perp \right\} Q_v = \overline{Q_v}\left\{ iv\cdot D - \frac{1}{2m}\not{D}_\perp\not{D}_\perp + \frac{1}{4m^2}\not{D}_\perp(iv\cdot D)\not{D}_\perp - \frac{1}{8m^3}\not{D}_\perp(iv\cdot D)^2\not{D}_\perp \right\} Q_v .$$

The field redefinition

$$Q_v \to \left(1 - \frac{D_\perp^2}{8m^2} - \frac{g\sigma_{\alpha\beta}G^{\alpha\beta}}{16m^2} + \frac{D_\perp^\alpha(iv\cdot D)D_{\perp\alpha}}{16m^3} + \frac{gv_\lambda D_{\perp\alpha}G^{\alpha\lambda}}{16m^3} - i\frac{\sigma_{\alpha\beta}D_\perp^\alpha(iv\cdot D)D_\perp^\beta}{16m^3} - i\frac{gv_\lambda\sigma_{\alpha\beta}D_\perp^\alpha G^{\beta\lambda}}{16m^3} \right) Q_v$$

[the σ matrices are understood to be $P_v\sigma P_v$, where $P_v=(1+\not{v})/2$] can be used to eliminate the time derivative terms, and put the Lagrangian into the ''NRQCD'' form. The result is Eq. (7) with $c_2=c_4=c_F=c_D=c_S=c_{W1}=c_{A1}=c_{B1}=1$, and $c_{W2}=c_{p'p}=c_M=c_{A2}=c_{A3}=c_{A4}=c_{B2}=0$. The c_A and c_B terms are quadratic in the field strengths, and are order g^2. The one-loop corrections to these terms will not be computed here.

IV. QUARK FORM FACTORS AND MATCHING CONDITIONS

A loop diagram in QCD is a function $F(\{p\},m,\mu,\epsilon)$ where $\{p\}$ are the external momenta, m is the quark mass, μ is the scale parameter of dimensional regularization, and the computation is done in $d=4-\epsilon$ dimensions. As an example, consider the diagram in Fig. 1 (in background field gauge), which gives radiative corrections to the quark-gluon three-point vertex. This vertex is conventionally expressed in terms of two form factors $F_1(q^2)$ and $F_2(q^2)$ defined by

$$\Gamma^{(3)} = -igT^a\overline{u(p')}\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]u(p), \tag{9}$$

where $q=p'-p$, and $\Gamma^{(3)}$ is the irreducible three-point function. In dimensional regularization, the diagram gives F_1 and F_2 as functions of the form $F_{1,2}(q^2/m^2,\mu/m,\epsilon)$. The F_1 form factor can be expanded as a power series in q^2/m^2 at fixed ϵ, followed by the limit $\epsilon\to 0$

$$F_1 = F_1(0)\left(\frac{A_0}{\epsilon_{UV}} + \frac{B_0}{\epsilon_{IR}} + (A_0+B_0)\ln\frac{\mu}{m} + D_0 \right) + q^2\frac{dF_1}{dq^2}(0)\left(\frac{A_1}{\epsilon_{UV}} + \frac{B_1}{\epsilon_{IR}} + (A_1+B_1)\ln\frac{\mu}{m} + D_1 \right) + \cdots, \tag{10}$$

and similarly for the F_2 form factor. It is conventional to label ϵ as either ϵ_{UV} or ϵ_{IR}, depending on whether the integral is ultraviolet or infrared divergent. Ultraviolet divergences are cancelled by renormalization counterterms. Infrared divergences cancel when a physically measurable quantity is computed. Expanding the form factor in q^2 and then taking the limit $\epsilon\to 0$ gives an expression that is analytic in q^2, and misses terms which are nonanalytic in q^2. The nonanalytic terms are not needed for the calculation of the coefficients in the effective theory, since the effective Lagrangian is analytic in momentum. The coefficients of the effective Lagrangian are determined by computing (for example) the difference of F_1 in the full theory and effective theories. The nonanalytic terms in F_1 cancel in the difference, and the analytic terms determine the unknown parameters $c_F\cdots c_{B2}$ in the effective Lagrangian.

Loop diagrams in HQET are functions $F(\{p\},\mu,\epsilon)$ times powers of the coefficients c_i in the effective Lagrangian, where $\{p\}$ are the external momenta. All on-shell loop graphs vanish when expanded in powers of p, followed by $\epsilon\to 0$. This is because the coefficient of any power of p is a dimensionally regulated integral of the form

$$\int \frac{d^dk}{(2\pi)^d} f(k^2,k\cdot v). \tag{11}$$

There is no dimensionful parameter in the integrand, so the integral vanishes. The matching condition is then trivial: one takes Eq. (10), and throws away the $1/\epsilon$ terms to obtain the difference of the graph in the full and effective theory. All the $1/\epsilon$ terms in the difference are ultraviolet divergences (which are cancelled by renormalization counterterms), since there are no infrared divergences in matching conditions. To see this more explicitly, one can evaluate integrals such as Eq. (11) by breaking them up into the sum of two integrals, one only ultraviolet divergent, and the other only infrared divergent. For example,

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4} = \int \frac{d^dk}{(2\pi)^d}\left[\frac{1}{k^2(k^2-m^2)} - \frac{m^2}{k^4(k^2-m^2)}\right] = \frac{i}{8\pi^2}\left[\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right] = 0, \tag{12}$$

since $\epsilon_{\rm UV}=\epsilon_{\rm IR}=\epsilon$. A given quantity in the effective theory is of the form

$$A_{\rm eff}\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right). \tag{13}$$

There can be no finite parts [the analogs of the $(A+B)\ln\mu/m$ and D terms in Eq. (10)], since the net integral is zero. A typical matching condition is of the form

$$\text{graphs in full theory} = \text{graphs in effective theory} + c_i, \tag{14}$$

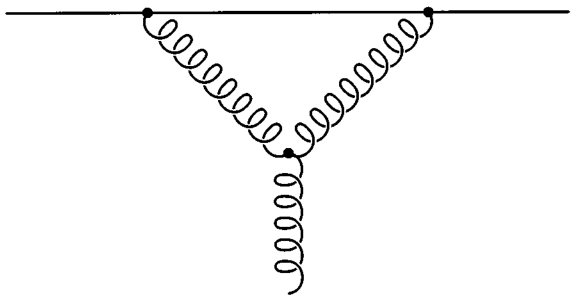

FIG. 1. Non-Abelian contribution to the one-loop vertex correction. The contribution of this diagram is denoted by a superscript (g).

where c_i is a coefficient in the effective Lagrangian. Using Eq. (10) and Eq. (13), the matching condition can be written as

$$\frac{B}{\epsilon_{\rm IR}}+(A+B)\ln\frac{\mu}{m}+D=-A_{\rm eff}\frac{1}{\epsilon_{\rm IR}}+c_i. \qquad (15)$$

The $A/\epsilon_{\rm UV}$ and $A_{\rm eff}/\epsilon_{\rm UV}$ terms are canceled by the renormalization counterterms in the full and effective theories, respectively. The coefficients in the effective Lagrangian have no infrared divergences, which implies $B=-A_{\rm eff}$ so that infrared divergences in the full theory are related to ultraviolet and infrared divergences in the effective theory, and

$$c_i=(A+B)\ln\frac{\mu}{m}+D. \qquad (16)$$

Thus c_i is obtained from Eq. (10) by keeping the finite pieces, and omitting the $1/\epsilon_{\rm UV}$ and $1/\epsilon_{\rm IR}$ terms. The procedure described here has been used previously in computing matching conditions [4,12,13].

The coefficients of the $Q^\dagger(\mathbf{D}^2/2m)Q$ and $Q^\dagger(\mathbf{D}^4/8m^3)Q$ are fixed by the dispersion relation $E^2=p^2+m^2$ of QCD, $c_2=c_4=1$. The other terms, which all contain at least one power of the gauge field A^μ, are obtained by computing the one-loop $Q^\dagger QA$ on-shell scattering amplitude. The wave-function renormalization graph Fig. 2 and the ''Abelian'' vertex correction Fig. 3 can be found in many textbooks on quantum field theory [14]. In dimensional regularization, one finds that the wave-function graph is

$$-i\Sigma(p)=-iC_F\frac{\alpha_s}{4\pi}[A(p^2)m+B(p^2)\not p], \qquad (17)$$

$$A(p^2)=\int_0^1 dx\,\Gamma(\epsilon/2)(4-\epsilon)[m^2x-p^2x(1-x)]^{-\epsilon/2}, \qquad (18)$$

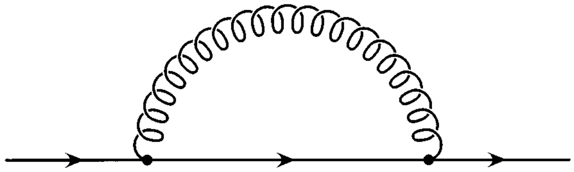

FIG. 2. One-loop wave-function renormalization.

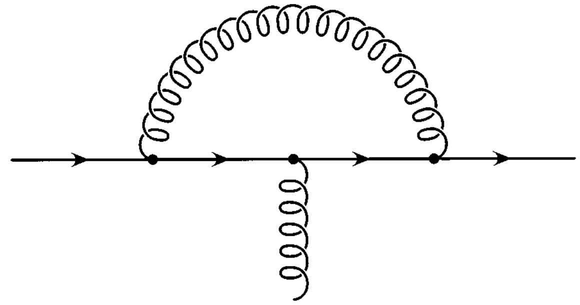

FIG. 3. Abelian one-loop vertex correction. The contribution of this diagram is denoted by a superscript (a).

$$B(p^2)=-\int_0^1 dx\,\Gamma(\epsilon/2)(2-\epsilon)\times(1-x)[m^2x-p^2x(1-x)]^{-\epsilon/2}, \qquad (19)$$

where

$$C_F=T^aT^a=\tfrac{4}{3}$$

is the quadratic Casimir of the quark representation (not to be confused with c_F, the coefficient of the Fermi interaction). The on-shell wave-function renormalization correction is

$$\delta Z=-C_F\frac{\alpha_s}{4\pi}\left[B(m^2)+2m^2\left(\frac{\partial A}{\partial p^2}+\frac{\partial B}{\partial p^2}\right)_{p^2=m^2}\right]$$
$$=C_F\frac{\alpha_s}{\pi}\left[\frac{1}{2\epsilon_{\rm UV}}+\frac{1}{\epsilon_{\rm IR}}+1-\frac{3}{2}\ln\frac{m}{\mu}\right]. \qquad (20)$$

The on-shell Abelian vertex correction from Fig. 3 can be expressed in terms of the form factors F_1 and F_2:

$$F_1^{(a)}(q^2)=\frac{\alpha_s}{2\pi}\left(C_F-\frac{1}{2}C_A\right)\left[\frac{1}{\epsilon_{\rm UV}}+\frac{1}{\epsilon_{\rm IR}}\left(2-\frac{q^2}{m^2}\right)I(q^2/m^2)+\left(3-\frac{q^2}{m^2}\right)I(q^2/m^2)-\frac{1}{2}J(q^2/m^2)-\left(1-\frac{q^2}{2m^2}\right)K(q^2/m^2)-1\right] \qquad (21)$$

and

$$F_2^{(a)}(q^2)=\frac{\alpha_s}{2\pi}\left(C_F-\frac{1}{2}C_A\right)I(q^2/m^2), \qquad (22)$$

where

$$I(q^2/m^2)=\int_0^1 dx\,\frac{m^2}{m^2-q^2x(1-x)}, \qquad (23)$$

$$J(q^2/m^2)=\int_0^1 dx\ln\frac{m^2-q^2x(1-x)}{\mu^2}, \qquad (24)$$

$$K(q^2/m^2)=\int_0^1 dx\,\frac{m^2}{m^2-q^2x(1-x)}\ln\frac{m^2-q^2x(1-x)}{\mu^2}, \tag{25}$$

and

$$C_A=3,$$

is the quadratic Casimir of the adjoint representation. Expanding to order q^2/m^2 gives

$$F_1^{(a)}=\frac{\alpha_s}{\pi}\left(C_F-\frac{1}{2}C_A\right)\left[\frac{1}{2\epsilon_{\rm UV}}+\frac{1}{\epsilon_{\rm IR}}+1-\frac{3}{2}\ln\frac{m}{\mu}+\frac{q^2}{m^2}\left(-\frac{1}{3}\frac{1}{\epsilon_{\rm IR}}-\frac{1}{8}+\frac{1}{3}\ln\frac{m}{\mu}\right)\right], \tag{26}$$

$$F_2^{(a)}=\frac{\alpha_s}{\pi}\left(C_F-\frac{1}{2}C_A\right)\left[\frac{1}{2}+\frac{q^2}{12m^2}\right]. \tag{27}$$

The final diagram is the non-Abelian vertex correction, Fig. 1. This is computed in background field Feynman gauge, which preserves gauge invariance. The resulting diagram can also be evaluated in terms of the F_1 and F_2 form factors:

$$\begin{aligned}F_1^{(g)}&=\frac{\alpha_s}{8\pi}C_A\int_0^1 dx\int_0^{1-x}dy\,\big(-\Gamma(1+\epsilon/2)\{2q^2(x+y)\\&\quad+2m^2(1-x-y)[2(x+y)+(4-\epsilon)(1-x-y)]\}\\&\quad\times[m^2(x+y-1)^2-q^2xy]^{-1-\epsilon/2}+(2-\epsilon)\Gamma(\epsilon/2)\\&\quad\times[m^2(x+y-1)^2-q^2xy]^{-\epsilon/2}\big)\\&=\frac{\alpha_s}{8\pi}C_A\left[\frac{2}{\epsilon_{\rm UV}}+\frac{4}{\epsilon_{\rm IR}}+4-6\ln\frac{m}{\mu}+\frac{q^2}{m^2}\right.\\&\quad\left.\times\left(-\frac{3}{\epsilon_{\rm IR}}-1+3\ln\frac{m}{\mu}\right)+\cdots\right],\end{aligned} \tag{28}$$

$$\begin{aligned}F_2^{(g)}&=-\frac{\alpha_s}{4\pi}C_Am^2\Gamma(1+\epsilon/2)\int_0^1 dx\int_0^{1-x}dy(1-x-y)[\epsilon\\&\quad+(2-\epsilon)(x+y)][m^2(x+y-1)^2-q^2xy]^{-1-\epsilon/2}\\&=\frac{\alpha_s}{8\pi}C_A\left[\frac{4}{\epsilon_{\rm IR}}+6-4\ln\frac{m}{\mu}+\frac{q^2}{m^2}\left(\frac{4}{\epsilon_{\rm IR}}+1-4\ln\frac{m}{\mu}\right)+\cdots\right].\end{aligned} \tag{29}$$

The total on-shell form factors at one loop are given by

$$\begin{aligned}F_1&=1-\delta Z+F_1^{(a)}+F_1^{(g)}\\&=1+\frac{\alpha_s}{\pi}\frac{q^2}{m^2}\left[\left(-\frac{1}{3\epsilon_{\rm IR}}-\frac{1}{8}+\frac{1}{3}\ln\frac{m}{\mu}\right)C_F\right.\\&\quad\left.+\left(-\frac{5}{24\epsilon_{\rm IR}}-\frac{1}{16}+\frac{5}{24}\ln\frac{m}{\mu}\right)C_A\right],\end{aligned} \tag{30}$$

$$\begin{aligned}F_2&=F_2^{(a)}+F_2^{(g)}\\&=\frac{\alpha_s}{\pi}\left[\frac{1}{2}C_F+\left(\frac{1}{2\epsilon_{\rm IR}}+\frac{1}{2}-\frac{1}{2}\ln\frac{m}{\mu}\right)C_A\right]+\frac{\alpha_s}{\pi}\frac{q^2}{m^2}\left[\frac{1}{12}C_F\right.\\&\quad\left.+\left(\frac{1}{2\epsilon_{\rm IR}}+\frac{1}{12}-\frac{1}{2}\ln\frac{m}{\mu}\right)C_A\right].\end{aligned} \tag{31}$$

The total form factor $F_1(0)$ is unity, since gauge invariance is preserved by the background field method.

The scattering amplitude for a low-momentum heavy quark off a background vector potential can be computed by expanding Eq. (9), and multiplying by $\sqrt{m/E}$ for the incoming and outgoing quarks. If $\mathbf{p}$ is the three-momentum of the incoming quark, $\mathbf{p}'$ is the three-momentum of the outgoing quark, and $\mathbf{q}=\mathbf{p}'-\mathbf{p}$, one finds that the effective interaction is

$$-igT^a u^\dagger_{\rm NR}(\mathbf{p}')[A^{0a}j^0-\mathbf{A}^a\cdot\mathbf{j}]u_{\rm NR}(\mathbf{p}), \tag{32}$$

where

$$\begin{aligned}j^0&=F_1(q^2)\left\{1-\frac{1}{8m^2}|\mathbf{q}|^2+\frac{i}{4m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})\right\}\\&\quad+F_2(q^2)\left\{-\frac{1}{4m^2}|\mathbf{q}|^2+\frac{i}{2m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})\right\},\end{aligned} \tag{33}$$

and

$$\begin{aligned}\mathbf{j}&=F_1(q^2)\left\{\frac{1}{2m}(\mathbf{p}+\mathbf{p}')+\frac{i}{2m}\boldsymbol{\sigma}\times\mathbf{q}-\frac{i}{8m^3}(|\mathbf{p}|^2+|\mathbf{p}'|^2)\boldsymbol{\sigma}\right.\\&\quad\times\mathbf{q}-\frac{i}{16m^3}(|\mathbf{p}'|^2-|\mathbf{p}|^2)\boldsymbol{\sigma}\times(\mathbf{p}+\mathbf{p}')\\&\quad\left.-\frac{1}{8m^3}(|\mathbf{p}'|^2+|\mathbf{p}|^2)(\mathbf{p}+\mathbf{p}')-\frac{1}{16m^3}(|\mathbf{p}'|^2-|\mathbf{p}|^2)\mathbf{q}\right\}\\&\quad+F_2(q^2)\left\{\frac{i}{2m}\boldsymbol{\sigma}\times\mathbf{q}-\frac{i}{16m^3}|\mathbf{q}|^2\boldsymbol{\sigma}\times\mathbf{q}\right.\\&\quad-\frac{1}{16m^3}|\mathbf{q}|^2(\mathbf{p}'+\mathbf{p})-\frac{1}{16m^3}(|\mathbf{p}'|^2-|\mathbf{p}|^2)\mathbf{q}\\&\quad-\frac{i}{8m^3}(|\mathbf{p}'|^2-|\mathbf{p}|^2)\boldsymbol{\sigma}\times(\mathbf{p}'+\mathbf{p})\\&\quad\left.+\frac{i}{8m^3}\boldsymbol{\sigma}\cdot(\mathbf{p}'+\mathbf{p})(\mathbf{p}'\times\mathbf{p})\right\}.\end{aligned} \tag{34}$$

Comparing Eqs. (33) and (34) with the scattering amplitude in the effective theory from the Lagrangian, Eq. (6), gives

$$c_F=F_1+F_2, \tag{35}$$

$$c_D=F_1+2F_2+8F_1', \tag{36}$$

$$c_S=F_1+2F_2, \tag{37}$$

$$c_{W1}=F_1+\tfrac{1}{2}F_2+4F_1'+4F_2', \tag{38}$$

$$c_{W2}=\tfrac{1}{2}F_2+4F_1'+4F_2', \tag{39}$$

$$c_{p'p}=F_2, \tag{40}$$

$$c_M=-\tfrac{1}{2}F_2-4F_1', \tag{41}$$

where

$$F_i\equiv F_i(0), \quad F_i'\equiv\frac{dF_i}{d(q^2/m^2)}\Big|_{q^2=0}. \tag{42}$$

Note that the nine parameters (including c_2 and c_4) in the effective Lagrangian, Eq. (6), are determined in terms of only three independent constants, F_2, F_1', and F_2', since $F_1=1$. Reparametrization invariance [16] gives six linear relations among the coefficients. This will be discussed in more detail in the next section.

The explicit expressions for the coefficients are obtained using Eqs. (30) and (31):

$$c_F=1+\frac{\alpha_s}{\pi}\left[\frac{1}{2}C_F+\left(\frac{1}{2}-\frac{1}{2}\ln\frac{m}{\mu}\right)C_A\right], \tag{43}$$

$$c_D=1+\frac{\alpha_s}{\pi}\left[\left(\frac{8}{3}\ln\frac{m}{\mu}\right)C_F+\left(\frac{1}{2}+\frac{2}{3}\ln\frac{m}{\mu}\right)C_A\right], \tag{44}$$

$$c_S=1+\frac{\alpha_s}{\pi}\left[C_F+\left(1-\ln\frac{m}{\mu}\right)C_A\right], \tag{45}$$

$$c_{W1}=1+\frac{\alpha_s}{\pi}\left[\left(\frac{1}{12}+\frac{4}{3}\ln\frac{m}{\mu}\right)C_F+\left(\frac{1}{3}-\frac{17}{12}\ln\frac{m}{\mu}\right)C_A\right], \tag{46}$$

$$c_{W2}=\frac{\alpha_s}{\pi}\left[\left(\frac{1}{12}+\frac{4}{3}\ln\frac{m}{\mu}\right)C_F+\left(\frac{1}{3}-\frac{17}{12}\ln\frac{m}{\mu}\right)C_A\right], \tag{47}$$

$$c_{p'p}=\frac{\alpha_s}{\pi}\left[\frac{1}{2}C_F+\left(\frac{1}{2}-\frac{1}{2}\ln\frac{m}{\mu}\right)C_A\right], \tag{48}$$

$$c_M=\frac{\alpha_s}{\pi}\left[\left(\frac{1}{4}-\frac{4}{3}\ln\frac{m}{\mu}\right)C_F-\left(\frac{7}{12}\ln\frac{m}{\mu}\right)C_A\right]. \tag{49}$$

The results for NRQED can be obtained by setting $C_A=0$ and $C_F=1$, and agree with those found in [9], with the replacement

$$\ln\mu\rightarrow\ln 2\Lambda-\tfrac{5}{6}. \tag{50}$$

The difference in finite parts is because the NRQED integrals were evaluated in Ref. [9] using a momentum space cutoff, instead of using dimensional regularization. The results for the $1/m$ operators agree with known results for HQET [4,5]. The $1/m^2$ matching conditions at tree level, and the μ dependence at one loop also agree with known results [6–8]. Note that c_F is independent of μ in QED. This is easy to see if one computes the renormalization of the magnetic moment operator in the effective theory in Coulomb gauge, in which all transverse photon interactions are suppressed by $1/m$. The renormalization is only due to vertex corrections from Coulomb photons. These vanish, because all poles in the loop integral over the energy k^0 are on one side of the real axis.

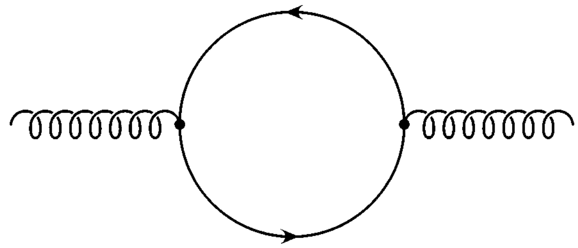

FIG. 4. Quark contribution to the vacuum polarization.

The effective Lagrangian has been computed using background field Feynman gauge and matching with on-shell quarks, but with an off-shell gluon. This fixes the coefficients in the Lagrangian Eq. (6). One can still redefine terms in the effective Lagrangian using the gluon equations of motion, which does not change S-matrix elements. For example, one can use the equations of motion to convert the Darwin term to a linear combination of four-quark operators. It is only the sum of Eq. (6) and the four-quark terms that gives convention independent S-matrix elements.

The discussion so far has concentrated on the fermion part of the effective Lagrangian. There is, in addition, the pure gauge field part of the effective action. The one-loop correction to the gluon propagator is shown in Figs. 4 and 5. The gluon diagram is the same in QCD and in HQET, so the one-loop matching condition is from the quark vacuum polarization diagram. This gives the effective action [17–19]

$$\mathcal{L}=-\frac{1}{4}d_1G^A_{\mu\nu}G^{A\mu\nu}+\frac{d_2}{m^2}G^A_{\mu\nu}D^2G^{A\mu\nu}+\frac{d_3}{m^2}gf_{ABC}G^A_{\mu\nu}G^{B\mu}{}_{\alpha}G^{C\nu\alpha}+\mathcal{O}\left(\frac{1}{m^4}\right), \tag{51}$$

with

$$d_1=1-\frac{\alpha_s}{3\pi}T_F\ln m^2/\mu^2, d_2=\frac{\alpha_s}{60\pi}T_F, d_3=\frac{13\alpha_s}{360\pi}T_F, \tag{52}$$

where

$$T_F=1/2$$

is the index of the quark representation. The identity

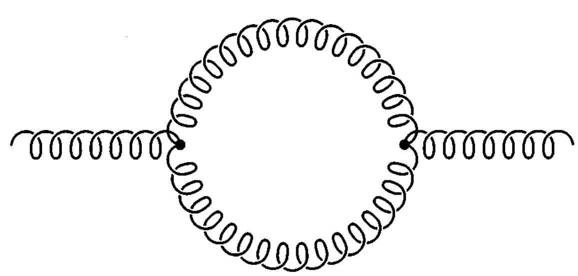

FIG. 5. Gluon contribution to the vacuum polarization.

$$0=\int 2D^{\mu}G^{A}_{\mu\alpha}D_{\nu}G^{\nu\alpha A}+2gf_{ABC}G^{A}_{\mu\nu}G^{B\mu}{}_{\alpha}G^{C\nu\alpha}+G^{A}_{\mu\nu}D^{2}G^{A\mu\nu}$$

has been used to eliminate $D^{\mu}G^{A}_{\mu\alpha}D_{\nu}G^{\nu\alpha A}$ from Eq. (51).

V. REPARAMETRIZATION INVARIANCE

The coefficients of operators in the HQET Lagrangian are constrained by reparametrization invariance [16]. The reparametrization invariant spinor field Ψ_v is given by

$$\Psi_v=\Lambda(w,v)\psi_v\,, \tag{53}$$

where ψ_v is the conventional heavy quark field that satisfies

$$\not{v}\,\psi_v=\psi_v\,, \tag{54}$$

$\Lambda(w,v)$ is the Lorentz transformation matrix

$$\Lambda(w,v)=\frac{1+\not{w}\not{v}}{\sqrt{2(1+w\cdot v)}} \tag{55}$$

and

$$w^{\mu}=\frac{v^{\mu}+iD^{\mu}/m}{|v^{\mu}+iD^{\mu}/m|}. \tag{56}$$

One needs to choose a particular operator ordering for the covariant derivatives; different orderings are related to each other by field redefinitions.

It is simplest to consider the consequences of reparametrization invariance when $D^{\mu}\to\partial^{\mu}$ in Eq. (53). Then there is no operator ordering ambiguity, and the field Ψ_v can be written as

$$\Psi_v=\left[\frac{1}{2\sqrt{(m+i\partial\cdot v)^2+(i\partial_{\perp})^2}\,(m+i\partial\cdot v+\sqrt{(m+i\partial\cdot v)^2+(i\partial_{\perp})^2})}\right]^{1/2}[m+i\partial\cdot v+i\not{\partial}_{\perp}+\sqrt{(m+i\partial\cdot v)^2+(i\partial_{\perp})^2}]\psi_v\,, \tag{57}$$

where

$$\partial^{\mu}_{\perp}=\partial^{\mu}-v^{\mu}\partial\cdot v. \tag{58}$$

If one uses Eq. (57), replaces ψ_v by the spinor $u_{\rm NR}e^{-ip\cdot x}$, with $p^2=m^2$, $\gamma^0u_{\rm NR}=u_{\rm NR}$, $\overline{u}^{\dagger}_{\rm NR}u_{\rm NR}=1$, and $v=(1,0,0,0)$, the field Ψ_v reduces to the spinor $ue^{-ip\cdot x}$, that satisfies the Dirac equation $\not{p}u=mu$, and is normalized so that $\overline{u}u=1$. This shows that reparametrization invariance will determine the coefficients of the $1/m$ suppressed operators which are fixed by relativistic invariance.

The reparametrization invariant kinetic term is

$$\overline{\Psi}_v i\not{\partial}\Psi_v=\overline{\psi}_v[\sqrt{(m+i\partial\cdot v)^2+(i\partial_{\perp})^2}-m]\psi_v\,. \tag{59}$$

This is not the same as the terms in the Lagrangian, Eq. (6). The reparametrization invariant field, Eq. (53), does not automatically produce a Lagrangian in the ''standard'' NRQCD form. However, one can convert Eq. (59) to this form by making a field redefinition

$$\psi_v=\left[\frac{\sqrt{(m+i\partial\cdot v)^2+(i\partial_{\perp})^2}+m}{m+i\partial\cdot v+\sqrt{-(i\partial_{\perp})^2+m^2}}\right]^{1/2}\psi'_v\,. \tag{60}$$

The kinetic energy term in the primed field is

$$\overline{\psi}'_v[m+i\partial\cdot v-\sqrt{-(i\partial_{\perp})^2+m^2}]\psi'_v\,, \tag{61}$$

which, when expanded, gives Eq. (6) with $c_2=c_4=1$. Thus $c_2=c_4=1$ follows from reparametrization invariance. The transformation factor in Eq. (60) when applied to on-shell spinors (instead of fields) reduces to $\sqrt{m/E}$. This is the same as the flux factor for the incoming and outgoing particles which was included in Eq. (32).

To determine the constraints of reparametrization invariance on the effective Lagrangian, consider Eq. (57) with the gauge fields included, i.e., with $\partial\to D$. Expanding to order $1/m^3$ gives

$$\Psi_v=\left[1+A+\frac{i\not{D}_{\perp}}{2m}B\right]\psi_v\,, \tag{62}$$

where

$$A=1-\frac{(iD_{\perp})^2}{8m^2}+\frac{(iD_{\perp})^2(iv\cdot D)}{4m^3},$$

$$B=1-\frac{iv\cdot D}{m}-\frac{3(iD_{\perp})^2}{8m^2}+\frac{(iv\cdot D)^2}{m^2}. \tag{63}$$

A particular ordering has been chosen for the operators in Eq. (62). A different ordering gives an effective Lagrangian that is related by a field redefinition.

The most general reparametrization invariant Lagrangian is a linear combination of invariant terms, such as $\overline{\Psi}_v(\not{v}+i\not{D}/m)\Psi_v$, $\overline{\Psi}_v\sigma^{\alpha\beta}G_{\alpha\beta}\Psi_v$, etc. The effective Lagrangian obtained in this way is not in the form Eq. (7), but it can be converted into that form by field redefinitions that preserve $\not{v}\,\psi_v=\psi_v$. One finds by a straightforward (but not very enlightening computation) that the effective Lagrangian is a linear combination of the invariant linear combinations

$$iv\cdot D+O_2+O_4+O_F+O_D+O_S+O_{W1},$$

$$2O_F+4O_D+4O_S+O_{W1}+O_{W2}+2O_{p'p}-O_M,$$

$$O_{W1}+O_{W2},2O_D+O_{W1}+O_{W2}-O_M,$$

$$O_{A1},O_{A2},O_{A3},O_{A4},O_{B1},O_{B2}, \tag{64}$$

up to terms of order $1/m^4$. Here O_F, etc., are the operator coefficients of c_F, etc., in Eq. (7). The above linear combinations imply the constraints

$$c_2=1,$$

$$c_4=1,$$

$$c_S=2c_F-1,$$

$$c_{W2}=c_{W1}-1,$$

$$c_{p'p}=c_F-1,$$

$$2c_M=c_F-c_D, \qquad (65)$$

which are satisfied by Eqs. (35) and (43).

VI. CONCLUSIONS

The HQET and NRQCD Lagrangian has been computed to one loop and order $1/m^3$, and has been shown to be reparametrization invariant to order α_s/m^3. The original form of the NRQCD propagator, Eq. (2), cannot be used to compute the effective Lagrangian by matching to QCD. Instead, one must treat the propagator as an infinite series, and resum the series *after* doing the loop integral. As a result, the matching computations for NRQCD and HQET are the same. It is straightforward to obtain the effective Lagrangian (in the one-quark sector) to higher orders in $1/m$ by expanding the form factors $F_{1,2}$ and the spinors in the computation of Eqs. (32) and (33) to higher orders. No further Feynman graphs need to be evaluated.

ACKNOWLEDGMENTS

I am indebted to M. Luke for numerous discussions about matching conditions in nonrelativistic effective field theories. This work was supported in part by U.S. Department of Energy Grant No. DOE-FG03-90ER40546.

[1] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990).
[2] W. E. Caswell and G. P. Lepage, Phys. Lett. **167B**, 437 (1986).
[3] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995).
[4] E. Eichten and B. Hill, Phys. Lett. B **243**, 427 (1990).
[5] A. F. Falk, B. Grinstein, and M. E. Luke, Nucl. Phys. **B357**, 185 (1991).
[6] C. Balzereit and T. Ohl, Phys. Lett. B **386**, 335 (1996).
[7] C. Bauer, Diplom thesis, Karlsruhe, 1996.
[8] B. Blok, J.G. Korner, D. Pirjol, and J.C. Rojas, Nucl. Phys. B (to be published).
[9] T. Kinoshita and M. Nio, Phys. Rev. D **53**, 4909 (1996).
[10] C. Morningstar, Phys. Rev. D **48**, 2265 (1993); **50**, 5902 (1994).
[11] H. D. Politzer, Nucl. Phys. **B172**, 349 (1980).
[12] M. Neubert, Phys. Rev. D **49**, 1542 (1994).
[13] D. J. Broadhurst and A. G. Grozin, Phys. Rev. D **52**, 4082 (1995).
[14] See, for example, S. Pokorski, *Gauge Field Theories* (Cambridge University Press, Cambridge, England, 1987).
[15] M. E. Luke and A. V. Manohar, Phys. Rev. D **55**, 4129 (1997).
[16] M. E. Luke and A. V. Manohar, Phys. Lett. B **286**, 348 (1992).
[17] P. Cho and E. H. Simmons, Phys. Rev. D **51**, 2360 (1995).
[18] V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, Fortschr. Phys. **32**, 585 (1984).
[19] S. Balk, J. G. Korner, and D. Pirjol, Nucl. Phys. **B428**, 499 (1994).