Point-splitting method of the commutator anomaly of Gauss law operators

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We analyze the generalized point-splitting method and Jo's result for the commutator anomaly. We find that certain classes of general regularization kernels satisfying integral conditions provide a *unique* result, which, however, *differs* from Faddeev's cohomological result. [S0556-2821(97)03516-9]

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I. INTRODUCTION

There is now an elegant cohomological theory—the so-called Stora-Zumino chain of descent equations [1,2]—established which describes the anomalies of quantum field theory (for a recent overview see [3,4]). The one-cocycle is identified with the anomaly in the covariant divergence of the non-Abelian chiral fermion current [1,2], the two-cocycle with the anomalous term—the Schwinger term [5]—in the commutator of the gauge group generators occurring in the same anomalous theory [6–8] (for an overview see [9,10]). It is this anomalous (equal time) commutator we are concerned with:

$$i[\mathcal{G}^a(\mathbf{x}),\mathcal{G}^b(\mathbf{y})]=f^{abc}\mathcal{G}^c\delta^3(\mathbf{x}-\mathbf{y})+S^{ab}(\mathbf{x}-\mathbf{y}). \quad (1.1)$$

The generator—Gauss-law operator—consists of two parts:

$$\mathcal{G}^a(\mathbf{x})=\delta^a(\mathbf{x})+\rho^a(\mathbf{x}), \quad (1.2)$$

the generator $\delta^a(\mathbf{x})$ of gauge transformations for the gauge potentials and the generator $\rho^a(\mathbf{x})$ of the gauge transformations for the fermionic fields,

$$\delta^a(\mathbf{x})=-(\mathbf{D}\cdot\mathbf{E})^a(\mathbf{x})=iD_i^{ba}\frac{\delta}{\delta A_i^b}(\mathbf{x}), \quad (1.3)$$

$$\rho^a(\mathbf{x})=-i\psi^\dagger(\mathbf{x})T^a\frac{1-\gamma_5}{2}\psi(\mathbf{x}). \quad (1.4)$$

E_i^a is the non-Abelian electric field, $D_i^{ab}=\delta^{ab}\partial_i+f^{abc}A_i^c$ the covariant derivative; the group matrices T^a are anti-Hermitian satisfying

$$[T^a,T^b]=f^{abc}T^c, \quad (1.5)$$

and finally γ_5 is chosen as $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3$.

The solution for this additional anomalous term in the commutator—which causes difficulties when quantizing the theory—has been found by Faddeev [7] on a cohomological basis:

$$S^{ab}(\mathbf{x},\mathbf{y})=-\frac{i}{24\pi^2}\varepsilon^{ijk}\mathrm{tr}\{T^a,T^b\}\partial_iA_j\partial_k\delta^3(\mathbf{x}-\mathbf{y}). \quad (1.6)$$

This cohomological result has been verified by computing the commutator with the Bjorken-Johnson-Low procedure [11–14], or by working with geometric methods [15–21]. However, as Jo [11] discovered, a generalized point-splitting method where the time is fixed *does not* provide Faddeev's cohomological result (1.6), contrary to claims in the literature [22]. Furthermore, Jo located an inherent ambiguity in the procedure due to the specific choice of the regularization kernels. (Note that we consider here the case of 1+3 dimensions; in 1+1 dimensions there occur no problems and all methods agree.) Reinvestigating the procedure we clarify this ambiguity and show how to overcome this problem. In fact, we find that a whole class of regularization kernels satisfying an integral condition provides a unique result.

II. GENERALIZED POINT-SPLITTING METHOD

In order to define an operator

$$\mathcal{J}(f)=-i\int d^3x\,\psi^\dagger(\mathbf{x})f(\mathbf{x})\frac{1-\gamma_5}{2}\psi(\mathbf{x}), \quad (2.1)$$

which has a singular behavior, we introduce a family of the smooth kernels:

$$F(\mathbf{x},\mathbf{y})=f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)f_{\mu_f}(|\mathbf{x}-\mathbf{y}|), \quad (2.2)$$

where

$$\lim_{\mu_f\to 0}f_{\mu_f}(|\mathbf{x}-\mathbf{y}|)=\delta^3(\mathbf{x}-\mathbf{y}). \quad (2.3)$$

The limit is understood in a distributional sense, so the $f_{\mu_f}(|\mathbf{x}-\mathbf{y}|)$ are δ-like functions and the $f[(\mathbf{x}+\mathbf{y})/2]$ contain matrices of the internal symmetry space. For each such kernel $F(\mathbf{x},\mathbf{y})$ we define the operator

$$\mathcal{J}(F)=-i\int d^3x d^3y\,\psi^\dagger(\mathbf{x})F(\mathbf{x},\mathbf{y})\frac{1-\gamma_5}{2}\psi(\mathbf{y}). \quad (2.4)$$

We also need the Fourier transformations

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$$\tilde{F}(\mathbf{p}',\mathbf{p})=\int d^3x d^3y e^{i\mathbf{p}'\cdot(\mathbf{x}+\mathbf{y})/2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}F(\mathbf{x},\mathbf{y}) \quad (2.5)$$

$$=\tilde{f}(\mathbf{p}')\tilde{f}_{\mu_f}(|\mathbf{p}|), \quad (2.6)$$

$$\tilde{f}(\mathbf{p}')=\int d^3x e^{i\mathbf{p}'\cdot\mathbf{x}}f(\mathbf{x}), \quad (2.7)$$

$$\tilde{f}_{\mu_f}(|\mathbf{p}|)=\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}}f_{\mu_f}(|\mathbf{x}|). \quad (2.8)$$

$\tilde{F}(\mathbf{p}',\mathbf{p})$ has the local limit

$$\lim_{\mu_f\to 0}\tilde{F}(\mathbf{p}',\mathbf{p})=\tilde{f}(\mathbf{p}'), \quad (2.9)$$

since

$$\lim_{\mu_f\to 0}\tilde{f}_{\mu_f}(|\mathbf{p}|)=1. \quad (2.10)$$

These smeared operators $\mathcal{J}(F)$ are well defined in Hilbert space and satisfy the familiar commutation relations

$$i[\mathcal{J}(F),\mathcal{J}(G)]=\mathcal{J}([F,G]), \quad (2.11)$$

where the commutator $[F,G]$ means

$$[F,G](\mathbf{x},\mathbf{y})=\int d^3z[F(\mathbf{x},\mathbf{z})G(\mathbf{z},\mathbf{y})-G(\mathbf{x},\mathbf{z})F(\mathbf{z},\mathbf{y})]. \quad (2.12)$$

However, in order to be able to perform the local limit we have to subtract the fixed-time vacuum expectation value (VEV) of $\mathcal{J}(F)$:

$$\langle\mathcal{J}(F)\rangle_A=\int d^3x d^3y \,\mathrm{tr}P(\mathbf{x},\mathbf{y})F(\mathbf{y},\mathbf{x})=\mathrm{Tr}FP, \quad (2.13)$$

where

$$P(\mathbf{x},\mathbf{y})=i\frac{1-\gamma_5}{2}\langle\psi(\mathbf{x})\psi^\dagger(\mathbf{y})\rangle_A. \quad (2.14)$$

In the local limit we need $P(\mathbf{x},\mathbf{y})$ for $\mathbf{x}\approx\mathbf{y}$, which diverges for $\mathbf{x}\to\mathbf{y}$. We extract $P^{\mathrm{inf}}=P(\mathbf{x},\mathbf{y})_{\mathbf{x}\to\mathbf{y}}$ so that $P-P^{\mathrm{inf}}$ has a local limit. Then we obtain the well-defined operator $\mathcal{J}(f)$ from the local limit of a such regularized quantity:

$$\mathcal{J}(f)=\lim_{\mu_f\to 0}\mathcal{J}_{\mathrm{reg}}(F), \quad (2.15)$$

with

$$\mathcal{J}_{\mathrm{reg}}(F)=\mathcal{J}(F)-\mathrm{Tr}FP^{\mathrm{inf}}. \quad (2.16)$$

On the other hand, we also need an operator $\mathcal{T}(f)$ defined by

$$\mathcal{T}(f)=\int d^3x f^a(\mathbf{x})\delta^a(\mathbf{x})=-i\int d^3x[D_i f(\mathbf{x})]^b\frac{\delta}{\delta A_i^b(\mathbf{x})}. \quad (2.17)$$

Now, in order to investigate the commutator (1.1) we have to consider

$$i[\mathcal{T}(f)+\mathcal{J}_{\mathrm{reg}}(F),\mathcal{T}(g)+\mathcal{J}_{\mathrm{reg}}(G)]$$
$$=\mathcal{T}([f,g])+\mathcal{J}_{\mathrm{reg}}([F,G])+S(F,G), \quad (2.18)$$

and we have to compute the Schwinger term in the local limit:

$$S(F,G)=i[\mathcal{T}(f),\mathcal{J}_{\mathrm{reg}}(G)]+i[\mathcal{J}_{\mathrm{reg}}(F),\mathcal{T}(g)]+\mathcal{J}([F,G])$$
$$-\mathcal{J}_{\mathrm{reg}}([F,G]), \quad (2.19)$$

$$S(f,g)=\lim_{\mu_f,\mu_g\to 0}S(F,G). \quad (2.20)$$

III. JO'S RESULT FOR THE COMMUTATOR ANOMALY

For symmetric regularization kernels the following commutators vanish:

$$[\mathcal{T}(f),\mathcal{J}_{\mathrm{reg}}(G)]=[\mathcal{J}_{\mathrm{reg}}(F),\mathcal{T}(g)]=0, \quad (3.1)$$

and we have, for the Schwinger term [11],

$$S(F,G)=\mathrm{Tr}[F,G]P^{\mathrm{inf}} \quad (3.2)$$

$$=\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\mathrm{tr}B^i(\mathbf{q})\int\frac{d^3p'}{(2\pi)^3}$$
$$\times[\chi^i(F,G;\mathbf{p}',\mathbf{q})$$
$$-\chi^i(G,F;\mathbf{p}',\mathbf{q})], \quad (3.3)$$

where $B^i(\mathbf{q})$ is the Fourier transformation of $B^i[(\mathbf{x}+\mathbf{y})/2]$ and

$$B^i(\mathbf{x})=\varepsilon^{ijk}(\partial_j A_k+A_j A_k)(\mathbf{x}). \quad (3.4)$$

The function

$$\chi^i(F,G;\mathbf{p}',\mathbf{q})=\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{|\mathbf{p}|^3}\tilde{F}\left(\mathbf{p}',\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{p}'}{2}\right)$$
$$\times\tilde{G}\left(-\mathbf{p}'-\mathbf{q},\mathbf{p}+\frac{\mathbf{p}'}{2}\right) \quad (3.5)$$

after expanding $\tilde{F}$ and $\tilde{G}$ can be rewritten as

$$\chi^i(F,G;\mathbf{p}',\mathbf{q})=\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{|\mathbf{p}|^3}[\tilde{F}(\mathbf{p}',\mathbf{p})\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})]$$
$$+\frac{p'^j}{2}\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{|\mathbf{p}|^3}\frac{\partial}{\partial p^j}$$
$$\times[\tilde{F}(\mathbf{p}',\mathbf{p})\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})]$$
$$+\frac{q^j}{2}\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{|\mathbf{p}|^3}\left[\frac{\partial}{\partial p^j}\tilde{F}(\mathbf{p}',\mathbf{p})\right]$$
$$\times\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})$$
$$+\text{higher-order derivative terms.} \quad (3.6)$$

The first integral is zero because the integrand is antisymetric under the change of $\mathbf{p}\to-\mathbf{p}$. The higher-order derivative terms vanish after the local limit.[1]

Then the function χ^i can be separated into two parts:

$$\chi^i(F,G;\mathbf{p}',\mathbf{q})=\chi_1^i(F,G;\mathbf{p}',\mathbf{q})+\chi_2^i(F,G;\mathbf{p}',\mathbf{q}), \quad (3.7)$$

where

$$\chi_1^i(F,G;\mathbf{p}',\mathbf{q})=\frac{p'^j}{2}\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{|\mathbf{p}|^3}\frac{\partial}{\partial p^j}\times[\tilde{F}(\mathbf{p}',\mathbf{p})\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})], \quad (3.8)$$

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q})=\frac{q^j}{2}\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{|\mathbf{p}|^3}\left[\frac{\partial}{\partial p^j}\tilde{F}(\mathbf{p}',\mathbf{p})\right]\times\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p}). \quad (3.9)$$

Whereas the first term is independent of the applied regularization kernels—the δ-like functions $\tilde{f}_{\mu_f}(|\mathbf{p}|)$, $\tilde{g}_{\mu_g}(|\mathbf{p}|)$—providing the unique result

$$\chi_1^i(F,G;\mathbf{p}',\mathbf{q})=-\frac{p'^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q}) \quad (3.10)$$

in the local limit $\mu_f,\ \mu_g\to0$, the second term is not. It strongly depends on the kernels and for Jo's choice of Gaussian regularization kernels

$$F(\mathbf{x},\mathbf{y})=f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\frac{1}{(4\pi\mu_f)^{3/2}}e^{-(\mathbf{x}-\mathbf{y})^2/4\mu_f}, \quad (3.11)$$

$$G(\mathbf{x},\mathbf{y})=g\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\frac{1}{(4\pi\mu_g)^{3/2}}e^{-(\mathbf{x}-\mathbf{y})^2/4\mu_g}, \quad (3.12)$$

or in momentum space

$$\tilde{F}(\mathbf{p}',\mathbf{p})=\tilde{f}(\mathbf{p}')e^{-\mu_f\mathbf{p}^2}, \quad (3.13)$$

$$\tilde{G}(\mathbf{p}',\mathbf{p})=\tilde{g}(\mathbf{p}')e^{-\mu_g\mathbf{p}^2}, \quad (3.14)$$

the result is

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q})=-\frac{1}{1+\mu}\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q}), \quad (3.15)$$

where we have introduced the parameter $\mu\equiv\mu_g/\mu_f$. Clearly, the local limit of χ_2^i depends on how μ_f and μ_g approach zero. With this ambiguity, the final expression for the Schwinger term becomes

$$S(F,G)=\frac{-i}{24\pi^2}\varepsilon^{ijk}\int d^3x\,\mathrm{tr}\left[(\partial_jA_k+A_jA_k)(\partial_ifg-\partial_igf)+\partial_i(A_jA_k)\left(\frac{1}{1+\mu}fg-\frac{\mu}{1+\mu}gf\right)\right]. \quad (3.16)$$

As emphasized by Jo using different regularization kernels may give rise to a different approach dependence; to a different dependence on μ. This is indeed the case as we shall demonstrate below.

IV. POWERLIKE REGULARIZATION KERNELS

Let us choose a new set of δ-like functions $\{f_{\mu_f}(|\mathbf{x}|,b)\}$, the power functions [23]

$$f_{\mu_f}(|\mathbf{x}-\mathbf{y}|,b)=\frac{1}{N}\frac{\mu_f^{2b-3}}{(|\mathbf{x}-\mathbf{y}|^2+\mu_f^2)^b}, \quad (4.1)$$

with the normalization (β function)

$$N=2\pi B(3/2,b-3/2)=2\pi\frac{\Gamma(3/2)\Gamma(b-3/2)}{\Gamma(b)}, \quad (4.2)$$

and $b\geqslant3$, $b\in\mathbf{R}$. The Fourier transforms are

$$\tilde{f}_{\mu_f}(|\mathbf{p}|,b)=\frac{1}{\tilde{N}}(\mu_f|\mathbf{p}|)^{b-3/2}K_{b-3/2}(\mu_f|\mathbf{p}|), \quad (4.3)$$

with

$$\tilde{N}=2^{b-5/2}\Gamma(b-3/2), \quad (4.4)$$

and $K_{b-3/2}(t)$ is a Bessel function. In this case we obtain for the ambiguous term [24]

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q})=-\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q})\frac{\mu^{2b-3}}{2}\times F(2b-3,b-1/2;2b-2;1-\mu^2), \quad (4.5)$$

where $F(a,b;c;z)$ denotes the hypergeometric function with the integral representation ($\mathrm{Re}c>\mathrm{Re}b>0$)

$$F(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 dt\,t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}. \quad (4.6)$$

Clearly, for the two parameter values[2] $\mu=0$ and $\mu\to\infty$ we recover—for all values of b—Jo's result (this must be the

[1]This is valid for all renormalization kernels and not only for the Gaussian ones used by Jo [11].

[2]For the case $\mu=0$ it is better to use the expression

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q})=-\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q})\times\frac{\mu^{b-3/2}}{2^{2b-5}\Gamma^2(b-3/2)}\cdot I(b,\mu), \quad (4.7)$$

where

$$I(b,\mu)=\int_0^\infty dt\,t^{2b-3}K_{b-5/2}(t)K_{b-3/2}(\mu t). \quad (4.8)$$

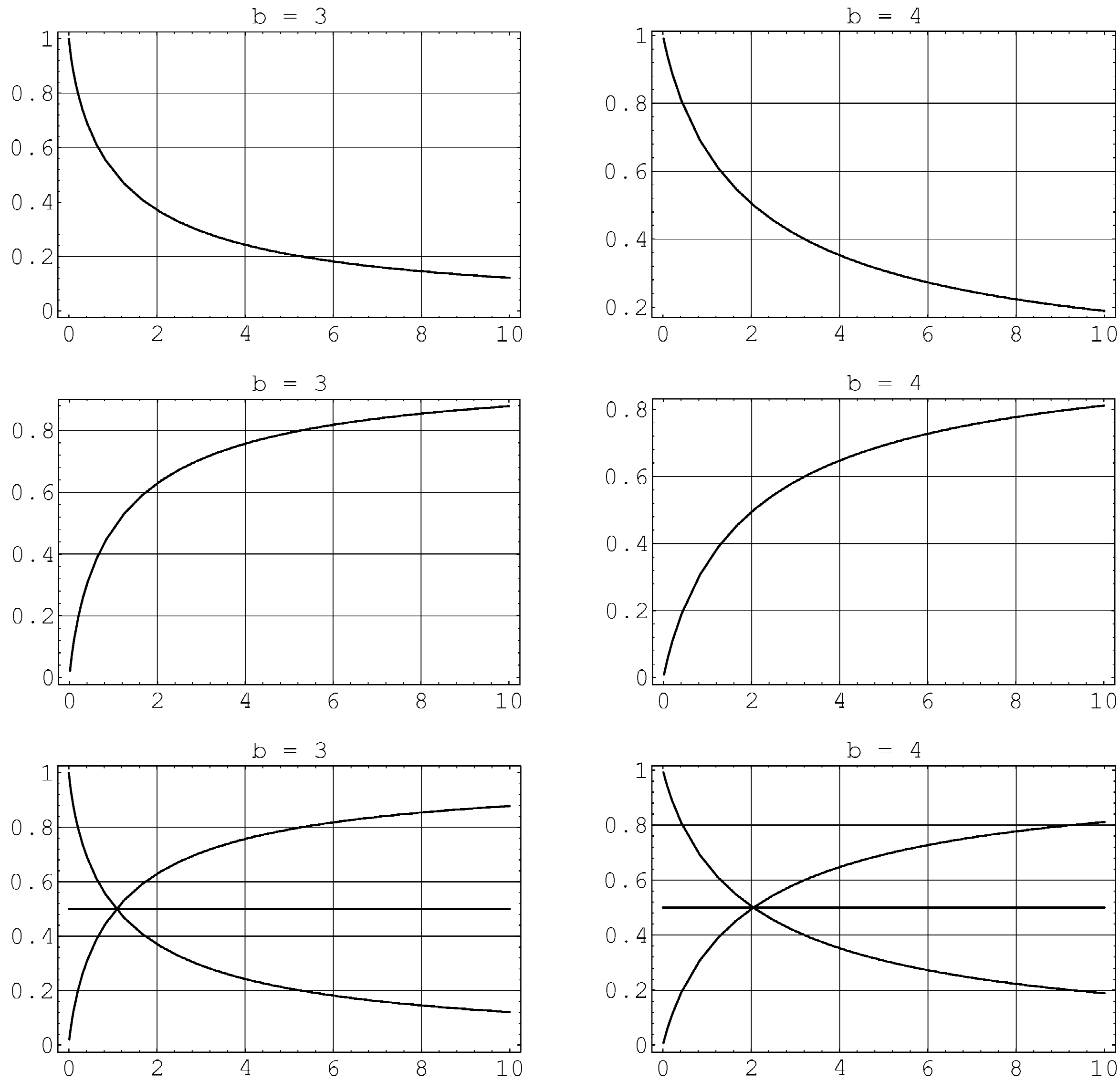

FIG. 1. The combination of Gaussian and power kernels, Eqs. (4.10) and (4.12) (see footnote 3), are plotted versus ξ for the values of $b=3$ and $b=4$.

case for general reasons as we shall demonstrate below). But also for $\mu=1$ the result (4.5) agrees with Jo's result derived with the Gaussian kernels. Of course, for a general value of μ this is not so. For example, for $b=3$ we get

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q}) = -\frac{3\mu+1}{(1+\mu)^3}\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q}). \tag{4.9}$$

Next we combine different δ-like functions. For example, let us choose the Gaussian kernel (3.11) to regularize the operator $\mathcal{J}(f)$ and the above power kernel (4.1) for $\mathcal{J}(g)$ then we obtain a different μ_f,μ_g dependence of the integral [24]

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q}) = -\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q}) \times\left(b-\frac{3}{2}\right)\xi^{b-3/2}U(b-1/2,b-1/2,\xi), \tag{4.10}$$

where $\xi\equiv\mu\mu_g/4$ and $U(a,b,z)$ denotes the Whittaker function with integral representation ($\mathrm{Re}\,a>0$)

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty dt e^{-zt}t^{a-1}(1+t)^{b-a-1}. \tag{4.11}$$

If we interchange the kernels then we obtain again an other μ_f,μ_g dependence

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q}) = -\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q}) \times\xi^{b-3/2}U(b-3/2,b-3/2,\xi). \tag{4.12}$$

We have plotted[3] the results (4.10) and (4.12) on Fig. 1 [25]. Again, for $\xi=0$ and $\xi\to\infty$ we recover the previous cases but now the desired agreement with the previous results, the

[3]Up to the common factor $-(q^i/12\pi^2)\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q})$.

TABLE I. The combination of Gaussian and power kernels. Values of ξ are given which show us how the limit procedures must be done for different values of b to satisfy integral conditions (5.6) and (5.7).

b	3	4	5	6	7
ξ	1.07748	2.04837	3.03504	4.02744	5.02194

value 1/2 where both functions (4.10) and (4.12) coincide (see footnote 3), is given at different ξ depending on the value of b. This corresponds to taking a different limit procedure for each value of b. The several ξ values we have collected in Table I [25]. Of course, for general values of ξ the results differ from the previous ones. So the above demonstrated dependence of the integral $\chi_2^i(F,G;\mathbf{p}',\mathbf{q})$ on the applied regularization kernels proves Jo's conjecture.

V. INTEGRAL CONDITION

We can overcome the above ambiguity in a quite natural way. Let us consider again the Schwinger term expression (3.3) for general regularization kernels. Since it is antisymmetric under interchange of f and g the final integral in the term

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q})=\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q}) \times \lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\frac{\partial}{\partial|\mathbf{p}|}\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\tilde{g}_{\mu_g}(|\mathbf{p}|) \tag{5.1}$$

must be invariant under this interchange, so

$$\lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\frac{\partial}{\partial|\mathbf{p}|}\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\tilde{g}_{\mu_g}(|\mathbf{p}|) = \lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\frac{\partial}{\partial|\mathbf{p}|}\tilde{g}_{\mu_g}(|\mathbf{p}|). \tag{5.2}$$

After partial integration,

$$2\lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\frac{\partial}{\partial|\mathbf{p}|}\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\tilde{g}_{\mu_g}(|\mathbf{p}|) = \lim_{\mu_f,\mu_g\to 0}[\tilde{f}_{\mu_f}(|\mathbf{p}|)\tilde{g}_{\mu_g}(|\mathbf{p}|)]_0^\infty=-1, \tag{5.3}$$

since δ-like functions satisfy

$$\tilde{f}_{\mu_f}(\infty)=\tilde{g}_{\mu_g}(\infty)=0$$

and

$$\lim_{\mu_f\to 0}\tilde{f}_{\mu_f}(0)=\lim_{\mu_g\to 0}\tilde{g}_{\mu_g}(0)=1. \tag{5.4}$$

Another way of getting the condition on the regularization is to use the normalization of the δ-like functions:

$$\lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\left[\frac{\partial}{\partial|\mathbf{p}|}\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\tilde{g}_{\mu_g}(|\mathbf{p}|) + \tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\frac{\partial}{\partial|\mathbf{p}|}\tilde{g}_{\mu_g}(|\mathbf{p}|)\right] = \lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\frac{\partial}{\partial|\mathbf{p}|}[\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\tilde{g}_{\mu_g}(|\mathbf{p}|)] = \lim_{\mu_f,\mu_g\to 0}[\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\tilde{g}_{\mu_g}(|\mathbf{p}|)]_0^\infty=-1 \tag{5.5}$$

and to respect the antisymmetry of the Schwinger term which implies the equality of the first two terms [see Eq. (5.2)]. So the antisymmetry of the Schwinger term already restricts the general possibilities for regularization and we are led to the following theorem.

Theorem 1. The classes of δ-like functions $\{f_{\mu_f}(|\mathbf{x}-\mathbf{y}|)\}$ and $\{g_{\mu_g}(|\mathbf{x}-\mathbf{y}|)\}$ which satisfy the integral conditions

$$\lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\frac{\partial}{\partial|\mathbf{p}|}\tilde{f}_{\mu_f}(|\mathbf{p}|)\cdot\tilde{g}_{\mu_g}(|\mathbf{p}|)=-\frac{1}{2}, \tag{5.6}$$

$$\lim_{\mu_f,\mu_g\to 0}\int_0^\infty d|\mathbf{p}|\frac{\partial}{\partial|\mathbf{p}|}\tilde{g}_{\mu_g}(|\mathbf{p}|)\cdot\tilde{f}_{\mu_f}(|\mathbf{p}|)=-\frac{1}{2}, \tag{5.7}$$

where both limits are of the same type, will provide a unique result for the Schwinger term $S(f,g)$.

This is the above-mentioned integral condition on the classes of regularization kernels *and* it also gives a condition on how μ_f and μ_g have to approach zero. For example, in the above described Gaussian or power kernel case the integral condition is satisfied for the value $\mu=1$, which is actually the most natural regularization, whereas in a combination of Gaussian and power kernels we must choose a special value of ξ depending on the value of b. Theorem 1 gives us the possibility to use every combination of regularization kernels and to define how μ_f and μ_g have to approach zero.

Finally, arriving in this way at a unique result, the Schwinger term of the Gauss-law commutator is given by

$$S^{ab}(\mathbf{x}-\mathbf{y})=\frac{-i}{24\pi^2}\varepsilon^{ijk}\mathrm{tr}\left\{(\partial_jA_k+A_jA_k)\{T^a,T^b\}\partial_i\delta^3(\mathbf{x}-\mathbf{y}) +\frac{1}{2}\partial_i(A_jA_k)\delta^3(\mathbf{x}-\mathbf{y})[T^a,T^b]\right\}. \tag{5.8}$$

Note that precisely the terms proportional to A_jA_k break Faddeev's cohomological result, Eq. (1.6) (as found by Jo [11]).

VI. CONCLUSION

When working with a generalized point-splitting method for the calculation of the Schwinger term in the commutator of Gauss-law operators, the occurring ambiguity due to the choice of regularization kernels can be overcome. The asymmetry of the Schwinger term restricts the possibilities for regularization allowing such that classes of regularization kernels which satisfy the integral conditions (5.6) and (5.7) lead to a unique result. A result, however, which differs from Faddeev's cohomology solution (1.6).

Note added. When calculating the commutator of the Gauss law operator and the Hamiltonian $[\mathcal{G},H]$, or equivalently the time derivative of the Gauss law operator, we should obtain the anomaly in the divergence of the chiral current [26]. However, the generalized point-splitting method used here does not work and must be altered. This we will present in a forthcoming publication [27].

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