New proof of the generalized second law

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The generalized second law of black-hole thermodynamics was proved by Frolov and Page [Phys. Rev. Lett. **71**, 3902 (1993)] for a quasistationary eternal black hole. However, realistic black holes arise from gravitational collapse, and in this case their proof does not hold. In this paper we prove the generalized second law for a quasistationary black hole that arises from gravitational collapse. [S0556-2821(97)03116-0]

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I. INTRODUCTION

The generalized second law of black-hole thermodynamics insists that the entropy of a black hole plus the thermodynamic entropy of fields outside the horizon does not decrease [1], where the black-hole entropy is defined as a quarter of the area of the horizon. Namely, it says that the entropy of the whole system does not decrease. It interests us in a quite physical sense since it links the world inside a black hole and our thermodynamic world. In particular it gives physical meaning to black-hole entropy indirectly since it concerns the sum of black-hole entropy and ordinary thermodynamic entropy and the physical meaning of the latter is well known by statistical mechanics.

Frolov and Page [2] proved the generalized second law for a quasistationary eternal black hole by assuming that the state of matter fields on the past horizon is thermal and that the set of radiation modes on the past horizon and that on the past null infinity are quantum-mechanically uncorrelated. The assumption is reasonable for the eternal case since a black hole emits a thermal radiation (the Hawking radiation). When we attempt to apply their proof to a noneternal black hole that arises from gravitational collapse, we might expect that things would go well by simply replacing the past horizon with a null surface at the moment of the formation of the horizon (the $v=v_0$ surface in Fig. 1). However, the expectation is disappointing since the assumption becomes problematic in this case. The reason is that on a collapsing background the thermal radiation is observed not at the moment of the horizon formation but at the future null infinity and that any modes on the future null infinity have a correlation with modes on the past null infinity located after the horizon formation. The correlation can be seen explicitly in Eq. (2.5) of this paper. Thus their proof does not hold for the case in which a black hole arises from gravitational collapse. Since astrophysically a black hole is thought to arise from gravitational collapse, we want to prove the generalized second law in this case.

In this paper we prove the generalized second law for a quasistationary black hole that arises from gravitational collapse. For this purpose we concentrate on an inequality between functionals of a density matrix since the generalized second law can be rewritten as an inequality between functionals of a density matrix of matter fields as shown in Sec. III. We seek a method to prove that a special functional of a density matrix cannot decrease under a physical evolution. (It is a generalization of a result by Sorkin [3].) To apply it to the system with a black hole and derive the generalized second law as its consequence we need to establish a property of physical evolution of matter fields around the black hole. Thus, for concreteness, we investigate a real massless scalar field semiclassically in a curved background that describes gravitational collapse and calculate conditional probabilities that, as a whole, have almost all the information about the behaviors of the scalar field after the formation of the horizon. (The probability we seek is a generalization of one calculated by Panangaden and Wald [4].) Using the result of the calculation, it is shown that a thermal density matrix of the scalar field at the past null infinity evolves to a thermal density matrix with the same temperature and the same chemical potential at the future null infinity, provided that the initial temperature and chemical potential are special values specified by the background geometry. Finally, we prove the generalized second law by using these results.

The rest of the paper is organized as follows. In Sec. II we consider a real massless scalar field in a background of a gravitational collapse to show a thermodynamic property of it. A thermal state with special values of temperature and chemical potential evolves to a thermal state with the same temperature and the same chemical potential. These special values are determined by the background geometry. In Sec. III, first the generalized second law is rewritten as an inequality that states that there is a nondecreasing functional of a density matrix of matter fields. After that we give a theorem that shows an inequality between functionals of density matrices. Finally, we apply it to the scalar field investigated in Sec. II to prove the generalized second law for the quasistationary background. In Sec. IV we summarize this paper.

II. MASSLESS SCALAR FIELD IN BLACK-HOLE BACKGROUND

In this section we consider a real massless scalar field in a curved background that describes the formation of a quasistationary black hole. Let us denote a past null infinity by $\mathcal{I}^-$, a future null infinity by $\mathcal{I}^+$, and a future event horizon by H^+. We introduce the usual null coordinates u,v and suppose that the formation of the event horizon H^+ is at $v=v_0$ (see Fig. 1). At $\mathcal{I}^-$ and $\mathcal{I}^+$, by virtue of the asymptotic

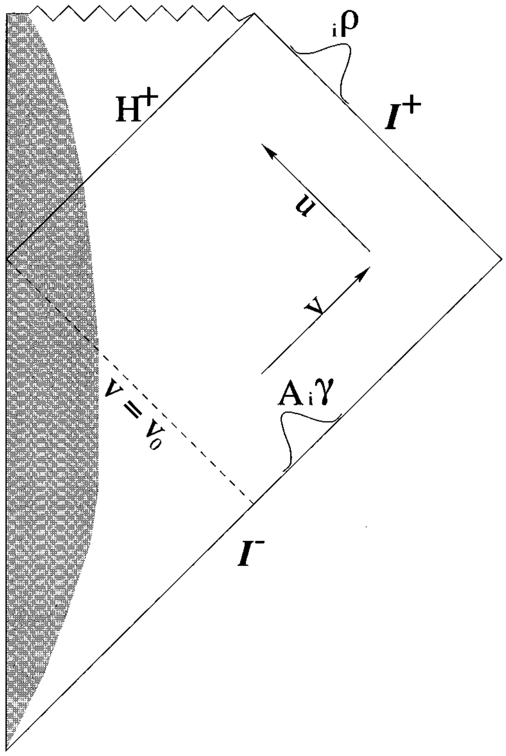

FIG. 1. Conformal diagram of a background geometry that describes a gravitational collapse. $\mathcal{I}^-$ and $\mathcal{I}^+$ are the past null infinity and the future null infinity, respectively, and H^+ is the future event horizon. The shaded region represents collapsing materials that form the black hole. In addition to the collapsing matter, we consider a real massless scalar field and investigate a scattering problem by the black hole after its formation $(v>v_0)$. Thus we specify possible initial states at $\mathcal{I}^-$ to those states that are excited from the vacuum by only modes whose support is within $v>v_0$ (elements of $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$) and possible mixed states constructed from them (elements of $\mathcal{P}$). In the diagram, $A\ {}_i\gamma$ $(i=1,2,\ldots)$ is a mode function corresponding to a wave packet whose peak is at $v>v_0$ on $\mathcal{I}^-$ and ${}_i\rho$ $(i=1,2,\ldots)$ is a mode function corresponding to a wave packet on $\mathcal{I}^+$.

flatness, there is a natural definition of the Hilbert spaces $\mathcal{H}_{\mathcal{I}^-}$ and $\mathcal{H}_{\mathcal{I}^+}$ of mode functions with positive frequencies [5]. The Hilbert spaces $\mathcal{F}(\mathcal{H}_{\mathcal{I}^\pm})$ of all asymptotic states are defined as follows with a suitable completion (symmetric Fock spaces):

$$\mathcal{F}(\mathcal{H}_{\mathcal{I}^\pm}) \equiv C \oplus \mathcal{H}_{\mathcal{I}^\pm} \oplus (\mathcal{H}_{\mathcal{I}^\pm} \otimes \mathcal{H}_{\mathcal{I}^\pm})_{\text{sym}} \oplus \cdots ,$$

where $(\ )_{\text{sym}}$ denotes the symmetrization $[(\xi\otimes\eta)_{\text{sym}}=\frac{1}{2}(\xi\otimes\eta+\eta\otimes\xi)$, etc.]. Physically, $\mathcal{C}$ denotes the vacuum state, $\mathcal{H}_{\mathcal{I}^\pm}$ denote one-particle states, $(\mathcal{H}_{\mathcal{I}^\pm}\otimes\mathcal{H}_{\mathcal{I}^\pm})_{\text{sym}}$ denote two-particle states, etc. We suppose that all our observables are operators on $\mathcal{F}(\mathcal{H}_{\mathcal{I}^\pm})$ since we observe a radiation of the scalar field radiated by the black hole at a place far away from it. In this sense $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})$ are quite physical. Next let us consider how to set an initial state of the scalar field. We want to see a response of the scalar field on the quasistationary black-hole background that arises from a gravitational collapse of other materials (dust, fluid, etc.). Hence the initial state at $\mathcal{I}^-$ must be such a state that it includes no excitations of modes located before the formation of the horizon (no excitation at $v<v_0$). A space of all such states is a subspace of $\mathcal{F}(\mathcal{H}_{\mathcal{I}^-})$ and we denote it by $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$. We would like to derive a thermal property of a scattering process of the scalar field by the quasistationary black hole. Hence we consider density matrices on $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$ and $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})$. Denote a space of all density matrices on $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$ by $\mathcal{P}$ and a space of all density matrices on $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})$ by $\tilde{\mathcal{P}}$.

Let us discuss an evolution of a state at $\mathcal{I}^-$ to the future. Since $\mathcal{I}^+$ is not a Cauchy surface because of the existence of H^+, $\mathcal{F}(\mathcal{H}_{\mathcal{I}^-})$ is mapped not to $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})$ but to $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})\otimes\mathcal{F}(\mathcal{H}_{H^+})$ by a unitary evolution, where $\mathcal{H}_{H^+}$ is a Hilbert space of mode functions on the horizon with a positive frequency and $\mathcal{F}(\mathcal{H}_{H^+})$ is a Hilbert space of all states on H^+ defined as a symmetric Fock space constructed from $\mathcal{H}_{H^+}$[see the definition of $\mathcal{F}(\mathcal{H}_{\mathcal{I}^\pm})$]. Although there is no natural principle to determine the positivity of the frequency (equivalently, there is no natural definition of the particle concept), the detailed definition of $\mathcal{H}_{H^+}$ does not affect the result since we shall trace out the degrees of freedom of $\mathcal{F}(\mathcal{H}_{H^+})$ [see Eq. (2.1)]. To describe the evolution of a quantum state of the scalar field from $\mathcal{F}(\mathcal{H}_{\mathcal{I}^-})$ to $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})\otimes\mathcal{F}(\mathcal{H}_{H^+})$ an S matrix is introduced [5]. For a given initial state $|\psi\rangle$ in $\mathcal{F}(\mathcal{H}_{\mathcal{I}^-})$, the corresponding final state in $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})\otimes\mathcal{F}(\mathcal{H}_{H^+})$ is $S|\psi\rangle$. Then the corresponding evolution from $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$ to $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})\otimes\mathcal{F}(\mathcal{H}_{H^+})$ is obtained by restricting S to $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$ and we denote it by S too. In this section we show a thermal property of the scalar field in the background by using the S-matrix elements given by Wald [5].

A. Definition of *T*

Let the initial state of the scalar field be $|\phi\rangle$ $[\in\mathcal{F}_{I^-(v>v_0)}]$ and observe the corresponding final state at $\mathcal{I}^+$ [see the argument after the definition of $\mathcal{F}(\mathcal{H}_{\mathcal{I}^\pm})$]. Formally the observation corresponds to a calculation of a matrix element $\langle\phi|S^\dagger OS|\phi\rangle$, where S is the S matrix that describes the evolution of the scalar field from $\mathcal{F}_{I^-(v>v_0)}$ to $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})\otimes\mathcal{F}(\mathcal{H}_{H^+})$ and O is a self-adjoint operator on $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})$ corresponding to the quantity we want to observe. The matrix element can be rewritten in a convenient fashion as

$$\langle\phi|S^\dagger OS|\phi\rangle=\text{Tr}_{\mathcal{I}^+}[O\rho_{\text{red}}],$$

where

$$\rho_{\text{red}}=\text{Tr}_{H^+}[S|\phi\rangle\langle\phi|S^\dagger]$$

and $\text{Tr}_{\mathcal{I}^+},\text{Tr}_{H^+}$ denote the partial traces over $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+}),\mathcal{F}(\mathcal{H}_{H^+})$, respectively. In viewing this expression we are lead to an interpretation that the corresponding final state at $\mathcal{I}^+$ is represented by the reduced density matrix ρ_{red}. Next we generalize this argument to a wider range of initial states, which includes all mixed states. For this case an initial state is represented not by an element of $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$ but by an element of $\mathcal{P}$ (a density matrix on $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$). Its evolution to $\mathcal{I}^+$ is represented as a map T induced by S followed by the partial trace Tr_{H^+}: let ρ ($\in \mathcal{P}$) be an initial density matrix; then the corresponding final density matrix $T(\rho)$ ($\in \widetilde{\mathcal{P}}$) is

$$T(\rho)=\mathrm{Tr}_{H^+}[S\rho S^\dagger]. \tag{2.1}$$

Note that T is a linear map from $\mathcal{P}$ into $\widetilde{\mathcal{P}}$.

B. Thermodynamic property of *T*

In this subsection we show a thermal property of the map T, which is summarized as Theorem 1. First let us calculate a conditional probability defined as

$$P(\{n_{i\rho}\}|\{n_{i\gamma}\})\equiv\langle\{n_{i\rho}\}|T(|\{n_{i\gamma}\}\rangle\langle\{n_{i\gamma}\}|)|\{n_{i\rho}\}\rangle, \tag{2.2}$$

where

$$|\{n_{i\gamma}\}\rangle\equiv\left[\prod_i \frac{1}{\sqrt{n_{i\gamma}!}}[a^\dagger(A\ {}_i\gamma)]^{n_{i\gamma}}\right]|0\rangle,$$

$$|\{n_{i\rho}\}\rangle\equiv\left[\prod_i \frac{1}{\sqrt{n_{i\rho}!}}[a^\dagger({}_i\rho)]^{n_{i\rho}}\right]|0\rangle. \tag{2.3}$$

$|\{n_{i\gamma}\}\rangle$ is a state in $\mathcal{F}_{\mathcal{I}^-(v>v_0)}$ characterized by a set of integers $n_{i\gamma}$ ($i=1,2,\dots$) and $|\{n_{i\rho}\}\rangle$ is a state in $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})$ characterized by a set of integers $n_{i\rho}$ ($i=1,2,\dots$). Therefore, $P(\{n_{i\rho}\}|\{n_{i\gamma}\})$ is a conditional probability for a final state to be $|\{n_{i\rho}\}\rangle$ when the initial state is specified to be $|\{n_{i\gamma}\}\rangle$. In the expressions A is a representation of a Bogoliubov transformation from $\mathcal{H}_{\mathcal{I}^+}\oplus\mathcal{H}_{H^+}$ to $\mathcal{H}_{\mathcal{I}^-}$ and ${}_i\gamma$ is such a unit vector in $\mathcal{H}_{\mathcal{I}^+}\oplus\mathcal{H}_{H^+}$ that $A_i\,\gamma$ corresponds to a wave pocket whose peak is located at a point on $\mathcal{I}^-$ later than the formation of the horizon ($v>v_0$) [5]. On the other hand, ${}_i\rho$ is a unit vector in $\mathcal{H}_{\mathcal{I}^+}$ and corresponds to a wave packet on $\mathcal{I}^+$ [5] (see Fig. 1). The probability (2.2) is a generalization of $P(k|j)$ investigated by Panangaden and Wald[1] [4]. It includes almost all the information[2] about a response of the scalar field on the quasistationary black hole that arises from the gravitational collapse while $P(k|j)$ does not, since any initial states on $\mathcal{I}^-$, which include no excitation before the formation of the horizon ($v<v_0$), can be represented by using the basis $\{|\{n_{i\gamma}\}\rangle\}$ and any final states on $\mathcal{I}^+$ can be expressed by the basis $\{|\{n_{i\rho}\}\rangle\}$, i.e., a set of all $|\{n_{i\gamma}\}\rangle$ generates $\mathcal{F}(\mathcal{H}_{\mathcal{I}^-(v>v_0)})$ and a set of all $|\{n_{i\rho}\}\rangle$ generates $\mathcal{F}(\mathcal{H}_{\mathcal{I}^+})$. This is the very reason why we generalize $P(k|j)$ to $P(\{n_{i\rho}\}|\{n_{i\gamma}\})$.

By using the S-matrix elements given in [5], the conditional probability is rewritten as (see Appendix A for its derivation)

$$\begin{aligned}P(\{n_{i\rho}\}|\{n_{i\gamma}\}) &= \prod_i\left[(1-x_i)x_i^{2n_{i\rho}}(1-|R_i|^2)^{n_{i\gamma}+n_{i\rho}}\sum_{l_i=0}^{\min(n_{i\gamma},n_{i\rho})}\sum_{m_i=0}^{\min(n_{i\gamma},n_{i\rho})}\frac{[-|R_i|^2/(1-|R_i|^2)]^{l_i+m_i}n_{i\gamma}!n_{i\rho}!}{l_i!(n_{i\gamma}-l_i)!(n_{i\rho}-l_i)!m_i!(n_{i\gamma}-m_i)!(n_{i\rho}-m_i)!}\right.\\ &\quad\left.\times\sum_{n_i=n_{i\rho}-\min(l_i,m_i)}^{\infty}\frac{n_i!(n_i-n_{i\rho}+n_{i\gamma})!}{(n_i-n_{i\rho}+l_i)!(n_i-n_{i\rho}+m_i)!}(x_i^2|R_i|^2)^{n_i-n_{i\rho}}\right],\end{aligned} \tag{2.4}$$

where R_i is a reflection coefficient for the mode specified by the integer i on the Schwarzschild metric (see Appendix A) and x_i is a constant defined by $x_i=\exp[-\pi(\omega_i-\Omega_{\mathrm{BH}}m_i)/\kappa]$. In the expression, ω_i and m_i are a frequency and an azimuthal angular momentum quantum number of the mode specified by the integer i, and Ω_{BH} and κ are the angular velocity and surface gravity of the black hole.

Now, the expression in the large square brackets in Eq. (2.4) appears also in the calculation of $P(k|j)$. Using the result of [4], it is easily shown that

$$P(\{n_{i\rho}\}|\{n_{i\gamma}\})=\prod_i\left[K_i\sum_{s_i=0}^{\min(n_{i\rho},n_{i\gamma})}\frac{(n_{i\rho}+n_{i\gamma}-s_i)!v_i^{s_i}}{s_i!(n_{i\rho}-s_i)!(n_{i\gamma}-s_i)!}\right], \tag{2.5}$$

where

$$K_i=\frac{(1-x_i)x_i^{2n_{i\rho}}(1-|R_i|^2)^{n_{i\gamma}+n_{i\rho}}}{(1-|R_i|^2x_i^2)^{n_{i\gamma}+n_{i\rho}+1}},$$

$$v_i=\frac{(|R_i|^2-x_i^2)(1-|R_i|^2x_i^2)}{(1-|R_i|^2)^2x_i^2}.$$

This is a generalization of the result of [4], and the following lemma is easily derived by using this expression.

[1]Their argument is restricted to the case when $n_{i\gamma}=n_{i\rho}=0$ for all i other than a particular value.

[2]All the information is included in $T^{\{n_{i\gamma}\}\{n'_{i\gamma}\}}_{\{n_{i\rho}\}\{n'_{i\rho}\}}$ defined in Lemma 2.

Lemma 1. For the conditional probability defined by Eq. (2.2) the following equality holds:

$$P(\{n_{i\,\rho}=k_i\}|\{n_{i\,\gamma}=j_i\})\exp\left(-\beta_{\rm BH}\sum_i j_i(\omega_i-\Omega_{\rm BH}m_i)\right)$$
$$=P(\{n_{i\,\rho}=j_i\}|\{n_{i\,\gamma}=k_i\})$$
$$\times\exp\left(-\beta_{\rm BH}\sum_i k_i(\omega_i-\Omega_{\rm BH}m_i)\right), \qquad (2.6)$$

where ω_i and m_i are the frequency and azimuthal angular momentum quantum number of the mode specified by i, $\Omega_{\rm BH}$ is the angular velocity of the horizon, and $\beta_{\rm BH}\equiv 2\pi/\kappa$. In this expression κ is a surface gravity of the black hole.

Note that $\beta_{\rm BH}^{-1}$ is the Hawking temperature of the black hole. This lemma states that a detailed balance condition holds.[3] Summing up about all k's, we expect that a thermal density matrix $\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})$ in $\mathcal{P}$ with a temperature $\beta_{\rm BH}^{-1}$ and a chemical potential $\Omega_{\rm BH}$ for azimuthal angular momentum quantum number will be mapped by T to a thermal density matrix $\widetilde{\rho}_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})$ in $\widetilde{\mathcal{P}}$ with the same temperature and the same chemical potential. To show that this expectation is true, we have to prove that all off-diagonal elements of $T[\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})]$ are zero. For this purpose the following lemma is proved in Appendix B.

Lemma 2. Denote a matrix element of T as

$$T^{\{n_{i\,\gamma}\}\{n'_{i\,\gamma}\}}_{\{n_{i\,\rho}\}\{n'_{i\,\rho}\}}\equiv\langle\{n_{i\,\rho}\}|T(|\{n_{i\,\gamma}\}\rangle\langle\{n'_{i\,\gamma}\}|)|\{n'_{i\,\rho}\}\rangle. \qquad (2.7)$$

Then

$$T^{\{n_{i\,\gamma}\}\{n'_{i\,\gamma}\}}_{\{n_{i\,\rho}\}\{n'_{i\,\rho}\}}=0, \qquad (2.8)$$

unless

$$n_{i\,\gamma}-n'_{i\,\gamma}=n_{i\,\rho}-n'_{i\,\rho} \qquad (2.9)$$

for all i.

Lemma 2 shows that all off-diagonal elements of $T(\rho)$ in the basis of $\{|\{n_{i\,\rho}\}\rangle\}$ vanish if all off-diagonal elements of ρ in the basis $\{|\{n_{i\,\gamma}\}\rangle\}$ are zero. Thus, combining it with lemma 1, the following theorem is easily proved. Note that a set of all $|\{n_{i\,\gamma}\}\rangle\langle\{n'_{i\,\gamma}\}|$ generates $\mathcal{P}$ and a set of all $|\{n_{i\,\rho}\}\rangle\langle\{n'_{i\,\rho}\}|$ generates $\widetilde{\mathcal{P}}$ [see the argument below Eq. (2.3)].

Theorem 1. Consider the linear map T defined by Eq. (2.1) for a real, massless scalar field on a background geometry that describes a formation of a quasistationary black hole. Then

$$T[\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})]=\widetilde{\rho}_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH}), \qquad (2.10)$$

where

$$\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})\equiv Z^{-1}\sum_{\{n_{i\,\gamma}\}}\exp\left(-\beta_{\rm BH}\sum_i n_{i\,\gamma}(\omega_i-\Omega_{\rm BH}m_i)\right)\times|\{n_{i\,\gamma}\}\rangle\langle\{n_{i\,\gamma}\}|,$$

$$\widetilde{\rho}_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})\equiv Z^{-1}\sum_{\{n_{i\,\rho}\}}\exp\left(-\beta_{\rm BH}\sum_i n_{i\,\rho}(\omega_i-\Omega_{\rm BH}m_i)\right)\times|\{n_{i\,\rho}\}\rangle\langle\{n_{i\,\rho}\}|,$$

$$Z\equiv\sum_{\{j_i\}}\exp\left(-\beta_{\rm BH}\sum_i j_i(\omega_i-\Omega_{\rm BH}m_i)\right). \qquad (2.11)$$

$\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})$ and $\widetilde{\rho}_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})$ can be regarded as "grand canonical ensembles" in $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, respectively, which have a common temperature $\beta_{\rm BH}^{-1}$ and a common chemical potential $\Omega_{\rm BH}$ for the azimuthal angular momentum quantum number. Thus the theorem says that the grand canonical ensemble at $\mathcal{I}^-$ $(v>v_0)$ with special values of the temperature and chemical potential evolves to a grand canonical ensemble at $\mathcal{I}^+$ with the same temperature and the same chemical potential. Note that the special values $\beta_{\rm BH}^{-1}$ and $\Omega_{\rm BH}$ are determined by the background geometry: $\beta_{\rm BH}^{-1}$ is the Hawking temperature and $\Omega_{\rm BH}$ is the angular velocity of the black hole formed. This result is used in Sec. III B to prove the generalized second law for the quasistationary black hole.

III. THE GENERALIZED SECOND LAW

The generalized second law is one of the most interesting conjectures in black-hole thermodynamics since it restricts ways of interaction between a black hole and ordinary thermodynamic matter. It can be regarded as a generalization both of the area law of black holes and of the second law of ordinary thermodynamics. The latter, which states that the total entropy of a system cannot decrease under the physical evolution of a thermodynamic system, can be proved for a finite-dimensional system if a microcanonical ensemble for the system does not change under the evolution [3].

In the preceding section we proved that the grand canonical ensemble of the scalar field does not change under the physical evolution on a background that describes the formation of a quasistationary black hole. So we expect that the generalized second law may be proved in a way similar to the proof of the second law of ordinary thermodynamics. For the purpose of the proof we rewrite the generalized second law as an inequality between functionals of a density matrix of matter fields.

The generalized second law of black-hole thermodynamics is

$$\Delta S_{\rm BH}+\Delta S_{\rm matter}\geqslant 0, \qquad (3.1)$$

where Δ denotes the change of quantities under the evolution of the system and $S_{\rm BH}$ and $S_{\rm matter}$ are the entropy of the black hole and the thermodynamic entropy of the matter fields,

[3] It guarantees that thermal distribution of any temperature is mapped to a thermal distribution of some other temperature closer to the Hawking temperature, as far as only the diagonal elements are concerned.

respectively. For a quasistationary black hole, using the first law of black-hole thermodynamics [6],

$$\Delta S_{\rm BH}=\beta_{\rm BH}(\Delta M_{\rm BH}-\Omega_{\rm BH}\Delta J_{\rm BH}),$$

the conservation of total energy

$$\Delta M_{\rm BH}+\Delta E_{\rm matter}=0,$$

and the conservation of total angular momentum

$$\Delta J_{\rm BH}+\Delta L_{\rm matter}=0,$$

it is easily shown that the generalized second law is equivalent to the inequality

$$\Delta S_{\rm matter}-\beta_{\rm BH}(\Delta E_{\rm matter}-\Omega_{\rm BH}\Delta L_{\rm matter})\geqslant 0, \tag{3.2}$$

where $\beta_{\rm BH}$, $\Omega_{\rm BH}$, $M_{\rm BH}$, and $J_{\rm BH}$ are the inverse temperature, angular velocity, mass, and angular momentum of the black hole; $E_{\rm matter}$ and $L_{\rm matter}$ are the energy and azimuthal component of the angular momentum of the matter fields. Equation (3.2) is of the form

$$U[\tilde{\rho}_0;\beta_{\rm BH},\Omega_{\rm BH})\geqslant U[\rho_0;\beta_{\rm BH},\Omega_{\rm BH}), \tag{3.3}$$

where U is the functional of the density matrix of the matter fields defined by

$$U[\rho;\beta_{\rm BH},\Omega_{\rm BH})\equiv -{\rm Tr}[\rho\ \ln\rho]-\beta_{\rm BH}({\rm Tr}[\mathbf{E}\rho] -\Omega_{\rm BH}\,{\rm Tr}[\mathbf{L}_z\rho]) \tag{3.4}$$

and ρ_0 and $\tilde{\rho}_0$ are the initial density matrix and the corresponding final density matrix, respectively. In the expression $\mathbf{E}$ and $\mathbf{L}_z$ are operators corresponding to the energy and the azimuthal component of the angular momentum. Note that Eq. (3.3) is an inequality between functionals of a density matrix of matter fields.[4] We will prove the generalized second law by showing that this inequality holds. Actually we do it in Sec. III B for a quasistationary black hole that arises from a gravitational collapse, using the results of Sec. II and a theorem given in the following subsection.

A. Nondecreasing functional

In this subsection we give a theorem that makes it possible to construct a functional that does not decrease by a physical evolution. It is a generalization of a result of [3]. In Sec. III B we derive Eq. (3.3) for a quasistationary black hole that arises from gravitational collapse, applying the theorem to the scalar field investigated in Sec. II.

Let us consider Hilbert spaces $\mathcal{F}$ and $\tilde{\mathcal{F}}$. First we give some definitions needed for the theorem.

Definition 1. A linear bounded operator ρ on $\mathcal{F}$ is called a density matrix if it is self-adjoint, positive semidefinite, and satisfies

$${\rm Tr}\rho=1.$$

[4]Information about the background geometry appears in the inequality as variables that parametrize the functional.

In the rest of this section we denote a space of all density matrices on $\mathcal{F}$ as $\mathcal{P}(\mathcal{F})$. Evidently, $\mathcal{P}(\mathcal{F})$ is a linear convex set rather than a linear set.

Definition 2. A map $\mathcal{T}$ of $\mathcal{P}(\mathcal{F})$ into $\mathcal{P}(\tilde{\mathcal{F}})$ is called linear if

$$\mathcal{T}[a\rho_1+(1-a)\rho_2]=a\mathcal{T}(\rho_1)+(1-a)\mathcal{T}(\rho_2)$$

for all a satisfying $0\leqslant a\leqslant 1$ and all ρ_1,ρ_2 $[\in\mathcal{P}(\mathcal{F})]$.

By this definition it is easily proved by induction that

$$\mathcal{T}\left(\sum_{i=1}^{N}a_i\rho_i\right)=\sum_{i=1}^{N}a_i\mathcal{T}(\rho_i) \tag{3.5}$$

if $a_i\geqslant 0$, $\Sigma_{i=1}^N a_i=1$, and $\rho_i\in\mathcal{P}(\mathcal{F})$.

Now we prove the following lemma, which concerns the $N\to\infty$ limit of the left-hand side of Eq. (3.5). We use this lemma in the proof of Theorem 2.

Lemma 3. Consider a linear map $\mathcal{T}$ of $\mathcal{P}(\mathcal{F})$ into $\mathcal{P}(\tilde{\mathcal{F}})$ and an element ρ_0 of $\mathcal{P}(\mathcal{F})$. For a diagonal decomposition

$$\rho_0=\sum_{i=1}^{\infty}p_i|i\rangle\langle i|,$$

define a series of density matrices of the form

$$\rho_n=\sum_{i=1}^{n}p_i/a_n|i\rangle\langle i|\quad(n=N,N+1,\dots), \tag{3.6}$$

where

$$a_n\equiv\sum_{i=1}^{n}p_i$$

and N is large enough that $a_N>0$. Then

$$\lim_{n\to\infty}\langle\Phi|\mathcal{T}(\rho_n)|\Psi\rangle=\langle\Phi|\mathcal{T}(\rho_0)|\Psi\rangle \tag{3.7}$$

for arbitrary elements $|\Phi\rangle$ and $|\Psi\rangle$ of $\tilde{\mathcal{F}}$.

This lemma says that $\mathcal{T}(\rho_n)$ has a weak-operator topology limit $\mathcal{T}(\rho_0)$.

Proof. By definition,

$$\rho_0=a_n\rho_n+(1-a_n)\rho_n', \tag{3.8}$$

where

$$\rho_n'=\begin{cases}\displaystyle\sum_{i=n+1}^{\infty}p_i/(1-a_n)|i\rangle\langle i| & (a_n<1)\\ \rho_n & (a_n=1).\end{cases}$$

Then the linearity of $\mathcal{T}$ shows that

$$\langle\Phi|\mathcal{T}(\rho_0)|\Psi\rangle=a_n\langle\Phi|\mathcal{T}(\rho_n)|\Psi\rangle+(1-a_n)\langle\Phi|\mathcal{T}(\rho_n')|\Psi\rangle.$$

Thus, if $\langle\Phi|\mathcal{T}(\rho_n')|\Psi\rangle$ is finite in the limit $n\to\infty$, then the lemma is established since

$$\lim_{n\to\infty}a_n=1.$$

For the purpose of proving the finiteness of $\langle\Phi|\mathcal{T}(\rho_n')|\Psi\rangle$, it is sufficient to show that $|\langle\Phi|\tilde{\rho}|\Psi\rangle|$ is bounded from above by $\|\Phi\|\,\|\Psi\|$ for an arbitrary element $\tilde{\rho}$ of $\mathcal{P}(\tilde{F})$. This is easy to prove as follows:

$$|\langle\Phi|\tilde{\rho}|\Psi\rangle| = \left|\sum_i \tilde{p}_i \langle\Phi|\tilde{i}\rangle\langle\tilde{i}|\Psi\rangle\right| \leq \sum_i |\langle\Phi|\tilde{i}\rangle\langle\tilde{i}|\Psi\rangle| \leq \|\Phi\|\,\|\Psi\|, \tag{3.9}$$

where we have used a diagonal decomposition

$$\tilde{\rho} = \sum_i \tilde{p}_i |\tilde{i}\rangle\langle\tilde{i}|.$$

■

Theorem 2. Assume the following three assumptions: (a) $\mathcal{T}$ is a linear map of $\mathcal{P}(\mathcal{F})$ into $\mathcal{P}(\tilde{\mathcal{F}})$, (b) f is a continuous function convex to below and there are nonnegative constants c_1, c_2, and c_3 such that $|f((1-\epsilon)x) - f(x)| \leq |\epsilon|[c_1|f(x)| + c_2|x| + c_3]$ for all x (≥ 0) and sufficiently small $|\epsilon|$, and (c) there are positive definite density matrices ρ_∞ $[\in\mathcal{P}(\mathcal{F})]$ and $\tilde{\rho}_\infty$ $[\in\mathcal{P}(\tilde{\mathcal{F}})]$ such that $\mathcal{T}(\rho_\infty) = \tilde{\rho}_\infty$. If $[\rho_\infty,\rho_0] = [\tilde{\rho}_\infty,\mathcal{T}(\rho_0)] = 0$ and $\mathrm{Tr}[\rho_\infty|f(\rho_0\rho_\infty^{-1})|] < \infty$, then

$$\tilde{\mathcal{U}}[\mathcal{T}(\rho_0)] \geq \mathcal{U}[\rho_0], \tag{3.10}$$

where

$$\mathcal{U}[\rho] \equiv -\mathrm{Tr}[\rho_\infty f(\rho\rho_\infty^{-1})], \qquad \tilde{\mathcal{U}}[\tilde{\rho}] \equiv -\mathrm{Tr}[\tilde{\rho}_\infty f(\tilde{\rho}\,\tilde{\rho}_\infty^{-1})]. \tag{3.11}$$

As stated in the first paragraph of this subsection, theorem 2 is used in Sec. III B to prove the generalized second law for a quasistationary black hole that arises from gravitational collapse.

Proof. First let us decompose the density matrices diagonally as

$$\rho_0 = \sum_{i=1}^{\infty} p_i |i\rangle\langle i|, \quad \rho_\infty = \sum_{i=1}^{\infty} q_i |i\rangle\langle i|, \qquad \mathcal{T}(\rho_0) = \sum_{i=1}^{\infty} \tilde{p}_i |\tilde{i}\rangle\langle i|, \quad \mathcal{T}(\rho_\infty) = \sum_{i=1}^{\infty} \tilde{q}_i |\tilde{i}\rangle\langle i|. \tag{3.12}$$

Then by lemma 3 and Eq. (3.5),

$$\tilde{p}_i = \langle\tilde{i}|\mathcal{T}(\rho_0)|\tilde{i}\rangle = \lim_{n\to\infty}\sum_{j=1}^{n} A_{ij}p_j/a_n, \tag{3.13}$$

where $a_n \equiv \Sigma_{i=1}^n p_i$ and $A_{ij} \equiv \langle\tilde{i}|\mathcal{T}(|j\rangle\langle j|)|\tilde{i}\rangle$. A_{ij} has the properties

$$\sum_{i=1}^{\infty} A_{ij} = 1, \quad 0 \leq A_{ij} \leq 1.$$

Similarly it is shown that

$$\tilde{q}_i = \lim_{n\to\infty}\sum_{j=1}^{n} A_{ij}q_j/b_n,$$

where $b_n \equiv \Sigma_{i=1}^n q_i$. By Eq. (3.13) and the continuity of f it is shown that

$$f(\tilde{p}_i/\tilde{q}_i) = \lim_{n\to\infty} f\left(\sum_{j=1}^{n} A_{ij}\frac{p_j/a_n}{\tilde{q}_i}\right). \tag{3.14}$$

Next define C_i^n and $\tilde{C}_i^n$ by

$$C_i^n \equiv \sum_{j=1}^{n} A_{ij}q_j/\tilde{q}_i, \quad \tilde{C}_i^n \equiv C_i^n/a_n; \tag{3.15}$$

then the convex property of f means

$$f(\tilde{p}_i/\tilde{q}_i) \leq \lim_{n\to\infty}\sum_{j=1}^{n} \frac{A_{ij}q_j}{C_i^n\tilde{q}_i} f(\tilde{C}_i^n p_j/q_j)$$

since

$$\sum_{j=1}^{n} \frac{A_{ij}q_j}{C_i^n\tilde{q}_i} = 1, \quad \frac{A_{ij}q_j}{C_i^n\tilde{q}_i} \geq 0.$$

Hence

$$-\tilde{\mathcal{U}}[\mathcal{T}(\rho_0)] = \sum_{i=1}^{\infty} \tilde{q}_i f(\tilde{p}_i/\tilde{q}_i) \leq \sum_{i=1}^{\infty} \lim_{n\to\infty}\sum_{j=1}^{n} \frac{A_{ij}q_j}{C_i^n} f(\tilde{C}_i^n p_j/q_j). \tag{3.16}$$

Since C_i^n and $\tilde{C}_i^n$ satisfy

$$\lim_{n\to\infty} C_i^n = \lim_{n\to\infty} \tilde{C}_i^n = 1,$$

it is implied by the assumption about f that

$$\left| f\left(\frac{\tilde{C}_i^n p_j}{q_j}\right) - f\left(\frac{p_j}{q_j}\right)\right| \leq |1-\tilde{C}_i^n|[c_1|f(p_j/q_j)| + c_2 p_j/q_j + c_3]$$

for sufficiently large n. Therefore,

$$\left|\sum_{j=1}^{n}\frac{A_{ij}q_j}{C_i^n}[f(\tilde{C}_i^n p_j/q_j)-f(p_j/q_j)]\right|$$
$$\leqslant\frac{|1-\tilde{C}_i^n|}{C_i^n}\left(c_1\sum_{j=1}^{n}A_{ij}q_j|f(p_j/q_j)|+c_2\sum_{j=1}^{n}A_{ij}p_j+c_3\sum_{j=1}^{n}A_{ij}q_j\right)$$
$$\leqslant\frac{|1-\tilde{C}_i^n|}{C_i^n}\left(c_1\sum_{j=1}^{n}q_j|f(p_j/q_j)|+c_2\sum_{j=1}^{n}p_j+c_3\sum_{j=1}^{n}q_j\right),$$

where we have used $0\leqslant A_{ij}\leqslant 1$ to obtain the last inequality. Since the first term in the large bracket in the last expression is finite in the limit $n\to\infty$ by the assumption of the absolute convergence of $\mathcal{U}[\rho_0]$ and all the other terms in the large bracket are finite,

$$\lim_{n\to\infty}\left|\sum_{j=1}^{n}\frac{A_{ij}q_j}{C_i^n}[f(\tilde{C}_i^n p_j/q_j)-f(p_j/q_j)]\right|=0.$$

Moreover, by the absolute convergence of $\mathcal{U}[\rho_0]$, it is easily shown that

$$\lim_{n\to\infty}\left|\left(\frac{1}{C_i^n}-1\right)\sum_{j=1}^{n}A_{ij}q_jf(p_j/q_j)\right|=0.$$

Thus

$$-\tilde{\mathcal{U}}[\mathcal{T}(\rho_0)]\leqslant\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}A_{ij}q_jf(p_j/q_j). \tag{3.17}$$

We can interchange the sum over i and the sum over j on the right-hand side of Eq. (3.17) since it converges absolutely by the absolute convergence of $\mathcal{U}[\rho_0]$. Hence

$$-\tilde{\mathcal{U}}[\mathcal{T}(\rho_0)]\leqslant\sum_{j=0}^{\infty}q_jf(p_j/q_j)=-\mathcal{U}[\rho_0].$$

■

B. Proof of the generalized second law

Let us combine theorem 1 with theorem 2 to prove the generalized second law. In theorem 2 set the linear map $\mathcal{T}$, the convex function $f(x)$, and the density matrices ρ_∞ and $\tilde{\rho}_\infty$ as

$$\mathcal{T}=T,$$
$$f(x)=x\ \ln x, \tag{3.18}$$
$$\rho_\infty=\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH}),$$
$$\tilde{\rho}_\infty=\tilde{\rho}_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH}).$$

Note that it is theorem 1 that makes such a setting possible. Hence, if an initial state ρ_0 and the corresponding final state $T(\rho_0)$ satisfy

$$[\rho_0,\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})]=[T(\rho_0),\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})]=0 \tag{3.19}$$

and $\mathcal{U}[\rho_0]$ converges absolutely, theorem 2 can be applied to the system of the quasistationary black hole and the scalar field around it. Now

$$\mathcal{U}[\rho_0]=-{\rm Tr}[\rho_0\ln\rho_0]-\beta_{\rm BH}({\rm Tr}[\mathbf{E}\rho_0]-\Omega_{\rm BH}{\rm Tr}[\mathbf{L}_z\rho_0])-\ln Z$$
$$=U[\rho_0;\beta_{\rm BH},\Omega_{\rm BH})-\ln Z, \tag{3.20}$$

$$\tilde{\mathcal{U}}[\mathcal{T}(\rho_0)]=-{\rm Tr}[\tilde{\rho}_0\ln\tilde{\rho}_0]-\beta_{\rm BH}({\rm Tr}[\tilde{\mathbf{E}}\tilde{\rho}_0]-\Omega_{\rm BH}{\rm Tr}[\tilde{\mathbf{L}}_z\tilde{\rho}_0])-\ln Z$$
$$=U[T(\rho_0);\beta_{\rm BH},\Omega_{\rm BH})-\ln Z,$$

where $\tilde{\rho}_0$ denotes $T(\rho_0)$,

$$\mathbf{E}\equiv\sum_{\{n_{i\gamma}\}}\left(\sum_i n_{i\gamma}\omega_i\right)|\{n_{i\gamma}\}\rangle\langle\{n_{i\gamma}\}|,$$
$$\mathbf{L}_z\equiv\sum_{\{n_{i\gamma}\}}\left(\sum_i n_{i\gamma}m_i\right)|\{n_{i\gamma}\}\rangle\langle\{n_{i\gamma}\}|$$

and

$$\tilde{\mathbf{E}}\equiv\sum_{\{n_{i\rho}\}}\left(\sum_i n_{i\rho}\omega_i\right)|\{n_{i\rho}\}\rangle\langle\{n_{i\rho}\}|,$$
$$\tilde{\mathbf{L}}_z\equiv\sum_{\{n_{i\rho}\}}\left(\sum_i n_{i\rho}m_i\right)|\{n_{i\rho}\}\rangle\langle\{n_{i\rho}\}|.$$

Thus the inequality (3.10) in this case is Eq. (3.3) itself, which in turn is equivalent to the generalized second law. Finally, theorem 2 proves the generalized second law for a quasistationary black hole that arises from gravitational collapse, provided that an initial density matrix ρ_0 of the scalar field satisfies the above assumptions. For example, it is guaranteed by lemma 2 that if ρ_0 is diagonal in the basis $\{|\{\eta_{i\,\gamma}\}\rangle\}$ then $T(\rho_0)$ is also diagonal in the basis $\{|\{n_{i\,\rho}\}\rangle\}$ and Eq. (3.19) is satisfied. The assumption of the absolute convergence of $U[\rho_0;\beta_{\rm BH},\Omega_{\rm BH})$ holds whenever initial state ρ_0 at $\mathcal{I}^-$ contains at most a finite number of excitations. Note that although $\rho_{\rm th}(\beta_{\rm BH},\Omega_{\rm BH})$ contains an infinite number of excitations by definition, ρ_0 does not. Therefore, the assumptions are satisfied when ρ_0 is diagonal in the basis $\{|\{n_{i\,\gamma}\}\rangle\}$ and contains at most a finite number of excitations.

IV. SUMMARY AND DISCUSSION

In summary we have proved the generalized second law for a quasistationary black hole that arises from gravitational collapse. To prove it we have derived the thermal property of the semiclassical evolution of a real massless scalar field on the quasistationary black-hole background and have given a method for searching for a nondecreasing functional. These are generalizations of the results of [4] and [3], respectively.

Now we comment on the Frolov-Page statement that their proof of the generalized second law may be applied to the case of the black hole formed by gravitational collapse [2]. Their proof for a quasistationary eternal black hole is based on the following two assumptions: (i) the state of matter fields on the past horizon is thermal and (ii) a set of radiation modes on the past horizon and one on the past null infinity are quantum-mechanically uncorrelated. These two assumptions are reasonable for the eternal case since black holes emit thermal radiation. In the case of a black hole that arises from gravitational collapse, we might expect that things would go well by simply replacing the past horizon with a null surface at the moment of the formation of the horizon (the $v=v_0$ surface in Fig. 1). However, the state of the matter fields on the past horizon is completely determined by the state of the fields before the horizon formation ($v<v_0$ in Fig. 1), in which there is no causal effect of the existence of the future horizon. Since the essential origin of the thermal radiation from a black hole is the existence of a horizon, the state of the fields on the null surface does not have to be a thermal one. Hence assumption (i) becomes problematic in this case. Although the above replacement may be the most extreme one, an intermediate replacement causes an intermediate violation of assumption (i) due to the correlation between modes on the future null infinity and modes on the past null infinity located after the horizon formation. The correlation can be seen explicitly in Eq. (2.5). Thus we conclude that Frolov and Page's proof cannot be applied to the case of the black hole formed by gravitational collapse.

Finally we discuss a generalization of our proof to a dynamical background. For the case of a dynamical background $\beta_{\rm BH}$ and $\Omega_{\rm BH}$ are changed from time to time by a possible backreaction. Thus, to prove the generalized second law for the dynamical background, we have to generalize theorem 1 to the dynamical case consistently with the backreaction. Once this can be achieved, theorem 2 seems useful to prove the generalized second law for the dynamical background.

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APPENDIX A: THE CONDITIONAL PROBABILITY

In this appendix we reduce Eq. (2.2) to Eq. (2.4). First the S matrix obtained by [5] is

$$S|0\rangle = N\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}\,(\overset{n}{\otimes}\epsilon),$$
$$Sa^\dagger(A\ {}_i\gamma)S^{-1}=R_i a^\dagger({}_i\rho)+T_i a^\dagger({}_i\sigma), \tag{A1}$$

where ϵ and N are a bivector and a normalization constant defined by

$$\epsilon = 2\sum_i x_i({}_i\lambda\otimes{}_i\tau)_{\rm sym},\quad N=\prod_i\sqrt{1-x_i},$$

where

$$x_i=\exp[-\pi(\omega_i-\Omega_{\rm BH}m_i)/\kappa].$$

In this expression ω_i and m_i are the frequency and azimuthal angular momentum quantum number of a mode specified by integer i, and $\Omega_{\rm BH}$ and κ are the angular velocity and surface gravity of the black hole, ${}_i\gamma$, ${}_i\rho$, ${}_i\sigma$, ${}_i\lambda$, and ${}_i\tau$ are unit vectors in $\mathcal{H}_{\mathcal{I}^+}\oplus\mathcal{H}_{H^+}$ defined in [5] and the first four are related as

$${}_i\gamma^a = T_i\,{}_i\sigma^a + R_i\,{}_i\rho^a,$$
$${}_i\lambda^a = t_i\,{}_i\rho^a + r_i\,{}_i\sigma^a, \tag{A2}$$

where t_i, T_i are transmission coefficients for the mode specified by the integer i on the Schwarzschild metric [5] and r_i, R_i are reflection coefficients. They satisfy[5]

$$|t_i|^2+|r_i|^2=|T_i|^2+|R_i|^2=1,$$
$$t_i=T_i,\quad r_i=-R_i^* T_i/T_i^*. \tag{A3}$$

By using the S matrix, we obtain

[5]The last two equations in (A3) are consequences of the time reflection symmetry of the Schwarzschild metric.

$$S|\{n_{i\,\gamma}\}\rangle = N\left[\prod_i \frac{1}{\sqrt{n_{i\,\gamma}!}}[R_i a^\dagger({}_i\rho)+T_i a^\dagger({}_i\sigma)]^{n_{i\,\gamma}}\right]\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}\left(\overset{n}{\otimes}\epsilon\right)_{\rm sym}$$

$$= N\sum_{n=0}^{\infty}\sum{}'\left[\prod_i \frac{1}{\sqrt{n_{i\,\gamma}!}}\binom{n_{i\,\gamma}}{m_i}R_i^{m_i}T_i^{n_{i\,\gamma}-m_i}\right]\sqrt{(2n)!}\left[\prod_i \frac{x_i^{n_i}}{n_i!}\binom{n_i}{l_i}t_z^{l_i}r_i^{n_i-l_i}\right]$$

$$\times\sqrt{\frac{(2n+\Sigma_i n_{i\,\gamma})!}{(2n)!}}\left(\prod_i \overset{n_i}{\otimes}{}_i\tau\overset{l_i+m_i}{\otimes}{}_i\rho\overset{n_i-l_i+n_{i\,\gamma}-m_i}{\otimes}{}_i\sigma\right)_{\rm sym}$$

$$= N\sum_{n_i=0}^{\infty}\sum_{m_i=0}^{n_{i\,\gamma}}\sum_{l_i=0}^{n_i}\sqrt{\left(\sum_i (2n_i+n_{i\,\gamma})\right)!}\prod_i\left[\frac{1}{\sqrt{n_{i\,\gamma}!}}\frac{x_i^{n_i}}{n_i!}\binom{n_{i\,\gamma}}{m_i}\binom{n_i}{l_i}R_i^{m_i}T_i^{n_{i\,\gamma}-m_i}t_i^{l_i}r_i^{n_i-l_i}\right]$$

$$\times\left(\prod_i \overset{n_i}{\otimes}{}_i\tau\overset{l_i+m_i}{\otimes}{}_i\rho\overset{n_i-l_i+n_{i\,\gamma}-m_i}{\otimes}{}_i\sigma\right)_{\rm sym}, \tag{A4}$$

where Σ' denotes a summation with respect to n_i, m_i, and l_i over the range $\Sigma_i n_i = n$, $n_i \geqslant 0$, $0 \leqslant m_i \leqslant n_{i\,\gamma}$, $0 \leqslant l_i \leqslant n_i$. In Eq. (A4), those orthonormal basis vectors in $|\{n_{i\,\rho}\}\rangle \otimes \mathcal{F}(\mathcal{H}_{H^+})$ that have a nonvanishing inner product with $S|\{n_{i\,\gamma}\}\rangle$ appear in the form[6]

$$\sqrt{\frac{(\Sigma_i(2n_i+n_{i\,\gamma}))!}{\Pi_i[n_i!n_{i\,\rho}!(n_i+n_{i\,\gamma}-n_{i\,\rho})!]}}\left(\prod_i \overset{n_i}{\otimes}{}_i\tau\overset{n_{i\,\rho}}{\otimes}{}_i\rho\overset{n_i+n_{i\,\gamma}-n_{i\,\rho}}{\otimes}{}_i\sigma\right)_{\rm sym}. \tag{A5}$$

Thus, when we calculate $(\langle\{n_{i\,\rho}\}|\otimes\langle H|)S|\{n_{i\,\gamma}\}\rangle$, the summation in Eq. (A4) is reduced to a summation with respect to n_i and m_i over the range $n_i \geqslant \max(0, n_{i\,\rho}-n_{i\,\gamma})$, $\max(0, n_{i\,\rho}-n_i) \leqslant m_i \leqslant \min(n_{i\,\rho}, n_{i\,\gamma})$ with[7] $l_i = n_{i\,\rho}-m_i$. Here $|H\rangle$ is an element of $\mathcal{F}(\mathcal{H}_{H^+})$. Paying attention to this fact, we can obtain the expression of the conditional probability

$$P(\{n_{i\,\rho}\}|\{n_{i\,\gamma}\}) = |N|^2\sum_{\{n_i\geqslant\max(0,n_{i\,\rho}-n_{i\,\gamma})\}}\left(\sum_i (2n_i+n_{i\,\gamma})\right)!\prod_i\left[\frac{x_i^{2n_i}}{n_{i\,\gamma}!(n_i!)^2}\left|\sum_{m_i=\max(0,n_{i\,\rho}-n_i)}^{\min(n_{i\,\rho},n_{i\,\gamma})}\binom{n_{i\,\gamma}}{m_i}\right.\right.$$

$$\left.\left.\times\binom{n_i}{n_{i\,\rho}-m_i}R_i^{m_i}T_i^{n_{i\,\gamma}-m_i}t_i^{n_{i\,\rho}-m_i}r_i^{n_i-n_{i\,\rho}+m_i}\right|^2\right]\left|\left\langle\sqrt{\frac{(\Sigma_i(2n_i+n_{i\,\gamma}))!}{\Pi_i[n_i!n_{i\,\rho}!(n_i+n_{i\,\gamma}-n_{i\,\rho})!]}}\right.\right.$$

$$\left.\left.\times\prod_i\left(\overset{n_i}{\otimes}{}_i\tau\overset{n_{i\,\rho}}{\otimes}{}_i\rho\overset{n_i+n_{i\,\gamma}-n_{i\,\rho}}{\otimes}{}_i\sigma\right)_{\rm sym},\prod_i\left(\overset{n_i}{\otimes}{}_i\tau\overset{n_{i\,\rho}}{\otimes}{}_i\rho\overset{n_i+n_{i\,\gamma}-n_{i\,\rho}}{\otimes}{}_i\sigma\right)_{\rm sym}\right\rangle\right|^2. \tag{A6}$$

The inner product in expression (A6) is equal to[8]

$$\sqrt{\frac{\Pi_i[n_i!n_{i\,\rho}!(n_i+n_{i\,\gamma}-n_{i\,\rho})!]}{(\Sigma_i(2n_i+n_{i\,\gamma}))!}}.$$

Finally, by using Eq. (A3) and exchanging the order of the summation suitably, we can obtain

---

[6]The number of the ''particle'' ${}_i\sigma$ in Eq. (A4) is $n_i+n_{i\,\gamma}-n_{i\,\rho}$, setting the number of the particle ${}_i\rho$ to $n_{i\,\rho}$.

[7]The range is obtained by inequalities $n_i \geqslant 0$, $0 \leqslant m_i \leqslant n_{i\,\gamma}$, $0 \leqslant l_i \leqslant n_i$, $l_i+m_i = n_{i\,\rho}$, and $n_i+n_{i\,\gamma}-n_{i\,\rho} \geqslant 0$.

[8]Equation (A5) is normalized to have unit norm.

$$P(\{n_{i\,\rho}\}|\{n_{i\,\gamma}\})=\prod_i\left[(1-x_i)x_i^{2n_{i\,\rho}}(1-|R_i|^2)^{n_{i\,\gamma}+n_{i\,\rho}}\sum_{l_i=0}^{\min(n_{i\,\gamma},n_{i\,\rho})}\sum_{m_i=0}^{\min(n_{i\,\gamma},n_{i\,\rho})}\frac{[-|R_i|^2/(1-|R_i|^2)]^{l_i+m_i}n_{i\,\gamma}!n_{i\,\rho}!}{l_i!(n_{i\,\gamma}-l_i)!(n_{i\,\rho}-l_i)!m_i!(n_{i\,\gamma}-m_i)!(n_{i\,\rho}-m_i)!}\right.$$
$$\left.\times\sum_{n_i=n_{i\,\rho}-\min(l_i,m_i)}^{\infty}\frac{n_i!(n_i-n_{i\,\rho}+n_{i\,\gamma})!}{(n_i-n_{i\,\rho}+l_i)!(n_i-n_{i\,\rho}+m_i)!}(x_i^2|R_i|^2)^{n_i-n_{i\,\rho}}\right].$$

This is what we have to show.

APPENDIX B: A PROOF OF LEMMA 2

In this appendix we give a proof of lemma 2.

Proof. Since a set of all ${}_i\tau$ and ${}_i\sigma$ generates $\mathcal{H}_{H^+}$ [5], the definition of $T^{\{n_{i\,\gamma}\}\{n'_{i\,\gamma}\}}_{\{n_{i\,\rho}\}\{n'_{i\,\rho}\}}$ leads to

$$T^{\{n_{i\,\gamma}\}\{n'_{i\gamma}\}}_{\{n_{i\,\rho}\}\{n'_{i\rho}\}}=\sum_{\{n_{i\sigma}\},\{n_{i\tau}\}}\langle\{n_{i\tau},n_{i\,\rho},n_{i\sigma}\}|S|\{n_{i\,\gamma}\}\rangle\langle\{n'_{i\,\gamma}\}|S|\{n_{i\tau},n'_{i\rho},n_{i\sigma}\}\rangle, \tag{B1}$$

where

$$|\{n_{i\tau},n_{i\,\rho},n_{i\sigma}\}\rangle\equiv\prod_i\left[\frac{1}{\sqrt{n_{i\tau}!n_{i\,\rho}!n_{i\sigma}!}}[a^\dagger({}_i\tau)]^{n_{i\tau}}[a^\dagger({}_i\rho)]^{n_{i\,\rho}}[a^\dagger({}_i\sigma)]^{n_{i\sigma}}\right]|0\rangle.$$

In this expression $S|\{n_{i\,\gamma}\}\rangle$ is given by Eq. (A4) and $S|\{n'_{i\,\gamma}\}\rangle$ is obtained by replacing $n_{i\,\gamma}$ with $n'_{i\,\gamma}$ in Eq. (A4). Now, those orthonormal basis vectors of the form $|\{n_{i\tau},n_{i\,\rho},n_{i\sigma}\}\rangle$ that have a nonzero inner product with $S|\{n_{i\,\gamma}\}\rangle$ must also be of the form (A5). Thus $T^{\{n_{i\,\gamma}\}\{n'_{i\,\gamma}\}}_{\{n_{i\,\rho}\}\{n'_{i\,\rho}\}}$ vanishes unless there exists a set of integers $\{n_i,n'_i\}$ $(i=1,2,\ldots)$ such that

$$n_i=n'_i,\quad n_i+n_{i\,\gamma}-n_{i\,\rho}=n'_i+n'_{i\,\gamma}-n'_{i\,\rho} \tag{B2}$$

for all i. The existence of $\{n_i\}$ and $\{n'_i\}$ is equivalent to the condition $n_{i\,\gamma}-n'_{i\,\gamma}=n_{i\,\rho}-n'_{i\,\rho}$ for all i. ■

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