Geometry of nonextreme black holes near the extreme state

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A nonextreme black hole in a cavity can achieve the extreme state with a zero surface gravity at a finite temperature on a boundary, the proper distance between the boundary and the horizon being finite. The classical geometry in this state is found explicitly for four-dimensional spherically symmetrical and $2+1$ rotating holes. In the first case the limiting geometry depends only on one scale factor and the *whole* Euclidean manifold is described by the Bertotti-Robinson spacetime. The general structure of a metric in the limit in question is also found with quantum corrections taken into account. Its angular part represents a two-sphere of a constant radius. In all cases the Lorentzian counterparts of the metrics are free from singularities. [S0556-2821(97)02716-1]

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One of the most intriguing issues in black hole physics is connected with the nature of the extreme state. In particular, it concerns the possibility of a thermodynamic description of such objects which is highly nontrivial because the Hawking temperature $T_H=0$ in the state under discussion. In Ref. [1] it was suggested to consider such objects as having an arbitrary temperature T_0 measured at infinity which determines the period in Euclidean time. This proposal is motivated by the qualitative difference between the topology of extreme and nonextreme black holes: the Euclidean manifold of an extreme one is regular irrespective of the value T_0 in contrast with nonextreme holes for which the absence of conical singularities demands $T_0=T_H$. Reasoning connected with topology [1,2] shows that one should ascribe the entropy $S=0$ to extreme holes instead of the Bekenstein-Hawking value $S=A/4$ (A is the surface area of a horizon).

However, further investigations showed that the deviation of T_0 from its Hawking value $T_H=0$ entails unavoidable divergences in the stress-energy tensor of quantum fields either for two-dimensional [3] or four-dimensional [4] extreme black holes and for this reason the possibility of $T_0\neq T_H$ is unacceptable physically. One can say that the principle of thermal equilibrium is more fundamental than the requirement of the regularity for the Euclidean manifold. The latter turns out to be simply irrelevant for extreme holes and cannot ensure a well-defined behavior of thermodynamic quantities for quantum fields in a corresponding background. In the case $T_0=T_H$ the difficulties indicated above are removed but the possibility to build a finite-temperature description of extreme holes is lost.

Meanwhile, recently it was traced [5] how a black hole can approach the extreme state in the topological sector of nonextreme configurations with $S=A/4$ when for the Reissner-Nordström (RN) metric $m\rightarrow e$ (m is a mass, e is a charge). In so doing, the thermodynamic equilibrium is fulfilled at every stage of the limiting transition, so difficulties due to $T_0\neq T_H$ cannot arise at all. In the grand-canonical ensemble approach when a system is characterized by the local temperature $T=\beta^{-1}$ and potential Φ at the boundary r_B it turns out that among all boundary data $(\beta,r_B,\Phi,)$ there exists such a subset $(\beta,r_B,\Phi(\beta,r_B))$ for which (i) the extreme state is realized at finite temperature, (ii) the horizon has the same area as the boundary surface but is situated at finite proper distance l_B from it (unlike the extreme topological sector where $l_B=\infty$). In some sense, a nonextreme hole imitates the extreme one, so $T_0\rightarrow 0$ (but T is finite).

The geometrical properties indicated above are rather unexpected and needed to be clarified. In this paper I show that for a wide class of black holes there exists the extreme state of nonextreme black holes which has the universal form of the limiting metric and find it explicitly. More exactly, I consider (i) spherically symmetrical configurations near the extreme state, (ii) the ultraextreme case and (iii) $(2+1)$-dimensional rotating black holes. It is remarkable that in spite of the variety of types of black hole solutions and their parameters, some shortening of description occurs (as we will see) near the state in question.

Consider the Euclidean black hole metric

$$ds^2=f(r)d\tau^2+dr^2/f+r^2d\omega^2,\quad d\omega^2=d\theta^2+d\phi^2\ \sin^2\theta. \tag{1}$$

The Euclidean time takes its values in the range $0\leq\tau\leq T_0^{-1}=T_H^{-1}$. Let us introduce the new variable $\tau_1=2\pi T_0\tau$ where now $0\leq\tau_1\leq 2\pi$. Then

$$ds^2=(\beta/2\pi)^2d\tau_1^2+dl^2+r^2d\omega^2. \tag{2}$$

$\beta[r(l)]=\beta_0[f(r)]^{1/2}$ is the inverse local temperature at an arbitrary point $r_+\leq r\leq r_B$, r_+ is the radius of the event horizon, l is the proper distance between r_+ and r.

For the spacetime (1) and (2) the equilibrium condition reads

$$\beta=\beta_0[f(r_B)]^{1/2},\quad T_0=T_H=f'(r_+)/4\pi, \tag{3}$$

where a prime denotes the derivative with respect to a corresponding argument, β is taken at the boundary r_B.

For the nearly extreme state $\beta_0\rightarrow\infty$ but there exists the

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subset of boundary data for which $r_+ \to r_B$ in such a way that the limit can be reached at finite β with the finite distance $l = l_B$ between r_+ and r_B [5]. In so doing, the coordinate r becomes ill defined and it is more convenient to use l or its dimensionless analog as a new coordinate. Let us choose this coordinate according to

$$r - r_+ = 4\pi T_0 b^{-1} (\sinh x/2)^2, \quad b = f'(r_+)/2. \tag{4}$$

In the limit in question the region $r_+ \leqslant r \leqslant r_B$ shrinks and we can expand $f(r)$ in a power series $f(r) = 4\pi T_0 (r - r_+) + b(r - r_+)^2 + \cdots$ near $r = r_+$. Then after substitution into Eqs. (2) and (3) we obtain

$$ds^2 = b^{-1}(d\tau_1 \sinh^2 x + dx^2) + r_B^2 d\omega^2, \quad x = lb^{1/2}. \tag{5}$$

For the RN hole $f = (1 - r_+/r)(1 - r_-/r)$ and in the extreme limit $r_+ = r_- = r_B$, $b = r_B^{-2}$. Then

$$ds^2 = r_B^2 (d\tau_1^2 \sinh^2 x + dx^2 + d\omega^2) \tag{6}$$

(Euclidean version) or

$$ds^2 = r_B^2 (-dt^2 \sinh^2 x + dx^2 + d\omega^2) \tag{7}$$

(Lorentzian version). This is nothing more than the Bertotti-Robinson (BR) spacetime [6,7].

That the BR metric is relevant for the description of the extreme RN hole was already pointed out in the literature [4,8,9] but in quite different context when the BR solution was considered in the topological sector corresponding to extreme black holes, $f = (1 - r_+/r)^2$ from the very beginning. Then, expanding the metric coefficients near $r = r_+$ and introducing the new radial variable according to $r - r_+ = r_B \rho^{-1}$ one obtains

$$ds^2 = r_B^2 \rho^{-2} (-dt_1^2 + d\rho^2 + \rho^2 d\omega^2). \tag{8}$$

Two metrics (7) and (8) look different but in fact are equivalent locally. This can be seen from the representation of the BR spacetime as the metric on the hyperboloid [7]. Putting $w^2 + v^2 - u^2 = 1$, $ds^2 = du^2 - dv^2 - dw^2$ and making substitution for the part of the hyperboloid $u \geqslant 0$

$$u = \cosh t \sinh x, \quad v = \sinh t \sinh x, \quad w = \cosh x, \tag{9}$$

we obtain Eq. (7). Another substitution

$$u = (2\rho)^{-1}(t_1^2 - \rho^2 + 1), \quad v = (2\rho)^{-1}(t_1^2 - \rho^2 - 1),$$
$$w = \rho^{-1} t_1 \tag{10}$$

(for simplicity we put $r_B = 1$ for a moment) gives us Eq. (8). Here two coordinate systems are connected by formulas

$$t_1/r_B = e^t \coth x, \quad \rho/r_B = e^t (\sinh x)^{-1}, \tag{11}$$

$$\cosh x = t_1/\rho, \quad e^{2t} = (t_1^2 - \rho^2)/r_B^2. \tag{12}$$

I stress that whereas the BR metric (8) gives only the approximate representation of the RN one of an extreme hole in a small vicinity of $r = r_+$ solution (6) refers to the *whole* Euclidean manifold whose four-volume in our case is finite. It is clear from Eqs. (6) and (7) that the proper distance between a horizon $x = 0$ and any other point with $x > 0$ is finite whereas the distance between $\rho = \infty$ and any other $\rho < \infty$ is infinite. It is this property, which gives rise to qualitative distinction between topologies of extreme and nonextreme holes and their entropies [1,2].

Thus, starting from the original RN metric we arrive in the limit $m \to e$ at two different versions of the BR spacetime corresponding to different Killing vectors $\partial/\partial t$ and $\partial/\partial t_1$ depending on a topological sector [10]. That it is Eq. (7) [but not Eq. (8)] which is relevant for our problem is not incidental. The Hawking effect is intimately connected with the existence of a horizon which makes the outer region geodesically incomplete. Either Eq. (7) or Eq. (8) possesses this property but Eq. (7) is more incomplete in the sense that only the part $t_1 > \rho$ of Eq. (8) is mapped to Eq. (7) as follows from Eqs. (11) and (12). As a result, $T_H = (2\pi)^{-1}$ for Eq. (7) in accordance with the finite boundary temperature in our problem whereas $T_H = 0$ for Eq. (8): roughly speaking, the more information lost, the more there is a temperature.

The BR metric is obtained above from the RN one and for this reason represents only classical geometry. Quantum effects will certainly change its form. (In particular, Riemann curvature $R \neq 0$ in general while $R = 0$ for the BR spacetime.) It is remarkable, however, that *all* effects of back reaction for the model (1) are encoded in the coefficient b only, so the metric is described by Eq. (5) instead of Eq. (6) where b takes its RN value r_B^{-2}. It is also worth paying attention that the explicit form of boundary data for which the state in question is achieved [$\beta = 4\pi r_B \Phi (1 - \Phi^2)^{-1}$ for the classical RN hole [5]] was not used in derivation at all as well as the form of field equations. Such universality is explained by that we consider the state for which *by definition* $T_0 \to 0$. Quantum effects can change the connection between T_H and black hole parameters as well as relationship between boundary data (say, between β and Φ) for a dressed hole as compared with a bare one. Nonetheless, if a theory admits the existence of an extreme state in the above sense, the limiting form of the metric with finite boundary conditions is described by Eq. (5).

Bearing in mind the possible role of quantum effects, of interest is a metric more general than Eq. (1):

$$ds^2 = U(r) d\tau^2 + V^{-1}(r) dr^2 + r^2 d\omega^2. \tag{13}$$

Then, repeating all manipulations step by step we arrive at

$$ds^2 = b^{-1}(a d\tau_1^2 + dx^2) + r_B^2 d\omega^2. \tag{14}$$

Here $a = 4 \sinh^2 x/2 (1 + c \sinh^2 x/2)$, $b = V''(r)/2$, $c = V'(r_+) U''(r_+) / V''(r_+) U'(r_+)$.

In general, if $c \neq 1$ the Riemann curvature is no longer constant but depends on a point due to effects of back reactions.

The developed approach is also applicable to two-dimensional metrics which can be obtained by discarding the term $d\omega^2$. It is known that in the topological sector of extreme holes there arise weak divergences of the stress-energy tensor of quantum fields on the event horizon discussed recently in the context of two-dimensional dilaton gravity [11]. These divergences are due to the property $T_H=0$ [4] and are absent in our case where $T_H\neq 0$ for the limiting form of the metric (6) and (14).

Let us discuss briefly the case of so-called ultraextreme black holes for which, in the extreme, state $b=0$ by definition [12]. One can check that finite boundary conditions are again possible due to $r_+\rightarrow r_B$ in spite of $T_H\rightarrow 0$. In this case the substitution $r-r_+=\pi T_H l^2$ shows that terms of the third order and higher in the expansion of $f(r)$ near $r=r_+$ are negligible and

$$ds^2=l^2 d\tau_1^2+dl^2+r_B^2 d\omega^2. \tag{15}$$

In other words, we obtain the direct product of the two-dimensional Rindler space and a two-sphere of a constant radius.

I stress that this result is by no means the trivial consequence of the known fact that the metric of a generic black hole can be represented by the Rindler one near the horizon. Such representation is approximate, valid only in a small region of spacetime near the event horizon, approximate to a nonextreme black hole and has nothing to do with the thermodynamic approach we deal with. Meanwhile, our black hole is ultraextreme and the four-dimensional volume of a Euclidean manifold is finite. The metric (15) appeared as a result of the limiting transition from the nonextreme state when $f'(r_+)$ is small but nonzero while $f''(r_+)=0$ by definition; had we started from the metric with $f=\text{const}(r-r_+)^3$ at once we would have obtained the metric which has nothing to do with Eq. (15).

Now we will analyze the case of $2+1$ black holes [13] which is achieved due to rotation. The metric reads [13]

$$ds^2=-N^2dt^2+N^{-2}dr^2+r^2(N^\phi dt+d\phi)^2,$$

$$N^2=-M+(r/l)^2+J^2/4r^2, \quad N^\phi=-J/2r^2. \tag{16}$$

As we will be interested in the metric near the horizon r_+ it is convenient to redefine the angular variable according to $\phi\rightarrow\phi-N^\phi(r_+)t$. Then

$$ds^2=-N^2dt^2+N^{-2}dr^2+r^2(N^\phi dt+d\phi)^2,$$

$$N^\phi=J(1/2r_+^2-1/2r^2). \tag{17}$$

Following the general approach for finite-size thermodynamics of $2+1$ holes (see [14,15] and references therein) let us consider the grand-canonical ensemble for which the set of boundary conditions includes in the case under discussion r_B, β and the angular velocity Ω of the heat bath with respect to zero angular momentum observers:

$$\Omega=JN^{-1}(1/2r_+^2-1/2r^2), \tag{18}$$

$$\beta=\beta_0 N, \quad \beta_0=2\pi r_+(M^2-J^2/l^2)^{-1/2}. \tag{19}$$

It is convenient to introduce dimensionless quantities $y=r_+/r_B$, $q=\Omega r$, $\sigma=\beta/2\pi l$, $z=J/Ml$. Then after some algebraic manipulations one obtains from Eqs. (18) and (19)

$$y=\{(1-q^2)[1+\sigma^2(1-q^2)]\}^{-1/2},$$

$$z=2\sigma q[q^2+\sigma^2(1-q^4)]^{-1}\{(1-q^2)[1+\sigma^2(1-q^2)]\}^{1/2}. \tag{20}$$

We are interested in the possibility of finding the finite-temperature solutions of these equations for the extreme state $(\sigma<\infty, z=1)$. It is seen from Eq. (20) directly that such a solution does exists if $y=1$ ($r_+=r_B$ as well as in the $3+1$ case) and boundary data are restricted by the condition

$$\sigma=q/(1-q^2). \tag{21}$$

The thermodynamic description needs the transition from the Lorentzian picture to the Euclidean one that for rotating holes gives rise to the complexification of a metric [16]. However, to avoid subtleties which are irrelevant for the issue under consideration, I will list a metric at once in the Lorentzian form. After expanding N^2 and N^ϕ in a power series near $r=r_+$ and using the substitution (4) one can obtain

$$ds^2=r_B^2[-dt_1^2\sinh^2 x+dx^2+(d\phi+\Omega_1 dt_1)^2],$$

$$\Omega_1=(\sinh^2 x/2)4r_B, \tag{22}$$

where time is normalized according to $t_1=2\pi T_0 t$. Now only two dimensionless parameters are independent among all boundary data—say $1/r_B$ and q, whereas σ is determined by Eq. (21).

Thus, we obtained the generalization of the BR metric to the case of rotation as the limiting form of $2+1$ holes in the state under discussion.

It is worth noting that if $\sigma\rightarrow 0$ $(T\rightarrow\infty)$ we obtain the extreme state not simply at finite temperature but even in the high-temperature limit. In so doing, $q\rightarrow 0$ according to Eq. (21), so the extreme hole $(J=Ml)$ is slowly rotating.

To summarize, it turned out that in the topological sector of nonextreme black holes a metric takes the universal form near the extreme state. Its classical geometry is described by a single parameter r_B which enters the metric simply as a scale factor in the static case and by two parameters for rotating $2+1$ holes. The essential feature of the obtained solutions is the absence of singularities of spacetime in contrast to the truly (topologically) extreme black holes for which singularities exist behind a horizon. Although in the thermodynamic approach only the region $r>r_+$ is relevant, it is possible that the above results shed light on the issue of singularities. Indeed, due to properly chosen boundary data all the range of a radial coordinate shrinks to the point $r=r_+$ in such a way that Lorentzian versions of metrics are free from singularities. In this sense it is thermodynamics which is responsible for removing singularities in the Lorentzian versions of metrics. Perhaps nature forbids the existence of truly extreme black holes as thermodynamic objects (see the beginning of the paper) but gives us instead the hint to another possible approach to the problems of either the extreme state or singularities in the framework of gravitational thermodynamics.

Of interest is to generalize the obtained results to nonspherical and rotating four-dimensional black holes.

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