Attractive or repulsive nature of the Casimir force for rectangular cavity

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The Casimir effect giving rise to an attractive or repulsive force between the configuration boundaries that confine the massless scalar field is reexamined for a $(D-1)$-dimensional rectangular cavity with unequal finite p edges and different spacetime dimensions D in this paper. With periodic or Neumann boundary conditions, the energy is always negative. The case of Dirichlet boundary conditions is more complicated. The sign of the Casimir energy satisfying Dirichlet conditions on the surface of a hypercube (a cavity with equal finite p edges) depends on whether p is even or odd. In the general case (a cavity with unequal p edges), however, we show that the sign of the Casimir energy does not only depend on whether p is odd or even. Furthermore, we find that the Casimir force is always attractive if the edges are chosen appropriately. It is interesting that the Casimir force may be repulsive for odd p cavity with unequal edges, in contrast with the same problem in a hypercube case. [S0556-2821(97)00116-1]

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I. INTRODUCTION

The study of vacuum fluctuations, as embodied in the Casimir effect [1], has been a subject of extensive research [2]. The Casimir effect is essentially a polarization of the vacuum of quantized fields which arises because of a change in the spectrum of vacuum oscillations when the quantization volume is bounded, or some background field (e.g., gravity) is presented. Early investigations of the effects of a gravitational background were performed by Utiyama and DeWitt [3], and work has continued on this important subject [4]. Historically, the first prediction of the effect of boundaries was performed by Casimir [1]: An attractive force $F=-(\pi^2/240a^4)$ should act on a unit area of two conducting plane-parallel plates in vacuum, where a is the distance between the plates. An attraction of this sort was subsequently observed experimentally [5]. For plates 1 cm^2 in area with $a=0.5$ μm, the force was ≈ 0.2 dyn, in accordance with the theoretical prediction. Another physical situation where the Casimir effect has been of great importance is in the bag models of QCD [6].

The question of whether the Casimir effect for a rectangular cavity with unequal edges and different spacetime dimensions gives rise to an attractive or repulsive force between the configuration boundaries that confine the field is up till now unsolved and will be discussed in this paper. In the general case of a D-dimensional spacetime, for a massless scalar field quantized inside a box with p edges of equal length $L_1=L_2=\cdots=L_p=L$ and $D-1-p$ edges with characteristic length $\lambda \gg L$, the sign of the Casimir energy E_p^D is always negative [7]. For Dirichlet boundary conditions, the sign of E_p^D depends on whether p is even or odd [8]. When the number of finite and equal edges of a rectangular box is odd it is analytically shown that $E_p^D<0$ for any D. On the contrary, when p is even there exists a particular critical value of spacetime dimension D_c, which depends on p, such that $E_p^D>0$ for $D<D_c$, and E_p^D is shown to be always negative for $D \geqslant D_c$. However, it is not surprising that existence of critical value D_c depends on whether p is even or odd, because authors of Ref. [8] have assumed $L_1=L_2=\cdots=L_p=L$ for simplicity. Actually, in the general case of a D-dimensional spacetime the sign of the Casimir energy may depend on (a) the spacetime dimensionality, (b) the type of boundary conditions, (c) the number p of independent directions with finite extension of the space region that constraints the field, (d) the relation between the lengths of these finite p edges, (e) the gravitational background, (f) the compactness of spacetime, and (g) the finite temperature. Lukosz [9] has argued that the Casimir energy associated with an electromagnetic field quantized inside a perfectly conducting box edges L_1, L_2, L_3, may change sign depending on the relative lengths. A similar behavior has been shown to occur in the case of a massless scalar field in a three-dimensional parallel-epipedal cavity with Dirichlet boundary conditions [10]. In this paper, the consequences of (e)–(g) will not be discussed, and we will discuss the Casimir effect of massless scalar field for a rectangular cavity with unequal edges and Dirichlet boundary conditions in D-dimensional Minkowski spacetime.

Few physicists would nowadays argue against the statement that the ζ-function regularization procedure has proven to be a very powerful and elegant technique. Rigorous extension of the proof of Epstein ζ-function [11] regularization has been obtained [12,13]. Recently, the generalized ζ-function has many interesting applications, e.g., in the piecewise string [14] and p-branes [15]. It may be worth emphasizing that the Riemann ζ function is the functional ζ function associated with the rectangular cavity of equal edges, and the Epstein ζ function is the functional ζ function associated with that of unequal edges. In this paper the sign of the Casimir energy and the nature of the Casimir force associated with a massless scalar field trapped inside a rectangular cavity (with p finite and unequal edges and $D-p-1$ infinite edges) in D-dimensional spacetime is discussed for different lengths of these finite edges. We find that the sign of the Casimir energy depends on the relative lengths of edges. Furthermore, we show that the Casimir force is always attractive if the lengths of edges are chosen appropriately. The Casimir force may be repulsive for odd-dimensional cavity ($p>1$) with unequal edges, in contrast with the same problem in a hypercube case.

II. CASIMIR ENERGY IN *D*-DIMENSIONAL MINKOWSKI SPACETIME

In calculations on the Casimir effect, extensive use is made of eigenfunctions and eigenvalues of the corresponding field equation. A Hermitian massless scalar field $\phi(t,x^a,x^T)$ defined in a D-dimensional Minkowski spacetime satisfies the free Klein-Gordon equation:

$$(\partial_t^2-\partial_i^2)\phi(t,x^a,x^T)=0, \tag{2.1}$$

where $i=1,\ldots,D-1$; $a=1,\ldots,p$; $T=p+1,\ldots,D-1$. The field is confined in the interior of $(D-1)$-dimensional rectangular cavity Ω with p edges of finite lengths $L_1,L_2,\ldots,L_p$ and $D-1-p$ edges with characteristic lengths of order $\lambda \gg L_a$. We consider the case of Dirichlet boundary conditions, i.e., $\phi(t,x^a,x^T)|_{\partial\Omega}=0$. The modes of the field are then

$$\phi_{\{n\}}=\sin\frac{n_1\pi x_1}{L_1}\sin\frac{n_2\pi x_2}{L_2}\cdots\sin\frac{n_p\pi x_p}{L_p}\,e^{ik_T\cdot x_T}e^{-i\omega_k t}, \tag{2.2}$$

$$\omega_k^2=k_T^2+\left(\frac{n_1\pi}{L_1}\right)^2+\left(\frac{n_2\pi}{L_2}\right)^2+\cdots+\left(\frac{n_p\pi}{L_p}\right)^2, \tag{2.3}$$

where $\{n\}$ stands for a short notation of $n_1,n_2,\ldots,n_p$, and n_a is a positive integer. In the ground state (vacuum) each of these modes contributes an energy $\omega_k/2$. The total energy of the field in the interior of Ω is thus given by

$$\varepsilon_p^D=\prod_{T=p+1}^{D-1}(L_T/2\pi)\int d^{D-p-1}k\sum_{\{n\}=1}^{\infty}\frac{1}{2}\,\omega_k. \tag{2.4}$$

We define the energy density $\varepsilon_p^D(L_1,L_2,\ldots,L_p)$ as a function of the finite length $L_1,L_2,\ldots,L_p$ of p edges:

$$\varepsilon_p^D=\frac{\varepsilon_p^D}{\prod_{T=p+1}^{D-1}L_T}. \tag{2.5}$$

Using the Eqs. (A4)–(A6), Eq. (2.5) becomes

$$\varepsilon_p^D(L_1,L_2,\ldots,L_p)=-\frac{\pi^{(D-p)/2}}{2^{D-p+1}}\Gamma\left(-\frac{D-p}{2}\right)\times E_p\left(\frac{1}{L_1^2},\frac{1}{L_2^2},\ldots,\frac{1}{L_p^2};-\frac{D-p}{2}\right), \tag{2.6}$$

where Epstein ζ function $E_p(a_1,a_2,\ldots,a_p;s)$ is defined as

$$E_p(a_1,a_2,\ldots,a_p;s)=\sum_{\{n\}=1}^{\infty}\left(\sum_{j=1}^{p}a_jn_j^2\right)^{-s}. \tag{2.7}$$

For $p=1$ the same result as published in Ref. [7] is obtained:

$$\varepsilon_1^D(L_1)=-\frac{\pi^{(D-1)/2}}{2^DL_1^{D-1}}\Gamma\left(\frac{1-D}{2}\right)\zeta(1-D), \tag{2.8}$$

where $\zeta(s)$ is the usual Riemann ζ function. Using the reflection formula

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\pi^{(s-1)/2}\zeta(1-s), \tag{2.9}$$

the energy density (2.8) can be written as

$$\varepsilon_1^D=-(2\sqrt{\pi})^{-D}L_1^{1-D}\Gamma\left(\frac{D}{2}\right)\zeta(D). \tag{2.10}$$

This result is finite for all positive D's, and is always negative. In the limit $L_1\to\infty$, Eq. (2.10) gives the energy density of the field in the absence of the plates.

III. THE SIGN OF THE CASIMIR ENERGY DENSITY OF $p=2$, 3 CAVITY

We consider first $p=2$ case in which the field is confined in the interior of $(D-1)$-dimensional rectangular cavity with two edges of finite lengths L_1, L_2 and $D-3$ edges with characteristic lengths of order $\lambda \gg L_1,L_2$. One can see from Eq. (2.6) that it is not straightforward to ascertain the sign of the Casimir energy density. For the $L_1=L_2=L$ case, Eq. (2.6) can be regularized using the techniques of Refs. [16, 8] giving the finite value for the Casimir energy:

$$\varepsilon_2^D(L,L)=\frac{L^{2-D}}{2^{D+1}}\left[2\pi^{(1-D)/2}\Gamma\left(\frac{D-1}{2}\right)A(1,1;D-1)-\pi^{-D/2}\Gamma\left(\frac{D}{2}\right)A(1,1;D)\right], \tag{3.1}$$

where

$$A(1,1;2r)=\sum_{m,n=-\infty}^{\infty}{}'\,(m^2+n^2)^{-r}, \tag{3.2}$$

where the prime means that the term $m=n=0$ has to be excluded. The sums of the type in Eq. (3.2) can be calculated efficiently with the help of the Jacobi θ function [17,18]. Using the Mellin transformation of exp($-b,\tau$),

$$\int_0^\infty \tau^{r-1}e^{-b\tau}d\tau=b^{-r}\Gamma(r), \tag{3.3}$$

we have

$$A(1,1;2r)=\frac{1}{\Gamma(r)}\int_0^\infty\tau^{r-1}\sum_{m,n=-\infty}^{\infty}{}'\exp[-(m^2+n^2)\tau]d\tau. \tag{3.4}$$

TABLE I. The Casimir energy densities for massless scalar fields satisfying Dirichlet boundary conditions inside a cavity with two unequal edges in a D-dimensional spacetime, where L_1 is the chosen unit length.

D	$L_2/L_1=1$	1.01	1.05	1.10	1.50	2.00	3.00	5.00
3	0.041 04	0.040 83	0.039 99	0.038 89	0.029 56	0.017 62	-6.3×10^{-3}	-5.4×10^{-2}
4	0.004 83	0.004 78	0.004 57	0.004 29	0.001 67	-1.8×10^{-3}	-8.6×10^{-3}	-2.2×10^{-2}
5	0.000 81	0.000 79	0.000 74	0.000 65	-2.6×10^{-4}	-1.5×10^{-3}	-3.9×10^{-3}	-8.9×10^{-3}
6	0.000 11	0.000 11	9.5×10^{-5}	6.4×10^{-5}	-3.1×10^{-4}	-8.2×10^{-4}	-1.8×10^{-3}	-3.9×10^{-3}
7	-1.9×10^{-5}	-1.8×10^{-5}	-2.3×10^{-5}	-3.3×10^{-5}	-2.0×10^{-4}	-4.4×10^{-4}	-9.2×10^{-4}	-1.9×10^{-3}
10	-2.6×10^{-5}	-2.5×10^{-5}	-2.3×10^{-5}	-2.3×10^{-5}	-4.9×10^{-5}	-8.7×10^{-5}	-1.6×10^{-4}	-3.2×10^{-4}
15	-6.9×10^{-6}	-6.5×10^{-6}	-5.5×10^{-6}	-5.2×10^{-6}	-8.7×10^{-6}	-1.4×10^{-5}	-2.5×10^{-5}	-4.6×10^{-5}
20	-3.1×10^{-6}	-2.8×10^{-6}	-2.3×10^{-6}	-2.1×10^{-6}	-3.4×10^{-6}	-5.2×10^{-6}	-8.9×10^{-6}	-1.6×10^{-5}

Now, using the definition of θ function,

$$\theta_3(0,q)\equiv\sum_{m=-\infty}^{\infty} q^{m^2}, \tag{3.5}$$

and setting $q=\exp(-\tau)$, we find, from Eq. (3.4),

$$A(1,1;2r)=\frac{1}{\Gamma(r)}\int_0^\infty \tau^{r-1}[\theta_3^2(0,e^{-r})-1], \tag{3.6}$$

where the subtrahend in the square brackets stems from the absence of a term with $m=n=0$ from Eq. (3.2). The known representation [19]

$$\frac{1}{4}[\theta_3^2(0,q)-1]=\sum_{i=1}^{\infty}\frac{q^i}{1+q^{2i}}=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(-1)^j q^{i(2j+1)}, \tag{3.7}$$

the Eq. (3.2) can be reduced as

$$A(1,1;2r)=4\zeta(r)\beta(r), \tag{3.8}$$

where $\zeta(r)$ is the Riemann ζ function and $\beta(r)$ is the Dirichlet series: namely,

$$\beta(r)\equiv\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^r}. \tag{3.9}$$

We thus see that the double series (3.3) has been transformed into a product of known one-dimensional series. Numerical calculations show that the energy density is positive for $D\leqslant 6$. However, it can be shown that the energy density becomes negative for integer values $D\geqslant 7$. The authors of Ref. [8] argued that there exists a particular critical value of spacetime dimension D_c, E_p^D shown to be always negative in p in even cases.

In the $L_1\neq L_2$ case, Eq. (2.6) is reduced to

$$\varepsilon_2^D=-\frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\right)^{D-2}\Gamma\left(\frac{2-D}{2}\right)E_2\left(\frac{1}{L_1^2},\frac{1}{L_2^2};\frac{2-D}{2}\right), \tag{3.10}$$

which can be regularized in elegant way by means of the bidmensional Epstein function. By making use of the Mellin transform

$$E_2\left(\frac{1}{L_1^2},\frac{1}{L_2^2};\frac{2-D}{2}\right)=\frac{1}{\Gamma\left(\frac{2-D}{2}\right)}\sum_{n_1,n_2=1}^{\infty}\int_0^\infty d\tau\,\tau^{D/2}\times\exp\left\{-\tau\left[\left(\frac{n_1}{L_1}\right)^2+\left(\frac{n_2}{L_2}\right)^2\right]\right\}, \tag{3.11}$$

and the modified Bessel functions of the second kind $K(z)$, one can obtain

$$E_2\left(\frac{1}{L_1^2},\frac{1}{L_2^2};\frac{2-D}{2}\right)=-\frac{1}{2}(L_1^{2-D}+L_2^{2-D})\zeta(2-D)+\frac{\sqrt{\pi}}{2}\left(\frac{L_2}{L_1^{D-1}}+\frac{L_1}{L_2^{D-1}}\right)\frac{\Gamma\left(\frac{1-D}{2}\right)\zeta(1-D)}{\Gamma\left(\frac{2-D}{2}\right)}$$
$$-\frac{\pi^{1-D}}{2\Gamma\left(\frac{2-D}{2}\right)}\frac{L_1}{L_2^{D-1}}\Gamma\left(\frac{D}{2}\right)\zeta(D)+\frac{\pi^{(3/2)D}\Gamma\left(\frac{D-1}{2}\right)\zeta(D-1)}{2\Gamma\left(\frac{2-D}{2}\right)L_2^{D-2}}$$
$$+\frac{2\pi^{(2-D)/2}}{\Gamma\left(\frac{2-D}{2}\right)L_1^{(D-1)/2}L_2^{(D-3)/2}}\sum_{n_1,n_2=1}^{\infty}\left(\frac{n_1}{n_2}\right)^{(D-1)/2}K_{(D-1)/2}\left(2\pi\frac{L_2}{L_1}n_1n_2\right). \tag{3.12}$$

From Eqs. (3.10) and (3.12), we have

$$\varepsilon_2^D(L_1,L_2)=\frac{1}{2^D L_1^{D-2}}\left[\frac{\Gamma\left(\frac{D-1}{2}\right)\zeta(D-1)}{\pi^{(D-1)/2}}-\frac{\Gamma\left(\frac{D}{2}\right)\zeta(D)}{\pi^{D/2}}\left(\frac{L_2}{L_1}\right)-4\left(\frac{L_2}{L_1}\right)^{(3-D)/2}\sum_{n_1,n_2=1}^{\infty}\left(\frac{n_1}{n_2}\right)^{(D-1)/2}K_{(D-1)/2}\left(\frac{2\pi n_1 n_2 L_2}{L_1}\right)\right]. \tag{3.13}$$

The sum of the converging series can be calculated efficiently with the help of the expansion

$$K_\nu(z)=\sqrt{\frac{\pi}{2z}}\,e^{-z}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{(8z)^k}\prod_{j=1}^{k}[4\nu^2-(2j-1)^2]. \tag{3.14}$$

Table I shows the Casimir energy density ε_p^D for $p=2$ and $D=3,4,\ldots,7,10,15,20$, where we have chosen L_1 unit. As expected, the result of Caruso, Neto, Svaiter, and Svaiter [8] is reconfirmed by this numerical calculation, taking into account $L_1=L_2$. The curves of the Casimir energy density for massless scalar fields satisfying Dirichlet boundary conditions in $D=3,4$ cases are shown in Fig. 1(a). For $D=5$, 6, and 7, the curves of the Casimir energy density are shown in Fig. 1(b). Figure 1(c) shows the curves of the Casimir energy density as functions of L_2/L_1 in $D=10$, 15, and 20 cases. There exists an evident minimum at $L_2/L_1=1$ as D increases. Numerical calculations also show that there exists a maximum at $L_2/L_1=\mu_{\max}(D)$. Table II shows the $\mu_{\max}(D)$ and $\varepsilon^D_{2\,\max}$ at $L_2/L_1=\mu_{\max}$ and $\varepsilon^D_{2\,\min}$ at $L_2/L_1=1$ for $D=6,7,\ldots,20$. Furthermore, numerical calculations show that there exists a particular critical ratio $\mu_c=2.737$ such that the energy density $\varepsilon_2^D<0$ if $L_2/L_1>\mu_c$ or $L_1/L_2>\mu_c$ for any dimension D. In the critical ratio $L_2/L_1=\mu_c$ or $L_1/L_2=\mu_c$, Eq. (3.13) gives the energy density of the field equal to that in the absence of the cavity. These are interesting, but they are rather weak results for we are using the curves in Fig. 1. However, we can prove that these results are exact in the D-dimensional Minkowski spacetime.

If we assumed $L_2=\lambda L_1$, where λ is the characteristic length of $L_T(T=p+1,\ldots,D-1)$, Eq. (3.13) can be written as

$$\varepsilon_2^D(L_1,L_2)=\frac{L_2}{2^D L_1^{D-1}}\left[\frac{\Gamma\left(\frac{D-1}{2}\right)\zeta(D-1)}{\lambda\,\pi^{(D-1)/2}}-\frac{\Gamma\left(\frac{D}{2}\right)\zeta(D)}{\pi^{D/2}}-4\lambda^{(1-D)/2}\sum_{n_1,n_2=1}^{\infty}\left(\frac{n_1}{n_2}\right)^{(D-1)/2}\times K_{(D-1)/2}(2\pi\lambda n_1 n_2)\right]. \tag{3.15}$$

In the limit $\lambda\to\infty$, we have

$$\varepsilon_2^D(L_1,L_2)=-L_2(2\sqrt{\pi})^{-D}L_1^{1-D}\Gamma\left(\frac{D}{2}\right)\zeta(D)=L_2\varepsilon_1^D(L_1). \tag{3.16}$$

Since ε_1^D is always negative and $\varepsilon_2^D(L_1,L_2)$ is a continued function for $L_2>0$, there exists a critical ratio μ_c, such that the energy density $\varepsilon_2^D(L_1,L_2)<0$ if $L_2/L_1>\mu_c$. Furthermore, there exists a Z_2 symmetry $L_1\leftrightarrow L_2$ for energy function. We now have proved exact argument that is stated as follows: There exists a critical ratio μ_c, such that the sign of the Casimir energy is negative if $L_2>\mu_c L_1$ or $0<L_2<\mu_c^{-1}L_1$ for $(D-1)$-dimensional rectangular cavity with two finite edges and $D-3$ edges with characteristic lengths of order $\lambda\gg L_1,L_2$. If $\mu_c^{-1}L_1<L_2<\mu_c L$, the sign of the Casimir energy is also negative if $D\geqslant 7$.

When p is odd the sign of the Casimir energy of a massless scalar field satisfying Dirichlet conditions is always negative whatever the value of D in the hypercube cases.

TABLE II. The maximum value of the Casimir energy densities at $L_2/L_1=\mu_{\max}$ for massless scalar fields satisfying Dirichlet boundary conditions inside a cavity with two unequal edges in a D-dimensional spacetime, where L_1 is the chosen unit length. Meantime, the values of ε_2^D at $L_1=L_2$ are listed to contrast with $\varepsilon^D_{2\max}$.

D	$\mu_{\max}$	$\varepsilon^D_{2\max}$	$\varepsilon_2^D\ (L_1,L_2)$
6	$1+(1\times10^{-7})$	0.000 114 640 7	0.000 114 640 8
7	1.0102	−0.000 019 239 4	−0.000 019 477 1
8	1.0375	−0.000 036 675 7	−0.000 038 696 2
9	1.0575	−0.000 031 107 2	−0.000 034 159 9
10	1.0724	−0.000 023 129 9	−0.000 026 376 2
11	1.0830	−0.000 016 709 7	−0.000 019 732 8
12	1.0911	−0.000 012 118 9	−0.000 014 779 5
13	1.0968	−0.000 008 940 1	−0.000 011 228 6
14	1.1008	−0.000 006 746 8	−0.000 008 704 2
15	1.1034	−0.000 005 221 2	−0.000 006 902 7
16	1.1049	−0.000 004 147 1	−0.000 005 605 9
17	1.1058	−0.000 003 380 8	−0.000 004 382 8
18	1.1063	−0.000 002 827 6	−0.000 003 973 1
19	1.1065	−0.000 002 424 8	−0.000 003 465 0
20	1.1067	−0.000 002 130 4	−0.000 003 091 6

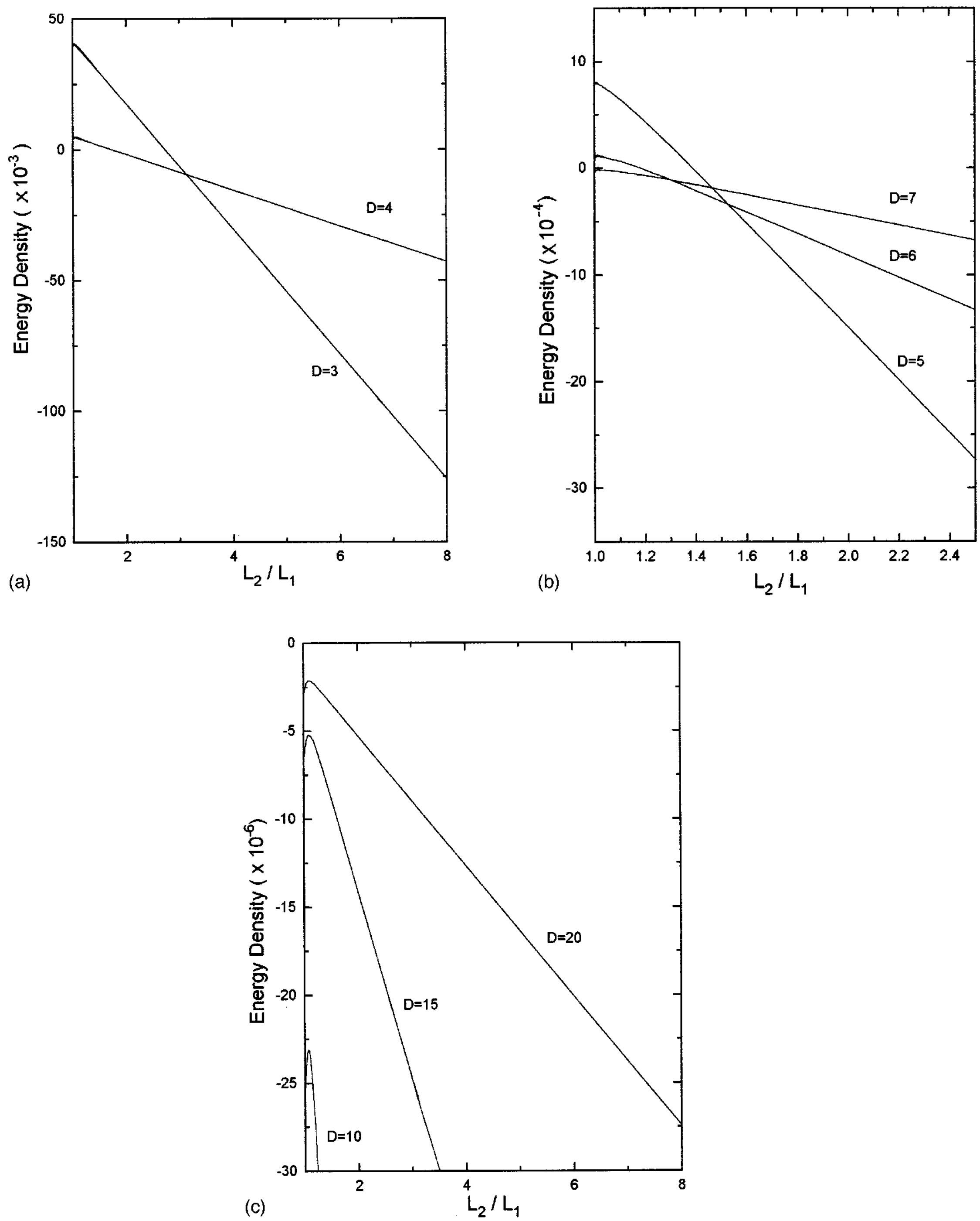

FIG. 1. The curves of the Casimir energy density as functions of L_2/L_1 in D-dimensional space. (a) $D=3,4$; (b) $D=5,6,7$; (c) $D=10,15,20$. According to the Casimir energy density, there exists a Z_2 symmetry $L_1 \leftrightarrow L_2$. As in this figure, we choose L_1 as unit, so using the value at $L_2/L_1>1$ we can find the value at $L_2/L_1<1$. For example, the value for $L_2/L_1=0.5$ clearly equals to 1/4 value for $L_2/L_1=2$.

The question whether the Casimir energy for the unequal finite p edges gives rise to a positive or a negative sign will be discussed here. For simplicity, we take $p=3$ in Eq. (2.6), ε_3^D can be written as

$$\varepsilon_3^D = -\frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\right)^{D-3}\Gamma\left(\frac{3-D}{2}\right)E_3\left(\frac{1}{L_1^2},\frac{1}{L_2^2},\frac{1}{L_3^2};\frac{3-D}{2}\right), \tag{3.17}$$

which can be regularized by means of the Mellin transform. We have

$$\varepsilon_3^D=\frac{1}{4}\left(\frac{\sqrt{\pi}}{2}\right)^{D-3}\Gamma\left(\frac{3-D}{2}\right)E_2\left(\frac{1}{L_2^2},\frac{1}{L_3^2};\frac{3-D}{2}\right)-\frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\right)^{D-2}L_1\Gamma\left(\frac{2-D}{2}\right)E_2\left(\frac{1}{L_2^2},\frac{1}{L_3^2};\frac{2-D}{2}\right)$$
$$-\left(\frac{1}{2}\right)^{D-2}L_1^{3-D/2}\sum_{k=0}^{\infty}\frac{1}{k!}(16\pi)^{-k}L_1^{-k}\prod_{j=0}^{k}[(D-2)^2-(2j-1)^2]\sum_{n_1,n_2,n_3=1}^{\infty}n_1^{(1-D-2k)/2}\left[\left(\frac{n_2}{L_2}\right)^2+\left(\frac{n_3}{L_3}\right)^2\right]^{(3-D+2k)/4}$$
$$\times\exp\left\{-2\pi L_1 n_1\left[\left(\frac{n_2}{L_2}\right)^2+\left(\frac{n_3}{L_3}\right)^2\right]^{1/2}\right\}. \tag{3.18}$$

We may use the same steps as above to write

$$\varepsilon_3^D(L_1,L_2L_3)=L_1\varepsilon_2^D(L_2,L_3) \tag{3.19}$$

in the limit $L_1 \gg L_2,L_3$. Since $\varepsilon_3^D(L_1,L_2,L_3)$ is continued for $L_1>0$, there exist critical values μ_c and D_c, such that the energy density $\varepsilon_3^D>0$ for $L_1>\mu_c L, L_2=L_3=L, D\leqslant D_c$; and $\varepsilon_3^D<0$ for $L_1>\mu_c L, L_2=L_3=L, D>D_c$. Numerical calculations show $\mu_c=5.287$ and $D_c=6$. Therefore, the behavior of $\varepsilon_2^D(L_1,L_2,L_3)$ looks to be $\varepsilon_2^D(L_2,L_3)$ when the edges are chosen appropriately. Similarly, we can prove

$$\varepsilon_3^D(L_1,L_2,L_3)=L_1L_2\varepsilon_1^D(L_3), \tag{3.20}$$

in the limit $L_1,L_2 \gg L_3$. It is obvious that $\varepsilon_3^D(L_1,L_2,L_3)$ is always negative if L_3/L_1 and L_3/L_2 are sufficiently small.

IV. ATTRACTIVE OR REPULSIVE NATURE OF CASIMIR FORCE

The preceding researches can be generalized to any p. After some work, we obtain recursion relations of the Casimir energy density

$$\varepsilon_p^D(L_1,L_2,\ldots,L_p)=\frac{\pi^{(D-p)/2}}{2^{D-p+2}}\Gamma\left(\frac{p-D}{2}\right)E_{p-1}\left(\frac{1}{L_2^2},\ldots,\frac{1}{L_p^2};\frac{p-D}{2}\right)$$
$$-\frac{\pi^{(D-p+1)/2}}{2^{D-p+2}}L_1\Gamma\left(\frac{p-D-1}{2}\right)E_{p-1}\left(\frac{1}{L_2^2},\ldots,\frac{1}{L_p^2};\frac{p-D-1}{2}\right)$$
$$-\frac{1}{2^{D-p+1}}L_1^{(p-D)/2}\sum_{k=0}^{\infty}\frac{1}{k!}(16\pi)^{-k}L_1^{-k}\prod_{j=1}^{k}[(p-D-1)^2-(2j-1)^2]$$
$$\times\sum_{\{n\}=1}^{\infty}n_1^{(p-D-2k-2)/2}\left[\left(\frac{n_2}{L_2}\right)^2+\cdots+\left(\frac{n_p}{L_p}\right)^2\right]^{(p-D+2k)/4}\exp\left\{-2\pi L_1 n_1\left[\left(\frac{n_2}{L_2}\right)^2+\cdots+\left(\frac{n_p}{L_p}\right)^2\right]^{1/2}\right\}. \tag{4.1}$$

If we assumed q finite edges $L_1,\ldots,L_q \gg L_{q+1},\ldots,L_p$ and repeat Eq. (4.1), we have

$$\varepsilon_p^D(L_1,\ldots,L_q,L_{q+1},\ldots,L_p)=L_1\ldots L_q\varepsilon_{p-q}^D(L_{q+1},\ldots,L_p). \tag{4.2}$$

From the continuity of function $\varepsilon_p^D(L_1,\ldots,L_q,L_{q+1},\ldots,L_p)$, we obtain the following argument: If the lengths of q finite edges are much longer than those of $p-q$ edges, the nature of the Casimir energy depends on the value of $p-q$. When $p-q=1$, ε_p^D is always negative. When $p-q=$even and $L_{q+1}=\cdots=L_p=L$, there exists a critical dimension for ε_3^D. When $D\leqslant D_c$, the sign of the Casimir energy is positive. When $D>D_c$, the sign of the Casimir energy is negative.

By using Eq. (2.10), the Casimir force per unit area in the $p=1$ case is attractive for any value of D and of magnitude

$$\frac{-\partial(\varepsilon_1^D/L)}{\partial L_1}=-\frac{D-1}{L_1^D}\Gamma\left(\frac{D}{2}\right)(2\sqrt{\pi})^{-D}\zeta(D). \tag{4.3}$$

The pressure of the vacuum between the plates is thus negative. Similarly, we have

$$-\frac{\partial \varepsilon_2^D}{\partial L_2}=\frac{1}{2^D}\left\{\frac{\Gamma\left(\frac{D}{2}\right)\zeta(D)}{\pi^{D/2}L_1^{D-1}}-4(D-2)\left(\frac{L_2}{L_1}\right)^{(1-D)/2}\sum_{n_1,n_2=1}^{\infty}\left(\frac{n_1}{n_2}\right)^{(D-1)/2}K_{(D-1)/2}\left(\frac{2\pi L_2 n_1 n_2}{L_1}\right)\right.$$
$$\left.+4\left(\frac{L_2}{L_1}\right)^{(3-D)/2}\sum_{n_1,n_2=1}^{\infty}\left(\frac{n_1}{n_2}\right)^{(D-1)/2}2\pi n_1 n_2 K_{(D-3)/2}\left(\frac{2\pi L_2 n_1 n_2}{L_1}\right)\right\} \quad (4.4)$$

in the $p=2$ case. Using Table II, we can find that the Casimir force is attractive at $1\leqslant L_2/L_1<\mu_{\max}$ and it is repulsive at $L_2/L_1>\mu_{\max}$. Therefore, the Casimir force may be repulsive for odd p cavity with unequal edges, in contrast with the same problem in a hypercube case. In fact, ε_p^D is very complicated in that its attractive or repulsive nature depends on the appropriate choice of edgelengths in the D-dimensional spacetime.

The results obtained in this paper permit us to discuss a possible application. We consider a $p=3$ rectangular cavity with walls of infinite conductivity. The electromagnetic field then satisfies the boundary condition $n\cdot B=0$ and $n\times E=0$. Following Ambjorn and Wolfram [7], the Casimir energy ${}^{(em)}\varepsilon_3^4(L_1,L_2,L_3)$ of the electromagnetic field can be written in terms of the massless scalar field as

$$ {}^{(em)}\varepsilon_3^4(L_1,L_2,L_3)=2\varepsilon_3^4(L_1,L_2,L_3)+\varepsilon_2^3(L_1,L_2)+\varepsilon_2^3(L_1,L_3)+\varepsilon_2^3(L_2,L_3). \quad (4.5)$$

Use will also be made of the well-known fact [9,20] that the order of magnitude of the electromagnetic zero-point energy does not change if one deforms a spherical shell of radius a into a cubic shell of length, with $L\approx 2a$. On the other hand, the Abraham-Lorentz model describes the electron as a conducting spherical shell of radius a. To guarantee the stability of the electron Poincare stresses had to be postulated. Casimir [21] proposed to extend the classical electron model by taking into account the zero-point fluctuations of the electromagnetic field inside and outside of the conducting shell. Unfortunately, the Casimir model of the electron fails, at least in the $L_1=L_2=L_3\approx 2a$ case, because the Casimir energy of an S^2 electron is positive from Eq. (4.5). Does this argument still hold for rectangular cavity? The answer is no, and it can be shown that the zero-point energy is negative when we choose lengths of edges, appropriately. We take, for example $L_1=1.6$ and $L_2=L_3=1$, then ${}^{(em)}\varepsilon_3^4\approx -2\times 10^{-3}$. Therefore, Casimir-like model of electron could be stable. Note that, in this case, the condition of stability will be satisfied only for a particular shape and size.

Finally, we shall give a brief discussion. When there is more than one finite length, there should be nontrivial quantum effects due to the corners and edges. By using the Epstein ζ-function technique, these effects have been included in this paper.

APPENDIX: SOME USEFUL DEFINITE INTEGRALS

This Appendix derives formulas for the Casimir energy in Sec. II. We consider first multiple integral

$$G=\int\cdots\int_{x_1^2+\cdots+x_n^2\leqslant R^2} f(\sqrt{x_1^2+\cdots+x_n^2})\,dx_1\cdots dx_n. \quad (A1)$$

The generalized polar coordinate transformation is

$$\begin{aligned} x_1&=r\cos\varphi_1,\\ x_2&=r\sin\varphi_1\cos\varphi_2,\\ x_3&=r\sin\varphi_1\sin\varphi_2,\\ x_{n-1}&=r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\cos\varphi_{n-1},\\ x_n&=r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}\end{aligned} \quad (A2)$$

and the Jacobian is

$$J\equiv\frac{D(x_1,x_2,\ldots,x_n)}{D(r,\varphi_1,\ldots,\varphi_{n-1})}=r^{n-1}\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\ldots\sin\varphi_{n-2}. \quad (A3)$$

Using Eqs. (A2) and (A3), we have

$$\begin{aligned} G&=\int_0^R r^{n-1}f(r)dr\cdot\int_0^\pi \sin^{n-2}\varphi_1 d\varphi_1\cdots\\ &\quad\times\int_0^\pi \sin\varphi_{n-2}d\varphi_{n-2}\cdot\int_0^{2\pi}d\varphi_{n-1}\\ &=\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^R r^{n-1}f(r)dr. \end{aligned} \quad (A4)$$

Let

$$B(p,q)=\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}},\quad p>0,q>0. \quad (A5)$$

The function B is called the beta function, one can show that

$$B(p,q)=B(q,p)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad (A6)$$

where Γ function $\Gamma(p)$ is defined

$$\Gamma(p)=\int_0^\infty x^{p-1}\exp(-x)dx, \quad p>0. \tag{A7}$$

In the terms of the Γ function, one can show

$$\Gamma(-n+\tfrac{1}{2})=(-1)^n\frac{2^n\sqrt{\pi}}{(2n-1)!!} \quad (n=1,2,\ldots). \tag{A8}$$

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