Implications of spontaneous glitches in the mass and angular momentum in Kerr space-time

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The outward-pointing principal null direction of the Schwarzschild Riemann tensor is null hypersurface forming. If the Schwarzschild mass spontaneously jumps across one such hypersurface, then the hypersurface is the history of an outgoing lightlike shell. The outward-pointing principal null direction of the Kerr Riemann tensor is asymptotically (in the neighborhood of future null infinity) null hypersurface forming. If the Kerr parameters of mass and angular momentum spontaneously jump across one such asymptotic hypersurface then the asymptotic hypersurface is shown to be the history of an outgoing lightlike shell and a wire singularity-free spherical impulsive gravitational wave. [S0556-2821(97)02416-8]

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I. INTRODUCTION

In a recent paper [1] the authors studied the physical consequences of abrupt changes occurring spontaneously in the multipole moments of a static axially symmetric isolated gravitating body. The conclusion was that a disturbance propagates with the speed of light away from the source which when analyzed near future null infinity is shown to consist of a spherical outgoing lightlike shell accompanied by a spherical impulsive gravitational wave. Motivated by the well-known phenomenon of glitches observed in pulsars [2], we examine in this paper the physical implications of glitches in the mass and angular momentum associated with the source of the Kerr space-time. We find that in the neighborhood of future null infinity a disturbance consisting of a spherical lightlike shell and a spherical impulsive gravitational wave can be identified. If the Kerr angular momentum vanishes (the Schwarzschild special case) then the gravitational wave does not exist. If the angular-momentum glitch includes an abrupt change in the direction of the angular momentum then the gravitational wave has the maximum two degrees of freedom of polarization. The spherical impulsive gravitational waves appearing in this paper (Sec. III) and in [1] are the only examples of such waves known to the authors which are free of unphysical directional (or wire) singularities (see Sec. I of [1] where this is discussed).

To study the physical properties of the disturbances mentioned above we use the Barrabès-Israel (BI) theory of lightlike shells and impulsive waves [3]. This theory is easily accessible and a useful description of part of it is also available in [1] where, in particular, the identification of the gravitational wave and the lightlike shell, when both exist, is given explicitly. To rederive the results stated in the present paper the reader must be familiar with the BI theory. For readers who wish to work through the calculations based on the BI theory we give some intermediate steps in the Appendix. The consequences of such calculations can be easily followed independently, however, in the main body of this paper.

To simplify the presentation and introduce our approach we present first in Sec. II the well-known (see [3], for example) Schwarzschild example in which the mass of the source spontaneously undergoes an abrupt but finite change. This is also useful as a special case of the corresponding Kerr example, which is the main point of the paper, given in Sec. III. This is followed by a brief discussion of our results in Sec. IV.

II. THE SCHWARZSCHILD EXAMPLE

Consider Schwarzschild space-time with the line element

$$ds^2=-\frac{2r^2d\zeta d\bar{\zeta}}{[1+(1/2)\zeta\bar{\zeta}]^2}+2du\,dr+\left(1-\frac{2m}{r}\right)du^2. \tag{2.1}$$

Here $u=\mathrm{const}$ are future-directed null hypersurfaces (null cones) generated by the geodesic integral curves of the null vector field $\partial/\partial r$. This vector field is also the outward-pointing principal null direction of the Riemann tensor of the space-time. We wish to consider this space-time undergoing a spontaneous abrupt change in the mass m of the source across one of the outward null hypersurfaces $u=0$ (say) and then ask: what are the physical properties of $u=0$? To do this we imagine the space-time divided into two halves M^+ corresponding to $u>0$ and M^- corresponding to $u<0$ both with boundary $u=0$ and then reattaching the halves on $u=0$ preserving, with the identity map, the induced line element on $u=0$:

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$$dl^2 = -r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2}$$

We denote the resulting space-time by $M^- \cup M^+$. For the space-time $M^- \cup M^+$ described above there is a stress-energy tensor concentrated on $u=0$ of the form

$$T^{\mu\nu} = S^{\mu\nu}\delta(u), \tag{2.3}$$

with $x^\mu = (\theta, \phi, r, u)$ and δ is the Dirac δ function. We refer to $S^{\mu\nu}$ as the surface stress-energy tensor of the lightlike shell with history $u=0$ (see [3]). The normal to $u=0$ is the null vector with components n^μ given via the one-form

$$n_\mu dx^\mu = du. \tag{2.4}$$

The BI theory [3] gives

$$16\pi S^{\mu\nu} = -\frac{4[m]}{r^2} n^\mu n^\nu, \tag{2.5}$$

where $[m]$ is the finite jump in the mass m across $u=0$. This means that there is no stress in the outgoing lightlike shell (as might be expected because the shell is spherical and expanding) and the surface energy density of the shell measured by a radially moving observer (discussed in [3]) is a positive multiple of

$$\sigma = -\frac{[m]}{4\pi r^2}, \tag{2.6}$$

and so it is natural to assume that $[m]<0$ for an outgoing shell. Thus we conclude that the space-time $M^- \cup M^+$ describes a Schwarzschild gravitational field (described by the space-time M^-) with an expanding spherical light-like shell propagating through it leaving behind a Schwarzschild field described by M^+ and with mass reduced compared to that of M^-.

In general in the type of situation described here (subdivision and re-attachment, or ''cut and paste,'' of a space-time on a null hypersurface) the space-time $M^- \cup M^+$ has a Weyl conformal curvature tensor containing a δ-function term singular on the null hypersurface and composed of a matter part (which is nonzero provided the stress in the shell is anisotropic) and a part describing an impulsive gravitational wave (see [1] where this is explicitly demonstrated and [3] for the calculation of these terms). However, due to the spherical symmetry of the Schwarzschild example above, the Weyl tensor of $M^- \cup M^+$ vanishes identically in this case. Thus, in particular, the shell above is unaccompanied by an impulsive gravitational wave.

III. THE KERR EXAMPLE

We consider here an analogous situation in the Kerr space-time to that considered in the Schwarzschild space-time in the previous section. We begin with a form of the Kerr space-time which makes it easy to identify the outgoing principal null direction of the Riemann tensor and which specializes to Eq. (2.1) when the Kerr angular-momentum parameter is put to zero. In addition it will be interesting not only to consider spontaneous changes in the magnitude of the Kerr angular momentum but also to include spontaneous changes in the direction of the angular momentum. We thus want to use a form of the Kerr solution which involves the mass parameter and three components of the angular momentum per unit mass. One such form can readily be obtained by first noting that the Kerr solution with mass m and angular momentum per unit mass A may be written in Kerr's [4] original coordinates $(\zeta, \bar{\zeta}, r, u)$ [with the simple replacement, as in Eq. (2.1), of the polar angles (θ,ϕ) with the complex coordinate $\zeta = \sqrt{2}e^{i\phi}\tan\theta/2$ and its complex conjugate $\bar{\zeta}$] in the form

$$ds^2 = -2\frac{r^2+P^2}{[1+(1/2)\zeta\bar{\zeta}]^2}d\zeta d\bar{\zeta} + 2d\Sigma(dr - iP_\zeta d\zeta + iP_{\bar{\zeta}}d\bar{\zeta} + Sd\Sigma), \tag{3.1}$$

where

$$P = A\left(\frac{1-(1/2)\zeta\bar{\zeta}}{1+(1/2)\zeta\bar{\zeta}}\right), \quad S = \frac{1}{2} - \frac{mr}{r^2+P^2}, \tag{3.2}$$

and the one-form $d\Sigma$ is given by

$$d\Sigma = du + iP_\zeta d\zeta - iP_{\bar{\zeta}}d\bar{\zeta}, \tag{3.3}$$

with $P_\zeta = \partial P/\partial\zeta$. The rotation

$$\zeta \to \frac{\sqrt{2}\sin(\theta_1/2) - \zeta\cos(\theta_1/2)}{e^{i\phi_1}\cos(\theta_1/2) + (\zeta/\sqrt{2})\sin(\theta_1/2)}, \tag{3.4}$$

where θ_1, ϕ_1 are constants, leaves the form of Eq. (3.1) invariant with P replaced by

$$P = \frac{a}{\sqrt{2}}\left(\frac{\zeta+\bar{\zeta}}{1+(1/2)\zeta\bar{\zeta}}\right) + \frac{b}{i\sqrt{2}}\left(\frac{\zeta-\bar{\zeta}}{1+(1/2)\zeta\bar{\zeta}}\right) + c\left(\frac{1-(1/2)\zeta\bar{\zeta}}{1+(1/2)\zeta\bar{\zeta}}\right), \tag{3.5}$$

where

$$a = A\sin\theta_1\cos\phi_1, \quad b = A\sin\theta_1\sin\phi_1, \quad c = A\cos\theta_1. \tag{3.6}$$

We thus obtain the Kerr solution with mass m and angular-momentum three-vector $\mathbf{J} = (ma, mb, mc)$ having the same magnitude but a different direction than the initial one $\mathbf{J} = (0,0,mA)$. Restoring the polar coordinates (θ,ϕ) as above we arrive at the line element [5]

$$ds^2 = -\rho^2(d\theta^2 + \sin^2\theta d\phi^2) + 2d\Sigma(dr - Nd\theta - M\sin\theta d\phi + Sd\Sigma), \tag{3.7}$$

with

$$\rho^2 = r^2 + P^2, \quad P = (a\cos\phi + b\sin\phi)\sin\theta + c\cos\theta, \tag{3.8}$$

$$d\Sigma = du + Nd\theta + M\sin\theta d\phi, \quad S = \frac{1}{2} - \frac{mr}{\rho^2}, \tag{3.9}$$

and

$$N = -a \sin\phi + b \cos\phi, \tag{3.10a}$$

$$M = -(a \cos\phi + b \sin\phi)\cos\theta + c \sin\theta. \tag{3.10b}$$

We note from Eq. (3.10) that

$$E^2 \equiv M^2 + N^2 = m^{-2}[|\mathbf{J}|^2 - (\mathbf{n}\cdot\mathbf{J})^2], \tag{3.11}$$

where the unit three-vector $\mathbf{n}$ is given by $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Also the outward-pointing principal null direction of the Riemann tensor is tangent to the vector field $\partial/\partial r$ or equivalently is given via the (nonexact) one-form $d\Sigma$ in Eq. (3.9). When this line element is written in a Kerr-Schild form in which the flat background is expressed in rectangular Cartesian coordinates and time, it can be shown—see [5] in which this is described in detail—that in the linear approximation it takes the form of the line element of the space-time outside the history of a slowly rotating sphere of mass m and angular momentum $\mathbf{J} = (ma, mb, mc)$, in exactly the same way as Kerr showed—see [4]—that his original form approximates the line element of the spacetime outside the history of such a sphere of mass m and angular momentum $\mathbf{J} = (0,0,mA)$.

In analogy with the Schwarzschild example in Sec. II we might expect that a spontaneous abrupt change in the parameters $\{m,a,b,c\}$ will result in a disturbance propagating through space-time along the outgoing principal null direction of the Kerr Riemann tensor. As this vector field is not surface forming for all values of r we cannot use the BI theory to study the disturbance for all r. However, for large r, specifically if the $O(r^{-2})$ terms are neglected, then $\partial/\partial r$ is tangent to null hypersurfaces $u=$const. The normal n^μ to $u=$const, given via the one-form $n_\mu dx^\mu = du$, satisfies $g_{\mu\nu}n^\mu n^\nu = O(r^{-2})$. In this approximation $u=$const are portions of future null cones having the integral curves of $\partial/\partial r$ as generators with r an affine parameter along them. These generators are, in the approximation under consideration, shear-free null geodesics with expansion r^{-1} and the induced line element on $u=$const is given approximately by

$$dl^2 = -r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.12}$$

This follows from Eq. (3.7) in which the ratio of the neglected terms to the retained terms in the induced line element is $O(r^{-2})$. Thus, if the disturbance is propagating in the direction of $\partial/\partial r$ then for sufficiently large values of r a front is formed (a null hypersurface is formed in space-time) with history $u=0$ (say). We now assume that *across the null portion* of $u=0$ a spontaneous jump in the parameters $\{m,a,b,c\}$ occurs from values $\{m,a,b,c\}$ to the past ($u<0$) of this null portion of $u=0$ to values $\{m_+, a_+, b_+, c_+\}$ to the future ($u>0$) of this null portion of $u=0$, and that the regions of space-time, $M^+(u>0)$ and $M^-(u<0)$, on either side of the null part of $u=0$ join on this null part with the identity map and thus preserving the induced line element (3.12). We can now use the BI theory to study the physical properties of the null part of $u=0$ by calculating the surface stress-energy tensor there and by calculating the δ-function term [the coefficient of $\delta(u)$] in the Weyl conformal curvature tensor. The results of our calculations for this Kerr example will naturally be a generalization of those for the Schwarzschild example in Sec. II. Again there is a stress-energy tensor of the form (2.3) on the null part of $u=0$ with surface stress energy described by the tensor $S^{\mu\nu}$ which in this case has components

$$16\pi S^{13} = \frac{[N]}{r^3} + O(r^{-4}), \tag{3.13a}$$

$$16\pi S^{23} = \frac{[M]\csc\theta}{r^3} + O(r^{-4}), \tag{3.13b}$$

$$16\pi S^{33} = -\frac{4[m]}{r^2} + \frac{2[E^2]}{r^3} + O(r^{-4}), \tag{3.13c}$$

with all other components small of order r^{-5}. Here as before the square brackets denote the jump across the null part of $u=0$ of the quantities contained therein. N, M, E^2 are given by Eqs. (3.10) and (3.11) and jump because the Kerr parameters jump. By Eq. (3.13) the null part of $u=0$ is the history of an outgoing lightlike shell. By Eqs. (3.13a) and (3.13b) there is an anisotropic stress in the shell (due to the jump in the Kerr angular momentum per unit mass). By Eq. (3.13c) the surface energy density of the shell measured by a radially moving observer is a positive multiple of

$$\sigma = -\frac{1}{4\pi r^2}\left([m] - \frac{[E^2]}{2r} + O(r^{-2})\right). \tag{3.14}$$

This is the generalization of Eq. (2.6) and $\sigma>0$ implies $[m]<0$ once again. It is interesting to note that an expanding lightlike shell sandwiched between two Reissner-Nordstrom space-times with different masses and charges (the charged version of the Schwarzschild example in Sec. II) has an exact surface energy density given by Eq. (3.14), without the error term, with $[E^2]$ replaced by $[e^2]$, the jump in the square of the charge across the history of the shell.

The BI theory enables us to calculate the coefficient of $\delta(u)$ in the Weyl conformal curvature tensor for the reattached space-time. This coefficient in general splits [1,3] into a matter part, which is present if there is anisotropic stress in the shell (as there is in the Kerr example), and a wave part describing an impulsive gravitational wave accompanying the lightlike shell. To display the components of this coefficient it is convenient to introduce the asymptotically null tetrad given via the one-forms $du, dr + S du$ [with S given in Eq. (3.9)], $(\sqrt{2})^{-1} r(d\theta + i \sin\theta d\phi)$ and its complex conjugate. This tetrad is asymptotically parallel transported along the integral curves of $\partial/\partial r$. By this we mean that the components on the tetrad of the covariant derivatives of the tetrad vectors in the direction of $\partial/\partial r$ are small of order r^{-2}. Denoting the Newman-Penrose components on this tetrad of the matter part of the coefficient of $\delta(u)$ by ${}^M\Psi_A$ ($A = 0,1,2,3,4$) and those of the wave part of this coefficient of $\delta(u)$ by ${}^W\Psi_A$, we find, for the matter part (see [1]),

$${}^M\Psi_0 = O(r^{-5}), \quad {}^M\Psi_1 = O(r^{-4}), \quad {}^M\Psi_2 = O(r^{-3}), \tag{3.15a}$$

$${}^M\Psi_3 = -\frac{1}{4\sqrt{2}r^2}[N - iM] + O(r^{-3}), \tag{3.15b}$$

$${}^M\Psi_4 = O(r^{-3}), \tag{3.15c}$$

and for the wave part all Newman-Penrose components vanish with the exception of

$$ {}^{W}\Psi_4 = \frac{1}{4r^4}\,[m(N-iM)^2] + O(r^{-5}). \tag{3.16} $$

Here again the square brackets denote the jump across the null part of $u=0$ of the quantity contained therein.

IV. DISCUSSION

We first notice that ${}^{M}\Psi_A$ is predominantly type III in the Petrov classification with n^μ as a degenerate principal null direction. The presence of ${}^{M}\Psi_A$ is due to the presence of anisotropic stress in the lightlike shell [see Eqs. (3.13a) and (3.13b)] which is a consequence of the nonvanishing Kerr angular-momentum parameters in this case. ${}^{W}\Psi_A$ is type N in the Petrov classification with n^μ as a fourfold degenerate principal null direction. This means that the shell is accompanied by a spherical impulsive gravitational wave whose presence is again due to the nonvanishing Kerr angular-momentum parameters. Since both N and M are smooth bounded functions of θ, ϕ for $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$ neither Eq. (3.15) nor Eq. (3.16) possess line singularities.

It is interesting to note that the predominant radial dependence of ${}^{M}\Psi_A$ and ${}^{W}\Psi_A$ [which is $O(r^{-2})$ for ${}^{M}\Psi_A$ and $O(r^{-4})$ for ${}^{W}\Psi_A$] is the same for ${}^{W}\Psi_A$ as in our earlier paper [1]. This is because the lightlike shell and the gravitational wave share the same null hypersurface history $u=0$ in space-time and are therefore in direct competition with each other. Hence it is no surprise that the matter part is more dominant than the wave part.

Finally we see that if the angular-momentum three-vector $\mathbf{J}$ introduced after Eq. (3.6) had for $u<0$, $\mathbf{J}=(0,0,mc)$ and for $u>0$, $\mathbf{J}=(0,0,m_+c_+)$ then $N^+=N=0$ and the spherical impulsive wave with amplitude (3.16) has one degree of freedom of polarization. Adding a change of direction to this change of magnitude of the angular momentum clearly adds the extra degree of freedom to the gravitational wave.

APPENDIX: USEFUL FORMULAS FOR SECS. II AND III

For readers who are familiar with the BI theory and wish to derive the results stated in Secs. II and III we list here the results of some useful intermediate calculations. These apply to the Kerr example of Sec. III. They all specialize to the Schwarzschild example of Sec. II when the Kerr angular momentum parameters are put to zero.

The jump $\gamma_{\mu\nu}$ in the transverse extrinsic curvature across the null part of $u=0$ is given by

$$ \gamma_{11} = -2[m] + \frac{[E^2]}{r} + O(r^{-2}), \tag{A1} $$

$$ \gamma_{22} = \gamma_{11}\sin^2\theta, \tag{A2} $$

$$ \gamma_{12} = -\frac{[mMN]}{r^2}\sin\theta + O(r^{-3}), \tag{A3} $$

$$ \gamma_{13} = -\frac{[N]}{r} + O(r^{-2}), \tag{A4} $$

$$ \gamma_{23} = -\frac{[M]}{r}\sin\theta + O(r^{-2}), \tag{A5} $$

$$ \gamma_{33} = 0, \quad \gamma_{\mu 4} \equiv 0. \tag{A6} $$

Along with the $O(r^{-2})$ leading term in γ_{12} given in Eq. (A3) we require the $O(r^{-2})$ leading term in $\gamma_{11}-\gamma_{22}\csc^2\theta$. This is neatly given along with Eq. (A3) by

$$ \gamma_{11} - \gamma_{22}\csc^2\theta - 2\gamma_{12}\csc\theta = -\frac{1}{r^2}[m(N-iM)^2] + O(r^{-3}). \tag{A7} $$

The stress-energy tensor $S^{\mu\nu}$ of the shell in terms of $\gamma_{\mu\nu}$ is [3]

$$ 16\pi\eta^{-1}S^{\mu\nu} = 2\gamma^{(\mu}n^{\nu)} - \gamma n^\mu n^\nu - \gamma^\dagger g^{\mu\nu} - q^{\mu\nu}, \tag{A8} $$

where in the present case $\eta = 1 + O(r^{-4})$,

$$ \gamma^\mu = \gamma^{\mu\nu}n_\nu, \quad \gamma^\dagger = \gamma^\mu n_\mu, \quad \gamma = g^{\mu\nu}\gamma_{\mu\nu}, \tag{A9} $$

and

$$ q^{\mu\nu} = \epsilon(\gamma^{\mu\nu} - \gamma g^{\mu\nu}), \tag{A10} $$

with $\epsilon = g_{\mu\nu}n^\mu n^\nu$. In the Kerr case $\epsilon = O(r^{-2})$ ($\epsilon=0$ in the Schwarzschild case) and

$$ q^{11} = O(r^{-6}), \quad q^{12} = O(r^{-7}), \quad q^{22} = O(r^{-6}), \tag{A11} $$

$$ q^{13} = O(r^{-5}), \quad q^{23} = O(r^{-5}), \quad q^{33} = O(r^{-4}), \tag{A12} $$

so that the $q^{\mu\nu}$ term in Eq. (A8) is absorbed into the $O(r^{-4})$ error in Eq. (3.13). This ensures that the accuracy given in Eq. (3.13) is the optimum consistent with $u=0$ being approximately null [in the sense that $\epsilon = O(r^{-2})$].

The δ-function term in the Weyl tensor is calculated from [3]

$$ C^{\kappa\lambda}{}_{\mu\nu} = \left\{ 2\eta n^{[\kappa}\gamma^{\lambda]}_{[\mu}n_{\nu]} - 16\pi\delta^{[\kappa}_{[\mu}S^{\lambda]}_{\nu]} + \frac{8\pi}{3}S^\alpha_\alpha\delta^{\kappa\lambda}_{\mu\nu} \right\}\delta(u). \tag{A13} $$

Care must be taken in identifying the wave part of the coefficient of $\delta(u)$ here. It is *not* given by the first term in Eq. (A13) but is *contained* in the first term (the reader must consult [1] to see this clearly).

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