Strong cosmic censorship and causality violation

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We investigate the instability of the Cauchy horizon caused by causality violation in the compact vacuum universe with the topology $B\times\mathbf{S}^1\times\mathbf{R}$, which Moncrief and Isenberg considered. We show that if the occurrence of curvature singularities are restricted to the boundary of the causality-violating region, the whole segments of the boundary become curvature singularities. This implies that strong cosmic censorship holds in the spatially compact vacuum space-time in the case of causality violation. This also suggests that causality violation cannot occur for a compact universe. [S0556-2821(97)06316-9]

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I. INTRODUCTION

Whether or not space-time is allowed to have causality-violating regions is an important problem in classical general relativity. Tipler [1,2] showed that any attempt to evolve closed timelike curves from an initial regular Cauchy data would cause singularities to form in a space-time. Kriele [3] showed that the causality-violating set has incomplete null geodesics if its boundary is compact. Maeda and Ishibashi [4] showed that singularities necessarily occur when a boundary of the causality-violating set exists in a space-time under the physically suitable assumptions. Thus, over the past few decades a considerable number of studies have been made to elucidate the relations between causality violation and singularities.

The appearance of causality violation causes the Cauchy horizon. This is closely related to the strong cosmic censorship. Its mathematical description is that the maximal globally hyperbolic development of a generic initial Cauchy data is inextendible [5]. In other words, it holds if a space-time admits no Cauchy horizon. So far only a few attempts have been made to study the instability of the Cauchy horizon due to causality violation in classical general relativity.

One can observe the appearance of causality violations and associated Cauchy horizons in compact universe models by extending the space-time maximally. The Taub-Newman-Unti-Tamburino (NUT) universe is one of such models. In the compact homogeneous vacuum universe case, Chruściel and Rendall [6] showed that the strong cosmic censorship holds in a few class of Bianchi types. Ishibashi *et al.* [7] showed that the strong cosmic censorship holds in compact hyperbolic inflationary universe models.

On the other hand Moncrief and Isenberg [8,9] showed that causality-violating cosmological solutions of Einstein equations are essentially artifacts of symmetries. They proved that there exists a Killing symmetry in the direction of the null geodesic generator on the Cauchy horizon if the Cauchy horizon is compact by using Einstein equations. We can easily understand this curious result by inspecting exact solutions which have causality-violating regions. For example, the Misner space-time and the Taub-NUT universe, which have compact Cauchy horizons, have Killing symmetries on the Cauchy horizons. However, the physically realistic universe is inhomogeneous and does not admit Killing symmetries. Thus we expect that a compact inhomogeneous universe does not have any Cauchy horizon or, if it does, the Cauchy horizon cannot be compact from the results of Moncrief and Isenberg. One often studies the inhomogeneous universe by adding perturbations on homogeneous models. Especially there are some works on the perturbative analysis of a spatially compact universe with a compact Cauchy horizon [10]. As one of such perturbative approaches, Konkowski and Shepley [11] studied two-dimensional cylindrical vacuum space-times. They demonstrated the tendency of the appearance of scalar curvature singularities on the Cauchy horizon (see also Ref. [12]). These investigations suggest that, if a spatially compact universe has a Cauchy horizon which divides the space-time into causality-preserving and -violating regions, scalar curvature singularities occur somewhere on the Cauchy horizon.

In this paper we present a theorem as our main result in which if a spatially compact universe satisfies generic condition and, if any, all the occurring curvature singularities are restricted to the boundaries of causality-violating regions, then whole segments of the Cauchy horizon become curvature singularities. Consequently it follows that such a universe cannot be extended to the causality-violating regions.

In the next section, we introduce a definition of the Cauchy horizon which is caused by a causality violation for discussing causal structure and singularities. In Sec. III, we observe the behavior of the null geodesic generators of the Cauchy horizon which satisfies the generic condition. We also review the dual null formalism of Hayward [13] for proving our theorem. We present our theorem in Sec. IV. Section V is devoted to a summary and discussion on the strong curvature singularity.

II. CHRONOLOGICAL CAUCHY HORIZON

The Cauchy horizons in the spatially compact space-times are characterized by the null geodesic generators of the Cauchy horizons. Figure 1(a) shows an example of the Cauchy horizon which is caused by a singularity, where there is no causality violating region. Figure 1(b) shows an

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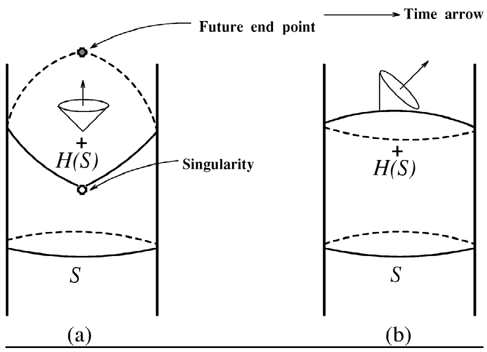

FIG. 1. Two types of the Cauchy horizons for the spatially compact universe, in which two spatial dimensions, B factors, are suppressed.

example of the Cauchy horizon which is caused by causality violation, where the light cone is tipped, allowing the existence of the closed or imprisoned null geodesic generators. The most distinct feature of the two Cauchy horizons is whether the null geodesic generator has future end point or not as shown in Fig. 1. Moncrief and Isenberg [8,9] have considered the compact Cauchy horizon of Fig. 1(b). In their papers, it has been shown that the space-time has a nontrivial Killing vector field. Their geometrical assumption in Ref. [8] is the following. Let the space-time be (M,g) such as $M=\Sigma\times\mathbf{R}$ where Σ is a compact three-manifold and the metric g is analytic. If there exists a Cauchy horizon $H^+(S)$ of a partial Cauchy surface S, then $H^+(S)$ is a compact null embedded hypersurface which is diffeomorphic to Σ and the topology of $H^+(S)$ is $B\times\mathbf{S}^1$, where B is a compact spacelike two-manifold and the $\mathbf{S}^1$ factor is generated by the closed null geodesic generator of $H^+(S)$. More precisely, they considered such a Cauchy horizon $H^+(S)$ with *a local product bundle structure* in the sense that *if* $H^+(S)$ *contains a closed null geodesic generator* γ, *there exists an open set* U_γ *containing* γ *such that* (i) $U_\gamma\cap H^+(S)$ *is diffeomorphic to* $B_\gamma\times\mathbf{S}^1$ *for some two-manifold* B_γ *and some diffeomorphism* $\phi_\gamma:U_\gamma\cap H^+(S)\to B_\gamma\times\mathbf{S}^1$, *and* (ii) *there is a smooth, surjective map* $\pi_\gamma:B_\gamma\times\mathbf{S}^1\to B_\gamma$ *such that, for any* $p\in B_\gamma$, $B_\gamma\times\mathbf{S}^1\approx B_\gamma\times\pi_\gamma^{-1}(p)$ *and the fiber* $\pi_\gamma^{-1}(p)$ *is diffeomorphic to a closed null generator lying in* $U_\gamma\cap H^+(S)$.

The set of the type $U_\gamma\cap H^+(S)$ is called *the elementary region of* $H^+(S)$. Under the above assumptions it is shown that a compact null hypersurface $H^+(S)$ has an analytic Killing field Y which is null and tangents to a null geodesic generator of $H^+(S)$. This interesting fact is due to the existence of the closed null geodesic generators of $H^+(S)$. Moncrief and Isenberg suggested that in generic space-time, the compact $H^+(S)$ the closed null geodesic generators cannot exist.

In generic space-times, it seems reasonable to suppose that the generic condition [14] is satisfied. We see that the generic condition contradicts the closedness of the null geodesic generator in Sec. III; the null geodesic generators cannot be closed if the generic condition is satisfied.

The nonexistence of $H^+(S)$ with closed null geodesic generators in the generic compact universe can be explained as the appearance of curvature singularities. As discussed in Ref. [8], if M has a compact Cauchy horizon $H^+(S)$, then M has a b-incomplete curve corresponding to a singular point which had been left out of space-time. This singularity is a quasiregular singularity. In this paper we are not concerned with such singularities but curvature singularities. We consider the case that the noncompactness of the Cauchy horizon is attributed to curvature singularities.

In general it is useful for describing singularities and causal structures to adopt *the b boundary*. Schmidt [15] has constructed the boundary ∂ of M which corresponds to singularities of (M,g) by using the b completeness. His construction is characterized by distinguishing between infinity and singular points at a finite distance. Hereafter we consider a large space $M^+:=M\cup\partial$.

Let us introduce the following definition, which is an extension of the above definition for a compact Cauchy horizon, so that one can treat both the compact and noncompact Cauchy horizons. Here, we should comment on the spatially compact space-time manifold $M=\Sigma\times\mathbf{R}$ considered in this paper. We want to consider the case that the compact spacelike three-manifold Σ has a local product bundle structure as mentioned above, replacing the closed null geodesic generator γ in the definition by a closed spacelike curve L which generates $\mathbf{S}^1$ factor of Σ. In this sense, we write $\Sigma\approx B\times\mathbf{S}^1$ throughout the paper. The simplest case is that Σ has a global product bundle structure; Σ is diffeomorphic to $B\times\mathbf{S}^1$ and the fibers coincide with closed spacelike curves lying in Σ. Examples of the type $\Sigma\approx\mathbf{T}^3\approx\mathbf{T}^2\times\mathbf{T}^1$ were constructed by Moncrief in Ref. [16]. The Taub-NUT universe is the nontrivial case; Σ is diffeomorphic to $\mathbf{S}^3$ with Hopf fibering $\mathbf{S}^3\to\mathbf{S}^2$ and fibers $\mathbf{S}^1$ coincide with closed spacelike curves in Σ.

Definition: Chronological Cauchy horizon. Consider a space-time (M,g) *with a partial Cauchy surface* S *of which Cauchy development* $D^+(S)$ *has compact spatial sections* $\Sigma\approx B\times\mathbf{S}^1$, *where* B *is a compact orientable two-manifold. We call a Cauchy horizon* $H^+(S)$ *the chronological Cauchy horizon if it satisfies the following conditions.*

(a) *Let* $\{q_n\}$ *be a sequence of points in* $D^+(S)$ *which converges to a point* p *in* $H^+(S)$. *There exists an infinite sequence* $\{L_n\}$ *of closed spacelike curves which generate* $\mathbf{S}^1$ *factors of* $D^+(S)$ *and each* L_n *passes through* q_n *such that, for every point* $r_n\in L_n$, *the tangent vector* K_n *of* L_n *at* r_n *approaches to null, i.e.,*

$$\lim_{q_n\to p} g(K_n,K_n)|_{r_n}=0.$$

(b) *If* $H^+(S)$ *contains a closed null geodesic generator* γ, *there exists an elementary region of the type* $U_\gamma\cap H^+(S)$ *with the local product bundle structure as mentioned above.*

(c) *There exists a compact spacelike orientable two-surface* B *on* $H^+(S)$ *such that there is no null geodesic generator of* $H^+(S)$ *which connects two different points* p *and* $q(\neq p)$ *on* B.

It is obvious that the chronological Cauchy horizon has no future end point in (M,g) from condition (a). This means that the segments of $\dot{D}^+(S,M^+)$ are null and, if exist, singularities are restricted to the boundary of the causality-violating region in M^+. As mentioned above, when we speak of a singularity in this paper, it means a curvature singularity. Thus, hereafter, ∂ denotes a curvature singularity.

The chronological Cauchy horizon is not required in general to be compact and hence it is a generalization of the Cauchy horizon which is considered in Ref. [8]. Indeed, the chronological Cauchy horizon can be noncompact if there are curvature singularities ∂. In that case, there are nonclosed incomplete null geodesic generators of $H^+(S)$, which terminate at ∂. Conditions (b) and (c) imply that such a noncompact Cauchy horizon can be regarded as one which is made of the compact Cauchy horizon considered in Ref. [8] by removing points which correspond to ∂. If ∂ is empty, the chronological Cauchy horizon is compact; i.e., it can be covered by a finite number of the elementary regions of the type $U_\gamma \cap H^+(S)$, and diffeomorphic to Σ.

III. PRELIMINARIES

In this section, we introduce the Gaussian null coordinates to observe the relation between the null geodesic generator of $H^+(S)$ and the generic condition for the discussion of our theorem. We concentrate our interest on the compact vacuum space-time $M \approx \Sigma \times \mathbf{R}$ which admits the chronological Cauchy horizon defined in the previous section. In addition, we introduce the dual null coordinates [13] and define a strong curvature singularity condition which is slightly different from Królak's one in Ref. [18] and weaker than it.

A. Gaussian null coordinates

We adopt the Gaussian null coordinates $\{t, x^3, x^a\}(a=1,2)$ in the neighborhood of the chronological Cauchy horizon (in detail, see Ref. [8]). In this coordinate system, the metric takes the form

$$g = 2dtdx^3 + \phi(dx^3)^2 + 2\beta_a dx^a dx^3 + h_{ab}dx^a dx^b. \tag{3.1}$$

The chronological Cauchy horizon $H^+(S)$ corresponds to the hypersurface $t=0$. The future development $D^+(S)$ is the region $t<0$. In $D^+(S)$, the vector field $\partial/\partial x^3$ has closed spacelike integral curves which generate an $\mathbf{S}^1$ factor and h_{ab} is the induced metric of the spacelike two-surface B. One can choose the coordinates $\{t, x^a, x^3\}$ such that $\phi=\beta_a=0$ on $H^+(S)$. Then, $\partial/\partial x^3|_{t=0}$ is tangent to the null geodesic generator of $H^+(S)$. In these coordinates, the Einstein equation $R_{33}=0$ is given by

$$(\ln\sqrt{h})_{,33} + \tfrac{1}{2}\phi_{,t}(\ln\sqrt{h})_{,3} + \tfrac{1}{4}h^{ac}h^{bd}h_{ab,3}h_{cd,3} = 0 \tag{3.2}$$

on $H^+(S)$. In the case that the integral curves of $\partial/\partial x^3$ are closed in the chronological Cauchy horizon $H^+(S)$, applying the maximum principle [17] for Eq. (3.2), we obtain $h_{,3}=0$ and consequently

$$h_{ab,3} = 0, \tag{3.3}$$

by substituting $h_{,3}=0$ into Eq. (3.2) again. This equation implies that the closed null geodesic generator of $H^+(S)$ must be shear free. This fact has been shown in Ref. [8]. As seen in Ref. [14] the shear-free congruence does not satisfy the generic condition; i.e., the null geodesic does not contain a point at which $K_{[\lambda}R_{\mu]\nu\rho[\sigma}K_{\tau]}K^\nu K^\rho \neq 0$, where K^μ is the tangent vector to the null geodesic. Therefore the null geodesic generators of $H^+(S)$ which satisfy the generic condition are not closed.

B. Dual null coordinates

In the dual null coordinates $\{u, v, x^a\}$, the metric is written as

$$g = -2e^{-\lambda}dudv + s^a s_a du^2 + 2s_a dx^a du + h_{ab}dx^a dx^b. \tag{3.4}$$

We choose the shift two-vector s^a such that $s^a=0$ on $H^+(S)$. One can easily understand that the dual null coordinates are transformed into the Gaussian null coordinates by taking

$$du = -e^\lambda dx^3, \quad dv = dt, \quad \phi = s^a s_a e^{2\lambda}, \quad \beta_a = -2s_a e^\lambda. \tag{3.5}$$

We introduce some quantities, of which the notation is as those in Ref. [19]. Introducing the null vectors

$$k^\mu = \left(\frac{\partial}{\partial v}\right)^\mu, \quad n^\mu = \left(\frac{\partial}{\partial u}\right)^\mu - s^a\left(\frac{\partial}{\partial x^a}\right)^\mu, \tag{3.6}$$

we define

$$\Sigma_{ab} := \mathcal{L}_k h_{ab}, \quad \tilde{\Sigma}_{ab} := \mathcal{L}_n h_{ab}, \tag{3.7}$$

where $\mathcal{L}_k$ represents the Lie derivative along the vector field k^μ. The expansions $\theta, \tilde{\theta}$, the shears $\sigma_{ab}, \tilde{\sigma}_{ab}$, and the twist vector ω_a are represented as

$$\theta = \tfrac{1}{2}h^{ab}\Sigma_{ab}, \quad \tilde{\theta} = \tfrac{1}{2}h^{ab}\tilde{\Sigma}_{ab}, \tag{3.8}$$

$$\sigma_{ab} = \Sigma_{ab} - \theta h_{ab}, \quad \tilde{\sigma}_{ab} = \tilde{\Sigma}_{ab} - \tilde{\theta}h_{ab}, \tag{3.9}$$

$$\omega_a = \tfrac{1}{2}e^\lambda h_{ab}\mathcal{L}_k s^b. \tag{3.10}$$

On $H^+(S)$, the null vector n^μ corresponds to a tangent vector of a null geodesic generator of $H^+(S)$. Equation (3.3) implies $\tilde{\sigma}_{ab}=0$.

C. Strong curvature singularity

As discussed in the previous section, we want to consider space-times which contain the chronological Cauchy horizon $H^+(S)$ and curvature singularity restricted to the boundary of the causality-violating region $\overline{H^+(S, M^+)}$. Thus such a singularity can be specified by, especially, the incomplete null geodesic generator of the chronological Cauchy horizon $H^+(S)$.

Definition: Strong curvature singularity. A future inextendible null geodesic generator l of $H^+(S)$ is said to terminate in a strong curvature singularity in the future if there exists a point p on l such that the expansion $\tilde{\theta}|_p$ is negative in the future direction.

We will discuss whether or not the strong curvature singularity can occur in the vacuum space-time by using the dual null formalism of Hayward in the last section.

IV. THEOREM

In this section, we present our theorem in which no spatially compact space-time can have a chronological Cauchy horizon under the seemingly physical assumptions.

Theorem. Let (M,g) *be a spatially compact vacuum space-time which admits a regular partial Cauchy surface* S *diffeomorphic to* $\Sigma \approx B \times \mathbf{S}^1$. *If* (M,g) *satisfies the following conditions: (i) the generic condition, i.e., every inextendible null geodesic contains a point at which* $K_{[\lambda}R_{\mu]\nu\rho[\sigma}K_{\tau]}K^{\nu}K^{\rho} \neq 0$, *where* K^{μ} *is the tangent vector to the null geodesic; (ii) the Cauchy horizon, if any, is the chronological Cauchy horizon* $H^+(S)$; *(iii) all occurring curvature singularities are the strong curvature singularities, then,* (M,g) *is globally hyperbolic.*

Proof. Suppose that space-time (M,g) was not globally hyperbolic. Then either $H^+(S)$ or $H^-(S)$ would exist. Let us consider only $H^+(S)$ without loss of generality and take the dual null coordinates $\{u,v,x^a\}$ defined in the previous section in some neighborhood of $H^+(S)$. The generator of $H^+(S)$ would have no past end point in (M,g) since S was a partial Cauchy surface. In addition every null geodesic generator of $H^+(S)$ would have no future end point in (M,g) from the definition of the chronological Cauchy horizon. Suppose that there was a closed null geodesic generator γ of $H^+(S)$. There would exist an elementary region $U_\gamma \cap H^+(S)$ which contains γ from condition (b) of the definition of the chronological Cauchy horizon. Since γ was closed in the elementary region, it would have to be shear free as discussed in Sec. III. On the other hand, because γ was inextendible, γ would have a point which satisfies generic condition (i). This contradicts the fact that γ is shear free. Therefore any null geodesic generator of $H^+(S)$ cannot be closed.

Since $H^+(S)$ is a chronological Cauchy horizon and $\overline{H^+(S,M^+)\cup\partial}$ generates an $\mathbf{S}^1$ factor, every null geodesic generator l of $\overline{H^+(S,M^+)}$ terminates at some points of ∂ both in future and past directions. In addition, such null geodesic generators do not intersect B more than once from condition (c). Here the curvature singularity ∂ is a strong curvature singularity from condition (iii). Thus the expansions of the future-directed null generators of $\overline{H^+(S,M^+)}$ become negative in the future direction somewhere near the future end points in ∂. On the other hand, the expansions of past-directed null generators also become negative in the past direction somewhere near the past end points in ∂.

Let Γ_u be a null hypersurface on which $u=\text{const}$ with tangent k^{μ} in the neighborhood and of which the intersection with $H^+(S)$, i.e., $\Gamma_u \cap H^+(S)$, coincides with B. From the ansatz of the dual null formalism, Γ_u is foliated by compact spacelike two-surfaces B_m and hence we can take an infinite sequence of the two-surfaces $\{B_m\}$ on Γ_u which converges to B. Let us consider the boundaries of the causal past sets $\dot{J}^-(B_m)$ (see Fig. 2). For each number m, $\dot{J}^-(B_m)$ is closed. Because the spatial section of $D^+(S)$ is compact and $\mathrm{int}D^+(S)$ is causally simple [14], there exists a null geodesic generator l_m of $\dot{J}^-(B_m)$ whose future and past end points, denoted by p_m and q_m, respectively, are on Γ_u and the tangent vector at p_m is n^{μ}. The limit points p and q of the infinite sequences $\{p_m\}$ and $\{q_m\}$, respectively, are on B.

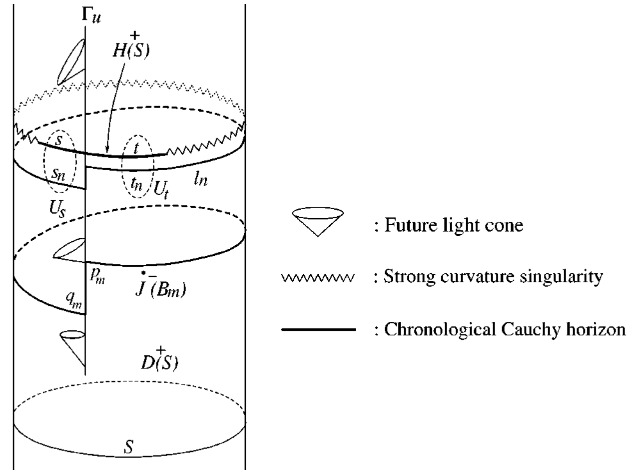

FIG. 2. A two-dimensional section of a spatially compact universe. The chronological Cauchy horizon $H^+(S)$, $\dot{J}^-(B_m)$, etc., are illustrated. The vertical line is Γ_u.

Here q does not necessarily coincide with p. Then, from the limit curve lemma in Ref. [20], for the infinite sequence of the null geodesic generators $\{l_m\}$, there exist limit null geodesic curves l_p and l_q on $H^+(S)$ which pass through the points p and q, respectively, and a subsequence $\{l_n\}$ which converges to l_p and l_q uniformly with respect to h. Here h is a complete Riemannian metric on the space-time (in detail see Ref. [20]).

As mentioned above, since every null geodesic generator of $H^+(S)$ terminates at ∂, the limit curves l_p and l_q also terminate at ∂ in the past and future, respectively, without intersecting B more than once. In addition, because ∂ is a strong curvature singularity, l_p and l_q have points t and s such that $\tilde{\theta}|_t > 0$ and $\tilde{\theta}|_s < 0$ in the future direction. Since l_n converges to l_p and l_q uniformly, for any neighborhoods U_s and U_t of the points s and t, respectively, there exists a natural number N such that all $l_n (n>N)$ intersect both U_s and U_t. It also can be taken the infinite sequences of points $\{s_n\}$ and $\{t_n\}$ such that, for each n, $s_n \in U_s \cap l_n$, $t_n \in U_t \cap l_n$, $s_n < t_n$, and these sequences converge to $s \in l_q$ and $t \in l_p$, respectively. Then, there exists a number N' such that, for any $n > N'$, $\tilde{\theta}|_{t_n} > 0$, $\tilde{\theta}|_{s_n} < 0$, in the future direction by continuity. This contradicts the fact that the expansion of l_n must decrease monotonically in the future direction. Consequently, $H^+(S)$ cannot exist. □

V. CONCLUSION AND DISCUSSIONS

We showed that in a compact universe, if the curvature singularity is restricted to the boundary of the causality-violating set, whole segments of the boundary become curvature singularities. Consequently the vacuum space-time with compact spatial section $\Sigma \approx B \times \mathbf{S}^1$ cannot be extended to the causality-violating region. This result means that strong cosmic censorship holds in such a space-time.

In our proof of the theorem, we use the strong curvature singularity condition, whose notion was first introduced by Tipler [21] and described in terms of expansions by Królak [18]. Our definition of the strong curvature singularity is slightly different from that by Królak. Therefore it is worth discussing whether or not our strong curvature singularity condition is reasonable in the vacuum space-time. Here we use the dual null coordinates of Hayward [13].

Let Γ_v be the null hypersurface $v=\text{const}$. On Γ_v, the Raychaudhuri equation can be written by

$$\frac{d\tilde{\theta}}{du}=-\frac{1}{2}\tilde{\theta}^2-\frac{d\lambda}{du}\tilde{\theta}-\frac{1}{4}\tilde{\sigma}_{ab}\tilde{\sigma}^{ab}. \tag{5.1}$$

Hayward noticed that Eq. (5.1) may be simplified by making use of coordinate freedom on Γ_v. Choosing the coordinate u on Γ_v as

$$\frac{d\lambda}{du}=-\frac{1}{2}\tilde{\theta}, \tag{5.2}$$

Eq. (5.1) is written by

$$\frac{d\tilde{\theta}}{du}=-\frac{1}{4}\tilde{\sigma}_{ab}\tilde{\sigma}^{ab}, \tag{5.3}$$

the ω_a, θ, and $\tilde{\theta}$ can be easily integrated along Γ_v and Γ_u, respectively, by using the vacuum Einstein equations (in detail, see Ref. [19]) as

$$\begin{aligned}\omega_a(u)=&\,\omega_a|_0\exp\left[-\int_{u_0}^{u}du'\,\tilde{\theta}(u')\right]\\&+\exp\left[-\int_{u_0}^{u}du'\,\tilde{\theta}(u')\right]\int_{u_0}^{u}du'\left(-\frac{1}{2}\Delta^b\tilde{\sigma}_{ab}\right.\\&\left.+\frac{3}{4}\Delta_a\tilde{\theta}+\frac{1}{2}\tilde{\theta}\Delta_a\lambda\right)\exp\left[\int_{u_0}^{u'}du''\,\tilde{\theta}(u'')\right],\end{aligned} \tag{5.4}$$

$$\begin{aligned}\theta(u)=&\,\theta|_0\exp\left[-\int_{u_0}^{u}du'\,\tilde{\theta}(u')\right]+\exp\left[-\int_{u_0}^{u}du'\,\tilde{\theta}(u')\right]\\&\times\int_{u_0}^{u}du'e^{-\lambda}K\exp\left[\int_{u_0}^{u'}du''\tilde{\theta}(u'')\right],\end{aligned} \tag{5.5}$$

$$\begin{aligned}\tilde{\theta}(v)=&\,\theta|_0\exp\left[-\int_{v_0}^{v}dv'\,\theta(v')\right]+\exp\left[-\int_{v_0}^{v}dv'\,\theta(v')\right]\\&\times\int_{v_0}^{v}dv'e^{-\lambda}L\exp\left[\int_{v_0}^{v'}dv''\theta(v'')\right].\end{aligned} \tag{5.6}$$

Here K and L are defined, respectively, as

$$\begin{aligned}K:=&-\tfrac{1}{2}{}^{(2)}R+\omega_a\omega^a-\tfrac{1}{2}\Delta^a\Delta_a\lambda+\tfrac{1}{4}\Delta^a\lambda\Delta_a\lambda-\omega^a\Delta_a\lambda\\&+\Delta_a\omega^a,\end{aligned} \tag{5.7}$$

$$\begin{aligned}L:=&-\tfrac{1}{2}{}^{(2)}R+\omega_a\omega^a-\tfrac{1}{2}\Delta^a\Delta_a\lambda+\tfrac{1}{4}\Delta^a\lambda\Delta_a\lambda+\omega^a\Delta_a\lambda\\&-\Delta_a\omega^a,\end{aligned} \tag{5.8}$$

and Δ_a and ${}^{(2)}R$ are, respectively, the covariant derivative, and Ricci scalar with respect to h_{ab}.

On the other null hypersurface Γ_u on which $u=\text{const}$, the Raychaudhuri equation is written by

$$\frac{d\theta}{dv}=-\frac{1}{2}\theta^2-\frac{d\lambda}{dv}\theta-\frac{1}{4}\sigma_{ab}\sigma^{ab}. \tag{5.9}$$

As well as on Γ_v, choosing the coordinate v on Γ_u such that

$$\frac{d\lambda}{dv}=-\frac{1}{2}\theta, \tag{5.10}$$

we can rewrite Eq. (5.9) as

$$\frac{d\theta}{dv}=-\frac{1}{4}\sigma_{ab}\sigma^{ab}. \tag{5.11}$$

With the help of Eqs. (5.2) and (5.10), we can express Eqs. (5.5) and (5.6) on each null hypersurface Γ_v, Γ_u such as

$$\theta(u)=\theta_0e^{2(\lambda-\lambda_{u_0})}+e^{2\lambda}\int_{u_0}^{u}du'e^{-3\lambda}K, \tag{5.12}$$

$$\tilde{\theta}(v)=\tilde{\theta}_0e^{2(\lambda-\lambda_{v_0})}+e^{2\lambda}\int_{v_0}^{v}dv'e^{-3\lambda}L. \tag{5.13}$$

In the vacuum space-time, the strong curvature singularities are caused by Weyl tensor only. The Weyl tensor produces the shear tensor $\sigma_{ab}(\tilde{\sigma}_{ab})$ and the square $\sigma_{ab}\sigma^{ab}(\tilde{\sigma}_{ab}\tilde{\sigma}^{ab})$, which can be interpreted as the gravitational energy. In the Kerr black hole case, Brady and Chambers [19] showed that only the quantity $\sigma_{ab}\sigma^{ab}$ diverges on the Cauchy horizon but $\tilde{\sigma}_{ab}\tilde{\sigma}^{ab}$ does not. In terms of expansions, this means that only the expansion θ diverges but $\tilde{\theta}$ does not. In generic space-times, however, it is expected that θ and $\tilde{\theta}$ behave similarly; both θ and $\tilde{\theta}$ diverge as they approach the curvature singularity. Indeed, from Eqs. (5.12) and (5.13), it turns out that, if λ diverges, both θ and $\tilde{\theta}$ diverge. If there exists a curvature singularity such that $\theta(u)$ diverges while $\tilde{\theta}(v)$ does not, then K must diverge but λ and L must not diverge. In the case, because L is different from K only the signature of the last two terms in Eq. (5.8), $+\omega^a\Delta_a\lambda-\Delta_a\omega^a$, the divergence of these two terms must cancel out that of all the other terms in L. Such a case is unlikely and cannot be considered as generic. This suggests that, in generic space-times, if at least either $\sigma_{ab}\sigma^{ab}$ or $\tilde{\sigma}_{ab}\tilde{\sigma}^{ab}$ diverges on the curvature singularity, both θ and $\tilde{\theta}$ diverge and hence our strong curvature singularity condition is satisfied. This means that the expansions of the null geodesic generators on $H^+(S)$ diverge on the singularity whenever the gravitational energy diverges. In addition, this suggests that the expansion of each incomplete causal geodesic diverges on the singularity independent of its tangent in generic vacuum space-times. A rigorous study of the discussion above will be given in future works.

One might consider that there exists a possibility to cause causality violation in the presence of matter. However, in the black hole case, the existence of matter does not change the property of the Cauchy horizon drastically as Brady and Smith [22] have shown by numerical investigation. Thus we believe that strong cosmic censorship in a compact universe also holds even if matter exists.

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