Gauss linking number and the electromagnetic uncertainty principle

Abhay Ashtekar* and Alejandro Corichi†

Center for Gravitational Physics and Geometry, Physics Department, Penn State, University Park, Pennsylvania 16802
(Received 27 January 1997)

It is shown that there is a precise sense in which the Heisenberg uncertainty between fluxes of electric and magnetic fields through finite surfaces is given by (one-half $\hbar$ times) the Gauss linking number of the loops that bound these surfaces. To regularize the relevant operators, one is naturally led to assign a framing to each loop. The uncertainty between the fluxes of electric and magnetic fields through a single surface is then given by the self-linking number of the framed loop which bounds the surface. [S0556-2821(97)02016-X]

I. INTRODUCTION

In 1833, Gauss noticed a striking fact about electromagnetism [1]. He considered a loop L_1 carrying a constant current I and computed the work W done in moving a magnetic monopole of strength m along a closed path L_2 in the magnetic field produced by the current:

$$W = \frac{mI}{4\pi} \oint_{L_1} ds \oint_{L_2} dt\, \epsilon_{abc}\, \dot{L}_1{}^a(s)\, \dot{L}_2{}^b(t)$$
$$\times \frac{L_1^c(s) - L_2^c(t)}{|L_1(s) - L_2(t)|^3}. \tag{1.1}$$

He then made a deep observation which can be stated in the modern mathematical terminology as follows: although the double integral

$$\mathcal{GL}(L_1, L_2) := \frac{1}{mI} W \tag{1.2}$$

makes use of Euclidean geometry in several ways, its value is in fact a topological invariant, a measure of the linking between the loops L_1 and L_2. In particular, even if one deforms the loops, the value of the double integral does not change so long as the loops do not touch or cross each other. This is a remarkable property and Gauss expressed the belief that the quantity $\mathcal{GL}(L_1, L_2)$ may have a fundamental significance. The view was shared by others. In particular, in his celebrated treatise on electricity and magnetism, Maxwell returns to this property and further elaborates on it [2,3].

It turns out that the double integral $\mathcal{GL}(L_1, L_2)$ does have a fundamental significance in electromagnetism, which however, could not have been guessed before the advent of quantum field theory. To see this, consider source-free Maxwell theory (in Minkowski space-time). In the Hamiltonian treatment, the vector potential $A_a(\bar{x})$ and the electric field $E^a(\bar{x})$ (on a constant time hyperplane) serve as the basic canonically conjugate fields with the Poisson brackets relations:

---
*Electronic address: ashtekar@phys.psu.edu
†Electronic address: corichi@phys.psu.edu

$$\{A_a(\bar{x}), E^b(\bar{y})\} = \delta_a^b \delta^3(\bar{x}, \bar{y}). \tag{1.3}$$

(Throughout this paper, curly brackets will denote Poisson brackets.) The vector potential itself is not an observable since it fails to be gauge invariant. However, we can integrate it over a closed loop α to obtain a gauge invariant functional:

$$B[\alpha] := \oint_\alpha A_a dl^a \equiv \int_{S_\alpha} B^a d^2 S_a, \tag{1.4}$$

where S_α is *any* two-surface bounded by the loop α. Similarly, given any two-surface S_β bounded by a closed loop β we can define the flux of the electric field,

$$E[\beta] := \int_{S_\beta} E^a d^2 S_a, \tag{1.5}$$

which depends only on the loop β (and not on the specific surface S_β with boundary β) because E^a is divergence free. The observables $B[\alpha]$ and $E[\beta]$ are (over)complete in the sense that their values at a point (B^a, E^a) of the physical phase space suffice to determine that point uniquely.

It is straightforward to compute the Poisson brackets between these observables if α and β have no point in common. The result is

$$\{B[\alpha], E[\beta]\} = \oint_\alpha dl^a(\bar{x}) \int_{S_\beta} d^2 S_a(\bar{y}) \delta^3(\bar{x}, \bar{y}) = I(\alpha, S_\beta), \tag{1.6}$$

where $I(\alpha, S_\beta)$ denotes the oriented intersection number between the loop α and the surface S_β. But, as a geometrical picture makes it clear (see Fig. 1), this intersection number is precisely the linking number between loops α and β. (An analytic calculation showing the equality of $I(\alpha, S_\beta)$ with $\mathcal{GL}(\alpha, \beta)$ is given in the Appendix.) Thus, we have

$$\{B[\alpha], E[\beta]\} = \mathcal{GL}(\alpha, \beta). \tag{1.7}$$

The fact that the Poisson bracket is metric independent may seem surprising at first. But note that, since the vector potential A_a is a one-form and the electric field E^a (being canonically conjugate to A_a) is naturally a vector density of weight one, neither the symplectic structure (1.3) nor the definitions
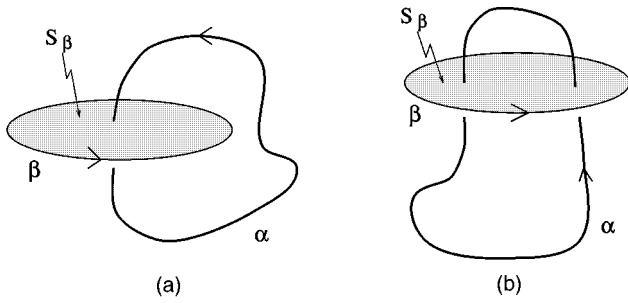
FIG. 1. The Gauss linking number $\mathcal{GL}(\alpha,\beta)$ equals the oriented intersection number $I(\alpha,S_\beta)$. In (a) this number is 1 and in (b) it is 0.

of the observables $E[\alpha]$, $B[\beta]$ require a metric (or any other background field) for their definitions. Hence, if well defined, the right side of Eq. (1.7) *has to be* a topological invariant of loops labeling the observables.

Let us pass to the quantum theory heuristically. One would expect that (if α and β have no point in common) the commutator between the magnetic and electric flux operators would be given by

$$[\hat{B}[\alpha],\hat{E}[\beta]]=i\hbar\ \mathcal{GL}(\alpha,\beta) \tag{1.8}$$

and hence the Heisenberg uncertainties should satisfy

$$(\Delta\hat{B}[\alpha])(\Delta\hat{E}[\beta])\geq\frac{\hbar}{2}|\mathcal{GL}(\alpha,\beta)|. \tag{1.9}$$

This implies that there is an intrinsic uncertainty in the simultaneous measurements of fluxes of electric and magnetic fields across finite surfaces if the Gauss linking numbers of the loops bounding the two surfaces is not zero (see Fig. 2). In this sense, as suspected by Gauss and others, the linking number does have a fundamental significance in electromagnetism.

A number of questions arise immediately. Can one make these quantum considerations precise? If the quantization is *based* on the algebra of operators generated by $\hat{B}[\alpha]$ and $\hat{E}[\beta]$, the answer is clearly in the affirmative. However, in such representations, the Maxwell Hamiltonian operator fails to be well defined. In the standard Fock representation where the Hamiltonian *is* well defined, operators $\hat{E}[\alpha]$ and $\hat{B}[\beta]$ fail to be well defined. (See, e.g., [5].) Can one nonetheless give meaning to the topological uncertainty relations (1.9) in the Fock representation *in a suitable limiting sense*? Second,
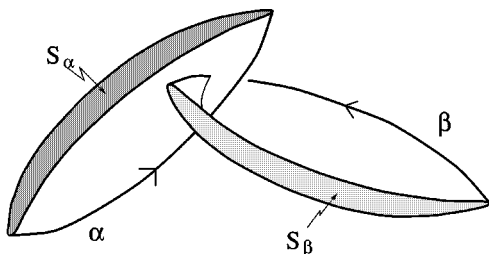

FIG. 2. The (absolute value of the) Gauss linking number between loops α and β is 1. In this case, the Heisenberg uncertainty between the magnetic flux through the surface S_α and the electric flux through S_β is $\hbar/2$.

the uncertainty relation given above fails to be meaningful if the loops α and β coincide. However, it is known that the Gauss self-linking number of a loop *is* well defined if the loop is framed [3]. Is there perhaps a framing that is naturally introduced in the process of regularization of the operators in question? If so, can one give meaning to the commutator of $\hat{B}[\alpha]$ and $\hat{E}[\alpha]$? The purpose of this work is to analyze these issues. We will find that the specific questions raised here can be answered affirmatively.

In Sec. II, we will consider the Fock space of photons and show that suitably ''thickened,'' regulated versions of the above ''flux operators'' are indeed well defined on the Fock space. This will enable us to regard $\hat{B}[\alpha]$ and $\hat{E}[\beta]$ as certain limits of well-defined operators. We will see that the commutation relations (1.8) also hold in a limiting sense. In this limit, the thickening of surfaces goes to zero. However, the loop that bounds the limiting surface carries the ''memory'' of the thickening in the form of a framing. We will see in Sec. III that the limits of the commutators of the regulated flux operators are functionals of these framed loops. In particular, the framing enables one to evaluate the (limits of) commutators without ambiguities even when the loops intersect and overlap. We should emphasize however that the level of rigor in this work is that generally used in theoretical physics rather than mathematical physics.

We will conclude this section with a few remarks.

(i) One often defines *dimensionless* observables: $B'[\alpha]=(c/e)B[\alpha]$ and $E'[\beta]=(1/e)E[\beta]$, which can be exponentiated to obtain Weyl commutation relations. [The exponential of $iB'[\alpha]\equiv i(c/e)\oint_\alpha dl^a A_a$, for example, is the U(1) holonomy.] In terms of these primed observables, the uncertainty principle reads $(\Delta B'[\alpha])(\Delta E'[\beta])\geq(1/2\alpha_{\text{fine}})\mathcal{GL}(\alpha,\beta)$, where α_{fine} is the fine structure constant.

(ii) In the non-Abelian case, one can replace $\exp iB'[\alpha]$ by the trace of the holonomy of the connection along the loop α. The analogue of $E'[\beta]$ is trickier. To ensure gauge invariance, one now has to thicken the loop to a ribbon. (See, e.g., [4,5].) The commutator is then again ''topological.'' However, the physical meaning of these observables is now less transparent.

(iii) Over the last five years, the uncertainty relation (1.9) was used by one of us (A.A.) to motivate the loop representation for gauge theories and gravity in a number of conferences (see, e.g., [4]). It was pointed out that a similar observation on the Maxwell uncertainties was made by Jürgen Löffelholz from the Leipzig mathematical physics group and also by condensed matter theorists interested in flux quantization. Unfortunately, however, we have not been able to find specific references.

II. REGULATED FLUX OPERATORS

A. Preliminaries

Let us begin by recalling a few facts about the Fock representation of photons. Since we are interested in (fluxes of) electric and magnetic operators, it will be convenient to adapt the discussion to a canonical framework. A vector V in the one-photon Hilbert space $\mathcal{H}$ is then represented by a pair

(A_a, E^a) of divergence-free vector fields on a constant time hyperplane and the Hilbert space norm is given by (see, e.g., [6]):

$$\langle V|V \rangle = \frac{1}{2\hbar} \int_\Sigma d^3x [A_a(\triangle^{1/2}A^a) + E^a(\triangle^{-1/2}E_a)].$$
(2.1)

Denote by $\mathcal{F}_\mathcal{H}$ the symmetric Fock space based on $\mathcal{H}$. Electric and magnetic fields are represented by operator valued distributions involving the standard linear combinations of creation and annihilation operators on $\mathcal{F}_\mathcal{H}$. Consider, for example, the smeared object

$$\hat{E}[f] := \int d^3x \hat{E}^a(x)f_a(x),$$
(2.2)

where f_a is a test field, i.e., vector field of compact support. Since E^a is divergence free, we have $E[f] = E[f + \partial g]$, for any test function g. This is a well-defined operator on $\mathcal{F}_\mathcal{H}$ provided the vector $V = (^Tf_a, 0)$ lies in the Hilbert space $\mathcal{H}$, i.e., provided the norm

$$\langle V|V \rangle = \frac{1}{2\hbar} \int d^3x \, [^Tf_a(\triangle^{1/2}\,{}^Tf^a)] \equiv \frac{1}{2\hbar} \int d^3k |k| |^T\widetilde{f}_a|^2$$
(2.3)

is finite, where ^Tf_a is the transverse part of f, and $^T\widetilde{f}$ its Fourier transform. (The transverse projection removes the "gauge freedom" of adding a gradient to f_a.) In that case, $\hat{E}[f]$ is expressible as a sum of the creation and annihilation operators associated with the state V.

The situation with the magnetic field operator is completely analogous. The commutator between smeared electric and magnetic fields is given by

$$[\hat{B}[f], \hat{E}[g]] = i\hbar \int d^3x \, \epsilon^{abc}[\partial_a f_b(\overline{x})]g_c(\overline{x})$$

$$= \hbar \int d^3k \, \epsilon^{abc}k_a(\widetilde{f}_b(\overline{k}))^\star \widetilde{g}_c(\overline{k}), \quad (2.4)$$

where $\widetilde{f}_b$ denotes the Fourier transform of f_b and a star denotes complex conjugation.

B. Regularization

Let us now consider the formal expression of the electric flux operator:

$$\hat{E}[\beta] = \int_{S_\beta} \hat{E}^a d^2S_a.$$
(2.5)

It can be expressed as a smeared electric field, $E[\beta] = \int d^3x E^a(\overline{x})f_a^{(\beta)}(\overline{x})$, where, however, the test field $f_a^{(\beta)}(\overline{x})$ is a *distribution* with support on S_β:

$$f_a^{(\beta)}(\overline{x}) = \int_{S_\beta} d^2S_a \delta^3(\overline{x}, \overline{s}_\beta),$$
(2.6)
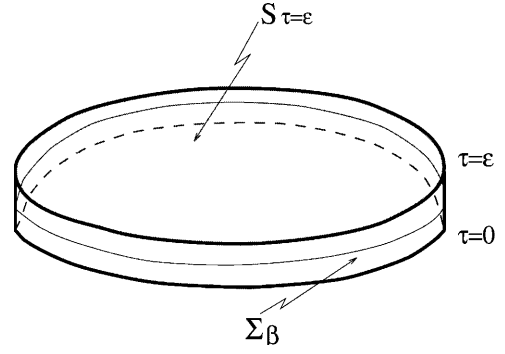
FIG. 3. The surface S_β with boundary β is thickened to a three-dimensional pillbox P_β bounded by the strip Σ_β. As ϵ tends to zero, P_β shrinks to S_β and Σ_β tends to the loop β. The "memory" of the strip is retained by the initial framing attached to β.

where $\overline{s}_\beta$ denotes a point on the surface S_β. Hence, the corresponding operator $\hat{E}[\beta]$ is only formal; it fails to be well defined on the Fock space. We must regulate it.

We will proceed in two steps (the first of which is the crucial one). Geometrically, the problem arises because $f_a^{(\beta)}$ is a distribution with two-dimensional support. We can remedy this situation by an appropriate "thickening" of the surface S_β. Let us therefore replace the loop β by a strip (or ribbon) Σ_β of height ϵ (see Fig. 3). More precisely, let us proceed as follows. Let us first equip β with a framing (i.e., let us introduce, at each point of β, a vector in a direction transverse to $\dot{\beta}^a$, the tangent to β). Then, for each $\tau \in [0, \epsilon]$, let us denote by β_τ the loop obtained by displacing β a distance τ along the framing. (Thus $\beta_0 \equiv \beta$). This construction uses the flat Euclidean metric on the spatial hyperplane. But the key final results will not depend on this flat metric.) Let S_τ denote a surface bounded by the loop β_τ (such that the assignment $\tau \rightarrow S_\tau$ is smooth.) The family of loops β_τ constitute the strip Σ_β and the three-dimensional region swept out by the family of surfaces S_τ constitutes a "pillbox" P_β with boundary Σ_β.

We can now consider the flux of the electric field through the *three-dimensional* pillbox region

$$E[P_\beta] = \int_0^\epsilon d\tau \int_{S_\tau} d^2S_a E^a(\overline{s}_\tau).$$
(2.7)

(From now on, loops will be assumed to be framed. However, for simplicity of notation, we will continue to denote them just by Greek letters $\alpha, \beta, \ldots$.) This yields a smearing of the electric field with a test field $f_a^{(P_\beta)}$ with support in *three* dimensions,

$$E[P_\beta] = \int d^3x E^a(x)f_a^{(P_\beta)}(x)$$

with

$$f_a^{(P_\beta)} = \int_0^\epsilon d\tau \int_{S_\tau} (dS_\tau)_a \delta^3(\overline{x}, \overline{s}_\tau),$$
(2.8)

and one can hope that the corresponding operator would be well defined in quantum theory. This completes our first step in regularization.

The key question now is the following: Is $V = ({}^T f_a^{(P_\beta)}, 0)$ normalizable with respect to the inner product (2.3)? This calculation is carried out in the Appendix. It turns out that, although the three-dimensional smearing softens the singularity of $f_a^{(\beta)}$ considerably, the norm $\langle V | V \rangle$ still has a logarithmic ultraviolet divergence. This arises because the three-dimensional pillbox P_β, on which $f_a^{(P_\beta)}$ is supported, has sharp boundaries. This is where the second step in the regularization procedure comes in. The problem can be handled in a number of ways. We will use the simplest one and just introduce an ultraviolet cutoff at $|\bar{k}| = \Lambda$. That is, we begin with $f_a^{(P_\beta)}$ as in Eq. (2.8), take the Fourier transform of its transverse part, multiply it by the step function which is unity if $|\bar{k}| \leq \Lambda$ and zero otherwise and consider the inverse Fourier transform $f_a^{(P_\beta, \Lambda)}$ of the resulting function. Then,

$\hat{E}[f^{(P_\beta, \Lambda)}]$ is a well-defined operator on the Fock space. This operator can be regarded as the regulated version of the heuristic expression $\hat{E}[\beta]$ since, in the classical theory, we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\lim_{\Lambda \to \infty} \int d^3 x E^a(x) f_a^{(P_\beta, \Lambda)}(x) \right) = E[\beta]. \quad (2.9)$$

The situation with the magnetic-flux operator, of course, is identical. Thus, given a framed loop α, we can introduce a strip Σ_α of height ϵ and consider a pillbox P_α it bounds. Then, $\hat{B}[f^{(P_\alpha, \Lambda)}]$ is a well-defined operator on the Fock space.

III. REMOVAL OF THE REGULATORS

Let us compute the commutator between the regularized flux operators using Eq. (2.4) and then, in the result, remove the regulators by taking appropriate limits. Removing the ultraviolet cutoff yields

$$\lim_{\Lambda \to \infty} \left[\hat{B}[f^{(P_\alpha, \Lambda)}], \ \hat{E}[f^{(P_\beta, \Lambda)}] \right] = \hbar \lim_{\Lambda \to \infty} \int_\Lambda d^3 k \ \epsilon^{abc} k_a \left[\tilde{f}^{(P_\alpha)}(\bar{k}) \right]^\star \tilde{f}^{(P_\beta)}(\bar{k}) = i\hbar \int d^3 x \ \epsilon^{abc} \left[\partial_a f_b^{(P_\alpha)}(\bar{x}) \right] f_c^{(P_\beta)}(\bar{x})$$

$$= i\hbar \int_0^\epsilon d\sigma \oint_{\alpha_\sigma} dl^a \int_0^\epsilon d\tau f_a^{(\beta_\tau)} = i\hbar \int_0^\epsilon d\sigma \int_0^\epsilon d\tau \{ B[\alpha_\sigma], \ E[\beta_\tau] \}, \quad (3.1)$$

where, in the first step, the subscript Λ denotes that the integration is carried out over the ball $|K| < \Lambda$. The final result is not surprising: the right side is just $i\hbar$ times the well-defined Poisson bracket between the classical ''thickened flux'' observables:

$$\lim_{\Lambda \to \infty} \left[\hat{B}[f^{(P_\alpha, \Lambda)}], \hat{E}[f^{(P_\beta, \Lambda)}] \right] = i\hbar \{ B[f^{(P_\alpha)}], E[f^{(P_\beta)}] \}. \quad (3.2)$$

Thus, as far as the commutator is concerned, the ultraviolet cutoff plays no essential role.

To remove the regulator ϵ, we have to compute

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left(\lim_{\Lambda \to \infty} \left[\hat{B}[f^{(P_\alpha, \Lambda)}], \hat{E}[f^{(P_\beta, \Lambda)}] \right] \right) = i\hbar \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left(\int_0^\epsilon d\sigma \int_0^\epsilon d\tau \{ B[\alpha_\sigma], E[\beta_\tau] \} \right) = i\hbar \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left(\int_0^\epsilon d\sigma \int_0^\epsilon d\tau \ \mathcal{P}(\alpha_\sigma, \ \beta_\tau) \right) \quad (3.3)$$

say. This calculation is more subtle. We will divide the discussion in four cases which bring out the role of framing in handling pathologies that arise when the loops intersect and overlap.

(i) The simplest case arises when the loops have no point in common. Then, for a sufficiently small ϵ, there is no intersection between any of the loops α_σ and β_τ. Hence, the Poisson bracket $\mathcal{P}(\alpha_\sigma, \beta_\tau)$ can be calculated exactly as in Sec. I. It is independent of σ and τ and equals $\mathcal{GL}(\alpha, \beta)$. Hence,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_0^\epsilon d\sigma \int_0^\epsilon d\tau \mathcal{P}(\alpha_\sigma, \beta_\tau) = \mathcal{GL}(\alpha, \beta). \quad (3.4)$$

Thus, in this case, the limiting procedure gives a precise meaning to the calculation of Sec. I: The uncertainty relation (1.9) holds in the sense that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left(\lim_{\Lambda \to \infty} \left[\hat{B}[f^{(P_\alpha, \Lambda)}], \hat{E}[f^{(P_\beta, \Lambda)}] \right] \right) = i\hbar \mathcal{GL}(\alpha, \beta). \quad (3.5)$$

(ii) Let us now consider the case when α and β intersect at a single point, say p. (Intersections at a finite number of points requires only a trivial extension of this case.) Now, the result depends on the thickening, or more precisely, on the framing at p initially chosen to carry out the thickening. Let $\dot{\alpha}^a$ and $\dot{\beta}^a$ denote the tangent vectors to the two loops at p. Consider the two-dimensional plane they span in the tangent space of p. Suppose that the frame vectors of the loops α and β lie on opposite sides of the plane. Then (for sufficiently small ϵ) among loops α_σ and β_τ, the only ones which intersect are α_0 and β_0, the original loops. Hence, $\mathcal{P}(\alpha_\sigma, \beta_\tau)$ is well defined in the σ, τ space except at the

single point, $(\sigma = 0, \tau = 0)$, which is of measure zero. Furthermore, at all other points, $\mathcal{P}(\alpha_\sigma, \beta_\tau)$ is independent of σ and τ. Its value is precisely the Gauss linking number $\mathcal{GL}(\alpha, \beta') = \mathcal{GL}(\alpha', \beta)$, where the primed loops are obtained by moving the unprimed ones slightly along the framing vectors. Thus, we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (\lim_{\Lambda \to \infty} [\hat{B}[f^{(P_\alpha, \Lambda)}], \hat{E}[f^{(P_\beta, \Lambda)}]]) = i\hbar \mathcal{GL}(\alpha, \beta')$$

$$= i\hbar \mathcal{GL}(\alpha', \beta), \tag{3.6}$$

which, in this case, is ($i\hbar$ times) the natural Gauss linking number associated with the *framed* loops.

(iii) Let us now consider the case where the two loops intersect at a single point p as before but where frame vectors at p (lie on the same side of the plane spanned by the two tangents and) are parallel. Then the two strips Σ_α and Σ_β intersect in a line rather than a single point. In this case the limit is more delicate. A loop α_σ on Σ_α intersects a loop β_τ on Σ_β if and only if $\sigma = \tau$. Thus, the calculation of Sec. I for computing $\mathcal{P}(\alpha_\sigma, \beta_\tau)$ goes through for the entire region of the parameter space $(\sigma, \tau) \in [0, \epsilon] \times [0, \epsilon]$ except for the diagonal. Again, since the diagonal is a set of measure zero, we can ignore it. However, now the integrand $\mathcal{P}(\alpha_\sigma, \beta_\tau)$ is no longer a constant on the entire parameter space. As a simple geometric picture reveals, it takes one value, $\mathcal{GL}(\alpha, \beta')$, on one side of the diagonal and another value, $\mathcal{GL}(\alpha', \beta)$, on the other, where as before the primed loops are obtained by displacing the unprimed loops slightly in the direction of framing. Hence, we now have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (\lim_{\Lambda \to \infty} [\hat{B}[f^{(P_\alpha, \Lambda)}], \hat{E}[f^{(P_\beta, \Lambda)}]]) = \frac{i\hbar}{2} [\mathcal{GL}(\alpha, \beta')$$

$$+ \mathcal{GL}(\alpha', \beta)]. \tag{3.7}$$

The right side is ($i\hbar$ times) the average of the two possible linking numbers one can obtain by displacing the loops α and β infinitesimally using the assigned framing, i.e., the "natural" (extension of the) Gauss linking number associated with the two given *framed* loops.

(iv) Finally, let us consider the commutator between fluxes of electric and magnetic associated with the *same* framed loop α. In this case the strips Σ_α and Σ_β as well as the "pillboxes" P_α and P_β coincide. Now, if $\sigma = \tau$ the loops α_σ and β_τ have no points in common. Hence, the integrand $\mathcal{P}(\alpha_\sigma, \beta_\tau)$ is well defined everywhere except along the diagonal. However, in this case (outside the diagonal which we can ignore) the value of the integrand is in fact constant, namely, the Gauss linking number $\mathcal{GL}(\alpha, \alpha')$, where α' is again obtained by displacing α slightly along the framing. Thus, in this case, we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (\lim_{\Lambda \to \infty} [\hat{B}[f^{(P_\alpha, \Lambda)}], \hat{E}[f^{(P_\alpha, \Lambda)}]]) = i\hbar \mathcal{GL}(\alpha, \alpha'). \tag{3.8}$$

The right side is precisely the *self-linking number* of the framed loop α [3]. Note in particular that if the framing is trivial (e.g., if all the frame vectors are parallel in the three-dimensional Euclidean space), the right side vanishes (even before taking the limit $\epsilon \to 0$). Thus, in this case, one can simultaneously measure the fluxes of electric and magnetic fields with *arbitrary* accuracy.

There are of course other cases one can analyze. For example, one can consider loops with isolated intersections, where, however, framing at the intersection points is not of the type considered in cases (ii) and (iii) above. Given the two framings, the calculation of the limit of the commutator is generally straightforward. Since the regularization is geometric and the final limiting procedure refers only to properties of ribbons obtained from framing, it is natural to interpret the result as the Gauss linking number of framed loops in those cases as well.

IV. DISCUSSION

In this work we have pointed out that there is a remarkable relation between the Gauss linking number, the simplest link invariant, and the Heisenberg uncertainty between fluxes of electric and magnetic fields, the basic observables of the quantum Maxwell theory. This uncertainty is intrinsic in that it arises because of the fundamental quantum fluctuations and persists even in the vacuum state.

The precise sense in which this relation holds is rather subtle especially if the loops in question intersect or overlap. In the classical theory, given any closed loop α, we can compute the fluxes $B[\alpha]$ and $E[\alpha]$ of magnetic and electric fields through any surface S_α bounded by the loop α. To obtain the corresponding quantum observables, however, we have to "thicken" the surfaces in question. A natural strategy is to frame the initial loop since a framing provides a canonical thickening. When this is done (and an ultraviolet cutoff is introduced) one obtains regulated flux operators which are well-defined on the Fock space. We can compute their commutators and *then* remove the regulators. The limit is the just ($i\hbar$ times) the Gauss link invariant of the framed loops. Even when the loops intersect or coincide [as in cases (ii), (iii), and (iv) considered in Sec. III], the limit of the commutator equals ($i\hbar$ times) the Gauss linking number of the *framed* loops.

For simplicity, we worked in Minkowski space-time. However, the entire discussion can be carried over without any difficulty to general stationary space-times (where the norm of the Killing field is bounded away from zero). In this case, one can use geodesics tangential to the framing to thicken the the loops and work in the canonical Fock representation selected by the Killing field [7,8]. In the nonstationary context, there is no canonical representation of the CCR. However, one can again construct the algebra of smeared flux operators and the basic results will hold on any Hilbert space on which this algebra can be represented. This is to be expected because the final results are topological and do not refer to the Minkowskian geometry used in the intermediate stages.

Finally, we wish to point out that the Gauss linking number also plays a key role in the expression of the measure which dictates the inner product on the photon Hilbert space

in the so-called self-dual representation (where states are appropriate functionals of self-dual connections) [9].

Note added. After this paper was submitted and posted on the LANL archives, we learned from Professor A. Widom that the formal relation (1.6) first appeared in [10]. Professor I. Birula pointed out that, in the analysis presented in this paper, it would be more appropriate to use the displacement vector D as the basic variable in place of the electric field E.

APPENDIX

1. From intersection to Gauss-linking number

In this subsection we shall show the analytic equivalence between the intersection number $I(\alpha, S_\beta)$ of Eq. (1.6) and the Gauss linking number of Eqs. (1.1) and (1.2). We start by rewriting the intersection number:

$$I(\alpha, S_\beta) = \{B[\alpha], E[\beta]\} = \int d^3x F^a[\alpha, \overline{x}] w_a[\beta, \overline{x}), \tag{A1}$$

where $F^a[\alpha, \overline{x})$ is the so-called *form factor* of the loop α,

$$F^a[\alpha, \overline{x}) = \oint_\alpha d\alpha^a \delta^3(\alpha, \overline{x}) \tag{A2}$$

and $w_a[\beta, \overline{x})$ is given by

$$w_a[\beta, \overline{x}) = \int_{S_\beta} dS_a \delta^3(\overline{x}, \overline{s}_\beta). \tag{A3}$$

Also, note that w_a is a potential for the form factor, since

$$F^a[\alpha, \overline{x}) = \epsilon^{abc} \partial_b w_c[\alpha, \overline{x}). \tag{A4}$$

There is an extra ''gauge freedom'' since w_a and $w'_a = w_a + \partial_a f$ give rise to the *same* form factor F^a.

So far, the intersection number does not depend on any background structure and is, therefore, topological in nature. Expression (1.1) is, however, written in terms of a Euclidean metric. Let us therefore introduce such a metric. Then,

$$\delta^3(\overline{x} - \overline{y}) = -\frac{1}{4\pi} \nabla^2_x \frac{1}{|\overline{x} - \overline{y}|} = -\frac{1}{4\pi} \partial_a \partial^a_x \left(\frac{1}{|\overline{x} - \overline{y}|}\right). \tag{A5}$$

We can now rewrite the intersection number as

$$I(\alpha, S_\beta) = -\frac{1}{4\pi} \int d^3x \int d^3y F^a[\alpha, \overline{x}) \, w_a[\beta, \overline{y})$$

$$\times \partial_d \partial^d_y \left(\frac{1}{|\overline{x} - \overline{y}|}\right)$$

$$= \frac{1}{4\pi} \int d^3x \int d^3y F^a[\alpha, \overline{x})$$

$$\times \partial_d w_a[\beta, \overline{y}) \, \partial^d_y \left(\frac{1}{|\overline{x} - \overline{y}|}\right), \tag{A6}$$

where in the second step we have integrated by parts. Now,

$$I(\alpha, S_\beta) = \frac{1}{4\pi} \int d^3x F^a[\alpha, \overline{x}) \int d^3y \left[2\partial_{[d} w_{a]}[\beta, \overline{y}) + (\partial_a w_d[\beta, \overline{y})) \partial^d_y \left(\frac{1}{|\overline{x} - \overline{y}|}\right) \right]. \tag{A7}$$

The last term can again be integrated by parts

$$\int d^3y (\partial_a w_d[\beta, \overline{y})) \partial^d_y \left(\frac{1}{|\overline{x} - \overline{y}|}\right) = -\int d^3y \frac{1}{|\overline{x} - \overline{y}|} (\partial_a \partial^d w_d[\beta, \overline{y})) = 0, \tag{A8}$$

where we have used the gauge freedom to select w_a such that $\partial^a w_a = 0$.

Finally, we have

$$I(\alpha, S_\beta) = \frac{1}{4\pi} \int d^3x \int d^3y \, F^a[\alpha, \overline{x}) F^b[\beta, \overline{y}) \epsilon_{abc} \partial^c_y \left(\frac{1}{|\overline{x} - \overline{y}|}\right)$$

$$= \frac{1}{4\pi} \int d^3x \int d^3y F^a[\alpha, \overline{x}) F^b[\beta, \overline{y}) \epsilon_{abc} \frac{x^c - y^c}{|\overline{x} - \overline{y}|^3}. \tag{A9}$$

Now, the definition of the form factor implies that: $\int d^3x F^a[\alpha, \overline{x}) f_a = \oint_\alpha d\alpha^a f_a$. Hence, we have the desired equality

$$I(\alpha, S_\beta) = \frac{1}{4\pi} \oint_\alpha ds \oint_\beta dt \, \epsilon_{abc} \dot{\alpha}^a(s) \dot{\beta}^b(t) \frac{\alpha^c(s) - \beta^c(t)}{|\alpha(s) - \beta(t)|^3}. \tag{A10}$$

2. Quantum operators

Recall from Sec. II that the smeared operators $\hat{E}[f^{(P_\beta)}]$ are well defined on the Fock space $\mathcal{F}$ if and only if the $\mathcal{H}$ norm of $V = ({}^T f_a^{(P_\beta)}, 0)$ is finite. In this subsection we will compute this norm and show that it has a logarithmic divergence thereby establishing the necessity of an ultraviolet cut-off.

Let us consider the simplest case. In cylindrical coordinates (ρ, ϕ, z), let the strip Σ_β defining the pillbox region P_β be circles of radius $\rho = R$ and $z = \tau$; thus $\vec{\alpha}_\tau(s) = (R, 2\pi s, -\epsilon/2 + \tau)$. We can take the S_τ surfaces to be parallel to the $z = \text{const}$ plane. In this geometry, the smearing covector field $f_a(x)$ is the "step function:" $f_a(x) = \nabla_a z$ if $\rho < R$ and $z \in [-\epsilon/2, \epsilon/2]$; $f_a(x) = 0$ otherwise. The Fourier transform will have a nonvanishing component only in the k_z direction:

$$\widetilde{f}_{k_z}(k) = \frac{1}{(2\pi)^{3/2}} \int_0^R \int_0^{2\pi} \int_{\epsilon/2}^{\epsilon/2} dz \, d\phi \rho d\rho e^{ik_\rho \rho \cos \phi} \, e^{ik_z z}. \tag{A11}$$

Using the identity

$$J_0(z) = \frac{1}{\pi} \int_0^\pi e^{iz \cos \theta} d\theta, \tag{A12}$$

and the recurrence formulas of Bessel functions we arrive at

$$\widetilde{f}_{k_z}(k) = \sqrt{\frac{2}{\pi}} R \frac{J_1(k_\rho R)}{k_\rho} \frac{\sin(k_z \epsilon)}{k_z}. \tag{A13}$$

The transverse part any field $\widetilde{f}_a^T(\overline{k})$ is the projection of the that field orthogonal to the radial vector k^a. Therefore,

$$|\widetilde{f}^T|^2 = |\widetilde{f}_{k_z}|^2 \frac{k_\rho^2}{k_\rho^2 + k_z^2}. \tag{A14}$$

The expression $\int_\Sigma d^3k \, |k| \, |\widetilde{f}_a^T(k)|^2$ now takes the form

$$\int_\Sigma d^3k \, |k| \, |\widetilde{f}_a(k)|^2 = 4R^2 \int_{-\infty}^\infty \int_0^\infty dk_z dk_\rho \frac{k_\rho^2}{[k_z^2 + k_\rho^2]^{1/2}}$$
$$\times \frac{J_1^2(\kappa R)}{k_\rho} \frac{\sin^2(k_z \epsilon)}{k_z^2}, \tag{A15}$$

where $|k| = [k_z^2 + k_\rho^2]^{1/2}$. It is now obvious that the integral diverges logarithmically since $J_1(x) \sim x^{-1/2}$ when $x \to \infty$.

[1] C. F. Gauss, *Werke* (Königliche Gesellschaft der Wissenschaften, Götingen, 1833), Vol. V, p. 605, note of Jan. 22.

[2] J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Oxford, Clarendon, England, 1873), pp. 40–42.

[3] R. Gambini and J. Pullin, *Loops, Knots, Gauge Theories and Quantum Gravity* (Cambridge University Press, Cambridge, England, 1996).

[4] A. Ashtekar, in *Mathematical Problems of Nonperturbative Quantum General Relativity*, Proceedings of the 1992 Les Houches School on Gravitation and Quantizations, edited by B. Julia and J. Zinn-Justin (Elsevier, Amsterdam, 1995).

[5] A Ashtekar and C. J. Isham, Phys. Lett. B **274**, 393 (1992); A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann, J. Math. Phys. **36**, 6456 (1995).

[6] I. Bialynicki-Birula and Z. Bialynicka-Birula, *Quantum Electrodynamics* (Pergamon, Oxford, England, 1975).

[7] A. Ashtekar and A. Magnon, Proc. R. Soc. London **A46**, 375 (1975).

[8] B. S. Kay, Commun. Math. Phys. **71**, 29 (1980).

[9] A. Ashtekar and A. Corichi, Class. Quantum Grav. **14**, A43 (1997).

[10] A. Widom and T. D. Clark, Phys. Lett. **90A**, 280 (1982).