Space for both no-boundary and tunneling quantum states of the Universe

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At the minisuperspace level of homogeneous models, the bare probability for a classical universe has a huge peak at small universes for the Hartle-Hawking ''no-boundary'' wave function, in contrast with the suppression at small universes for the ''tunneling'' wave function. If the probability distribution is cut off at the Planck density (say), this suggests that the former quantum state is inconsistent with our observations. For inhomogeneous models in which stochastic inflation can occur, it is known that the idea of including a volume factor in the observational probability distribution can lead to arbitrarily large universes being likely. Here, this idea is shown to be sufficient to save the Hartle-Hawking proposal even at the minisuperspace level (for suitable inflaton potentials) by giving it enough space to be consistent with observations. [S0556-2821(97)03716-8]

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Various remarkable features of our observed Universe (large size, low curvature, approximate isotropy and homogeneity, and the second law of thermodynamics) strongly suggest that its state is not random but highly special. Two leading proposals for special quantum states of the Universe are the Hartle-Hawking ''no-boundary'' proposal [1–7] and the ''tunneling'' proposal of Vilenkin, Linde, and others [8–13]. In toy models incorporating presumed approximations for these proposals, they both seem to lead to low-entropy early universes and so might explain the second law of thermodynamics. If a suitable inflaton is present in the effective low-energy dynamical theory, and if sufficient inflation occurs, both proposals seem to lead to a large universe with low curvature and approximate homogeneity and isotropy today. However, it has been controversial whether both proposals do indeed predict sufficient inflation.

In particular, in the minisuperspace approximation of using only Robertson-Walker geometries and a single homogeneous inflaton scalar field, the tree-level or zero-loop probability densities for the two proposals have the opposite signs in the exponent of the Euclidean action S_{E} [itself inversely proportional to the inflaton potential $V(\phi_0)$ at the nucleation value ϕ_0 of the inflaton field ϕ, when the nucleation is via a Euclidean four-dimensional hemisphere]:

$$P_{\mathrm{NB}}=e^{-2S_{\mathrm{E}}}=e^{\pi a_0^2}=\exp\left(\frac{3}{8V(\phi_0)}\right) \tag{1}$$

for the Hartle-Hawking no-boundary proposal, and

$$P_{\mathrm{T}}=e^{-2|S_{\mathrm{E}}|}=e^{+2S_{\mathrm{E}}}=e^{-\pi a_0^2}=\exp\left(-\frac{3}{8V(\phi_0)}\right) \tag{2}$$

for the tunneling proposal, where

$$a_0=a_0(\phi_0)=[8\pi V(\phi_0)/3]^{-1/2} \tag{3}$$

is the radius of the Euclidean four-dimensional hemisphere that is a solution of the Einstein equations with a stress-energy tensor $T_{\mu\nu}=-V(\phi_0)g_{\mu\nu}$, with Planck units being used throughout ($\hbar=c=G=1$).

During inflation the inflaton potential $V(\phi)$ decreases to some particular value where inflation ends, so if one presumes that one has a realization of the universe configuration in which the probability density is roughly maximized, then at this level the tunneling proposal seems to favor the maximum amount of inflation possible, whereas the no-boundary proposal seems to favor the minimum amount. Typically, the maximum amount of inflation is infinite (e.g., when the inflaton field in unbounded, or when the inflaton potential has a maximum), so in this regard the tunneling proposal seems consistent with observations, which are themselves apparently consistent with an arbitrarily large universe. However, the minimum amount of inflation, just sufficient for it to be called inflation, is very small [14], leading to a universe that would recollapse long before it got large enough to be consistent with our observations of the Universe.

One might conclude that the no-boundary proposal has thus been refuted by observations. However, before rejecting it by such a simple-minded argument, one should look for possible correction factors. For example, if one considers the total probability rather than just the probability density, it has been shown [15] that the no-boundary proposal might be as viable as the tunneling proposal. For, although the unnormalized probability density of the no-boundary proposal has an utterly enormous peak at tiny amounts of inflation, one can easily see that if the nucleating value of the inflaton field is unbounded above, and if loop [16] or other effects do not damp the probability density at these large values (where the zero-loop approximation has the probability density tending to a constant, assuming that the potential either diverges or tends to a constant in this limit), then the integral of the probability density over this infinite range of the value of the inflaton field gives a diverging unnormalized probability for sufficient inflation. This swamps the exponentially large but finite unnormalized probability for insufficient inflation, giving a prediction that the no-boundary proposal leads to sufficient inflation (actually, an arbitrarily large amount of inflation) with unit normalized probability.

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The usual objection to this argument for saving the no-boundary proposal is that for it to work for an inflaton potential that rises indefinitely (and not unnaturally slowly) for arbitrarily large values of the inflaton field, one must assume that the probability density is not unduly damped for large values of the potential that greatly exceed the Planck density, which is roughly the largest density where the zero-loop approximation might be expected to be rather reliable. It is sometimes said that one should put a cutoff on the probability distribution at the Planck density, in which case the integral over the probability density at lower values is grossly insufficient to overbalance the huge peak at minimal values for inflation in the no-boundary proposal.

The first part of this objection is indeed valid, that the total-probability solution to the apparent difficulty of the no-boundary proposal does seem to require suitable physics in the Planck regime, which we certainly do not yet understand. Thus this solution is only a possible solution, not definitely a viable one.

On the other hand, the claim that the probability distribution should be cut off at the Planck density is at least equally *ad hoc* and unjustified at present, so, just as one cannot yet be sure that the total-probability solution does work, neither can one be sure that it does not work. In other words, it is not definitely not viable.

Another correction is the use of the selection principle called the weak anthropic principle (perhaps somewhat misleadingly, since it is not meant to refer just to mankind) [17], that what we observe about the universe is conditioned on where we as observers exist within the Universe. Here, I will take the ''where'' to mean not only where we are spatially or temporally within the Universe, but also where we are within the quantum state of the Universe (e.g., where we are within the probability distribution for different universe configurations). This principle can save the no-boundary proposal when one considers inhomogeneous inflaton fluctuations and their back reaction on the metric, a process called ''stochastic inflation,'' which can lead to a ''self-reproducing universe'' with ''eternal inflation'' that occurs for an indefinitely long time and hence makes the volume of the Universe arbitrarily large [9,18–22]. The idea is based on the observation that typical observers or civilizations are more likely to occur in spaces of larger volume, other factors being equal [19,20,26,22,23].

The first application of the inhomogeneous stochastic evolution of the inflaton field to eternal inflation was made by Vilenkin [9] for new inflation, when he showed that one could get an arbitrarily large amount of inflation even within a bubble (which could encompass the entire Universe in Vilenkin's picture, as in that of Hawking and Moss [24]), and not just outside the bubbles as in previous analyses. After Linde discovered the scenario of ''chaotic inflation'' [25] (inflation from an inflaton potential without a maximum, or in a region where there is no maximum), which seems more realistic than new inflation, he discovered that it also leads to eternal stochastic inflation and what he called a ''self-regenerating universe'' [19,26]. (Linde was actually the first to use the phrase ''eternal inflation'' [19], and he has been the leading researcher of it since that time [19,26,22].)

Eternal stochastic inflation occurs when the rms ''stochastic'' change in the scalar field from the freezing out of inhomogeneous modes during one Hubble time $\Delta t = H^{-1}$,

$$\delta\phi = \frac{H}{2\pi} = \sqrt{\frac{2V}{3\pi}}, \tag{4}$$

is greater than the slow-roll change of the field during that same time:

$$\Delta\phi = -\dot{\phi}\Delta t = \frac{V'}{3H^2} = \frac{V'}{8\pi V}. \tag{5}$$

This occurs when the stochastic inflation condition

$$V'^2 < \frac{128\pi}{3} V^3 \tag{6}$$

is satisfied [26].

As a result of both the stochastic and slow-roll changes in the inflaton field, in some regions the field decreases, and in others it increases. Although the amount of comoving volume in which the field decreases is greater than the amount in which the field increases (because of the slow-roll change $\Delta\phi$, which is toward smaller fields), the back reaction of the inflaton potential on the metric causes the physical volume to increase more in the regions in which the field increases. Therefore, when one weights the regions by their physical volumes rather than by their comoving volumes, the dominant behavior is for the inflaton field to increase. This process allows the inflaton field to remain large for an arbitrarily long time, thereby leading to an arbitrarily large amount of inflation [9,19–22]. The results of eternal stochastic inflation are claimed to be independent of the initial conditions [22] (a claim which seems to me implicitly to assume some strong restriction on the allowed quantum states, perhaps analogous to a claim that in nongravitational quantum field theory in classical Minkowski spacetime, suitable states all asymptotically approach that of the vacuum in each local region as the excitations disperse with time).

Now, the main point of the present paper is the conceptual or pedagogical point that even at the crude level of using only the zero-loop homogeneous minisuperspace approximation, the anthropic-principle idea of weighting by the physical volume can save the Hartle-Hawking no-boundary proposal from appearing inconsistent with our observations of an expanding universe, at least for a wide range of inflaton potentials, even if the probability distribution is damped or cut off at the Planck density.

To use Vilenkin's language [23], suppose we start with what he calls the ''principle of mediocrity,'' that our civilization is average, ''randomly picked in the metauniverse.'' This leads to a probability distribution for various observed results that is proportional not only to what I shall call the ''bare'' probability distribution of universe configurations having these results, but also to the number of civilizations occurring within the corresponding configuration. (Note that what I am calling a universe configuration, Vilenkin calls a universe, and what I call the Universe, Vilenkin calls the metauniverse.)

The bare probability distribution is that given by the appropriate probability interpretation for the correct quantum state of the Universe that does not make reference to observers or civilizations. Then, with other factors being equal, one would expect the number of civilizations to be proportional to the volume of space at the time at which the civilizations occur. (I myself might prefer [27] to focus on conscious perceptions rather than civilizations, and someone else might prefer to focus on observers, but one would expect any of these to be proportional to the volume of space, other conditions being the same.) The volume of space at the time of the civilizations would itself be proportional to the volume of space at the end of inflation (assuming that inflation occurred, and that there is a fixed volume expansion factor between the end of inflation and the time of the civilizations, as there would be for approximately $k=0$ Friedmann-Robertson-Walker (FRW) parts of the Universe with the same density at the end of inflation, the same density at the time of the civilizations, or the same post-inflation age then, and the same equation of state at the intermediate densities).

Thus, one would expect that the probability distribution for observed results to be roughly proportional to the bare probability distribution for these results, multiplied by the volume of space at the end of inflation in the universe configuration that has these results. I shall call this product the ''observational'' probability distribution.

Here, I shall focus on the probability distribution, in the minisuperspace approximation, of one ''constant of motion,'' ϕ_0, of an approximate classical universe model that matches a universe configuration. (The probability distribution of ϕ_0 in stochastic inflation has been studied in [22].) In particular, I shall focus on universe configurations in which the effective constants of nature are the same as in our configuration and in which the large-scale configuration is approximately that of a classical $k=1$ Friedmann-Robertson-Walker universe which evolved from a period of single inflation starting with a moment of time symmetry at which the size of that universe was a minimum and the inflaton field had the homogeneous value ϕ_0.

According to the Hartle-Hawking no-boundary proposal in the minisuperspace approximation being used here, the zero-loop approximation for the bare probability distribution gives the unnormalized approximate probability density of Eq. (1) for ϕ_0, but with $P_{\rm NB}$ there replaced by $P_{\rm bare}$ here to emphasize that it is the approximate bare probability: namely,

$$P_{\rm bare}(\phi_0)d\phi_0 = e^{\pi a_0^2(\phi_0)}d\phi_0 = \exp\left(\frac{3}{8V(\phi_0)}\right)d\phi_0. \quad (7)$$

[By ''approximate,'' I mean, e.g., ignoring loop effects, Jacobians, or other prefactors of the exponential of twice the negative of the real part of the Euclidean action. I also mean that this Euclidean action is itself given here only in the approximation that variations in the potential and in the energy density are negligible during a Euclidean regime that is assumed to be a Friedmann-Robertson-Walker four-dimensional hemisphere of radius a_0 given by Eq. (3), bounded by a totally geodesic round equatorial three-sphere, that is, the three-space of a universe at its moment of nucleation out of the Euclidean regime and into the Lorentzian regime of inflation.]

As noted above, the bare probability rises sharply for smaller values of $V(\phi_0)$ when this is much smaller than unity (the Planck density). However, we need to multiply the bare probability by the volume of space at the end of inflation to get the observational probability.

The volume of three-space at the moment of nucleation, which is the beginning of a Lorentzian period of inflation that would have had a moment of time symmetry then if the Lorentzian evolution were analytically continued backward as well as forward in real Lorentzian time from this moment of nucleation, is

$$\mathcal{V}_0 = 2\pi^2 a_0^3 = \left[\frac{27\pi}{128V^3(\phi_0)}\right]^{1/2}. \quad (8)$$

Then we need to multiply by the volume expansion factor during inflation. Let us assume that the inflaton potential has $0<V'(\phi)\equiv dV/d\phi \ll V(\phi)\ll 1$ for $\phi_{\rm e}\ll\phi\ll\phi_{\rm Pl}$, with $\phi_{\rm e}$ being the value of ϕ where

$$V'(\phi_{\rm e}) = V(\phi_{\rm e}) \quad (9)$$

(which will be taken to be the point at which inflation ends, since this is roughly the point at which the slow-roll approximation breaks down), and with $\phi_{\rm Pl}$ being the value of the inflaton field that leads to a potential of the Planck density,

$$V(\phi_{\rm Pl}) = 1 \quad (10)$$

in the Planck units I am using. Such a potential leads to slow-roll inflation for ϕ in this range (which is assumed to include ϕ_0). For simplicity, use the slow-roll approximation all the way to the end of inflation at $\phi=\phi_{\rm e}$. Then, the slow-roll approximations to the FRW-inflaton equations lead to a volume expansion factor during inflation of roughly

$$\frac{\mathcal{V}_{\rm e}}{\mathcal{V}_0} = \left(\frac{a_{\rm e}}{a_0}\right)^3 = \exp\left(24\pi\int_{\phi_{\rm e}}^{\phi_0}\frac{V(\phi)d\phi}{V'(\phi)}\right)$$
$$= \exp\left(12\pi\int_{\phi_0}^{\phi_{\rm e}}\frac{a(\phi)d\phi}{a'(\phi)}\right). \quad (11)$$

Multiplying this volume expansion factor by the volume of space at the beginning of inflation and by the bare probability distribution gives the unnormalized observational probability distribution

$$P_{\rm obs}(\phi_0)d\phi_0 = \mathcal{V}_{\rm e}P_{\rm bare}(\phi_0)d\phi_0$$
$$= \mathcal{V}_0\left(\frac{a_{\rm e}}{a_0}\right)^3 P_{\rm bare}(\phi_0)d\phi_0$$
$$= \left[\frac{27\pi}{128V^3(\phi_0)}\right]^{1/2}\exp\left(24\pi\int_{\phi_{\rm e}}^{\phi_0}\frac{V(\phi)d\phi}{V'(\phi)}\right)$$
$$\times\exp\left(\frac{3}{8V(\phi_0)}\right)d\phi_0$$

$$
\begin{aligned}
&= 2\pi^2 a_0^3(\phi_0)\exp\left(12\pi\int_{\phi_0}^{\phi_e}\frac{a(\phi)d\phi}{a'(\phi)}\right) \\
&\quad\times\exp(\pi a_0^2(\phi_0)) \\
&= e^{p(\phi_0)}d\phi_0,
\end{aligned} \tag{12}
$$

where the logarithm of the unnormalized observational probability density is

$$
\begin{aligned}
p(\phi_0) &= 24\pi\int_{\phi_e}^{\phi_0}\frac{V(\phi_0)d\phi}{V'(\phi_0)}+\frac{3}{8V(\phi_0)}-\frac{3}{2}\ln V(\phi_0) \\
&\quad+\frac{1}{2}\ln\left(\frac{27\pi}{128}\right) \\
&= 12\pi\int_{\phi_0}^{\phi_e}\frac{a_0(\phi)d\phi}{a_0'(\phi)}+\pi a_0^2(\phi_0)+3\ln a_0(\phi_0) \\
&\quad+\ln(2\pi^2) \\
&\approx 24\pi\int_{\phi_e}^{\phi_0}\frac{V(\phi_0)d\phi}{V'(\phi_0)}+\frac{3}{8V(\phi_0)} \\
&= 12\pi\int_{\phi_0}^{\phi_e}\frac{a_0(\phi)d\phi}{a_0'(\phi)}+\pi a_0^2(\phi_0).
\end{aligned} \tag{13}
$$

I shall generally use one of these last two approximate expressions, dropping the logarithm of the volume at the beginning of inflation as a relatively unimportant term, and keeping only the logarithm of the volume expansion factor during inflation and minus twice the Euclidean action in the zero-loop approximation to the bare probability, since those two terms generally dominate when $V(\phi_0)\ll 1$, or, equivalently, when $a_0(\phi_0)\gg 1$ (nucleating universe much larger than the Planck size).

Now we need to put in the fact that the observational probability density is cut off for $\phi_0<\phi_m$, where ϕ_m is the minimum value of the nucleating inflaton field to lead to enough inflation for the existence of civilizations. If civilizations can only occur when the Universe is old enough for some nucleosynthesizing stars to have burned out and yet for other heat-producing stars to still be burning, then if the constants of nature take the values that they do in our universe configuration, one would need at least N_m (roughly 60 [26,28]) e-folds of inflation, and this makes ϕ_m the solution of the equation

$$
N_m = 8\pi\int_{\phi_e}^{\phi_m}\frac{V(\phi)d\phi}{V'(\phi)}. \tag{14}
$$

If, for the sake of argument, we also cut off the probability distribution for $\phi_0>\phi_M$ (e.g., with $\phi_M=\phi_{Pl}$), then unless ϕ_M is utterly enormous [which would require that $V(\phi)$ be extremely flat at large ϕ if $V(\phi_M)\ll 1$], then most of the total (integrated) observational probability for $\phi_m<\phi_0<\phi_M$ will occur near the maximum of the approximate unnormalized observational probability density, or of its logarithm $p(\phi_0)$. If the maximum is at or very near ϕ_m, then one would predict that a typical civilization would see such a universe recollapsing, which is contrary to our observations. But if the maximum is at a sufficiently higher value of ϕ_0, there would be enough inflation that the Universe today would be much larger than what we can see and hence very nearly spatially flat on a scale corresponding to its present age (under the assumption that it is approximately Friedmann-Robertson-Walker), thus agreeing with observations.

Now the analysis depends on the qualitative form of the inflaton potential $V(\phi)$. Because a period of slow-roll inflation has $V(\phi)$ monotonically decreasing with time while ϕ itself also changes monotonically, I shall assume that within the entire range $\phi_m<\phi<\phi_M$, $V'(\phi)$ is bounded away from zero and hence has a single sign (which without loss of generality is herein taken to be positive, since one could replace ϕ by $-\phi$ if necessary). I shall also assume that $V(\phi)$ is sufficiently smooth to be at least twice differentiable as a function of ϕ.

The first two derivatives of the approximate expression for the logarithm of the probability density, $p(\phi_0)$, then have the form

$$
p' \equiv \frac{dp}{d\phi_0} = \frac{24\pi V}{V'}-\frac{3V'}{8V^2} = 2\pi a_0\left(a_0'-\frac{6}{a_0'}\right), \tag{15}
$$

$$
\begin{aligned}
p'' \equiv \frac{d^2p}{d\phi_0^2} &= 24\pi\left(1-\frac{VV''}{V'^2}\right)-\frac{3}{8}\left(\frac{V''}{V^2}-2\frac{V'^2}{V^3}\right) \\
&= 2\pi\left[a_0\left(1+\frac{6}{a_0'^2}\right)a_0''+a_0'^2-6\right] \\
&= 2\pi a_0\left(1+\frac{6}{a_0'^2}\right)a_0''+\frac{a_0'}{a_0}p',
\end{aligned} \tag{16}
$$

where the primes on the V's and a_0's, just as on the p's, mean derivatives with respect to the independent variable ϕ_0, of which they are functions.

For a fairly general class of potentials $V(\phi_0)$, which I shall call class 1, $p(\phi_0)$ has no local maximum between ϕ_m and ϕ_M. For example, at a local extremum of p, one can see from Eqs. (15) and (16) that the second derivative of p with respect to ϕ_0 has the same sign as the second derivative of a_0 or of $V^{-1/2}$, so if these functions are concave upward (as they are, for example, if V is a positive power of ϕ_0 or is exponentially increasing with ϕ_0), then $p(\phi_0)$ has no local maximum.

Within this class 1, the maximum of $p(\phi_0)$ occurs at one of the end points, at ϕ_m or ϕ_M, and so it is simply a question of whether $p(\phi_m)$ or $p(\phi_M)$ is larger, assuming that the difference is greater than the generally less-important factors we have dropped. If $p(\phi_m)$ is the larger, then the observational probability for the Hartle-Hawking proposal (at least within the minisuperspace and zero-loop approximations) would be dominated by cases in which observers occurred almost entirely very late within a recollapsing universe, which is contrary to observations. But if $p(\phi_M)$ is the larger, then the observational probability would be dominated by cases in which a universe is expanding near the $k=0$ borderline when observers occur within it, which is consistent with our observations.

For the complementary class, which I shall call class 2, $p(\phi_0)$ does have one or more local maxima between ϕ_m and

ϕ_M, where $a_0' = -\sqrt{6}$ and $a_0'' < 0$. In this case one needs to compare the values of $p(\phi_0)$ at these local maxima as well as at the end points ϕ_m and ϕ_M. Still assuming that ignored factors are insignificant, and assuming that no local maximum occurs so close to ϕ_m that it would give insufficient inflation to be consistent with observations, the only case in which the Hartle-Hawking proposal would apparently (i.e., if our approximations are valid) give typical results inconsistent with our observations of the expansion of the Universe would be the case in which $p(\phi_m)$ is larger than the value of $p(\phi_0)$ at the other end point or any of the local maxima in between.

One *sufficient* (but not *necessary*) condition for $p(\phi_m)$ not to be the global maximum of $p(\phi_0)$ for $\phi_m < \phi_0 < \phi_M$, and hence for the Hartle-Hawking proposal to be consistent with our observations of the expansion of the Universe (assuming, as always, that other corrections factors are negligible), is that $p'(\phi_m) > 0$. In terms of the potential V and its derivative V', the sufficient condition for the Hartle-Hawking proposal to pass this test is

$$V'^2 < 64\pi V^3, \tag{17}$$

when evaluated at $\phi_0 = \phi_m$. Or, in terms of the derivative a_0' (still with respect to ϕ_0) of $a_0 = (8\pi V/3)^{-1/2}$ at $\phi_0 = \phi_m$, it is

$$-\sqrt{6} < a_0' < 0. \tag{18}$$

On the other hand, for $p(\phi_0)$ to be greater than $p(\phi_m)$ for some larger value of ϕ_0, one has the *necessary* condition that Eq. (17) or (18) be true when evaluated in at least some range of ϕ_0 greater than ϕ_m.

Consider the class 1 example of a power-law potential with positive (constant) exponent n,

$$V(\phi) = \frac{\lambda}{n}\phi^n, \tag{19}$$

where λ is a coupling constant for the field, which in the Planck units we are using is a number that shall be assumed to be small. (For example, for $n=2$, it is the square of the mass of the inflaton field ϕ, which is then a free massive field, minimally coupled to gravity.)

Again, making the approximation of keeping only the volume expansion factor and the zero-loop bare probability factor, this leads to the logarithm of the observational probability density varying roughly as

$$p(\phi_0) = p(\phi_m) + \frac{12\pi}{n}(\phi_0^2 - \phi_m^2) - \frac{3n}{8\lambda}(\phi_m^{-n} - \phi_0^{-n}). \tag{20}$$

For $\phi_0 \gg \phi_m$, one gets, roughly,

$$p(\phi_0) - p(\phi_m) = \frac{12\pi}{n}\phi_0^2 - \frac{3n}{8\lambda\phi_m^n}. \tag{21}$$

Since there are are no local maxima of $p(\phi_0)$ for this class 1 potential, it would allow the Hartle-Hawking proposal (in the minisuperspace approximation under consideration) to be consistent with observations if and only if $p(\phi_M) > p(\phi_m)$, or, roughly,

$$\phi_M^2 > \frac{n^2}{32\pi\lambda\phi_m^n}. \tag{22}$$

To express this condition as a condition on the coupling constant λ for a given exponent n, we need to write ϕ_M and ϕ_m in terms of these two parameters of the potential. If one takes $\phi_M = \phi_{Pl}$, the value where $V=1$, one gets

$$\phi_M = \phi_{Pl} = \left(\frac{n}{\lambda}\right)^{1/n}. \tag{23}$$

Furthermore, ϕ_m is determined by the need for N_m (roughly 60 [26,28]) e-folds of inflation before ϕ decreases to ϕ_e, where the slow-roll approximation ends and inflation ends. For the power-law potential given by Eq. (19), Eq. (10) gives

$$\phi_e = n, \tag{24}$$

and then for N_m e-folds of inflation, we need

$$N_m = \ln(a_e/a_0) = \frac{4\pi}{n}(\phi_m^2 - \phi_e^2), \tag{25}$$

so

$$\phi_m = \sqrt{\phi_e^2 + \frac{nN_m}{4\pi}} = \sqrt{n^2 + \frac{nN_m}{4\pi}} \approx \left(\frac{nN_m}{4\pi}\right)^{1/2} \sim 2\sqrt{n}, \tag{26}$$

where the first approximation is for $n \ll N_m/4\pi$, and the second [26] uses the fact that $N_m/4\pi \sim 4$. [The fact that this number is not very large suggests that even the first approximation is not very good, but in a more careful analysis [26] the slow-roll approximation breaks down at a ϕ_e that is actually something like $n/\sqrt{16\pi}$, which would make the first term inside the square root of Eq. (26) about 50 times smaller than the crude estimate above.]

Now, if one inserts ϕ_M from Eq. (23) and the last approximation for ϕ_m from Eq. (26) into the inequality (22), one finds that it becomes

$$\lambda^{(n-2)/n} > 2^{-n(n+5)/2}\pi^{-n/2}n^{-[(n-2)/2]^2}, \tag{27}$$

the condition for the Hartle-Hawking proposal (in the minisuperspace and zero-loop approximations) to have most observers see a nearly flat universe, consistent with our observations, rather than a recollapsing universe, for the power-law potential (19).

We have already assumed that $\phi_m \ll \phi_M$, which implies that the coupling constant must be small,

$$\lambda \ll 2^{-n}n^{(2-n)/n}, \tag{28}$$

so the inequality (27) is automatically true for $n \leq 2$ (recall that we are assuming that $n > 0$). In particular, for a free massive inflaton ($n=2$), with mass much less than the Planck mass, the Hartle-Hawking proposal (even at the minisuperspace level being considered) would be consistent with our observations in predicting that a typical observer would see a nearly flat universe on large scales.

However, for exponents $n>2$, the consistency of the Hartle-Hawking proposal is not automatic at this minisuperspace level. There is always a range of values of the coupling constant λ that is consistent with both inequalities (27) and (28), but for sufficiently large values of n, the allowed range for λ is at values too large to be consistent with the observed density fluctuations of the Universe (which one can calculate only by going outside the minisuperspace approximation, at least for the fluctuations).

For example, one may use the approximate expression Linde [26] gives (on p. 185) for the coupling constant of a power-law potential from the density fluctuations of the universe:

$$\lambda \sim 2.5\times 10^{-13} n^2 (4n)^{-n/2}. \tag{29}$$

Then, one can readily calculate, using the approximations above,

$$p(\phi_0)-p(\phi_{\mathrm{m}}) \approx \frac{12\pi}{n}\phi_{\mathrm{M}}^2 - \frac{3n}{8\lambda\phi_{\mathrm{m}}^n} \sim 48\pi\left(\frac{4\times 10^{12}}{n}\right)^{2/n} - \frac{3}{8}\left(\frac{4\times 10^{12}}{n}\right), \tag{30}$$

and this last expression is positive if and only if

$$n < 2.543\,007\,534\,8. \tag{31}$$

Of course, the crudeness of the approximations above does not justify the precision given here for the value of n at which the last expression of Eq. (30) vanishes; it merely suggests that for n greater than roughly 5/2, the minisuperspace and zero-loop approximations seem to make the Hartle-Hawking proposal be in conflict with observations if one cuts off the distribution of nucleating universes at the Planck density. Such a conflict does not occur for any power-law potential with a suitably small coupling constant λ if one goes beyond the minisuperspace approximation to eternal stochastic inflation [22].

Thus, we see that for a power-law potential, when one includes the volume factor in the distribution of observers (or of civilizations, or simply of conscious beings), the minisuperspace and zero-loop approximations for the Hartle-Hawking no-boundary proposal give results consistent with our observations of a universe expanding near the critical density, even when an *ad hoc* cutoff is imposed on the minisuperspace toy model at the Planck density, if the exponent of the power law is smaller than roughly 2.5. This includes the simple case of a free massive field but excludes the case of a quartic potential (though the latter is allowed in inhomogeneous models giving eternal stochastic inflation [22]).

Of course, there are other forms of the potential that would also make the Hartle-Hawking proposal consistent with observations by the approximations above. These include the cases in which the potential has a smooth maximum (below the Planck density) at some finite field value, and the case in which the potential continues to rise for arbitrarily large field values but asymptotically approaches a finite limit (also assumed to be below the Plank density so that no cutoff need be made).

In both of these cases, one can get an arbitrarily large volume by having the field nucleate arbitrarily near the maximum of the potential in the first case, or at an arbitrarily large field value in the second case. Then the slow-roll approximation will give an arbitrarily large amount of inflation, so the volume factor can become arbitrarily large and hence dominate over any large (but necessarily finite) peak in the bare probability distribution. (For this peak to be finite, I am assuming that the cosmological constant is zero or positive, so that the potential is bounded below by zero, and that inflation occurs only when the potential has a positive value, strictly bounded away from zero.)

It is interesting that the inequality (17) is the same (up to a small change in the coefficient that is not important at the level of the approximations being employed here) as the inequality (6) that occurs for some range within an inflaton potential allowing eternal inflation. Therefore, when the Hartle-Hawking approximation in the minisuperspace approximation is consistent with our observations of an expanding universe, then at the level of considering inhomogeneous fluctuations, it leads to stochastic inflation and a large expanding universe also consistent with such observations. However, the converse is not true, since potentials obeying the inequality (6) somewhere within the allowed range, and leading to stochastic inflation within this range, need not in the minisuperspace approximation necessarily have the peak in the observational probability distribution be at a nucleating inflaton value ϕ_0 that is higher than the minimum value ϕ_{m}. [The inequality (6) merely implies that the observational probability density is rising with ϕ_0 there, but not that it is necessarily higher there than it is at ϕ_{m}.]

In particular, for any power-law potential (19) with a small coupling constant λ, the inequality (6) is true for [26]

$$\phi > \left(\frac{3n^3}{128\pi\lambda}\right)^{1/(n+2)} \tag{32}$$

(which is a value below ϕ_{Pl} and hence within the allowed range if $\phi_{\mathrm{M}} \geqslant \phi_{\mathrm{Pl}}$). Thus, stochastic inflation can occur for any power-law potential (with a positive exponent and a sufficiently small coupling constant), giving a Hartle-Hawking state consistent with our observations of a large expanding universe, even though the minisuperspace approximation, for exponents n greater than about 2.5, would suggest that the state would give typical observations of a recollapsing universe, if one cut off the probability distribution at the Planck density $V(\phi_{\mathrm{Pl}})=1$.

In summary, when one includes the volume of space in converting from bare probabilities to observational probabilities, then the Hartle-Hawking no-boundary proposal for the quantum state of the Universe, as well as the tunneling proposal, both seem to have enough space to be consistent with our observations of a nearly flat expanding universe (rather than a contracting universe), at least for a wide class of inflaton potentials that obey the inequality (6) somewhere within the allowed range of the inflaton field, even if one cuts off the probability distribution for universes nucleating above the Planck density. This fact has been known to be the case for eternal stochastic inflation [21,22], and here the pedagogical point is made that the consistency of both proposals with the aforementioned observations occurs even within the minisuperspace approximation for a certain subset of the potentials that allow eternal inflation (e.g., for a massive scalar field, though not for a quartic potential, despite the fact that the latter does allow eternal stochastic inflation, and hence consistency with observations in the realistic case in which one allows inhomogeneous metrics).

On the other hand, there are inflaton potentials (such as the power-law potentials with exponents larger than roughly 2.5) that would make the Hartle-Hawking no-boundary proposal, at the zero-loop minisuperspace level with the probability distribution for nucleating universes cut off at the Planck density, appear to be inconsistent with our observations of an expanding universe, even though a calculation invoking eternal stochastic inflation would show that it is actually consistent.

One might ask whether it is the zero-loop or the minisuperspace approximation (or both) that in these cases makes such a large difference from eternal stochastic inflation. I would conjecture that although stochastic inflation requires one to go beyond the homogeneous minisuperspace approximation, it may not require one to go beyond the zero-loop approximation. Very preliminary evidence suggests to me that one should be able to get something like stochastic inflation simply by considering inhomogeneous complex classical solutions of the Einstein-matter field equations that obey the no-boundary conditions (when these conditions are expressed as analytic equations that may be satisfied by complex solutions). In the zero-loop approximation, the bare probabilities would then be given simply by the exponential of minus twice the real part of the Euclidean action (the imaginary part of the Lorentzian action), but then to get the observational probabilities one would need to multiply by the volume of space on hypersurfaces where the local conditions are suitable for observers or civilizations.

In conclusion, when one tests theories of quantum cosmology against our observations of a large universe, even within the zero-loop minisuperspace approximation, there is often space for both the no-boundary and the tunneling proposals.

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[1] S. W. Hawking, in *Astrophysical Cosmology: Proceedings of the Study Week on Cosmology and Fundamental Physics*, edited by H. A. Brück, G. V. Coyne, and M. S. Longair (Pontificiae Academiae Scientiarum Scripta Varia, Vatican, 1982).

[2] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[3] S. W. Hawking, Nucl. Phys. **B239**, 257 (1984); in *Relativity, Groups and Topology II*, edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984); Phys. Rev. D **32**, 2489 (1985); in *Field Theory, Quantum Gravity and Strings*, Proceedings of the Seminar Series, Meudon and Paris, France, 1984-1985, edited by H. J. De Vega and N. Sanchez, Lecture Notes in Physics Vol. 246 (Springer, New York, 1986), p. 1; Phys. Scr. **T15**, 151 (1987).

[4] J. J. Halliwell and S. W. Hawking, Phys. Rev. D **31**, 1777 (1985).

[5] D. N. Page, Phys. Rev. D **32**, 2496 (1985); **34**, 2267 (1986); in *Quantum Concepts in Space and Time*, edited by R. Penrose and C. J. Isham (Clarendon, Oxford, 1986), p. 274; in *Gravitation: A Banff Summer Institute*, edited by R. B. Mann and P. Wesson (World Scientific, Singapore, 1991), p. 135.

[6] J. J. Halliwell, in *Quantum Cosmology and Baby Universes*, edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991), p. 159; Sci. Am. (Int. Ed.) **265** (6), 28 (1991).

[7] S. W. Hawking, R. Laflamme, and G. W. Lyons, Phys. Rev. D **47**, 5342 (1993).

[8] A. Vilenkin, Phys. Lett. **117B**, 25 (1982).

[9] A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).

[10] A. D. Linde, Zh. Éksp. Teor. Fiz. **87**, 369 (1984) [Sov. Phys. JETP **60**, 211 (1984)]; Lett. Nuovo Cimento **39**, 401 (1984).

[11] Ya. B. Zel'dovich and A. A. Starobinsky, Pis'ma Astron. Zh. **10**, 323 (1984) [Sov. Astron. Lett. **10**, 135 (1984)].

[12] V. A. Rubakov, Phys. Lett. **148B**, 280 (1984).

[13] A. Vilenkin, Phys. Rev. D **30**, 509 (1984); Nucl. Phys. **B252**, 141 (1985); Phys. Rev. D **33**, 3560 (1986); **37**, 888 (1988); T. Vachaspati and A. Vilenkin, *ibid.* **37**, 898 (1988); A. Vilenkin, *ibid.* **39**, 1116 (1989); **50**, 2581 (1994).

[14] L. P. Grishchuk and L. V. Rozhanskii, Phys. Lett. B **208**, 369 (1988).

[15] S. W. Hawking and D. N. Page, Nucl. Phys. **B264**, 185 (1986); D. N. Page, Int. J. Theor. Phys. **25**, 545 (1986).

[16] A. O. Barvinskii and A. Yu. Kamenshchik, Class. Quantum Grav. **7**, L181 (1990); A. O. Barvinskii, A. Yu. Kamenshchik, I. P. Karmazin, and I. V. Mishakov, *ibid.* **9**, L27 (1992); A. O. Barvinskii, A. Yu. Kamenshchik, and I. P. Karmazin, Ann. Phys. (N.Y.) **219**, 201 (1992); A. O. Barvinskii and A. Yu. Kamenshchik, Phys. Lett. B **332**, 270 (1994).

[17] B. Carter, in *I.A.U. Symposium*, edited by M. S. Longair (Reidel, Dordrecht, 1974), Vol. 63; Philos. Trans. R. Soc. London, Ser. A **310**, 347 (1983); B. J. Carr and M. J. Rees, Nature (London) **278**, 605 (1979); J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle* (Clarendon, Oxford, 1986).

[18] A. A. Starobinskii, in *Field Theory, Quantum Gravity and Strings* [3], p. 107.

[19] A. D. Linde, Mod. Phys. Lett. A **1**, 81 (1986); Phys. Lett. B **175**, 395 (1986); Phys. Scr. **T15**, 169 (1987); Phys. Today **40**, 61 (1987); in *300 Years of Gravitation*, edited by S. W. Hawk-

ing and W. Israel (Cambridge University Press, Cambridge, England, 1987), p. 604.
[20] M. Aryal and A. Vilenkin, Phys. Lett. B **199**, 351 (1987).
[21] M. Mijić, Phys. Rev. D **42**, 2469 (1990); Int. J. Mod. Phys. A **6**, 2685 (1991).
[22] A. D. Linde, D. A. Linde, and A. Mezhlumian, Phys. Rev. D **49**, 1783 (1994).
[23] A. Vilenkin, Phys. Rev. Lett. **74**, 846 (1995); Phys. Rev. D **52**, 3365 (1995); in *String Gravity and Physics at the Planck Energy Scale*, Proceedings of the International School of Astrophysics ''D. Chalonge,'' 4th Course, Erice, Sicily, 1995, edited by N. Sanchez and A. Zichichi, NATO ASI Series, Series C: Mathematical and Physical Sciences, Vol. 476 (Kluwer, Dordrecht, 1996); talk given at the 1st RESCEU International Symposium on the Cosmological Constant and the Evolution of the Universe, Tokyo, Japan, 1995 (unpublished), gr-qc/9512031.
[24] S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).
[25] A. D. Linde, JETP Lett. **38**, 149 (1983); Phys. Lett. **129B**, 177 (1983).
[26] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland, 1990).
[27] D. N. Page, ''Sensible Quantum Mechanics: Are Only Perceptions Probabilistic?'' University of Alberta Report Alberta-Thy-05-95, 1995, quant-ph/9506010; Int. J. Mod. Phys. D **5**, 583 (1996).
[28] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).