PHYSICAL REVIEW D VOLUME 56, NUMBER 4 15 AUGUST 1997

Age of the universe: Influence of the inhomogeneities on the global expansion factor

Heinz Russ,[1] Michael H. Soffel,[2] Masumi Kasai,[3,4] and Gerhard Börner[4]
[1]*Institut für Astronomie und Astrophysik, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany*
[2]*Institut für Planetare Geodäsie, Universität Dresden, Mommsenstrasse 13, D-01082 Dresden, Germany*
[3]*Department of Physics, Hirosaki University, 3 Bunkyo-cho, Hirosaki, 036 Japan*
[4]*Max-Planck Institut für Astrophysik, Karl-Schwarzschild Strasse 1, D-85740 Garching, Germany*
(Received 23 December 1996)

For the first time we calculate quantitatively the influence of inhomogeneities on the global expansion factor by averaging the Friedmann equation. In the framework of the relativistic second-order Zel'dovich-approximation scheme for irrotational dust we use observational results in the form of the normalization constant fixed by the Cosmic Background Explorer results and we check different power spectra, namely, for adiabatic cold dark matter (CDM), isocurvature CDM, hot dark matter, warm dark matter, strings, and textures. We find that the influence of the inhomogeneities on the global expansion factor is very small. So the error in determining the age of the universe using the Hubble constant in the usual way is negligible. This does not imply that the effect is negligible for local astronomical measurements of the Hubble constant. Locally the determination of the redshift-distance relation can be strongly influenced by the peculiar velocity fields due to inhomogeneities. Our calculation does not consider such effects, but is constrained to comparing globally homogeneous and averaged inhomogeneous matter distributions. In addition we relate our work to previous treatments. [S0556-2821(97)00816-3]

PACS number(s): 98.80.Hw, 04.25.Nx, 98.62.Ai

I. INTRODUCTION

Lower limits of the age of the universe are observationally determined in many ways. Measurements of isotopic ratios of radioactive nuclei determine the ages of meteorites by 4.5 Gyr [1,2]. Studies of the cooling of white dwarfs [3] lead to an age of our galaxy of at least 10 Gyr. Galactic ages could be determined to lie in a range of 12.6 to 19.6 Gyr by measuring the abundance ratio of different isotopes of elements [1,4]. Measurements of the luminosity of stars located at the turn-off point of the Hertzsprung Rusell diagram determine the ages of globular clusters to lie in a range of 12 to 18 Gyr [5–7]. On the other hand, upper limits of the age of the universe can be derived using cosmological models. Using a standard Friedmann-Lemaître-Robertson-Walker (FLRW) model with vanishing cosmological constant ($\Lambda=0$), the inverse of the Hubble parameter as measured today, H_0^{-1}, provides an upper limit of the age of the universe (the index 0 indicates values at the present time). Recent measurements of cepheid variables in the virgo cluster [8,9] lead to a Hubble parameter of about $H_0=80$ km/(s Mpc) (upper limit $H_0^{-1}\approx 12.2$ Gyr), leading to an age of 8.15 Gyr for a flat universe with vanishing cosmological constant. This is far below the observational lower limits cited above. There are several ways out of this dilemma.

The first is believing in a lower value of the Hubble parameter. There are two reasons for that: first there exist other observational results [10]; secondly the redshift-distance relation can be influenced by the inhomogeneities, which will influence the Hubble constant.

The second is to believe that the high value of the Hubble constant comes from the fact that we live in an underdensed region of the universe, whereas on average over the whole universe the expansion parameter is smaller [11].

The third is believing in some nonvanishing cosmological constant [8,12], where under some circumstances no upper limit can be derived ($t_0 > H_0^{-1}$), and the age of the universe can be about 30 Gyr or even higher [12].

In this paper we want to investigate still another way. In the usual calculation of its age, the universe is assumed to be exactly isotropic and homogeneous. This might be a good approximation due to the high isotropy of the microwave-background radiation, so that the FLRW description might be valid in some averaged sense. On the other hand, the inhomogeneities are large, even on large scales, for example, at a scale of ~ 10 Mpc the density constrast $\delta=\delta\rho/\rho$ might reach unity. In the early universe deviations from homogeneity and isotropy were small, but after the deviations became nonlinear, these inhomogeneities could influence the global expansion factor. This effect is called by us the backreactions of the inhomogeneities. As a result of these backreactions the value of the Hubble parameter cannot be taken in the usual way for a determination of the age of the universe. We will calculate quantitatively the effect of the backreactions and we will see how large the deviations from the usual age determinations are. This paper is organized as follows. In Sec. II we present the basic equations and the averaged Friedmann equation in a general form. In Sec. III we use the results of the relativistic Zel'dovich-type approximation to second order (Russ *et al.* [13]) based on the tetrad formalism in cosmology (Kasai [14]) and calculate the backreactions using different power spectra and the normalization constant fixed by the Cosmic Background Explorer (COBE) results. This paper was influenced by the pioneering paper from Bildhauer and Futamase [15]; we compare our results with theirs and others (Buchert and Ehlers [16,17], Futamase [18,19]) in Sec. IV. Section V is devoted to conclusions.

II. THE FRIEDMANN EQUATION IN AN INHOMOGENEOUS UNIVERSE

In this section, we summarize a general relativistic treatment to describe the nonlinear evolution of an inhomogeneous irrotational[1] universe [14,20,21]. The models we consider contain irrotational dust with energy density ρ and four-velocity u^μ. We will neglect the curvature constant k and a possible cosmological constant Λ. Neglecting the fluid pressure and the vorticity is a reasonable assumption in a cosmological context. In comoving synchronous coordinates, the line element can be written in the form (indices $\mu,\nu,\ldots,$ run from 0 to 3 and indices $i,j,\ldots,$ run from 1 to 3)

$$ds^2=-c^2dt^2+g_{ij}dx^idx^j \tag{2.1}$$

and $u^\mu=(c,0,0,0)$. Then, Einstein's field equations read

$$\frac{1}{2}[{}^3R^i{}_ic^2+(u^i{}_{;i})^2-u^i{}_{;j}u^j{}_{;i}]=\frac{8\pi G}{c^2}\rho, \tag{2.2}$$

$$u^i{}_{;j\|i}-u^i{}_{;i\|j}=0, \tag{2.3}$$

$$\dot{u}^i{}_{;j}+u^k{}_{;k}u^i{}_{;j}+{}^3R^i{}_jc^2=\frac{4\pi G}{c^2}\rho\delta^i{}_j, \tag{2.4}$$

where ${}^3R^i{}_j$ is the three-dimensional Ricci-tensor,

$$u^i{}_{;j}=\frac{1}{2}g^{ik}\dot{g}_{jk} \tag{2.5}$$

is the extrinsic curvature, $\|$ denotes the covariant derivative with respect to the three metric g_{ij}, and an overdot denotes $\partial/\partial t$. We introduce the conformal factor $a(t)$ as

$$g_{ij}=a(t)^2\gamma_{ij} \tag{2.6}$$

and we introduce the quantity $V^i{}_j$, describing the deviation from a homogeneous and isotropic expansion

$$V^i{}_j\equiv u^i{}_{;j}-\frac{\dot{a}(t)}{a(t)}\delta^i{}_j=\frac{1}{2}\gamma^{ik}\dot{\gamma}_{jk}. \tag{2.7}$$

Then we can write two of the Einstein equations in the following form, which we call the Friedmann equations:

$$\frac{\dot{a}(t)^2}{a(t)^2}=\frac{8\pi G}{3c^2}\rho-\frac{c^2}{6}{}^3R-\frac{1}{6}[(V^k{}_k)^2-V^l{}_kV^k{}_l]-\frac{2\dot{a}(t)}{3a(t)}V^k{}_k \tag{2.8}$$

and

$$\frac{\ddot{a}(t)}{a(t)}=-\frac{4\pi G}{3c^2}\rho-\frac{1}{3}V^l{}_kV^k{}_l-\frac{1}{3}\dot{V}^k{}_k-\frac{2\dot{a}(t)}{3a(t)}V^k{}_k. \tag{2.9}$$

We introduce the averaging procedure [22]

$$\langle A\rangle=\frac{1}{V}\int_V A\sqrt{g}d^3x, \tag{2.10}$$

[1]We do not consider here the effect of rotation, which might turn the effect of the inhomogeneities in the opposite direction, i.e., tending to increase the age of the universe.

where $g\equiv\det g_{ij}$. V is the comoving volume of a compact domain $\mathcal{D}(t)$ of the fluid [16,17]. V should be sufficiently large so that we can assume periodic boundary conditions. The scale factor $a_D(t)$ describes the expansion of this volume. Therefore the expansion rate of the universe is defined by

$$3\frac{\dot{a}_D(t)}{a_D(t)}\equiv\frac{\dot{V}}{V}=3\frac{\dot{a}(t)}{a(t)}+\langle V^k{}_k\rangle. \tag{2.11}$$

We then average the Friedmann equations, apply the commutation rule, and neglect higher order terms (see Appendix A) to get the averaged Friedmann equations in the form

$$\frac{\dot{a}_D(t)^2}{a_D(t)^2}=\frac{8\pi G}{3c^2}\langle\rho\rangle-\frac{c^2}{6}\langle{}^3R\rangle-\frac{1}{6}\langle(V^k{}_k)^2-V^l{}_kV^k{}_l\rangle \tag{2.12}$$

and

$$\frac{\ddot{a}_D(t)}{a_D(t)}=-\frac{4\pi G}{3c^2}\langle\rho\rangle+\frac{1}{3}\langle(V^k{}_k)^2-V^l{}_kV^k{}_l\rangle. \tag{2.13}$$

These are the general equations for the evolution of the expansion factor of an inhomogeneous universe. They do not depend on a specific model of the universe. Equation (2.13) has already been discovered by Buchert and Ehlers [16,17] (see Sec. IV).

III. MODEL OF THE INHOMOGENOUS UNIVERSE

We use the solution of the relativistic Zel'dovich approximation to second order (Russ *et al.* [13]) based on the tetrad formalism (Kasai [14]) to get

$$\frac{\dot{a}_D(t)^2}{a_D(t)^2}=\frac{8\pi G}{3c^2}\rho_b(t)-\frac{1}{t_{\rm in}^2}\int_V\frac{100}{243a_D^2c^2t_{\rm in}^2}\Psi^{,k}\Psi_{,k}d^3x. \tag{3.1}$$

The function $\Psi(\mathbf{x})$ is related to the initial displacements of the particles, to first order it represents the potential of the density fluctuations $-\Psi^{,k}{}_{,k}=\delta(\mathbf{x},t_{\rm in})$. Here we put $a_D(t_{\rm in})=1$ and $V(t_{\rm in})=1$. For a justification of Eq. (3.1) see Appendix B. We use the background relationships

$$\rho_b(t_{\rm in})=\frac{c^2}{6\pi Gt_{\rm in}^2}$$

and

$$t_{\rm in}=\frac{2}{3H_0(1+z_{\rm in})^{3/2}}, \tag{3.2}$$

where H_0 is the present value for the Hubble parameter

$$H_0^2\equiv\frac{\dot{a}_D^2(t_0)}{a_D^2(t_0)}=\frac{8\pi G}{3c^2}[\rho_b(t_0)+\rho_{\rm corr}(t_0)]$$

$$\equiv\frac{8\pi G}{3c^2}\rho_b(t_0)[1+\delta_{\rm corr}(t_0)]. \tag{3.3}$$

We then get

$$\frac{\dot{a}_D^2(t)}{a_D^2(t)}=\frac{H_0^2}{1+\delta_{\rm corr}(t_0)}\left(\frac{a_D^3(t_0)}{a_D^3(t)}+\frac{a_D^2(t_0)}{a_D^2(t)}\,\delta_{\rm corr}(t_0)\right) \tag{3.4}$$

with

$$\delta_{\rm corr}(t_0)=-\frac{25}{108}10^{-6}h_0^2(1+z_{\rm in})^4\ {\rm Mpc}^{-2}\int_V \Psi^{,m}\Psi_{,m}d^3x \tag{3.5}$$

and $H_0=100h_0$ km/(s Mpc). We integrate the Friedmann equation (3.4) to get the age of the universe:

$$t_0=\frac{2}{3}\,\frac{\sqrt{1+\delta_{\rm corr}(t_0)}}{H_0}\left(\frac{3}{2}\int_0^1\frac{\sqrt{x}dx}{\sqrt{1+x\,\delta_{\rm corr}(t_0)}}\right). \tag{3.6}$$

Since $\delta_{\rm corr}(t_0)$ has a negative sign the age of the inhomogeneous universe is less than the age of a corresponding homogeneous one calculated with a given Hubble constant. Here, we want to estimate these differences quantitatively. The flat background allows a Fourier decomposition $\delta(\mathbf{x},t_{\rm in})=\Sigma_{\mathbf{k}}\delta_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$, so we get

$$\int_V \Psi^{,m}\Psi_{,m}d^3x=\sum_{\mathbf{k}}\frac{1}{\mathbf{k}^2}|\delta_{\mathbf{k}}|^2, \tag{3.7}$$

where $|\delta_{\mathbf{k}}|^2$ is the power spectrum of density fluctuations

$$|\delta_{\mathbf{k}}|^2=P(k). \tag{3.8}$$

What we need is to know the power spectrum at initial time $t_{\rm in}$, where the fluctuations are still linear and just start to move into the nonlinear regime. We choose $z_{\rm in}=8$ [23]. The power spectrum evolves according to [24]

$$P(k,t_{\rm in})=\frac{1}{(1+z_{\rm in})^2}P(k,t_{\rm pr})T^2(k), \tag{3.9}$$

where $T(k)$ is a transfer function and the primordial power spectrum is assumed to be

$$P(k,t_{\rm pr})=Ak^n. \tag{3.10}$$

The normalization constant is fixed by the COBE results

$$A_{\rm COBE}=\left(\frac{96\pi^2}{5}\right)\Omega_0^{-1.54}c^4H_0^{-4}\left(\frac{Q_{\rm rms}}{T_{\gamma_0}}\right)^2. \tag{3.11}$$

The COBE rms fluctuation is given to be $Q_{\rm rms}=9.3\ \mu$K [25], if we take $n=1$ (scale invariant primordial power spectrum), which is the most reasonable value [26]. The temperature of the microwave background radiation is $T_{\gamma_0}=2.73$ K. The volume will be taken large enough so that we can convert the sum into an integral

$$\sum_{\mathbf{k}}\to\frac{1}{(2\pi)^3}\int d^3k \tag{3.12}$$

to get

$$\delta_{\rm corr}(t_0)=-\frac{25}{216\pi^2}10^{-6}h_0^2(1+z_{\rm in})^2A\int kT^2(k)dk. \tag{3.13}$$

In the following we will assume $h_0=0.8$ and $\Omega_{\rm tot}=1$. For the normalization constant one finds $A\approx 4.35\times 10^5$ Mpc^4. In the following we will calculate the age of the universe using different transfer functions $T(k)$ and power spectra $P(k)$, where $AkT(k)^2=P(k,t_0)$.

A. Adiabatic cold dark matter fluctuations

The transfer function for adiabatic cold dark matter (CDM) fluctuations is given by [27]

$$T_{\rm CDM,\,ad}(k)=\frac{\ln(1+2.34q)}{2.34q}[1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4]^{-1/4}, \tag{3.14}$$

where

$$q\equiv\frac{k\theta^{1/2}}{\Omega_{\rm CDM}h_0^2\ {\rm Mpc}^{-1}}. \tag{3.15}$$

Here $\theta=\rho_{\rm er}/(1.68\rho_\gamma)$ is a measure of the ratio of the energy density in relativistic particles (photons plus neutrinos) to that contained in photons. We will set $\theta=1$, corresponding to three flavors of relativistic neutrinos plus the photons, and we will take $\Omega_{\rm CDM}=1$. Varying $\Omega_{\rm CDM}$ will change the result only slightly. The result is $\delta_{\rm corr}(t_0)=-2.50\times 10^{-3}$ and the age of the universe becomes

$$t_0\approx 0.9995\times\frac{2}{3H_0}. \tag{3.16}$$

B. Isocurvature cold dark matter fluctuations

The transfer function for isocurvature CDM fluctuations reads [27]

$$T_{\rm CDM,\,isoc}(k)=\left[1+\frac{(40q)^2}{1+215q+(16q)^2(1+0.5q)^{-1}}+(5.6q)^{8/5}\right]^{-5/4}, \tag{3.17}$$

where

$$q\equiv\frac{k}{\Omega_{\rm CDM}h_0^2\ {\rm Mpc}^{-1}}. \tag{3.18}$$

We again will take $\Omega_{\rm CDM}=1$. The result is $\delta_{\rm corr}(t_0)=-5.63\times 10^{-4}$ and the age of the universe reads

$$t_0\approx 0.99989\times\frac{2}{3H_0}. \tag{3.19}$$

C. Hot dark matter + adiabatic CDM

If we assume only one species of massive neutrinos, adiabatic fluctuations give [27]

$$T_{\nu,\mathrm{ad}}(k)=\exp[-0.16(kR_{f\nu})-(kR_{f\nu})^2/2]\times[1+1.6q+(4.0q)^{3/2}+(0.92q)^2]^{-1}, \tag{3.20}$$

where

$$q\equiv\frac{k}{\Omega_\nu h_0^2\ \mathrm{Mpc}^{-1}}$$

and

$$R_{f\nu}=2.6(\Omega_\nu h_0^2)^{-1}\ \mathrm{Mpc}. \tag{3.21}$$

We will take $\Omega_\nu=0.3$ and in the adiabatic CDM transfer function we set $\Omega_{\mathrm{CDM}}=0.7$. The total power spectrum is then given by $P(k)=[0.3\sqrt{P_\nu(k)}+0.7\sqrt{P_{\mathrm{CDM}}(k)}]^2$. The result is $\delta_{\mathrm{corr}}(t_0)=-6.95\times10^{-4}$ and the age of the universe becomes

$$t_0\approx0.99986\times\frac{2}{3H_0}. \tag{3.22}$$

D. Warm dark matter fluctuations

Adiabatic fluctuations of warm dark matter give [27]

$$T_{\mathrm{warm,\,ad}}(k)\approx\exp\left[-\frac{kR_{\mathrm{fw}}}{2}-\frac{(kR_{\mathrm{fw}})^2}{2}\right]T_{\mathrm{CDM,\,ad}}(k), \tag{3.23}$$

where

$$T_{\mathrm{CDM,\,ad}}(k)=[1+1.7q+(4.3q)^{3/2}+q^2]^{-1} \tag{3.24}$$

and

$$q\equiv\frac{k}{\Omega_{\mathrm{CDM}}h_0^2\ \mathrm{Mpc}^{-1}}$$

and

$$R_{\mathrm{fw}}=0.2\left(\frac{g_{\mathrm{CDM,\,dec}}}{100}\right)^{-4/3}(\Omega_{\mathrm{CDM}}h_0^2)^{-1}\ \mathrm{Mpc}. \tag{3.25}$$

Here $g_{\mathrm{CDM,\,dec}}$ is the effective number of particle degrees of freedom when the CDM particles decoupled, values range from 60–300, we will set $g_{\mathrm{CDM,\,dec}}=300$ and $\Omega_{\mathrm{CDM}}=1$. The result is $\delta_{\mathrm{corr}}(t_0)=-2.90\times10^{-3}$ corresponding to an age of the universe of

$$t_0\approx0.99942\times\frac{2}{3H_0}. \tag{3.26}$$

E. String and texture models

The power spectrum of a cosmic string network evolving in a flat universe dominated by CDM is given by [28,29]

$$P(k)=\frac{Ak}{[1+\alpha_2k+(\alpha_3k)^2+(\alpha_4k)^3][1+(\alpha_5k)^2]^2}, \tag{3.27}$$

where $\alpha_2=7.57h_0^{-2}$, $\alpha_3=5.89h_0^{-2}$, $\alpha_4=1.93h_0^{-2}$, and $\alpha_5=0.000357h_0^{-2}$. This power spectum assumes that the string network is characterized by a scaling solution and that the power is dominated by the coherent motions of loop strings; perturbations induced by string loops are neglected [28].

The power spectrum of a CDM universe with perturbations seeded by textures is given by [30,31]

$$P(k)=\frac{Ak}{\{1+[\alpha k+(\beta k)^{3/2}+(\gamma k)^2]^\nu\}^{2/\nu}}, \tag{3.28}$$

with $\nu=1.2$, $\alpha=19.4h_0^{-2}$, $\beta=6.6h_0^{-2}$, and $\gamma=3.0h_0^{-2}$, where we still set $\Omega_0=1$. Although the non-Gaussian nature of the string and texture models means that the power spectrum does not provide a full description of the density field even in the linear regime, the power spectrum is still a well-defined quantity, and it is meaningful to compare it to observations [28] and give excellent fits to it [31]. The result for the string and texture models are almost exactly the same and give $\delta_{\mathrm{corr}}(t_0)=-8.88\times10^{-3}$. The age of the universe in this case becomes

$$t_0\approx0.9982\times\frac{2}{3H_0}. \tag{3.29}$$

IV. COMPARISON WITH PREVIOUS WORKS

A. Newtonian treatment

For a comparison with the Newtonian treatment by Buchert and Ehlers [16,17] we have to identify their $\nabla\mathbf{v}$ with our $u^i{}_{;i}$ and their $\nabla\mathbf{u}$ with our $V^k{}_k$. Their result

$$\frac{\ddot{a}_D}{a_D}=-\frac{4\pi G}{3}\frac{M}{V}+\frac{1}{3}\langle(u^k{}_{,k})^2-u^k{}_{,m}u^m{}_{,k}\rangle \tag{4.1}$$

is found to agree exactly with ours in Eq. (2.13), except for the derivatives, which are covariant in our case. Equation (23) in [17] is an extension to describe a globally anisotropic universe. They concluded, that their Eq. (9) in [16] and Eq. (19) in [17] must also hold in general relativity, because they were derived by averaging the Raychaudhuri equation. In our treatment the Raychaudhuri equation can be derived by combining Eq. (2.2) and the trace of Eq. (2.4), replacing $u^i{}_{;j}=(1/3)\Theta\,\delta^i{}_j+\sigma^i{}_j$. So it is possible to recover all their results, exept for the vorticity, which we assume to vanish in our treatment.

B. Futamase's approximation scheme

Futamase [18,19] calculated the backreactions based on his approximation scheme, where he introduced two small parameters representing the amplitude of the metric fluctuations and the ratio between the scale of the variation of this metric fluctuations and the scale of $a(t)$ and the background metric. Then in [18] he used a cosmological post-Newtonian approximation. In [19] he employed the 3+1 splitting of space time, then the Isaacson averaging [32] is performed on the background spatial hypersurface. His results in his Eq. (3.16) [18] or in Eq. (68) [19] are of the same order as ours, but the factors are different. There are several reasons for that discrepancy: first in his approximation scheme he neglected some terms, we do not use such an approximation. Secondly he did not introduce a scale factor $a_D(t)$ defined by the expansion of a comoving volume. He introduced the conformal factor $a(t)$, then he rescaled this scale factor by neglecting terms such as $\langle \overline{h^k{}_k} \rangle$. Thirdly his averaging process in [18] is not defined using the square root of the real metric under the integral, rather he used the square root of the background metric, which is essentially unity for a flat background. He also used this averaging process at the end of [19] to recover the results of [15].

C. Bildhauer and Futamase

Bildhauer and Futamase [15] calculated the backreactions of the inhomogeneities based on the work of Futamase [18] and the Newtonian Zel'dovich approximation (Buchert [33]). Their result [Eq. (25), see also Eq. (84) in [19]] reads, with $b \equiv \delta_{\rm corr}(t_0)$,

$$\delta_{\rm corr}(t_0) = \frac{19}{36} 10^{-6} h_0^2 (1+z_{\rm in})^4 \frac{1}{\mu^2} \langle |\vec{U}|^2 \rangle, \tag{4.2}$$

where we want to indicate a typing error: M_1 defined in their Eq. (25) is not the same as in their Eq. (22), the factor k^2/μ^2 is incorporated into M_1. With $\mu^{-2}\langle |\vec{U}|^2 \rangle = \mu^{-2}\langle |\nabla s_{\rm in}|^2 \rangle = \langle \Psi^{,m}\Psi_{,m} \rangle$ this is of the same order as our result, only the factor is different. The reasons are the same as those in the previous subsection. Another error was found by Futamase [19]: $M_1 = 57\pi^3$ should read $M_1 = 57/8$, the mistake comes from the use of the wrong integration region $[0,2\pi]$ instead of $[0,d]$. They derived at the conclusion that the underestimation of the age of the universe is approximately 30%, which is not correct since they just assumed $\delta_{\rm corr}(t_0)$ to be of order unity instead of calculating it quantitatively as we did here.

V. CONCLUDING REMARKS

We have calculated quantitatively the influence of the inhomogeneities on the global expansion factor of a flat universe with vanishing cosmological constant in the framework of a Zel'dovich-type relativistic approximation scheme using the results from COBE. The first result is that the backreactions act as an additional energy density, which is proportional to a_D^{-2}, so we can interpret the averaged expansion as Friedmannian with a small positive spatial curvature. The second result is that this influence is very small. As a consequence of this the modification of the age of the universe calculated in the usual way (i.e., assuming a homogeneous universe) with a given Hubble constant is negligible. In all models considered here relative differences were less than $\approx 2\times 10^{-3}$. This does not imply that the inhomogeneities are negligible for local astronomical measurements of the Hubble constant. Locally the determination of the redshift-distance relation can be strongly influenced by the peculiar velocity fields due to inhomogeneities $[d = H_z^{-1} + O(2)]$. Our calculation does not consider such effects, but is constrained to comparing globally homogeneous and averaged inhomogeneous matter distributions. Calculating the modification of the redshift-distance relation will be the subject of future investigations. As a result the age problem of the universe that arises in high-density models can only be solved either with a lower Hubble constant, with a nonzero cosmological constant, or with a reduced age of globular clusters.

ACKNOWLEDGMENTS

H.R. wants to thank H. Riffert and A. Geyer for stimulating and helpful discussions. The authors would like to thank T. Buchert and J. Ehlers for valuable comments. H.R. was supported by the ''Deutsche Forschungsgemeinschaft.''

APPENDIX A: COMMUTATION RULE

The time derivative of an averaged quantity reads

$$\frac{d}{dt}\langle A \rangle = -\frac{\dot{V}}{V}\langle A \rangle + \frac{1}{V}\int_V (\dot{A}\sqrt{g} + A\dot{\sqrt{g}}) d^3x. \tag{A1}$$

This leads to the commutation rule [22,16,17]

$$\frac{d}{dt}\langle A \rangle - \langle \dot{A} \rangle = -\langle \theta \rangle \langle A \rangle + \langle A\theta \rangle, \tag{A2}$$

where

$$\theta = \frac{\dot{\sqrt{g}}}{\sqrt{g}}$$

and

$$\langle \theta \rangle = 3\frac{\dot{a}_D(t)}{a_D(t)}. \tag{A3}$$

To convert Eqs. (2.8) and (2.9) to the Eqs. (2.12) and (2.13) we used

$$\frac{d}{dt}\langle V^k{}_k \rangle - \langle \dot{V}^k{}_k \rangle = \langle (V^k{}_k)^2 \rangle - \langle V^k{}_k \rangle^2 \tag{A4}$$

and neglected the term $\langle V^k{}_k \rangle^2$, because it is a higher order quantity.

APPENDIX B: MODEL OF THE INHOMOGENEOUS UNIVERSE

The result of the relativistic Zel'dovich approximation to second order [13] is the following metric tensor:

$$\gamma_{ij}=\left(1+\frac{20}{9c^2t_{\rm in}^2}\Psi\right)\delta_{ij}+2a_D(t)\Psi_{,ij}+\frac{a_D(t)}{c^2t_{\rm in}^2}\left[-\frac{20}{3}\Psi_{,i}\Psi_{,j}-\frac{40}{9}\Psi\Psi_{,ij}+\frac{10}{9}\Psi^{,k}\Psi_{,k}\delta_{ij}\right]+a_D^2(t)\left[\frac{19}{7}\Psi^{,k}{}_{,i}\Psi_{,kj}-\frac{12}{7}\Psi^{,k}{}_{,k}\Psi_{,ij}+\frac{3}{7}[(\Psi^{,k}{}_{,k})^2-\Psi^{,k}{}_{,\ell}\Psi^{,\ell}{}_{,k}]\delta_{ij}\right], \tag{B1}$$

where we set $\alpha=-50/81$. Since their difference is only of second order, we could replace $a(t)$ by $a_D(t)$. The determinant we only need, to first order,

$$\sqrt{\gamma}=1+\frac{10}{3c^2t_{\rm in}^2}\Psi+a_D(t)\Psi^{,k}{}_{,k}. \tag{B2}$$

This leads to

$$\langle{}^3R\rangle=\frac{1}{a_D^2c^2t_{\rm in}^2V_{\rm in}}\int_V\left[-\frac{40}{9}\Psi^{,m}{}_{,m}+a_D(t)\frac{20}{9}[\Psi^{,k}{}_{,m}\Psi^{,m}{}_{,k}-(\Psi^{,k}{}_{,k})^2]+\frac{1}{c^2t_{\rm in}^2}\left(\frac{400}{81}\Psi\Psi^{,m}{}_{,m}+\frac{600}{81}\Psi^{,k}\Psi_{,k}\right)\right]d^3x \tag{B3}$$

and

$$\langle(V^k{}_k)^2-V^k{}_mV^m{}_k\rangle=\frac{4}{9a_Dt_{\rm in}^2V_{\rm in}}\int_V[(\Psi^{,m}{}_{,m})^2-\Psi^{,k}{}_{,m}\Psi^{,m}{}_{,k}]d^3x. \tag{B4}$$

The averaged Friedmann equations read

$$\frac{\dot a_D^2}{a_D^2}=\frac{8\pi G}{3c^2}\langle\rho\rangle+\frac{1}{t_{\rm in}^2V_{\rm in}}\int\left[\frac{20}{27a_D^2}\Psi^{,m}{}_{,m}+\frac{8}{27a_D}[(\Psi^{,m}{}_{,m})^2-\Psi^{,k}{}_{,m}\Psi^{,m}{}_{,k}]-\frac{1}{a_D^2c^2t_{\rm in}^2}\left(\frac{300}{243}\Psi^{,m}\Psi_{,m}+\frac{200}{243}\Psi\Psi^{,m}{}_{,m}\right)\right]d^3x \tag{B5}$$

and

$$\frac{\ddot a_D}{a_D}=-\frac{4\pi G}{3c^2}\langle\rho\rangle+\frac{1}{t_{\rm in}^2V_{\rm in}}\int\frac{4}{27a_D}[(\Psi^{,m}{}_{,m})^2-\Psi^{,k}{}_{,m}\Psi^{,m}{}_{,k}]d^3x. \tag{B6}$$

Integration and the assumption of periodic boundary conditions lead to

$$\int_V\Psi^{,m}{}_{,m}d^3x=0. \tag{B7}$$

The Fourier transformation $\delta(\mathbf{x},t_{\rm in})=\Sigma_{\mathbf{k}}\delta_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$ together with $\int_V\exp[i(\mathbf{k}-\mathbf{k}')\mathbf{x}]d^3x=\delta_{\mathbf{kk}'}$ leads to

$$\int_V(\Psi^{,m}{}_{,m})^2d^3x=\int_V\Psi^{,m}{}_{,k}\Psi^{,k}{}_{,m}d^3x \tag{B8}$$

and

$$\int_V\Psi\Psi^{,m}{}_{,m}d^3x=-\int_V\Psi^{,m}\Psi_{,m}d^3x. \tag{B9}$$

The averaged density is treated as

$$\frac{8\pi G}{3c^2}\langle\rho\rangle=\frac{8\pi G}{3c^2}\frac{1}{V}\int\rho(t_{\rm in})\sqrt{g(t_{\rm in})}d^3x=\frac{8\pi G}{3}\frac{M}{V}\equiv\frac{8\pi G}{3c^2}\rho_b(t), \tag{B10}$$

where we used the conservation law $\rho(t_{\rm in})\sqrt{g(t_{\rm in})}=\rho(t)\sqrt{g(t)}$. Note that even in a case where the averaged second scalar invariant would not vanish our treatment would still be consistent, since in every case

$$\frac{d}{dt}\left(\frac{\dot a_D^2}{a_D^2}\right)=2\frac{\dot a_D}{a_D}\left(\frac{\ddot a_D}{a_D}-\frac{\dot a_D^2}{a_D^2}\right) \tag{B11}$$

is satisfied.

[1] G. Börner, *The Early Universe* (Springer-Verlag, Berlin, 1988).
[2] T. Kirsten, in *The Origin of the Solar System*, edited by S. F. Dermott (Wiley, New York, 1978).
[3] R. E. Nather and D. E. Winget, Sky Telesc. **4**, 374 (1992).
[4] E. M. Burbidge, G. R. Burbidge, W. A. Fowler, and F. Hoyle, Rev. Mod. Phys. **29**, 547 (1957).
[5] I. Jr. Ibn and A. Renzini, Phys. Rep. **105**, 331 (1984).
[6] B. Chaboyer and Y. C. Kim, Astrophys. J. **454**, 767 (1995).
[7] M. Bolte and C. J. Hogan, Nature (London) **376**, 399 (1995).
[8] W. L. Freedman *et al.*, Nature (London) **371**, 757 (1994).
[9] M. J. Pierce *et al.*, Nature (London) **371**, 385 (1994).
[10] A. Saha *et al.*, Astrophys. J. **438**, 8 (1995).
[11] X. Shi, L. M. Widrow, and L. J. Dursi, Mon. Not. R. Astron. Soc. **281**, 565 (1996).
[12] W. Priester, J. Hoell, and H. J. Blome, Phys. Bl. **45** (2), 51 (1989).
[13] H. Russ, M. Morita, M. Kasai, and G. Börner, Phys. Rev. D **53**, 6881 (1996).
[14] M. Kasai, Phys. Rev. D **52**, 5605 (1995).

[15] S. Bildhauer and T. Futamase, Gen. Relativ. Gravit. **23**, 1251 (1991).
[16] T. Buchert and J. Ehlers, Astron. Astrophys. (to be published).
[17] J. Ehlers and T. Buchert, Gen. Relativ. Gravit. (to be published).
[18] T. Futamase, Mon. Not. R. Astron. Soc. **237**, 187 (1989).
[19] T. Futamase, Phys. Rev. D **53**, 681 (1996).
[20] M. Kasai, Phys. Rev. Lett. **69**, 2330 (1992).
[21] M. Kasai, Phys. Rev. D **47**, 3214 (1993).
[22] Fortunately we only have to average scalars. The averaging process with tensors is much more complicated, for problems about averaging tensors see, R. M. Zalaletdinov, Gen. Relativ. Gravit. **24**, 1015 (1992); **25**, 673 (1993). He also already found a more general version of the commutation rule.
[23] Y. P. Jing, H. J. Mo, G. Börner, and L. Z. Fang, Astron. Astrophys. **284**, 703 (1994).
[24] G. Börner, H. J. Mo, and Y. P. Jing, in *Galaxies in the Young Universe*, Proceedings of a Workshop held at Ringberg Castle, edited by H. Hipperlein, K. Meisenheimer, and H. J. Röser, 1994 (unpublished).
[25] P. B. Stark, Astrophys. J. **408**, L73 (1993).
[26] R. Scaramella and N. Vottorio, Mon. Not. R. Astron. Soc. **263**, L17 (1993).
[27] J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, Astrophys. J. **304**, 15 (1986).
[28] K. B. Fisher *et al.*, Astrophys. J. **402**, 42 (1993).
[29] A. Albrecht and A. Stebbins (unpublished).
[30] N. Turok, Phys. Scr. **T36**, 135 (1991).
[31] J. A. Peacock, Mon. Not. R. Astron. Soc. **253**, 1P (1991).
[32] R. A. Isaacson, Phys. Rev. **166**, 1272 (1968).
[33] T. Buchert, Mon. Not. R. Astron. Soc. **254**, 729 (1992).