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ARTICLES

Accuracy of estimating the multipole moments of a massive body from the gravitational waves of a binary inspiral

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If the gravitational field of a massive, compact body is stationary, axially symmetric, and reflection symmetric across the equatorial plane, and if a much less massive compact object (such as a neutron star or a small black hole) were to orbit in a circle on the equatorial plane of the central, compact body, then the produced gravitational waves would carry the values of the central body's multipole moments. By detecting those waves and extracting from them the central body's lowest few moments, gravitational-wave detectors have the potential to test the black-hole no-hair theorem and search for exotic objects such as naked singularities and boson or soliton stars. This paper estimates how accurately we can expect to measure the central body's moments. The measurement errors are estimated using a combination of, first, the leading-order (of a post-Newtonian series) contribution of each moment to the gravitational-wave phase, second, an *a priori* probability distribution that constrains each moment's magnitude to a range appropriate for a compact body, and third, any relations that the multipole moments satisfy among themselves, which reduce the number of degrees of freedom for the waves (this is useful in cases when one is searching for a specific type of compact body). We find that the Earth-based LIGO detector cannot provide sufficiently precise measurements of enough multipole moments to search for exotic objects, but the space-based LISA detector can do so. [S0556-2821(97)01016-3]

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I. INTRODUCTION

Construction has begun on several ''high-frequency'' Earth-based laser-interferometer gravitational-wave detectors, including the Laser Interferometer Gravitational-Wave Observatory (LIGO) [1]; and hopefully the European Space Agency's ''low-frequency'' Laser Interferometer Space Antenna (LISA) [2] will be flying by the year 2014. Among the promising observable events for Earth-based detectors are neutron stars or small black holes spiraling into massive black holes, $M_\odot \ll M \lesssim 300 M_\odot$; and similarly promising for space-based detectors are white dwarfs, neutron stars, or small black holes spiraling into $\sim 10^5 M_\odot$ to $\sim 3\times 10^7 M_\odot$ black holes [3,4]. For LISA, binary inspirals where the smaller mass is a $\sim 10 M_\odot$ black hole would be of especial interest: at $\sim 10 M_\odot$, the orbiting object is less likely to be perturbed by other orbiting objects than would be a solar-mass compact object, and it would generate stronger gravitational waves than its solar-mass counterpart [2,5–7]. Although the calculation of event rates depends on many assumptions, we might expect [2] to detect such events with a signal-to-noise ratio of ~ 40 for one year searches with LISA, for binaries at cosmological redshift $z=1$.

The waves from such types of inspiral carry information of the central black hole's spacetime geometry, encoded in two sets of multipole moments [8]. If the central body is indeed a black hole, then the moments should satisfy a relation as dictated by the black-hole no-hair theorem [9]. Any discrepancies would signal either a violation of the no-hair theorem and general relativity, or that the central body is not a black hole but actually another type of compact object, such as a naked singularity or a boson or soliton star [10,11]. In this paper, we will take the viewpoint of general relativity being correct, and that we are searching for non-black-hole, massive, compact bodies.

Of course, a discrepancy can either be actually physical or be merely a statistical error from the data analysis of the detected gravitational waves. The latter effect would be due to the matched filtering analysis measuring best-fit parameters for the waves' source which differ from the actual parameters by detector-noise-induced stochastic errors. In this paper, we will attempt to *estimate* the errors [we use the word ''error'' to mean the standard error (the rms difference between the actual and measured values of a parameter)] that we can expect for measurements of the first few multipole moments of the central body.

Finn [12], Finn and Chernoff [13], Cutler and Flanagan [14], and Poisson and Will [15] have established the data analysis formalism (that we shall use and extend to measuring multipole moments) and have analyzed measurement accuracies for the mass and spin parameters of compact binaries for Earth-based detectors such as LIGO. Poisson [16] has used the same type of data analysis formalism for space-based detectors. His analysis shows how one can use measurements of binary inspirals to look for violations of general relativity or indications that the central body is not a black hole. In that analysis, the gravitational-wave phase is expanded in a power series around the frequency of gravitational waves at the last stable circular orbit (the orbit at which a test particle can no longer move on a circular geodesic and must plunge into the central body). The measured parameters in that power series become the parameters that describe the central body (or the theory of gravity, if one chooses the viewpoint that general relativity may not be correct). This type of parametrization would be suitable when assuming that the central body is a black hole or some type of similar object; however, it would be less useful when searching for objects vastly different from a black hole. For example, a spinning boson star [11] may have a radius larger than the last stable circular orbit for a black hole of the same mass. Such a case would not be well suited for the power series expansion around the frequency of gravitational waves at the last stable circular orbit.

Our analysis differs from Ref. [16] in that our classification of the central body is in terms of the multipole moment expansion of its external gravitational field around radial infinity. Correspondingly, our analytical expansion of the produced gravitational-wave phase is performed around the gravitational-wave frequency of zero. The multipole moment parametrization covers a much broader range of possible central bodies.

We have mentioned that we only provide estimates of the measurement errors in this paper. An exact calculation would be very difficult. By confining our analysis to a simple case and using a simplified model gravitational waveform, we can calculate errors which should give some indication of what the errors would be in the exact calculation.

Here is a summary of our idealizing assumptions.

(1) For simplicity, we will deal only with central bodies whose external gravitational field is stationary, axisymmetric, reflection symmetric across the equatorial plane, and asymptotically flat. With this assumption, the multipole moments can be described by two sets of scalars [8,17,18]: There are the mass multipole moments consisting of the mass M and higher-order multipole moments M_l (the mass quadrupole moment M_2, M_4, $M_6, \ldots$). In our units which we use here and throughout, we set $G=c=1$, so that M_l has units of $(\text{mass})^{l+1}$. Since it will be more useful to deal with dimensionless quantities, we define the dimensionless moments $m_l \equiv M_l/M^{l+1}$. There are also the mass-current multipole moments S_l (the spin angular momentum S_1, the current octopole moment S_3, S_5, $S_7, \ldots$), for which we can define dimensionless counterparts $s_l \equiv S_l/M^{l+1}$.

(2) The sizes of the errors that we will compute for the moments m_l and s_l, as well as for the other binary parameters such as the masses, will be functions of the values of those moments. However, for simplicity and for the sake of being able to present the results in a concise form, we will only compute the errors for the case in which the moments m_l and s_l are either zero or small enough that terms appearing in the gravitational-wave phase which are quadratic in these moments can be ignored in our analysis. Making this approximation miscalculates the errors for the moments by amounts which scale linearly with the moments, while it miscalculates the errors for all other parameters by amounts which scale quadratically with the moments [see Eq. (3.26) of Ref. [14] and surrounding discussion]. The errors as computed for this spherical or almost spherical case should be reasonable estimates for the errors in the general case. As we will see below in Sec. IV, there will be some cases when the errors on the moments become $\gtrsim 1$, so that even if the best-fit measurement for a moment has it equal to zero, the actual value of the moment may not be small at all and our approximation of dropping quadratic terms becomes poor. Even in such cases, the errors should still be good as order of magnitude approximations. This is because terms quadratic in the multipole moments occur at a higher order in a post-Newtonian series and typically affect the waveform by a smaller amount.

(3) Besides its mass and multipole moments, the central body has other parameters that can affect the waveforms. In this paper, we will not consider how such effects enter into the analysis. For example, the waveforms will depend on whether or not the central body has a horizon, but we will assume that the waves generated if the central body does not have a horizon are the same as those if the body does have one. Similarly, the waveforms will depend on whether or not the central body absorbs energy through tidal heating; we will assume it does not absorb energy.

(4) We will assume that the inspiraling compact object has a sufficiently small mass μ ($\mu \ll M$) that its orbital path is close to being a geodesic of the central body's unperturbed spacetime geometry, and that this is true throughout the inspiral, up to a point just before the last stable circular orbit when the object plunges into the central body.

(5) In general, the orbit will be both elliptical and out of the equatorial plane. The eccentricity e is probably small for the smaller mass binaries that Earth-based interferometers can detect, because gravitational radiation reaction tends to circularize orbits [19]. However, for $\sim 10^6 M_\odot$ central bodies studied by space-based detectors, the orbit may be highly eccentric due to recent perturbations by other orbiting objects [5,7]. Unlike the case with eccentricity, the inclination angle ι between the orbital axis and the central body's symmetry axis is not driven to be small by radiation reaction [20]. Therefore, in general, the orbital motion will be very complicated, consisting of the orbiting object traveling (approximately) in an ellipse, while that ellipse precesses in its plane, and while that plane itself precesses around the central body's symmetry axis. For this paper, to avoid these complications, we will only solve the problem in the ideal situation of the compact object traveling in the equatorial plane in a slowly shrinking, circular orbit.

(6) We will assume that the inspiraling object travels through vacuum. This may not be a good assumption if an accretion disk surrounds the central body. We will also assume that any other orbiting objects do not significantly perturb the orbit of the object whose waves we are measuring. We will only consider the case when the equatorial size of the central body is smaller than the radius of the last stable circular orbit, although it is easy to modify our analysis below to account for the waves cutting off at a larger radius than where such a last stable circular orbit would be.

(7) The predicted templates for the gravitational waveforms are not yet known. This is because the computation of the waves from the inspiral of a compact object around a body with arbitrary multipole moments is complicated by two-dimensional differential equations which are not separable and have not yet been solved. However, we do know how each multipole moment affects the phase of the gravitational waves to leading order in a post-Newtonian expansion [8]. With this information we should be able to get at least a good order of magnitude estimate for the errors.

(8) We will assume a large signal-to-noise ratio S/N (as

defined below). The limit of large S/N is necessary to simplify the analysis and so that we can be certain of detection in the first place.

(9) We will use the noise curves for LIGO and LISA to compute the measurement accuracies. We will assume the noise is stationary and Gaussian. The noise curves for these detectors are only the expected ones; the actual curves might turn out to be different when the detectors are fully operational. Furthermore, space-based detectors will revolve around the sun [21], thereby having changing angular sensitivity patterns. We will not incorporate this revolution effect in our analysis. However, it is partially taken into account through the fact that the signal-to-noise ratio will be reduced through the angular averaging that the revolution creates (see Ref. [2], Fig. 1.3, and associated discussion).

(10) The errors depend on how we model the possible values that the multipole moments can take. Below, we will give a model for the *a priori* probability distribution that is appropriate for a compact body of characteristic size (radius) r. Although this value of r is arbitrary, we will select a particular value ($r=3M$) in our calculations.

This is certainly a long list of approximations and restrictions; however, it is reasonable to expect that they will not seriously compromise the primary intent of this paper: to find out the prospects for measuring multipole moments so as to determine whether or not it is worthwhile for theorists to pursue this calculation in greater depth. We will see that for LISA it is worthwhile.

Hopefully, many of the above restrictions will be removed in future, more sophisticated analyses, so that experimenters will have a complete set of numerically generated templates with which to work. These numerically generated templates will be accurate not only where the gravitational-wave frequency is near zero (where our analysis is valid) or near the frequency at the last stable circular orbit (where the analysis of Ref. [16] is valid), but also at all frequencies in between.

We will use the convention that the orbital angular momentum vector of the orbiting object points in the direction relative to which the mass-current moments are defined. For example, if the central body were a Kerr black hole and spinning in the same direction as the object revolves, then s_1 would be positive; if spinning in the opposite direction, then s_1 would be negative.

The binary will generally be at distances where the cosmological redshift z cannot be neglected. Therefore, the frequencies of the gravitational waves as measured are a factor of $(1+z)^{-1}$ of those that would be measured at the source. Similarly, it is $(1+z)\mu$ and $(1+z)M$ that are measured, as opposed to μ and M. The dimensionless multipole moments m_l and s_l are not affected by the redshift factor. To make our equations easier to read, we will not write down these factors of $(1+z)$, although the conversion should be remembered that where below we write μ or M, we imply $(1+z)\mu$ or $(1+z)M$, respectively.

In Sec. II, we will briefly review the method for computing the errors. In Sec. III, we will construct a model of the gravitational waves in the time and frequency domains. We will also try to quantitatively understand the validity of using only the leading-order contribution of each moment [assumption (7) above]. In Sec. IV, we will compute the errors for several different situations, and will deduce their implications for LIGO and for LISA.

II. DATA ANALYSIS

The data analysis formalism used in this section is described in much greater detail in Refs. [12–15]. In this section, we will summarize that formalism as needed for our purpose, and show how the multipole moments can be given an *a priori* probability distribution.

In the presence of a time-dependent gravitational-wave strain $h(t)$, the gravitational-wave detectors measure a signal $s(t)=h(t)+n(t)$, where $n(t)$ is noise which we assume to be Gaussian. We assume that the waveform $h(t)$ is one of many possible waveforms $h(t,\theta^i)$ for which we have theoretically predicted templates, with θ^i being the parameters that describe the waves, including the multipole moment parameters. We do not know, from the gravitational-wave measurements, exactly what are the true values of θ^i. Rather, all we know is that if we have measured the signal $s(t)$, the probability distribution function for θ^i to be the correct values can be written in terms of the prior probability distribution and the inner product as (see Appendix A of Ref. [14])

$$p(\theta^i)\propto p^{(0)}(\theta^i)e^{-(n|n)/2}=p^{(0)}(\theta^i)e^{-(s-h|s-h)/2}. \quad (1)$$

Here, $p^{(0)}(\theta^i)$ is our *a priori* probability distribution of the parameters θ^i, and $(n|n)$ is the inner product (defined below) of $n(t)$ with itself. Although $p^{(0)}(\theta^i)$ can be modeled rather arbitrarily, we choose a particular fairly unrestrictive model. We assume that the prior probability satisfies

$$p^{(0)}(\theta^i)=\prod_{\text{even } l} p^{(0)}(m_l)\prod_{\text{odd } l} p^{(0)}(s_l); \quad (2)$$

that is, we assume that there is a uniform prior probability distribution function for all parameters except the multipole moments, and that each moment is *a priori* independent of the other moments. Our assumption that the central body is compact (at least it has a small equatorial plane circumference since the inspiraling object is able to make tightly bound orbits) suggests that the magnitude of each moment m_l or s_l cannot be much greater than $(r/M)^l$, where r is some parameter that can be thought of as the characteristic size or radius of the compact body. This parameter is not necessarily associated with some physical radius of the central body; rather, it is just some parameter that we have to choose which restricts the multipole moments. More specifically, we assume prior probability distributions of the form

$$p^{(0)}(m_l)\propto \exp\left[-\frac{1}{2}\left(m_l\frac{M^l}{r^l}\right)^2\right], \quad (3a)$$

$$p^{(0)}(s_l)\propto \exp\left[-\frac{1}{2}\left(s_l\frac{M^l}{r^l}\right)^2\right]. \quad (3b)$$

We should not choose $r\gtrsim 6M$, since then the central body would not be compact and in such case we would not be able to measure the moments accurately anyway. On the other hand, we should not choose $r\lesssim M$, because we wish to consider a class of possible compact bodies broader than just black holes, and black holes themselves have moments satisfying $|m_l|\leqslant 1$ and $|s_l|\leqslant 1$, as a result of the no-hair theorem [8,9,18],

$$m_l+is_l=(is_1)^l, \tag{4}$$

and the restriction $|s_1|\leqslant 1$. Below, when we calculate values for the errors, we will mainly use the choice $r=3M$, but also we we will show how the errors change when we change r.

Equations (3) state that the *a priori* probability distribution for each moment is centered around zero and has a width of $(r/M)^l$. One might raise the objection that centering the distribution around zero is not the best choice. For example, most spinning objects have a negative mass quadrupole moment m_2 due to the equator's centrifugal bulge. However, our assumption is easy to work with, and involves a minimum of theoretical prejudices. One might also decide to model the central body as a spinning object which cannot rotate faster than the point at which it would centrifugally breakup and therefore might restrict s_l to be constrained as $(r/M)^{l-(1/2)}$ instead of $(r/M)^l$. We will not use this alternative model.

The inner product $(\cdot|\cdot)$ between two signals (or a signal and a template) is defined by [Ref. [14], Eq. (2.3)]

$$(h_1|h_2)=2\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f)+\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)}df, \tag{5}$$

where a tilde represents the Fourier transform, and $S_n(f)$ is the detector's noise spectral density: For Earth-based detectors, we follow Cutler and Flanagan [Ref. [14], Eq. (2.1)] and use the following approximate analytic fit to the expected LIGO noise curve for advanced interferometers [1]:

$$S_n(f>10\ \text{Hz})=\frac{3\times 10^{-48}}{\text{Hz}}\left[\left(\frac{f}{f_0}\right)^{-4}+2+2\left(\frac{f}{f_0}\right)^2\right],$$

$$S_n(f<10\ \text{Hz})=\infty, \tag{6}$$

where $f_0=70$ Hz. For space-based detectors, we use the following fit to the LISA noise curve

$$S_n(f)=\frac{8\times 10^{-42}}{\text{Hz}}\left[\left(\frac{f}{f_a}\right)^{-14/3}+2+2\left(\frac{f}{f_\lambda}\right)^2\right], \tag{7}$$

where $f_a=0.0015$ Hz and $f_\lambda=0.03$ Hz. This fit is only valid for 10^{-4} Hz$<f<10^{-1}$ Hz, to which we restrict our analysis. We choose this fit as it agrees with the noise curve in Fig. 1.3 of Ref. [2]. The term scaling as $f^{-14/3}$ is due to acceleration noise (see Table 3.3 of Ref. [2]), the term constant in f is due to optical-path noise (such as shot noise; see Table 3.2 of Ref. [2]), and the term scaling as f^2 is due to the gravitational waves having shorter wavelengths than LISA's round-trip arm lengths.

The overall prefactors in these two noise curves do not affect our analysis. Rather, those prefactors affect the signal-to-noise ratio, as computed by [Ref. [14], Eq. (2.5)]

$$S/N=(h|h)^{1/2}. \tag{8}$$

We assume the S/N to be a given number, and the overall amplitude of the signal h is normalized to give that number. This normalization can be done because the amplitude of h is inversely proportional to the distance to the binary [as we will see below in Eq. (15)]. Therefore, we assume that the binary is at the distance required to get the assumed S/N.

Denoting by $\overline{\theta^i}$ the best-fit values for the parameters θ^i, the probability that the true set of parameters is $\overline{\theta^i}+\Delta\theta^i$ is [Ref. [15], Eqs. (2.8) and (2.9)]

$$p(\overline{\theta^i}+\Delta\theta^i)\propto p^{(0)}(\overline{\theta^i}+\Delta\theta^i)e^{-(1/2)\Gamma_{ij}\Delta\theta^i\Delta\theta^j}, \tag{9}$$

where

$$\Gamma_{ij}=\left(\frac{\partial h}{\partial\theta^i}\middle|\frac{\partial h}{\partial\theta^j}\right). \tag{10}$$

The partial derivatives are evaluated at $\theta^i=\overline{\theta^i}$.

With our assumption that the best-fit parameters $\overline{\theta^i}$ have $m_l=0$ and $s_l=0$, then our prior probabilities are

$$p^{(0)}(\overline{\theta^i}+\Delta\theta^i)\propto e^{-(1/2)\Gamma_{ij}^{(0)}\Delta\theta^i\Delta\theta^j}, \tag{11}$$

where

$$\Gamma^{(0)}_{m_lm_l}=(M/r)^{2l}\quad \text{for even } l>2, \tag{12a}$$

$$\Gamma^{(0)}_{s_ls_l}=(M/r)^{2l}\quad \text{for odd } l>1, \tag{12b}$$

and all other components of the $\Gamma^{(0)}$ matrix are zero.

The error matrix can be computed by taking the inverse of the Fisher information matrix $\Gamma+\Gamma^{(0)}$ (see Appendix 6 of Ref. [14]):

$$\Sigma=(\Gamma+\Gamma^{(0)})^{-1}. \tag{13}$$

The error $\delta\theta^i$ (that is, the standard error, or the root-mean-square error) for each parameter θ^i is [Ref. [14], Eq. (2.8)]

$$\delta\theta^i=(\Sigma^{ii})^{1/2}. \tag{14}$$

We have to know the template forms $h(t,\theta^i)$; that is, how the waveform depends on all the parameters for which we are fitting. As we will see in the next section, these parameters consist of the overall signal amplitude, two integration constants (the time and phase of signal arrival), the masses μ and M, and the multipole moments m_l and s_l. Since we only need $\partial h/\partial m_l$ and $\partial h/\partial s_l$ (both evaluated around the spherical case) for Eq. (10), then we only need to know the waveform accurate to linear order in each of the moments with m_l and s_l. That is, we need to know the waveform assuming the central body is spherical, and how the waveform varies when each moment is varied.

III. THE GRAVITATIONAL WAVEFORMS

In this section, we will first construct a model waveform as a function of the just mentioned parameters. We will then compute its Fourier transform. Finally, we will examine the validity of our assumption of only including the leading-order (in a post-Newtonian series) contribution of each moment.

The gravitational-wave strain $h(t)$ that a detector measures is very complicated: $h(t)$ is a linear combination of the waveforms $h_+(t)$ and $h_\times(t)$ that come from the source, with the coefficients in that combination being functions of the orientations of the detector's axes and the direction to the binary. Although these orientations change during the duration of the signal for space-based detectors which revolve around the sun, we ignore this slow modulation [21]. The waveforms $h_+(t)$ and $h_\times(t)$ themselves are complicated functions of the angles between the binary's axes and the line from the binary to the detector. All of the angular factors which go into determining the amplitude for the waves are combined into some function Q of the angles. For the purpose of trying to estimate the errors, the fact that Q is really a slowly changing function of time is not important. Nor are we too concerned with the form of Q as a function of the angles for the same reason that we were not too concerned with the prefactors in the noise curves (6) and (7): the distance to the binary is adjusted as appropriate to give us the assumed S/N.

Not only is the overall amplitude of the signal not too important for our analysis, but also the exact form of the amplitude as a function of time is not nearly as important as the phase of the oscillating waveform. This is because there is a large number of cycles in the signal, so that the effect of a slight change in the parameters on the phase is on the order of that large number times greater than the effect of that slight change in parameters on the amplitude. Because of this, we can approximate [13–15] the waveform as having an amplitude as computed in the Newtonian limit for a spherical body. In addition, it is a sufficiently good approximation to examine only the dominant frequency F of the gravitational waves, which is twice the orbital frequency.

As the orbit shrinks, F is a slowly varying function of time t. The model waveform $h(t)$ can then be written as a function of F as [see Ref. [14], Eq. (2.12)]

$$h(t)=\left(\frac{384}{5}\right)^{1/2}\pi^{4/3}Q\frac{\mu}{D}(MF)^{2/3}\cos\left(\int 2\pi F\,dt\right), \quad (15)$$

where D is distance to the source, chosen to give the assumed S/N.

The $F(t)$ appearing in Eq. (15) should be computed carefully, for if $F(t)$ were off by a small fraction, then after a number of cycles equal to half the reciprocal of that fraction, the template would go from in-phase to out-of-phase with the gravitational wave. Instead of dealing with $F(t)$, we use the dimensionless quantity $\Delta N(F)$, the number of cycles that the dominant gravitational-wave frequency spends in a logarithmic interval of frequency:

$$\Delta N \equiv \frac{F^2}{dF/dt}=F^2\frac{dE/dF}{dE/dt}, \quad (16)$$

where E is the orbital energy of the binary.

For dE/dt, we add the exact post^4-Newtonian series expansion for a spherical black hole [Ref. [23], Eq. (43)], rounded off to six digit accuracy, and the leading-order (in a post-Newtonian series) contribution of each moment [Ref. [8], Eq. (55)], resulting in

$$\begin{aligned}\frac{dE}{dt}=&-\frac{32}{5}\left(\frac{\mu}{M}\right)^2(\pi MF)^{10/3}\Bigg[1-3.71131(\pi MF)^{2/3}\\&+12.5664(\pi MF)-4.92846(\pi MF)^{4/3}\\&-38.2928(\pi MF)^{5/3}+[115.732-5.43492\ln(\pi MF)]\\&\times(\pi MF)^2-101.510(\pi MF)^{7/3}\\&+[-117.504+17.5810\ln(\pi MF)](\pi MF)^{8/3}\\&+\sum_{\text{even } l\geq 2}\frac{4(-1)^{l/2}(l+1)!!m_l(\pi MF)^{2l/3}}{3l!!}\\&-\sum_{\text{odd } l\geq 3}\frac{8(-1)^{(l-1)/2}l!!s_l(\pi MF)^{(2l+1)/3}}{3(l-1)!!}\\&-\frac{11}{4}s_1(\pi MF)\Bigg]. \quad (17)\end{aligned}$$

For dE/dF, we add the exact expression for the spherical case [Ref. [22], Eq. (4)] to the leading-order and linear contribution of each moment [Ref. [8], Eq. (56)], resulting in

$$\begin{aligned}\frac{dE}{dF}=&-\frac{\pi M\mu}{3(\pi MF)^{1/3}}\Bigg[\frac{1-6(\pi MF)^{2/3}}{[1-3(\pi MF)^{2/3}]^{3/2}}\\&-\sum_{\text{even } l\geq 2}\frac{(-1)^{l/2}(4l-2)(l+1)!!m_l(\pi MF)^{2l/3}}{3l!!}\\&+\sum_{\text{odd } l\geq 1}\frac{(-1)^{(l-1)/2}(8l+12)l!!s_l(\pi MF)^{(2l+1)/3}}{3(l-1)!!}\Bigg]. \end{aligned}\quad (18)$$

Combining Eqs. (16)–(18), we calculate ΔN as a post-Newtonian series, which, as with $F(t)$, has to be calculated accurately. However, we do not expand out the $[1-6(\pi MF)^{2/3}]$ factor that came from dE/dF for the spherical case. This nonexpansion was shown to greatly improve the accuracy of the template [see Ref. [22], Eq. (18)]. The result is [see Ref. [8], Eqs. (57)]

$$\Delta N=\frac{5}{96\pi}\left(\frac{M}{\mu}\right)(\pi MF)^{-5/3}\times\Bigg[\sum_{n=0,2,3,4,5,6,7,8}a_n(\pi MF)^{n/3}[1-6(\pi MF)^{2/3}]+\sum_{n=6,8}b_n(\pi MF)^{n/3}\ln(\pi MF)[1-6(\pi MF)^{2/3}]-\sum_{\text{even } l\geqslant 2}\frac{(-1)^{l/2}(4l+2)(l+1)!!m_l(\pi MF)^{2l/3}}{3l!!}+\sum_{\text{odd } l\geqslant 3}\frac{(-1)^{(l-1)/2}(8l+20)l!!s_l(\pi MF)^{(2l+1)/3}}{3(l-1)!!}+\frac{113}{12}s_1(\pi MF)\Bigg]. \tag{19}$$

Above, the a and b coefficients are those that describe the post-Newtonian expansion around a spherical black hole: $a_0=1$, $a_2=8.21131$, $a_3=-12.5664$, $a_4=52.2782$, $a_5=-111.531$, $a_6=335.734$, $a_7=-716.863$, $a_8=1790.54$, $b_6=5.43492$, and $b_8=47.2175$. Even though Eq. (19) is really just part of a post-Newtonian series, we treat it as exact for our model of the waveform.

We need, for Eq. (5), the Fourier transform of $h(t)$. Following Refs. [13–15], we compute $\tilde{h}(f)$ from Eq. (15) using the stationary phase approximation

$$\tilde{h}(f)=\mathcal{A}f^{-7/6}\exp[i\psi(f)], \tag{20}$$

where $\mathcal{A}=(Q/D)\mu^{1/2}M^{1/3}$ and

$$\psi(f)=2\pi ft(f)-\phi(f)-\frac{\pi}{4}, \tag{21}$$

with

$$t(f)=\int\frac{dt}{dF}dF=\int\frac{\Delta N}{F^2}dF \tag{22}$$

and

$$\phi(f)=\int 2\pi F\frac{dt}{dF}dF=\int 2\pi\frac{\Delta N}{F}dF. \tag{23}$$

Equation (20) has to be modified to account for the waves' shutting off at the last stable circular orbit. We have to set $\tilde{h}(f)$ to zero for $f>6^{-3/2}(\pi M)^{-1}$, the gravitational-wave frequency when the orbiting object plunges into the central body [14]. [Technically, this frequency changes with the mass M and with the multipole moments, and such variations enter into the $\partial h/\partial\theta^i$ terms in Eq. (10), but these variations can be ignored because they affect the amplitude of the signal, which is measured to far less accuracy than the phase.] Substituting our expression (19) for ΔN in Eqs. (21)–(23) and keeping the expression only to linear order in m_l and s_l yields

$$\psi(f)=2\pi ft_*-\phi_*-\frac{\pi}{4}+\frac{3}{128}\left(\frac{M}{\mu}\right)(\pi Mf)^{-5/3}\Bigg\{\sum_{n=0,2,3,4,6,7}\frac{40a_n}{(n-8)(n-5)}(\pi Mf)^{n/3}-\sum_{n=0,2,4,5,7,8}\frac{240a_n}{(n-6)(n-3)}(\pi Mf)^{(n+2)/3}-\tfrac{40}{9}(a_5-6a_3)(\pi Mf)^{5/3}\ln(\pi Mf)+\tfrac{40}{9}(a_8-6a_6)(\pi Mf)^{8/3}\ln(\pi Mf)-20b_6(\pi Mf)^2[\ln(\pi Mf)-\tfrac{1}{3}]+\tfrac{40}{9}(b_8-6b_6)(\pi Mf)^{8/3}[\tfrac{1}{2}\ln^2(\pi Mf)-\ln(\pi Mf)]-24b_8(\pi Mf)^{10/3}[\ln(\pi Mf)-\tfrac{21}{10}]-\sum_{\text{even } l\neq 4}\frac{(-1)^{l/2}40(2l+1)(l+1)!!m_l(\pi Mf)^{2l/3}}{3(2l-5)(l-4)l!!}+\sum_{\text{odd } l\geqslant 3}\frac{(-1)^{(l-1)/2}80(2l+5)l!!s_l(\pi Mf)^{(2l+1)/3}}{3(l-2)(2l-7)(l-1)!!}+\frac{113}{3}s_1(\pi Mf)-50m_4(\pi Mf)^{8/3}\ln(\pi Mf)\Bigg\}, \tag{24}$$

where t_* and ϕ_* are integration constants from the integrals in Eqs. (22) and (23). These two parameters must be included in the list of parameters θ^i for which errors are computed. They can be *called* the time and phase of the signal, although such names are rather arbitrary since we only defined t_* and ϕ_* up to the addition of constants. One could easily redefine, by adding constants, t_* and ϕ_* to be the time and phase when F reaches some fiducial frequency F_*. Since we are interested more in the multipole moment parameters than in t_* and ϕ_*, we do not bother to do such redefinitions.

For the remainder of this section, we will try to understand how good or poor is our approximation of using only the leading-order contribution (of a post-Newtonian series) of each multipole moment. The relevance of this discussion depends mainly on the frequency range through which we measure the inspiral. For example, if the frequency at the beginning of the measured inspiral were a factor of 10 less than at the end of the inspiral at the last stable circular orbit, then most of the cycles would be at frequencies where the post-Newtonian expansion would be good even to leading order. Consequently, our estimates of the errors on the low-

TABLE I. The values of g_l and h_l, used in the formulas for the last stable circular orbit as calculated to leading order and to higher order, respectively.

l	g_l	h_l
1	3.08×10^{-1}	7.48×10^{-1}
2	6.69×10^{-2}	2.34×10^{-1}
3	-2.00×10^{-2}	-7.44×10^{-2}
4	-4.18×10^{-3}	-2.13×10^{-2}
5	9.48×10^{-4}	5.32×10^{-3}
6	1.96×10^{-4}	1.47×10^{-3}
7	-3.89×10^{-5}	-3.26×10^{-4}
8	-7.99×10^{-6}	-8.81×10^{-5}
9	1.47×10^{-6}	1.83×10^{-5}
10	3.02×10^{-7}	4.89×10^{-6}
11	-5.28×10^{-8}	-9.73×10^{-7}
12	-1.08×10^{-8}	-2.58×10^{-7}
13	1.82×10^{-9}	4.97×10^{-8}
14	3.73×10^{-10}	1.31×10^{-8}
15	-6.13×10^{-11}	-2.47×10^{-9}
16	-1.25×10^{-11}	-6.45×10^{-10}
17	2.02×10^{-12}	1.20×10^{-10}
18	4.12×10^{-13}	3.12×10^{-11}
19	-6.52×10^{-14}	-5.70×10^{-12}
20	-1.33×10^{-14}	-1.48×10^{-12}
21	2.08×10^{-15}	2.68×10^{-13}
22	4.24×10^{-16}	6.94×10^{-12}
23	-6.55×10^{-17}	-1.24×10^{-14}
24	-1.34×10^{-17}	-3.21×10^{-15}

est few moments would be good. Another case in which our approximation would be valid is if the gravitational-wave frequency at the last stable circular orbit were high and in a region of poor detector sensitivity, so that most of our information would come from the lower frequency portion of the waves. If, however, a significant portion of our information were to come from near the last stable circular orbit, then we would have to examine how well our approximation serves.

Our main concern is that our form for including the multipole moments in ΔN might for some reason be extremely poor near the last stable circular orbit, for example, due to poor convergence of some series. It is not so much our concern that higher-order post-Newtonian terms which are comparable to the leading-order terms might make a difference. In fact, we have experimented by ‘‘making up’’ higher-order post-Newtonian terms (with coefficients on the order of the leading-order term) for each moment, and found little differences in the errors as computed in the next section.

If there were some strong effect on ΔN and consequently $\psi(f)$ [note that $\psi(f)$ is derived exactly from ΔN; if the latter were exactly correct, then so would be the former], it would likely show up in the location of the last stable circular orbit where, by virtue of dE/dF going to zero, ΔN goes to zero. We can test the validity of the leading-order approximation by assuming small values of all moments m_l and s_l and computing the gravitational-wave frequency at the last stable circular orbit as computed by two ways, listed in the next two paragraphs. This test should be a sensitive indicator as to the accuracy of our approximation, because of the strong dependence of the gravitational-wave phase on the last stable circular orbit [22]; if we have calculated the last stable circular orbit's dependence on the moments to some accuracy then we are likely to have calculated the high-frequency gravitational-wave phase's dependence on the moments to a similar accuracy.

The first way of computing the last stable circular orbit is by looking at where the ΔN in Eq. (19) ‘‘thinks’’ it is; that is, where that ΔN goes to zero. Solving $\Delta N=0$ gives the last stable circular orbit at frequency

$$F=\frac{1}{6^{3/2}\pi M}\left(1+\sum_{\text{even } l\geq 2} g_l m_l+\sum_{\text{odd } l\geq 1} g_l s_l\right), \qquad (25)$$

where the values of the g_l coefficients are listed in Table I.

The second way is by computing where ΔN of the actual gravitational waves [as opposed to our model gravitational waves and the ΔN in Eq. (19)] would go to zero. This can be computed because we can compute where the actual dE/dF goes to zero. We did this by writing a computer program to calculate out the metric with all the m_l and s_l up to $l=32$ set to zero except one slightly nonzero moment. This program uses a method that we will not discuss in this paper because, first, the discussion would be lengthy, second, a future paper [24] will give details of the method, and third,

TABLE II. The error $\delta\theta^i$ for each parameter θ^i, when fitting up to the $l_{\max}$th moment, using LIGO. We use the abbreviation $L(\cdots)\equiv\log_{10}(\cdots)$. We assume $\mu=0.2M_\odot$, $M=30M_\odot$, $r=3M$, and $S/N=10$.

$l_{\max}$	$L(\delta t_*/\text{sec})$	$L(\delta\phi_*)$	$L(\delta\mu/\mu)$	$L(\delta M/M)$	$L(\delta s_1)$	$L(\delta m_2)$	$L(\delta s_3)$	$L(\delta m_4)$	$L(\delta s_5)$	$L(\delta m_6)$	$L(\delta s_7)$	$L(\delta m_8)$	$L(\delta s_9)$	$L(\delta m_{10})$
0	−1.96	0.57	−3.22	−3.14										
1	−1.07	1.04	−2.35	−2.12	−1.94									
2	−0.45	2.16	−1.63	−1.47	−1.30	−0.25								
3	−0.14	3.19	−1.00	−0.80	−0.45	0.24	0.83							
4	0.34	3.25	−0.92	−0.72	−0.44	0.38	1.18	1.83						
5	0.70	3.28	−0.64	−0.45	−0.44	0.56	1.19	1.83	2.36					
6	0.75	3.28	−0.57	−0.39	−0.44	0.60	1.19	1.83	2.36	2.85				
7	0.77	3.28	−0.53	−0.34	−0.43	0.61	1.19	1.83	2.36	2.85	3.33			
8	0.78	3.29	−0.51	−0.33	−0.42	0.61	1.19	1.83	2.36	2.85	3.33	3.81		
9	0.78	3.29	−0.51	−0.33	−0.42	0.61	1.19	1.83	2.36	2.85	3.33	3.81	4.29	
10	0.78	3.29	−0.51	−0.32	−0.42	0.61	1.19	1.83	2.36	2.85	3.33	3.81	4.29	4.77

TABLE III. The error $\delta\theta^i$ for each parameter θ^i, when fitting up to the $l_{\max}$th moment, using LISA. We use the abbreviation $L(\cdots)\equiv\log_{10}(\cdots)$. We assume $\mu=10M_\odot$, $M=10^5M_\odot$, $r=3M$, and $S/N=10$.

$l_{\max}$	$L(\delta t_*/\mathrm{sec})$	$L(\delta\phi_*)$	$L(\delta\mu/\mu)$	$L(\delta M/M)$	$L(\delta s_1)$	$L(\delta m_2)$	$L(\delta s_3)$	$L(\delta m_4)$	$L(\delta s_5)$	$L(\delta m_6)$	$L(\delta s_7)$	$L(\delta m_8)$	$L(\delta s_9)$	$L(\delta m_{10})$
0	1.74	0.75	−4.90	−4.80										
1	2.71	1.33	−3.92	−3.70	−3.53									
2	3.37	2.41	−3.17	−3.01	−2.89	−1.82								
3	3.73	3.52	−2.49	−2.28	−1.94	−1.27	−0.66							
4	5.43	4.69	−1.51	−1.34	−1.07	0.09	0.96	1.61						
5	6.01	4.86	−0.72	−0.54	−0.95	0.50	0.99	1.62	2.35					
6	6.08	4.94	−0.64	−0.46	−0.90	0.58	1.08	1.71	2.36	2.75				
7	6.08	4.95	−0.64	−0.46	−0.89	0.58	1.14	1.80	2.36	2.84	2.98			
8	6.08	4.95	−0.64	−0.46	−0.89	0.58	1.14	1.81	2.36	2.84	3.18	3.72		
9	6.08	4.95	−0.64	−0.46	−0.89	0.58	1.14	1.81	2.36	2.84	3.22	3.75	4.23	
10	6.08	4.95	−0.64	−0.46	−0.89	0.58	1.14	1.81	2.36	2.84	3.23	3.76	4.24	4.75

the method is simply the inverse process of Fodor, Hoenselaers, and Perjés' method of computing the multipole moments given the metric [25]: we guess a metric, compute its moments using Ref. [25], and then modify the metric order-by-order in a power series in $M/(\text{radius})$. After we computed the metric, we calculated the frequency where a circular orbit's energy is minimal. The result is that this method predicts a last stable circular orbit of

$$F=\frac{1}{6^{3/2}\pi M}\left(1+\sum_{\text{even } l\geq 2} h_l m_l+\sum_{\text{odd } l\geq 1} h_l s_l\right), \qquad (26)$$

where the values of these h_l are also given in Table I. It is easy to verify the first coefficient analytically ($h_1=11/6^{3/2}$), since the last stable circular orbit of a slowly rotating Kerr black hole can be computed (see Sec. 61 of Ref. [27]).

Examining Table I, we see that for low values of l, the g_l and h_l are of the same order of magnitude, but at higher values they begin to differ greatly. For low values of l, then, our approximation of using only the leading order contribution of each moment m_l or s_l in ΔN should be adequate, for two reasons. First, with a low value of l, the moment has a large fraction of its effect on ΔN at frequencies much less than the frequency at the last stable circular orbit where the post-Newtonian expansion is good and the leading order term should be sufficient. Second, even at higher frequencies close to the frequency at the last stable circular orbit, the approximation should be at the correct order of magnitude. On the other hand, for higher values of l, neither of these two reasons is applicable. As it will turn out, however, the errors for the higher-order moments will be primarily determined by the *a priori* information $\Gamma^{(0)}$, so that the severe disagreement between the g_l and the h_l for large l is unimportant.

IV. RESULTS

In this section, we will discuss the errors $\delta\theta^i$ for a variety of situations. All numbers were computed using MATHEMATICA to numerically evaluate the $\delta\theta^i$ in Eq. (14) from Eq. (13), which uses the *a priori* information matrix in Eqs. (12) and the inspiral information matrix in Eq. (10), which in turn relies upon Eq. (5), either noise curve (6) or (7), and Eqs. (20) and (24).

While the diagonal terms in the error matrix give the errors, the off-diagonal terms contain information of the correlation coefficients [see Ref. [15], Eq. (2.12)]. We will not list these terms for lack of space. Usually, there is a strong correlation (correlation coefficient close to $+1$ or -1) between the time and phase parameters, the two mass parameters, and the first few (lowest order) multipole moment parameters. On the other hand, there is usually a weaker correlation be-

TABLE IV. The error $\delta\theta^i$ for each parameter θ^i, when fitting up to the $l_{\max}$th moment, using LISA. We use the abbreviation $L(\cdots)\equiv\log_{10}(\cdots)$. We assume $\mu=10M_\odot$, $M=10^5M_\odot$, $r=3M$, and $S/N=100$.

$l_{\max}$	$L(\delta t_*/\mathrm{sec})$	$L(\delta\phi_*)$	$L(\delta\mu/\mu)$	$L(\delta M/M)$	$L(\delta s_1)$	$L(\delta m_2)$	$L(\delta s_3)$	$L(\delta m_4)$	$L(\delta s_5)$	$L(\delta m_6)$	$L(\delta s_7)$	$L(\delta m_8)$	$L(\delta s_9)$	$L(\delta m_{10})$
0	0.74	−0.25	−5.90	−5.80										
1	1.71	0.33	−4.92	−4.70	−4.53									
2	2.37	1.41	−4.17	−4.01	−3.89	−2.82								
3	2.73	2.52	−3.49	−3.28	−2.94	−2.27	−1.66							
4	4.54	3.80	−2.40	−2.23	−1.97	−0.81	0.07	0.72						
5	5.99	4.74	−0.73	−0.55	−1.12	0.47	0.59	0.95	2.35					
6	6.05	4.87	−0.68	−0.50	−1.00	0.54	0.94	1.53	2.35	2.58				
7	6.07	4.88	−0.66	−0.48	−0.99	0.56	0.94	1.53	2.35	2.81	2.68			
8	6.07	4.88	−0.65	−0.47	−0.98	0.57	0.96	1.56	2.35	2.81	3.16	3.68		
9	6.08	4.91	−0.65	−0.47	−0.96	0.58	1.04	1.67	2.35	2.82	3.20	3.74	4.10	
10	6.08	4.92	−0.64	−0.46	−0.95	0.58	1.05	1.69	2.35	2.82	3.20	3.75	4.17	4.70

TABLE V. The error $\delta\theta^i$ for each parameter θ^i, when fitting up to the $l_{\max}$th moment, using LISA. We use the abbreviation $L(\cdots)\equiv\log_{10}(\cdots)$. We assume $\mu=10M_\odot$, $M=10^5M_\odot$, $r=2M$, and $S/N=10$.

$l_{\max}$	$L(\delta t_*/\text{sec})$	$L(\delta\phi_*)$	$L(\delta\mu/\mu)$	$L(\delta M/M)$	$L(\delta s_1)$	$L(\delta m_2)$	$L(\delta s_3)$	$L(\delta m_4)$	$L(\delta s_5)$	$L(\delta m_6)$	$L(\delta s_7)$	$L(\delta m_8)$	$L(\delta s_9)$	$L(\delta m_{10})$
0	1.74	0.75	−4.90	−4.80										
1	2.71	1.33	−3.92	−3.70	−3.53									
2	3.37	2.41	−3.17	−3.01	−2.89	−1.82								
3	3.73	3.52	−2.49	−2.28	−1.94	−1.27	−0.66							
4	4.96	4.23	−1.95	−1.78	−1.51	−0.38	0.49	1.15						
5	5.23	4.28	−1.54	−1.36	−1.48	−0.22	0.50	1.15	1.50					
6	5.26	4.28	−1.49	−1.31	−1.48	−0.20	0.50	1.15	1.50	1.80				
7	5.26	4.28	−1.48	−1.30	−1.48	−0.20	0.50	1.15	1.50	1.80	2.10			
8	5.26	4.28	−1.48	−1.30	−1.48	−0.20	0.50	1.15	1.50	1.80	2.10	2.41		
9	5.26	4.28	−1.48	−1.30	−1.48	−0.20	0.50	1.15	1.50	1.80	2.10	2.41	2.71	
10	5.26	4.28	−1.48	−1.30	−1.48	−0.20	0.50	1.15	1.50	1.80	2.10	2.41	2.71	3.01

tween higher-order moments, because the errors for higher-order moments are primarily dependent upon the (uncorrelated) *a priori* probability distribution, as we will soon see. We will now focus only on the diagonal terms.

In computing the errors, the amplitude parameter $\mathcal{A}$ can be computed easily and separately from the others, because a glance at Eqs. (5), (10), and (20) shows that $\Gamma_{\mathcal{A}j}=0$, where j is any other parameter besides $\mathcal{A}$. Thus $\mathcal{A}$ is uncorrelated with any other parameter. Its error is given by

$$\delta\mathcal{A}=\frac{\mathcal{A}}{S/N}. \tag{27}$$

We now consider the errors for the other parameters only.

The errors depend on how many multipole moment parameters $l_{\max}$ are included in the fit. For example, the error on s_1 would be greater if fitting for moments up to m_4 ($l_{\max}=4$) than if fitting up to s_3 ($l_{\max}=3$). As $l_{\max}$ increases, the dependence is at first very strong, but eventually the errors begin to approach certain values. If we are measuring the moments of an unknown object, then $l_{\max}$ has to be chosen to be infinite or, in practice, large enough such that the errors stop growing. If, on the other hand, we are trying to determine whether an object is a black hole or not then we only have to fit up to $l_{\max}=2$. This is because we can perform the fit assuming all the moments with $l>l_{\max}$ (s_3, $m_4,\ldots$) are given by Eq. (4), and then the test of whether m_2 satisfies Eq. (4) or not serves to check whether the object is a black hole or not. As another example, searching for a spinning boson star would require fitting up to $l_{\max}=3$ [11].

When one knows any such relation that gives the higher-order moments (with $l>l_{\max}$) as functions of the lower-order moments (with $1\le l\le l_{\max}$), all occurrences of the higher-order moments in Eq. (19) should be replaced with the functions of the lower-order moments. For the present though, we want to perform an analysis without specifying the relation. We can do this by replacing all the higher-order moments with zero, instead of with the (presently unspecified) functions of the lower-order moments. This is a good approximation, because a lower-order moment has a much stronger effect on ΔN (and consequently on the waveform and on the errors) through where it normally occurs at a low order in the post-Newtonian expansion than where it occurs in the replacement of a higher-order moment.

In Tables II–VII (described in more detail in the following paragraphs), each column shows the error of the parameter listed at the top of the column. We actually give the base-10 logarithm, so that large negative numbers correspond to precise measurements. Each row corresponds to a different value of $l_{\max}$, the number of moments being measured. As $l_{\max}$ increases, the error on the $l_{\max}$th moment (10 to the power of the rightmost number in that row) ap-

TABLE VI. The error $\delta\theta^i$ for each parameter θ^i, when fitting up to the $l_{\max}$th moment, using LISA. We use the abbreviation $L(\cdots)\equiv\log_{10}(\cdots)$. We assume $\mu=10M_\odot$, $M=10^6M_\odot$, $r=3M$, and $S/N=10$.

$l_{\max}$	$L(\delta t_*/\text{sec})$	$L(\delta\phi_*)$	$L(\delta\mu/\mu)$	$L(\delta M/M)$	$L(\delta s_1)$	$L(\delta m_2)$	$L(\delta s_3)$	$L(\delta m_4)$	$L(\delta s_5)$	$L(\delta m_6)$	$L(\delta s_7)$	$L(\delta m_8)$	$L(\delta s_9)$	$L(\delta m_{10})$
0	3.92	1.92	−4.35	−4.74										
1	4.92	2.84	−4.24	−3.28	−2.83									
2	6.05	4.93	−2.00	−2.53	−1.01	−0.31								
3	7.07	6.32	−1.79	−1.19	0.08	0.84	0.89							
4	7.60	6.32	−1.51	−1.02	0.09	0.84	1.19	1.82						
5	8.04	6.32	−0.70	−0.47	0.11	0.84	1.20	1.83	2.38					
6	8.12	6.34	−0.57	−0.35	0.12	0.84	1.20	1.83	2.38	2.85				
7	8.14	6.39	−0.50	−0.29	0.15	0.86	1.21	1.85	2.38	2.86	3.16			
8	8.14	6.40	−0.50	−0.29	0.15	0.86	1.21	1.85	2.38	2.86	3.23	3.73		
9	8.14	6.40	−0.50	−0.29	0.16	0.87	1.21	1.85	2.38	2.86	3.28	3.77	4.20	
10	8.15	6.40	−0.50	−0.28	0.16	0.87	1.21	1.85	2.38	2.86	3.29	3.78	4.22	4.73

TABLE VII. The error $\delta\theta^i$ for each parameter θ^i, when fitting up to the $l_{\rm max}$th moment, using LISA. We use the abbreviation $L(\cdots)\equiv\log_{10}(\cdots)$. We assume $\mu=100M_\odot$, $M=10^5M_\odot$, $r=3M$, and $S/N=10$.

$l_{\rm max}$	$L(\delta t_*/{\rm sec})$	$L(\delta\phi_*)$	$L(\delta\mu/\mu)$	$L(\delta M/M)$	$L(\delta s_1)$	$L(\delta m_2)$	$L(\delta s_3)$	$L(\delta m_4)$	$L(\delta s_5)$	$L(\delta m_6)$	$L(\delta s_7)$	$L(\delta m_8)$	$L(\delta s_9)$	$L(\delta m_{10})$
0	1.46	0.48	−4.30	−4.15										
1	2.39	1.16	−3.24	−3.05	−2.98									
2	3.28	1.99	−2.29	−2.11	−2.60	−1.12								
3	3.58	2.56	−1.87	−1.68	−1.98	−1.03	−0.41							
4	4.63	3.77	−1.19	−1.01	−1.11	0.21	1.09	1.78						
5	5.02	3.92	−0.72	−0.55	−0.93	0.53	1.12	1.78	2.33					
6	5.06	3.95	−0.68	−0.50	−0.90	0.57	1.13	1.78	2.34	2.81				
7	5.06	3.95	−0.67	−0.49	−0.90	0.57	1.14	1.80	2.35	2.83	3.23			
8	5.06	3.95	−0.67	−0.49	−0.90	0.57	1.14	1.81	2.35	2.83	3.25	3.78		
9	5.06	3.95	−0.67	−0.49	−0.90	0.57	1.14	1.81	2.35	2.83	3.26	3.78	4.28	
10	5.06	3.95	−0.67	−0.49	−0.90	0.57	1.14	1.81	2.35	2.83	3.26	3.78	4.28	4.76

proaches the *a priori* error for the $l_{\rm max}$th moment $(r/M)^{l_{\rm max}}$. We should expect this, since the gravitational-wave measurement does not have enough information to make the error significantly smaller. When we get to such a value of $l_{\rm max}$ that the error is close to $(r/M)^{l_{\rm max}}$, then adding another row to the table (fitting for an extra moment) ceases to increase the errors for all the parameters.

For LIGO, we find discouraging results. In Table II, we show the results of a best-case, albeit unrealistic, scenario: We assume that the small mass is only $\mu=0.2M_\odot$, the smallest mass a neutron star can theoretically have, so that by being small there are many cycles falling in the LIGO band. We assume that the large mass is $M=30M_\odot$ so that the final plunge occurs at 147 Hz and the final inspiral waves are near the frequency of greatest detector sensitivity. (There are over 11 000 gravitational-wave cycles from when the gravitational-wave frequency enters the LIGO band at 10 Hz until the plunge.) With a signal-to-noise ratio $S/N=10$, we still find only a marginal capability of searching for black holes, as can be seen by examining the results for $l_{\rm max}=2$. Say we made measurements of an inspiral giving best-fit measurements of $\overline{s_1}=0.8$ and $\overline{m_2}=-0.64$. Keeping with our assumption (made above in the Introduction) that the errors on the moments are approximately independent of the best-fit values of the moments, then from the $l_{\rm max}=2$ row in Table II, we see that the error on s_1 is 0.05, and the error on m_2 is 0.56. Hence, we would not be able to confidently say that the compact body is a black hole, despite the best-fit parameters satisfying Eq. (4). Other choices of masses besides $0.2M_\odot$ and $30M_\odot$ usually give slightly worse results. Our ''best-case'' choice of masses was made assuming that the S/N for all cases is 10. Of course, one can always get better results by assuming a larger S/N; that is, assuming the binary is closer. However, unless we observe a binary at a much larger S/N, it appears that LIGO will not allow us to search for exotic objects or test the black-hole no-hair theorem.

On the other hand, we find encouraging results for the space-based detector LISA. The LISA mission is designed to last for two years, although the spacecraft may be functional for over a decade [2]. We therefore chose two year observation times in computing results—that is, we assume that we measure the inspiral from two years before the last stable circular orbit until the last stable circular orbit.

In Table III, we see the results of a binary with $\mu=10M_\odot$, $M=10^5M_\odot$, and $S/N=10$. In the two years of observation, the gravitational-wave frequency F sweeps from 4.3×10^{-3} Hz to 4.4×10^{-2} Hz. In that time period, there are about 4.2×10^5 gravitational-wave cycles. For the same situation (except with different values of μ and M) described above where with LIGO we could not determine whether the body was a black hole or not, we can with LISA measure s_1 to within 0.0013 instead of 0.05, and we can measure m_2 to within 0.015 instead of 0.56, and hence enable a determination of whether the object is a black hole or not.

In Table IV, we see the results for the same situation as in Table III, but now with $S/N=100$. Note that for small values of $l_{\rm max}$ where all errors in Table III are much less than the *a priori* errors, then increasing the signal-to-noise by a factor of 10 as in Table IV simply decreases the size of each error by a factor of 10. In Table V, we see the same situation as in Table III, but now with r set to $2M$ instead of its value in all the other tables, $3M$. Note that this makes little difference in rows in which $l_{\rm max}$ is small enough that all the errors are much less than the *a priori* errors anyway. In Table VI, we see the same situation as in Table III, except now with the large mass $M=10^6M_\odot$. In this case, the gravitational-wave frequency F sweeps from 2.2×10^{-3} Hz to 4.4×10^{-3} Hz, in about 1.8×10^5 cycles. In Table VII, we see the same situation as in Table III, except now with the small mass $\mu=100M_\odot$. In this case, the gravitational-wave frequency F sweeps from 1.8×10^{-3} Hz to 4.4×10^{-2} Hz, in about 1.8×10^5 cycles. Although these two larger mass cases have greater errors as shown in Tables VI and VII than those in Table III, the larger mass cases will have a larger S/N than the smaller mass case, assuming the distance and angles between the binary and the detector are the same in all cases.

We should remember that the above results are only valid in the case when all our assumptions made in the Introduction hold. In removing these assumptions, perhaps the most difficult step will be generalizing the results to eccentric and nonequatorial orbits. An orbit with high eccentricity e radiates very strongly and is very strongly affected by the spin of the central body [7], and by all of the multipole moments [8]. Similarly, a nonequatorial (with a large inclination angle ι)

orbit around a spinning central body precesses [26], and the precession depends on the multipole moments [8]. If we knew the values of e and ι, and we knew that $e^2 \ll 1$ and $\iota^2 \ll 1$, then our analysis would still be valid, because the gravitational-wave phase evolution depends on small eccentricity or small inclination angle like e^2 or ι^2, respectively. (The gravitational-wave amplitude *does* have modulations of order e or ι at the precession frequencies associated with an elliptical orbit or a nonequatorial orbit [8]. However, because this precession frequency is different than the gravitational-wave frequency, then a cross-correlation of two signals with precession using Eqs. (5) and (10) has the terms linear in e or ι mostly canceling out due to the integration of a highly oscillatory term.) In reality, we do not know *a priori* the values of e and ι or whether or not they are small, thus necessitating an analysis with general orbits rather than our circular orbits.

Despite the great number of assumptions we have made, we believe our results will still be accurate enough to convey the main message: that the prospect of using space-based detectors to search for non-black-hole, massive, stellar objects is promising and deserving of future efforts to remove our simplifying assumptions and enable a more careful analysis.

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