# Model-independent representation of electroweak data

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(Received 16 December 1996)

General model-independent expressions are developed for the polarized and unpolarized cross sections for $e^+e^- \to f\bar{f}$ near the $Z^0$ resonance. The expressions assume only the analyticity of $S$-matrix elements. Angular dependence is included by means of a partial wave expansion. The resulting simple forms are suitable for use in fitting data or in Monte Carlo event generators. A distinction is made between model-independent and model-dependent QED corrections and a simple closed expression is given for the effect of initial-final state bremsstrahlung and virtual QED corrections. [S0556-2821(97)04715-2]

PACS number(s): 12.15.Lk, 11.55.Bq

## I. INTRODUCTION

The purpose of an experiment such as the CERN $e^+e^-$ collider (LEP) is twofold. First and foremost it should measure and record experimental results with a minimum of theoretical input or prejudice and without presupposing that the data are described by a particular theoretical model. Only in this way can the hard-won experimental data be of use should our present understanding of the physics change or the standard model be supplanted.

The second purpose is to test the correctness of the various candidate theoretical models that describe the physics of the processes involved. Such theoretical models will contain parameters, such as $\sin^2\theta_W$, that can be extracted by fitting the data with predictions of the model. A given model is ruled out when the values obtained for the extracted parameters differ depending on the measurement or physical process from which it was obtained. The values of model parameters should be extracted and recorded for comparison between past and future experiments. However, these parameters may become meaningless once the model to which they pertain is ruled out or modified. They therefore cannot perform the function of recording experimental data for posterity.

While the high-energy physics community has been saturated with analyses that confront the standard model and its possible extensions with the experimental data, much less attention has been given to preserving the data in an unambiguous model-independent form.

One way to do this would be to make a complete set of raw data available. The sheer volume of data makes this impractical. In addition, ''raw'' data is seldom truly raw having been subjected to on-line triggering. It therefore bears the stamp of the on-line selection procedure.

Another possible way is that experimentalists provide plots of raw cross sections as a function of, say, center-of-mass energy or scattering angle. Such cross sections again suffer from the problem that they have been subjected to on-line selection. To produce such plots, experimental cuts generally need to be introduced.

Both of the above possibilities require that the potential user, who wishes to test a given model for consistency with experiment, have a fairly sophisticated machinery in place for treating the QED and other background effects. Details of experimental cuts, detector geometry, etc., should all be meticulously recorded.

Fortunately a third possibility exists by which the experimental results can forever be recorded in a way that makes them straightforwardly available for testing theoretical models as they appear. That is to identify and extract the model-independent physical observables that are inherent in the data and record those. The resulting set of physical parameters is small and convenient to use having all detector-dependent effects removed from it. In order to test the consistency of a candidate model one needs only calculate the given physical observable in terms of the parameters of that model and compare it to the recorded value. There is no convention, model, or scheme dependence in the physical observables so that the comparison can be safely and unambiguously made.

Part of the reason why experimental data has not generally been recorded in this way may be the lack of understanding of the distinction between model-independent physical observables and parameters specific to a given model. The distinction between the two is considered in Sec. II.

To be fair to experimentalists, the results of LEP experiments are extracted in a relatively model-independent way. Semiempirical expressions are fitted to the data in order to extract quantities such as the mass, total and partial width of the $Z^0$ boson [1]. The shortcoming is that these expressions are indeed semiempirical and at some level of accuracy they will fail to describe the data correctly. Being semiempirical they are also somewhat arbitrary and one needs to have the detailed expressions that were used in the data analysis in order to interpret the experimental results.

The first serious attempt to consistently describe LEP data in a fundamentally model-independent way was made by Borrelli *et al.* [2]. They clearly recognized the inadequacies inherent in model-dependent analyses of the data. Their approach involved expressing cross sections to $O(\alpha)$ in terms of five independent physical observables: $M$, $\Gamma$, $B$, $R$, and $\mathcal{I}$. These observables represent the mass, width, branching ratio, nonresonant, and absorptive pieces of the matrix element, respectively. This work was extended by Isidori [3]. An important consequence of Ref. [2] is the clear statement

of the need for five independent measurements to fully describe LEP data. Motivated by this Consoli and Piccolo [4] suggested that the final LEP scan at the $Z^0$ be extended to include five energies rather than just three.

The shortcoming of the analysis performed in Ref. [2] is that at a certain level of accuracy it becomes unclear as to just what the physical observables are. These authors followed the conventional wisdom and expanded about a real mass $M$ chosen to coincide with the renormalized mass in the popular on-shell renormalization scheme. The choice of this $M$ as an expansion point is arbitrary and hence the physical observables extracted using it will also be arbitrary.

To elucidate the difficulties of defining physical observables further, consider the problem of determining the total width of the $Z^0$ resonance. In principle an energy scan can be performed for the cross section for $e^+e^-$ producing some final state determined. The full width at half maximum of the resulting resonance curve can then be read off. A raw resonance curve will, of course, wear a radiative tail generated by initial-state photon radiation. That being removed the exact shape of the resonance curve will depend, via final-state vertex corrections, on which final state has been selected for the measurement. Thus the width of the resonance curve does not provide a way of directly determining a unique model-independent total width for the $Z^0$ boson.

The mass is equally problematic to define. Even in the most naive of analyses the resonance peak lies far from what is assumed to be the mass. A discussion of the issues involved can be found in Ref. [5] and in the following section.

As discussed in Ref. [5] similar problems exist for the definition of partial widths. Fortunately a way does exist to define the physical properties of the $Z^0$ boson and describe LEP data in a simple and truly model-independent way. Prior to 1991 most calculations of physics at the $Z^0$ resonance where demonstrably gauge-dependent. It was shown in Ref. [6] how the gauge-dependence could be removed by appealing to the known properties of the analytic $S$-matrix near resonance. The solution involved starting from the known structure of the complete $S$-matrix element and then performing a Laurent expansion about its complex pole, $s_p$. It was pointed out there, and independently in Ref. [7] that the physical mass, traditionally used for unstable particles and defined from $S$-matrix theory, differed significantly from that being extracted by LEP. The analytic $S$-matrix seems to provide the only way of defining the properties of the $Z^0$ boson in a simple and truly model-independent way. As such it is the most appropriate and robust way of preserving LEP data.

In Ref. [8], a paper concerned with the general renormalization of the pole expansion, it was shown that the pole expansion could be used to obtain a simple general expression for the $S$-matrix element near the $Z^0$ resonance for $e^+e^- \rightarrow f\bar{f}$, with $f$ being a generic fermion species. Provided one is not too close to a production threshold the general matrix element takes the form of a Laurent expansion

$$A(s) = \frac{R}{s - s_p} + \sum_{n=0}^{\infty} B_n (s - s_p)^n \tag{1}$$

for fixed scattering angle. It was clearly stated that this expression was applicable to two-particle final states thereby excluding bremsstrahlung diagrams. It easily follows that neglecting terms $O(\Gamma_Z^2/M_Z^2)$ and higher,

$$A(s) = \frac{R}{s - s_p} + B_0, \tag{2}$$

and therefore depends on three complex numbers: $s_p$, $R$, and $B_0$. Here and in what follows $O(\Gamma_Z/M_Z) \equiv O(N_f \alpha)$, where $N_f$ is the number of fermions species into which the $Z^0$ can decay. The cross section will thus depend on five real parameters, in agreement with Borrelli et al. [2], since the overall phase is lost. The difference here is that expansion is made about the pole which is a fundamental property of the $S$-matrix element. The resulting coefficients and definitions of physical observables are therefore not dependent on an arbitrary choice of real expansion point $M$. If it were not for the tree-level photon exchange diagram we could drop $B_0$ in Eq. (2) to obtain the stated level of accuracy. $A(s)$ would then depend on three real numbers only. The question of the model-independent parametrization of the angular dependence of the scattering amplitude was not considered in Ref. [8].

Subsequently Leike et al. [9] repeated the analysis of Ref. [8] explaining how the Laurent expansion could be implemented in practice. They carried out an actual fit to data including QED corrections. They suggested a parametrization that made the presence of photon exchange diagrams explicit rather than absorbing them into background:

$$A(s) = \frac{R_Z}{s - s_p} + \frac{R_\gamma}{s} + B(s), \tag{3}$$

where $B(s)$ is a function having no poles. This parametrization was incorporated into the computer program SMATASY [10].

Including the photon contribution $R_\gamma/s$ and background together in this way may lead to difficulties however. Whereas Eq. (1) is a self-consistent Laurent expansion about a simple pole, $s_p$, and valid within some radius of convergence, Eq. (3) is not. Hence the coefficients $R_\gamma$ and $B_i$ are not independent quantities. In particular the photon exchange term may be written as a Taylor series expansion about $s_p$:

$$\frac{R_\gamma}{s} = \frac{R_\gamma}{s_p} - \frac{R_\gamma}{s_p^2}(s - s_p) + \frac{R_\gamma}{s_p^3}(s - s_p)^2 + \cdots. \tag{4}$$

Thus any change in $R_\gamma$ can be exactly compensated by a corresponding change in the coefficients $B_i$ which is an undesirable feature for fitting. Whereas finite truncations of the series $\Sigma B_i(s - s_p)^i$ may produce adequate fits, as more and more terms are included in the series the coefficients become indeterminate.

The L3 Collaboration [11] performed an analysis of their data based on Eq. (3), curiously without a citation to Ref. [8] that was the original source for the $S$-matrix approach. They truncated their amplitude at

$$A(s) = \frac{R_Z}{s - s_p} + \frac{R_\gamma}{s} \tag{5}$$

so that the coefficients are indeed independent and thus amenable to fitting.

In this paper we look in more detail at the model-independent $S$-matrix description of LEP data as a way of preserving the experimental results in a transparent and natural way that will continue to be understandable and useful for many years to come. In Sec. III the basic $S$-matrix formalism is reviewed and extended. It is shown how to describe the angular dependence of the scattering amplitude in a model-independent way. In Sec. IV the inclusion of QED corrections is discussed. A distinction is made between model-independent and model-dependent bremsstrahlung. We also give a simple exact formula for initial-final state interference corrections to the resonant term.

## II. PHYSICAL OBSERVABLES VS MODEL-DEPENDENT PARAMETERS

In this paper the term model-independent physical observable will be taken to mean a quantity that can be directly defined in terms of some set of experimental measurements without the need for input from some theoretical model. Thus the electromagnetic coupling constant $\alpha$ is exactly defined from the result of a Thomson scattering experiment:

$$\sigma_T = \frac{8\pi}{3} \frac{\alpha^2}{m_e^2}. \tag{6}$$

Similarly the muon decay constant, $G_\mu$, is exactly defined from the experimental measurement of the muon lifetime, $\tau_\mu$ through the relation

$$\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3}\left(1 - \frac{8m_e^2}{m_\mu^2}\right)\left[1 + \frac{3}{5}\frac{m_\mu^2}{M_W^2} + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right. $$
$$\left. \times\left(1 + \frac{2\alpha}{3\pi}\ln\frac{m_\mu}{m_e}\right)\right]. \tag{7}$$

The relation (7) is quite complicated but nevertheless, once $\tau_\mu$ is measured, $G_\mu$ is unambiguously defined. The complexity of Eq. (7) arises from an attempt to factor out QED corrections. There may exist a more convenient or pragmatic way of defining $G_\mu$ from $\tau_\mu$ but the one given is well-established and in common use. Note that, in principle at a certain level of precision, what one means by the lifetime of an unstable particle becomes unclear because the decay curve is not precisely exponential. These considerations are relevant for the $Z^0$ boson but, because of its extremely long lifetime, are unlikely to ever be of concern for the muon.

The position of the pole, $s_p$, may be regarded as a model-independent physical observable because its existence depends only on the analyticity of the $S$-matrix which is ultimately believed to derive from causality. Its value can, in principle, be extracted from measurements of the cross section $\sigma(e^+e^- \rightarrow f\bar{f})$ over a large energy range using analytic continuation onto the second Riemann sheet. The same value of $s_p$ will be obtained for any process involving an intermediate $Z^0$.

In a similar way the residue $R_{if}$ at the pole for a given $f\bar{f}$ final state can be extracted from experiment without detailed model-dependent input. It is known to factorize, $R_{if} = R_i \cdot R_f$ and the $R_f$ can form the basis for a model-independent definition of the partial width [5].

The essential point about a model-independent physical observable is that once a set of experimental measurements is available its value is fixed. In the case of Thomson scattering or the measurement of the muon lifetime, a single number is returned by the experiment and what one means by a model-independent physical observable is clear-cut. Things become less obvious for observables, such as $s_p$ and $R_{if}$, that need to be extracted by fitting experimental data over a certain energy range but they still represent viable model-independent physical observables.

By contrast, model-dependent parameters require the Lagrangian of the underlying model be known and specified. A detailed calculation is required in order to fit the experimental data. The values obtained will be sensitive to which renormalization scheme was used in the calculation and will be subject to what *ad hoc* modifications (improved Born approximations, effective mixing angles, and the like) one makes above and beyond a consistent truncated perturbation series.

A good example of a model-dependent parameter is $\sin^2\theta_W$. From a theoretical point of view, in the standard model, $\theta_W$ is the angle of rotation that diagonalizes the mass matrix of the neutral $W_3$ and $B$ boson that appears in the lowest-order renormalized Lagrangian. In all renormalization schemes the relation

$$\sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2} \tag{8}$$

holds provided $M_W$ and $M_Z$ are the renormalized masses in the particular scheme that has been chosen. In the modified minimal subtraction ($\overline{\text{MS}}$) renormalization scheme $M_W$ and $M_Z$ depend on an arbitrary scale and hence so does $\sin^2\theta_W$. Once a particular renormalization scheme has been chosen, experimental results may be used as input to determine the values of the renormalized parameters. These renormalized parameters generally have no physical meaning outside of the particular model or renormalization scheme that has been chosen and are eminently unsuitable for recording experimental results. They can, however, still be used to test a given model by using it to make predictions for other physical observables. Thus the $\overline{\text{MS}}$ renormalization scheme has renormalized masses that are clearly unphysical but is still a viable and convenient scheme to use in many situations.

At the risk of blurring the distinction between physical quantities and (unphysical) renormalized parameters, one can try to define a renormalization scheme that sets the renormalized parameters to be equal to physical observables as was done in Ref. [12]. Again the problem of just what are the physical observables arises. Furthermore in gauge theories, with their interrelated coupling-constant and mass counterterms, one must take care not to violate Ward identities. In its original incarnation the on-shell renormalization scheme [12] used a definition for the physical mass that was subsequently shown [13,14] to be gauge-dependent. In the same works it was suggested to modify the on-shell renormalization scheme in such a way that the renormalized mass is

identified with the manifestly gauge-invariant, but arbitrary, quantity (11) constructed from the pole of the $S$ matrix. In principle, since the renormalized mass is not a physical quantity, it is not required to be gauge-invariant *a priori* although it is of great practical convenience to have it so. Consistency of this renormalization scheme with Ward identities has yet to be explored.

It is important to keep in mind the clear distinction between the physical mass and the unphysical renormalized mass. The former is a model-independent physical observable but the latter is not. An unstable particle is associated with a pole in the $S$-matrix element $s = s_p$ lying below the real axis. It is this complex number as a whole that is physically significant. For convenience two real numbers can be extracted from $s_p$ and identified with the mass, $M$, and width, $\Gamma$. There is no fundamental way of doing this and any such decomposition will be arbitrary. Two long-standing conventions are to define

$$s_p = M^2 - iM\Gamma, \tag{9}$$

$$s_p = \left( M - \frac{i}{2}\Gamma \right)^2. \tag{10}$$

It was noted [6,7] that in the case of the $Z^0$ the definition (9) produces a value that is 34 MeV below the value being extracted by LEP. The latter is based on the use of the on-shell renormalization scheme. In Ref. [13] it was suggested that, rather than employ the traditional definitions, (9) and (10), the $Z^0$ boson mass should be defined as

$$M_Z^2 = \mathrm{Re}\, s_p + \frac{(\mathrm{Im}\, s_p)^2}{\mathrm{Re}\, s_p}. \tag{11}$$

This is reasonable because it turns out to be numerically close to the value being extracted by LEP and hence requires minimal modification of existing analyses and existing experimental results. However it is no more nor less fundamental than (9) or (10) and no more nor less deserving of the title of physical mass.

Attempts have been made to give physical definitions to $\sin^2\theta_W$. Llewellyn-Smith and Wheater [15] defined an experimental $\sin^2\theta_W^{\mathrm{exp}}$ from the ratios of charged- and neutral-current neutrino scattering experiments. Although this definition constitutes a model-independent physical observable it is not the parameter that appears in the standard model Lagrangian. It is really some convenient way of encapsulating the result of cross-section measurements that once extracted from experiment must be corrected by means of a detailed model-dependent calculation, in order to yield a value for $\sin^2\theta_W$ in some particular renormalization scheme.

### III. MODEL-INDEPENDENT LINE SHAPE

The general matrix element for the process $e^+e^- \to f\bar{f}$ near resonance is a sum over current-current interactions

$$A(s,t) = \sum_{i,f} \mathcal{M}_{if}(s,t) J_i \cdot J_f \tag{12}$$

and is a function of the usual Mandelstam variables $s$ and $t$. The form factors $\mathcal{M}_{if}(s,t)$ are analytic functions of $s$ and

$t$. For massless electrons and final-state fermions only vector and axial-vector currents can appear and thus

$$A(s,t) = \sum_{i,f=L,R} \mathcal{M}_{if}(s,t) [\bar{v}(p_{e^+}) \gamma_\mu \gamma_i u(p_{e^-})]$$

$$\times [\bar{u}(p_f) \gamma_\mu \gamma_f v(p_{\bar{f}})], \tag{13}$$

where $u$ and $v$ are the fermion wave functions and $\gamma_L$, $\gamma_R$ are the usual helicity projection operators $\gamma_{L,R} = (1 \pm \gamma_5)/2$. For massive final-state fermions, magnetic moment terms, $\sigma_{\mu\nu}q^\nu\gamma_L$ and $\sigma_{\mu\nu}q^\nu\gamma_R$, are possible but these can be at most $O(\alpha m_f^2)$ in the final cross section and will be dropped in the later analysis. We will therefore discard them at the outset. At this point it is convenient to go to the center-of-mass frame and express $\mathcal{M}_{ij}$ as a function of $s$ and $\cos\theta$ the cosine of the scattering angle. These may be expanded as a Laurent series about the complex pole of the scattering amplitude,

$$\mathcal{M}_{if}(s,\cos\theta) = \frac{R_{if}}{s - s_p} + B_{0,if}(\cos\theta) + B_{1,if}(\cos\theta)(s - s_p)$$

$$+ \cdots, \tag{14}$$

that is valid within a radius of convergence defined by the position of the nearest branch point which corresponds to a production threshold. For the $Z^0$ resonance the low-order thresholds for fermion production lie sufficiently far from the resonance so as to be unlikely to ever to be of concern. However, some interesting physical consequences arise in the case of nearby thresholds [5,16].

As a consequence of Fredholm theory it is known that the residue at the pole factorizes and we may write $R_{if} = R_i \cdot R_f$ where $R_i$ does not depend on the properties of the final-state particle and $R_f$ is independent of those of the initial-state particle. The functions $B_{n,if}$ can be expanded in partial waves:

$$B_{n,if}(\cos\theta) = \sum_{m=0}^{\infty} B_{if}^{nm} P_m(\cos\theta), \tag{15}$$

where $P_m(x)$ is the Legendre polynomial of order $m$. All constants $s_p$, $R_{if}$, and $B_{if}^{nm}$ are, in principle, complex numbers.

The general differential cross section in the center-of-mass frame for the process $e^+e^- \to f\bar{f}$ with massless incoming electrons of polarization, $P$, colliding with unpolarized positrons is

$$\frac{d\sigma}{d\Omega} = \left( \frac{1+P}{2} \right)^2 \left( \frac{d\sigma_{LL}}{d\Omega} + \frac{d\sigma_{LR}}{d\Omega} \right) + \left( \frac{1-P}{2} \right)^2 \left( \frac{d\sigma_{RR}}{d\Omega} + \frac{d\sigma_{RL}}{d\Omega} \right) \tag{16}$$

with

$$\frac{d\sigma_{ij}}{d\Omega} = \frac{s\beta}{64\pi^2}\left\{\left(\frac{1\pm\beta\cos\theta}{2}\right)^2|\mathcal{M}_{if}|^2\right.$$

$$\left.+\left(\frac{1-\beta^2}{4}\right)\mathrm{Re}\,\mathcal{M}_{if}\mathcal{M}^*_{i,-f}\right\}, \qquad (17)$$

where $\beta=\sqrt{1-4m_f^2/s}$ and $m_f$ is the mass of the final state fermion. The upper sign pertains for $i=f$ and the lower for $i\neq f$. The second term is a helicity flip term and is needed only for massive final-state fermions. There the index $-f$ means the opposite helicity to $f$. Performing the angular integrations over the full solid angle it becomes clear that cross sections must take the general form

$$\sigma_{if}=\frac{s\beta}{32\pi}\left\{\frac{c^{if}_{-2}}{|s-s_p|^2}+\mathrm{Re}\left(\frac{c^{if}_{-1}}{s-s_p}\right)+c^{if}_0\right\} \qquad (18)$$

keeping terms up to order $O(\Gamma_Z^2/M_Z^2)$ relative to the leading one and the leading term proportional to $1-\beta^2=4m_f^2/s$. This last term only contributes in the case of the $b$-quark. The above form is likely to be adequate for all practical purposes in the foreseeable future. The constants $c^{if}_{-2}$ and $c^{if}_0$ are real and $s_p$ and $c^{if}_{-1}$ are complex and hence to this level of accuracy the cross section depends only on six real constants: $\mathrm{Re}\,s_p$, $\mathrm{Im}\,s_p$, $c^{if}_{-2}$, $\mathrm{Re}\,c^{if}_{-1}$, $\mathrm{Im}\,c^{if}_{-1}$, and $c^{if}_0$. Because $s_p$ is the same for all $\sigma_{if}$, summation over initial or final states will lead to a cross section of the same overall form. Using the usual normalization for the Legendre polynomials

$$\int_{-1}^{1}[P_n(x)]^2dx=\frac{2}{2n+1},$$

and the results

$$\int_{-1}^{1}\left(\frac{1\pm\beta x}{2}\right)^2 P_n(x)dx=\begin{cases}\dfrac{3+\beta^2}{6}, & n=0,\\[2mm] \pm\dfrac{\beta}{3}, & 1,\\[2mm] \dfrac{\beta^2}{15}, & 2,\end{cases}$$

we have

$$c^{if}_{-2}=\frac{3+\beta^2}{6}|R_{if}|^2+\left(\frac{1-\beta^2}{4}\right)\mathrm{Re}\,R_{if}R^*_{i,-f}, \qquad (19)$$

$$c^{if}_{-1}=2R_{if}\left(\frac{3+\beta^2}{6}B^{00*}_{if}\pm\frac{\beta}{3}B^{01*}_{if}+\frac{\beta^2}{15}B^{02*}_{if}\right)+2R_{if}$$

$$\times(s_p-s_p^*)\left(\frac{3+\beta^2}{6}B^{10*}_{if}\pm\frac{\beta}{3}B^{11*}_{if}+\frac{\beta^2}{15}B^{12*}_{if}\right), \qquad (20)$$

$$c^{if}_0=2R_{if}\left(\frac{3+\beta^2}{6}B^{10*}_{if}\pm\frac{\beta}{3}B^{11*}_{if}+\frac{\beta^2}{15}B^{12*}_{if}\right)+\int_{-1}^{1}d(\cos\theta)$$

$$\times\left(\frac{1\pm\beta\cos\theta}{2}\right)^2|B_{0,if}(\cos\theta)|^2. \qquad (21)$$

Using the properties of Legendre polynomials

$$\int_{-1}^{1}d(\cos\theta)\left(\frac{1\pm\beta\cos\theta}{2}\right)^2|B_{0,if}(\cos\theta)|^2$$

$$=\frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{1}{(2m+1)}+\beta^2\frac{(2m^2+2m-1)}{(2m-1)(2m+1)(2m+3)}\right)$$

$$\times|B^{0m}_{if}|^2\pm\beta\sum_{m=0}^{\infty}\frac{(m+1)}{(2m+1)(2m+3)}\mathrm{Re}B^{0m}_{if}B^{0,m+1*}_{if}$$

$$+\beta^2\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{(2m+1)(2m+3)(2m+5)}\mathrm{Re}B^{0m}_{if}B^{0,m+2*}_{if}. \qquad (22)$$

To order $O(\Gamma_Z/M_Z)$ the last term in Eq. (18) can be dropped and the cross section then depends on five real constants as first pointed out by Borrelli *et al.* [2]. This same total was obtained in Ref. [8] using arguments based on analyticity similar to those above. The form (18) is extremely general and robust being valid for polarized and unpolarized matrix elements as well as for angular integrations over less than the full solid angle.

To this level of accuracy one further assumption can be made to further simplify the expressions (19)–(21). Since no new particles have been found with a mass $m\lesssim M_Z/2$ we may be sure that the first terms that give rise to corrections that depend on the scattering angle $\theta$ are box Feynman diagrams corresponding to two $W$ or two $Z$ exchange. These diagrams are one-loop and nonresonant. They therefore represent corrections of $O(\alpha\Gamma_Z/M_Z)$ relative to the lowest order and may be dropped by setting $B^{1n}_{if}=0$.

The constants appearing in Eq. (18) can be extracted from data in a model-independent manner and can be calculated in any theoretical model. By comparing their measured and predicted value candidate models may be tested. The form (18) assumes only the analyticity of the $S$-matrix element which is a consequence of causality. It may happen that some of the constants in the above parametrization may seem to be poorly determined especially those in higher orders. This is a fact of life and represents a limit on how well the model-independent parameters can be determined from a given experiment. The parameters in this formulation, once determined, remain valid even in the face of dramatic changes to theoretical models. They can only be adjusted by improved experiments and remain a fundamental and meaningful description of the data. They are thus largely impervious even to possible profound changes to theoretical understanding. The temptation should be resisted to inject more detailed model-dependent assumptions into the data analysis even when this seems to yield tighter bounds on the parameters. If it is done it should be in addition rather, than instead of the extraction of the above constants.

The complete generality of Eq. (18) means that Monte Carlo event generators can be set up assuming that the underlying cross section is of this form. The constants $s_p$, $R_{if}$, and $B_{if}^{mn}$ corresponding to the predictions of an particular model may then be input from independent sources and consistency with experimental results studied. This provides an efficient method of parametrizing corrections that depend on the scattering angle, $\theta$. Typically these come from box diagrams whose analytic structure is complicated and cumbersome for Monte Carlo simulations.

## IV. QED CORRECTIONS

Ultimately in the confrontation of theoretical predictions with experimental data QED corrections must be taken into account. These QED corrections may be grouped into two classes; model-independent and model-dependent. Model-independent QED corrections are those in which the photon is attached only to external fermion legs. Such corrections sense nothing of the detailed structure of the underlying model. They may therefore be accounted for using a structure function approach [17–19] applied to a general cross section of the form (18) or by Monte Carlo methods. In general the Feynman diagrams contributing to the model-independent QED corrections are infrared divergent and can therefore give rise to large logarithms particularly if strong cuts are applied to the photon energy.

Model-dependent QED corrections are those in which photons are connected to internal charged particles. For example a photon that is produced by bremsstrahlung off an internal $W$ is sensitive to the charged current structure of the underlying model. Such corrections cannot be treated in a model-independent manner. They are however always infrared finite and so do not give rise to anomalously large corrections. In the case where final-state photons are individually detected an $S$-matrix motivated form analogous to Eqs. (13) and (14) could be developed for the process $e^+e^-\to f\bar{f}\gamma$. The size of the model-dependent QED corrections represent the level of accuracy to which a given model-independent analysis is valid.

Applying the structure function approach to treat the initial-state QED corrections leads to a corrected cross section in the form of a convolution integral

$$\sigma_T(s) = \int ds' \sigma(s') \rho_{\text{ini}}(1-s'/s), \qquad (23)$$

where $\rho_{\text{ini}}$ is a known structure function.

Final-state corrections may be treated similarly but for most purposes result in an overall multiplicative factor being applied to the cross section. Detailed descriptions can be found in the literature [9,20]. QED corrections to asymmetries may also be handled in this manner [21].

The initial-state and final-state QED corrections are remarkable in their compactness and their relative simplicity. In the case of initial-final state corrections, i.e., diagrams having a photon connected to both the initial-state and final-state fermion lines, similar convolution integral forms have been derived [20], however, a simple exact expression exists for first order QED corrections to the resonant part of the cross section. This expression can be obtained directly from Refs. [22,23]. The $O(\alpha)$ corrections to the cross section $\sigma_{if}(s)$ coming from initial-final state bremsstrahlung and virtual corrections applied to the resonant parts of the amplitude are exactly given by

$$\Delta\sigma_{if}(s) = \pm\frac{3\alpha}{\pi}Q_iQ_fI_2\left(\frac{s_p}{s}, k_{\max}\right)\sigma_{if}^R(s) \qquad (24)$$

and

$$\sigma_{if}^R(s) = \frac{s\beta}{32\pi}\frac{c_{-2}^{if}}{|s-s_p|^2}. \qquad (25)$$

The upper sign in Eq. (24) pertains if the polarization $i=f$ and the lower if $i\neq f$. $Q_i$ and $Q_f$ are the electric charges of the initial- and final-state fermions. $k_{\max}$ is the maximum allowed photon energy in the bremsstrahlung contribution expressed as a fraction of the center-of-mass energy, $2E_\gamma/\sqrt{s}<k_{\max}$:

$$I_2(z,k_{\max}) = \text{Re}\left\{z(z+1)\ln\frac{k_{\max}+z-1}{z} + (z-1)(1-k_{\max})\right\}$$
$$-\ln|z| - 2\ln k_{\max}. \qquad (26)$$

For rather loose cuts on the photon energy, i.e., $k_{\max}$ near 1, the function $I_2$ passes through zero somewhere near the resonance. This behavior was explained recently [24] on physical grounds. Near resonance a physical unstable $Z^0$ with a finite lifetime is created. Its finite propagation length means that the virtual photon must itself propagate a finite distance in order to connect the initial and final states and the amplitude is therefore reduced. Alternatively the finite propagation length may be regarded as resulting in a loss of correlation between initial and final states. The upshot is that initial-final state QED corrections have a rather small but manageable contribution to the resonant line shape when cuts are loose.

[1] A. Blondel, in *Perspectives in the Standard Model*, TASI Lectures, edited by R. K. Ellis *et al.* (World Scientific, Singapore, 1992).

[2] A. Borrelli, M. Consoli, L. Maiani, and R. Sisto, Nucl. Phys. **B333**, 357 (1990).

[3] G. Isidori, Phys. Lett. B **314**, 139 (1993).

[4] M. Consoli and M. Piccolo, Report No. hep-ph/9505261 (unpublished).

[5] R. G. Stuart, Phys. Rev. Lett. **70**, 3193 (1993).

[6] R. G. Stuart, Phys. Lett. B **262**, 113 (1991).

[7] S. Willenbrock and G. Valencia, Phys. Lett. B **259**, 373 (1991).

[8] R. G. Stuart, Phys. Lett. B **272**, 353 (1991).

[9] A. Leike, T. Riemann, and J. Rose, Phys. Lett. B **273**, 513 (1991).

[10] S. Kirsch and T. Riemann, Comput. Phys. Commun. **88**, 89 (1995).

[11] L3 Collaboration, O. Adriani *et al.* Phys. Lett. B **315**, 494 (1993).

[12] A. Sirlin, Phys. Rev. D **22**, 971 (1980).

[13] A. Sirlin, Phys. Rev. Lett. **67**, 2127 (1991).

[14] A. Sirlin, Phys. Lett. B **267**, 240 (1991).

[15] J. F. Wheater and C. H. Llewellyn-Smith, Nucl. Phys. **B228**, 547 (1983).

[16] T. Bhattacharya and S. Willenbrock, Brookhaven Report No. BNL-56481 (unpublished).

[17] E. A. Fadin and V. S. Kuraev, Sov. J. Nucl. Phys. **41**, 446 (1985).

[18] O. Nicrosini and L. Trentadue, Phys. Lett. B **196**, 551 (1987).

[19] O. Nicrosini and L. Trentadue, Z. Phys. C **39**, 479 (1988).

[20] D. Bardin *et al.*, Nucl. Phys. **B351**, 1 (1991).

[21] T. Riemann, Phys. Lett. B **293**, 451 (1992).

[22] J. H. Kühn and R. G. Stuart, Phys. Lett. B **200**, 360 (1988).

[23] S. Jadach, J. H. Kühn, R. G. Stuart, and Z. Wąs, Z. Phys. C **38**, 609 (1988); **45**, 528(E) (1990).

[24] R. G. Stuart, Nucl. Phys. B (to be published).