Physics from Breit-frame regularization of a lattice Hamiltonian

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We suggest a Hamiltonian formulation on a momentum lattice using a physically motivated regularization using the Breit frame which links the maximal parton number to the lattice size. This scheme restricts parton momenta to positive values in each spatial direction. This leads to a drastic reduction in the number of degrees of freedom as compared to a regularization in the rest frame (center at zero momentum). We discuss the computation of physical observables such as (i) the mass spectrum in the critical region, (ii) structure and distribution functions, (iii) the S matrix, and (iv) finite temperature and finite density thermodynamics in the Breit-frame regularization. For the scalar (3+1)-dimensional ϕ^4 theory we present numerical results for the mass spectrum in the critical region. We observe scaling behavior for the mass of the ground state and for some higher-lying states. We compare our results with renormalization group results by Lüscher and Weisz. Using the Breit frame we calculate, for QCD, the relation between the $W^{\mu\nu}$ tensor, structure functions (polarized and unpolarized), and quark distribution functions. We use the improved parton model with a scale dependence and take into account a nonzero parton mass. In the Bjorken limit we find the standard relations between F_1, F_2, g_1, and the quark distribution functions. We discuss the role of helicity. We present numerical results for parton distribution functions in the scalar model. For the ϕ^4 model we find no bound state with internal parton structure. [S0556-2821(97)03015-4]

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I. INTRODUCTION

The standard model of strong interaction physics (QCD) has been confirmed very successfully through comparison between experiment with perturbative and nonperturbative (mostly lattice) calculations. There are a number of observables, which need to be computed nonperturbatively. In some cases nonperturbative computational progress seems very hard to come by. Let us mention the following examples: (a) S matrix for hadron-hadron scattering; (b) structure functions of the proton, in particular at small values of the Bjorken variable x_B; (c) excited states in the hadron mass spectrum; (d) finite density thermodynamics of hadronic matter.

In deep-inelastic lepton-hadron scattering (DIS), one is interested in structure functions and their interpretation in terms of distribution functions. In order to define distribution functions, one has to specify a reference frame. Possible choices are the rest frame, the infinite momentum frame, light-cone coordinates, or the Breit frame. Conventionally, the infinite momentum frame and light-cone coordinates have been used. The Breit frame (characterized by $q_0=0$) is a particular frame in the sense that $Q=\sqrt{-q^2}$, the momentum of the exchanged photon is the resolution ability of the photon to resolve the proton structure. This does not hold in any other frame with $q_0\neq 0$.

The Breit frame is not only conceptionally attractive, but we suggest here that it is useful also for nonperturbative numerical calculations. We have introduced a new regularization scheme for a lattice Hamiltonian, based on the Breit frame. Its construction is guided by the kinematical variables which play a role in deep-inelastic scattering (DIS). We have introduced a momentum lattice based on the Breit frame. It is centered around the proton momentum (in the case of DIS proton scattering). The scheme restricts parton momenta to positive values in each spatial direction and links the maximal parton number to the lattice size. This leads to a drastic reduction in the number of degrees of freedom as compared to a regularization in the rest frame with the center at zero momentum. We have applied the scheme to the scalar (3+1)-dimensional ϕ^4 (ϕ^4_{3+1}) model. We have computed the mass spectrum and extracted physics close to the critical line (second order phase transition). We have found very good agreement with the predictions of the renormalization group by Lüscher and Weisz [1]. To our knowledge, critical behavior of a (3+1)-dimensional field theory has been extracted for the first time from a Hamiltonian formulation.

The successfulness of this method with respect to the relatively small numerical effort (diagonalization of matrices on the order of 50) makes us cautiously optimistic that other physical observables or other models could be treated as well. In this work we want to elaborate on these ideas. In Sec. II we present the Hamiltonian formulation in the Breit-frame regularization. We explain physical and mathematical reasons for the method working. The calculation of the mass spectrum for the ϕ^4_{3+1} model with numerical results near the critical line are presented in Sec. III. In Sec. IV we discuss the structure functions and distribution functions in the Breit frame. For QCD we compute analytically the relation between the hadronic tensor $W^{\mu\nu}$, the structure functions F_1, F_2, g_1, g_2, and the quark distribution functions. We present numerical results for the parton distribution function for the scalar model ϕ^4_{3+1}. The usefulness of this method eventually depends on its potential in numerical calculations of gauge theories. Thus the Breit-frame regularization for lattice gauge

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theories is given in Sec. V. In Sec. VI, we discuss advantages of the Breit-frame regularization for the purpose to compute the S matrix of a scattering reaction from a Hamiltonian lattice formulation. Finally, the computation of thermodynamical observables at finite temperature *and* finite density from the Breit frame is the topic of Sec. VII. A summary is given in Sec. VIII.

II. FORMALISM

A. Hamiltonian formulation

Considering nonperturbative methods in many-body physics, statistical mechanics, and field theory, most successful techniques are sum-rule techniques and lattice field theory in the Lagrangian formulation using Monte Carlo methods to compute functional integrals. Although there is a Hamiltonian formulation of lattice field theory, i.e., the Kogut-Susskind Hamiltonian in lattice gauge theory [2], Hamiltonian methods have not been mainstream in the domain of nonperturbative methods. One of the basic reasons was that in order to treat adequately the physical degrees of freedom, very many virtual particles have to be taken into account. As a function of particle number the dimension of Hilbert space increases exponentially. Nevertheless (maybe due to shortcomings or slow progress in Lagrangian lattice field theory), over recent years several workers have explored Hamiltonian methods, by trying to work with effective Hamiltonians having a small number of degrees of freedom. Examples are the work by Lüscher [3] and Koller and van Baal [4], the exp[S] method coming from nuclear physics [5], applications of the Kogut-Susskind Hamiltonian to compute glueball masses and string tension in QCD [6], the Hamiltonian approach in light-cone quantization [7], and quite recently a Hamiltonian renormalization group approach [8]. These approaches have employed quite different strategies to cope with the problem of a large number of degrees of freedom: E.g., Wilson and co-workers have pursued the idea of the renormalization group, i.e., thinning out degrees of freedom and constructing a new (renormalized) Hamiltonian with a sufficiently small number of effective degrees of freedom. Lüscher [3] and Koller and van Baal [4] have discovered that much physics of the low-lying QCD spectrum can be described by zero-momentum dynamics plus a suitable treatment of the remaining degrees of freedom. The idea of the exp[S] method [5] is that the linked cluster structure underlying a ground state in a many-body theory can be generated by a suitable operator S, and it automatically guarantees the correct infinite volume singularity of the ground state energy. In the applications of the Kogut-Susskind Hamiltonian to QCD [2], several groups have developed clever ways to take into account the high number of plaquettes and closed loop variables, e.g., via the t-expansion method by Horn and co-worker [9]. Finally, an advantage of the regularized (discretized) light-cone Hamiltonian method is that light-cone momenta p^+ of all partons are positive and add up. Thus a total light-cone momentum P^+ drastically limits the number of degrees of freedom. However, this does not hold for the perpendicular momentum $p_\perp$.

Herein, we pursue the following alternative. Let us consider as an example the scalar model given by the Hamiltonian

$$H=\int d^3x\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}(\vec{\nabla}\phi)^2+\frac{m_0^2}{2}\phi^2+\frac{g_0}{4!}\phi^4, \qquad (1)$$

where m_0 and g_0 are the bare mass and coupling constant, respectively. We construct the corresponding Hamiltonian in momentum space:

$$\begin{aligned}H=&\sum_{\vec{k}}\ \omega(\vec{k})a^\dagger(\vec{k})a(\vec{k})+\sum_{\vec{k}\vec{l}\vec{m}}\frac{g_o}{4(2\pi)^3}\\&\times\left[4\frac{a^\dagger(\vec{k})a(\vec{l})a(\vec{m})a(\vec{k}+\vec{l}+\vec{m})}{\sqrt{\omega(\vec{k})\omega(\vec{l})\omega(\vec{m})\omega(\vec{k}+\vec{l}+\vec{m})}}+6\frac{a^\dagger(\vec{k})a^\dagger(\vec{l})a(\vec{m})a(\vec{k}+\vec{l}-\vec{m})}{\sqrt{\omega(\vec{k})\omega(\vec{l})\omega(\vec{m})\omega(\vec{k}+\vec{l}-\vec{m})}}\right.\\&\left.+4\frac{a^\dagger(\vec{k}+\vec{l}+\vec{m})a^\dagger(\vec{l})a^\dagger(\vec{m})a(\vec{k})}{\sqrt{\omega(\vec{k}+\vec{l}+\vec{m})\omega(\vec{l})\omega(\vec{m})\omega(\vec{k})}}\right]+12g_0\sum_{\vec{k}}\ \frac{1}{4(2\pi)^3\omega(\vec{k})}a^\dagger(\vec{k})a(\vec{k})\sum_{\vec{l}}\ \frac{1}{\omega(\vec{l})}.\end{aligned} \qquad (2)$$

We did not do normal ordering, but we have subtracted the vacuum energy. We have written the Hamiltonian in discretized form by introducing a lattice in momentum space with a momentum resolution Δk and a momentum cutoff Λ. Conventionally, one would choose a regular lattice, symmetrical with respect to zero momentum (rest frame). This would be suitable to compute the vacuum state with the quantum number $P=0$. We propose the following alternative: We choose the same regularization, however, we retain only those lattice momenta which correspond to fast moving partons going in the same direction as the hadron (proton). This will be referred to as Breit-frame regularization in what follows. As will be shown below, this yields a drastic reduction in the effective degrees of freedom as compared to the rest frame.

B. Breit-frame regularization

The most important experiment in order to probe the structure of hadrons is deep-inelastic scattering (DIS). Its simplest form involves inclusive scattering of an unpolarized lepton off a hadronic target. Let us recall some basic notation

[10]. The hadron in its ground state interacts with the probing lepton by the exchange of a virtual photon (or neutrino). The hadron (proton) carries momentum P before the collision and goes over to a hadronic state X with momentum P_X. Correspondingly, the electron has momenta k and k'. The exchanged photon carries momentum $q=k-k'$. One defines $Q^2=-q^2$. In Feynman's parton model it is assumed that the proton is made up of constituents, i.e., the partons. They are weakly bound, i.e., the binding energy is small compared to the resolution ability $Q:=\sqrt{-q_\mu q^\mu}$ of the probing photon. Let p denote a parton momentum. Conventionally, one introduces the Bjorken variable $x_B:=Q^2/2P_\mu q^\mu=(q\cdot p)/(q\cdot P)=p^{(L)}/P^{(L)}$. The superscript L denotes the longitudial direction, i.e., the direction of $\vec{P}$. The second equation results from the impulse approximation, i.e., the partons are on the mass shell. The last identity holds in the Breit frame. The *Breit frame* of the hadron is defined by the requirements that the photon energy q_0 be zero and that the photon momentum $\vec{q}$ be antiparallel to the hadron momentum $\vec{P}$. In this frame, the longitudinal component of the parton momentum obeys $p_L=Q/2$. The rationale for this particular choice of frame is that QCD structure functions $F(x_B,Q)$ can be interpreted as a linear combination of parton momentum distribution functions $f(x_B,Q)$, which have a more intuitive interpretation. This relation holds for *leading twist* (higher twists are suppressed for large Q^2). Structure functions are another way of expressing scattering cross sections. The distribution function of a parton counts the number of those partons with a given momentum fraction x_B in the proton. For a precise definition see Ref. [10].

If the hadron is in its ground state, then the longitudinal momentum p_L of the parton can be neither negative nor can it be greater than P_L:

$$0\leqslant p^{(L)}\leqslant P^{(L)}. \tag{3}$$

Thus follows the well-known constraint $0\leqslant x_B\leqslant 1$. Equation (3) can be viewed as a regularization of the longitudinal parton momenta. However, it does not restrict the transverse components of parton momenta. In the Bjorken limit ($Q^2\to\infty$, $P\cdot q\to\infty$, $x_B=\text{const}$) combined with the Breit frame, the hadron is a fast moving object. In momentum space, an object which is spherical in the rest frame becomes prolate in a fast moving frame. Hence, it is physically justified to restrict the transverse parton momenta to a finite region of ellipsoidal (prolate) shape. In particular, we have chosen a sphere centered at the midpoint of the interval $[0,P^{(L)}]$ (any prolate ellipsoid lies within this sphere), given by

$$(\vec{p}-\vec{P}/2)^2\leqslant(\vec{P}/2)^2. \tag{4}$$

One should note that this constraint also follows directly from $0\leqslant x_B\leqslant 1$ and going into the ''parton Breit frame'' (defined by $q^0=0$ and $\vec{q}$ being antiparallel to the *parton* momentum $\vec{p}$) where x_B takes the form $x_B=(\vec{p}\cdot\vec{p})/(\vec{P}\cdot\vec{p})$.

Because we are working in the Hamiltonian approach we need to define a basis in Hilbert space. We construct the Hilbert space as a Fock space of free particles and select (parton) momenta $\vec{p}$ from a bounded domain corresponding to DIS, as given by Eq. (4). This is an *assumption* based on the physical intuition that the experimentally observable parton momenta are those which dominate the quantum dynamics. This assumption has been tested by computing critical behavior of renormalized masses and a good agreement with analytical scaling behavior has been observed (see below).

Now, we introduce a momentum lattice regularization: In order to have a practically convenient lattice we further constrain the parton momenta from Eq. (4), namely, by selecting a regular cube centered at $\vec{P}/2$ and located inside the ball given by Eq. (4). I.e., the parton momenta $\vec{p}$ are restricted to the domain

$$0\leqslant p_i\leqslant\Lambda=\frac{|\vec{P}|}{\sqrt{3}}\quad\text{for}\quad i=x,y,z. \tag{5}$$

Inside this domain we define lattice momenta $\vec{p}:=\vec{n}\Delta p$ where $\vec{n}$ is an integer vector and Δp is the momentum lattice resolution. One notices that all lattice momenta are nonnegative. Contrary to regularization in the rest frame which does *not* limit the particle number, our approach has the following important property: The effective Hilbert space is built from the Fock states $|(a^\dagger_{\vec{k}_1})^{n_1}\cdots(a^\dagger_{\vec{k}_N})^{n_N}|0\rangle$, with the conditions that the total momentum be $n_1\vec{k}_1+\cdots+n_N\vec{k}_N=\vec{P}$, which is located on the surface of the Breit domain, *and* that each parton momentum $\vec{k}_i$ be inside the Breit domain, as given by Eq. (5). Thus, the regularized Hamiltonian is given by Eq. (2), restricted to the effective Hilbert space.

C. Reasons for reduction in the number of effective degrees of freedom

Why does this regularization scheme lead to a Hamiltonian with a small number of effective degrees of freedom? First, for any given state from the effective Hilbert space, the Fock space particle numbers are bounded, if one considers only nonzero parton momenta. This follows from $n_1\vec{k}_1+\cdots+n_N\vec{k}_N=\vec{P}$. Thus an upper bound on the total particle number is $|\vec{P}|/\Delta p$. This does, however, not limit the zero-mode particle number. The zero mode has to be taken into account explicitly. The zero mode only determines the vacuum expectation value of the field $\langle\phi\rangle$. In this work we only consider the symmetric phase of the model $\langle\phi\rangle=0$. In the ϕ^4 model the vacuum expectation value $\langle\phi\rangle$ is an order parameter for symmetry breaking and thus the field has zero fluctuation in the infinite volume limit. In this limit it becomes a classical variable. Thus, for sufficiently large volume it is justified to set the zero mode to zero. In addition, in models where the zero mode cannot be dropped, the zero-mode describes only one degree of freedom, which can be treated like a quantum mechanical oscillator. In summary, the ultraviolet cutoff Λ given by Eq. (5) implies a total particle number cutoff and thus drastically reduces the dimension of the Hilbert space.

Second, if one wishes to compute the mass spectrum of a physical particle, but does not want to compute the vacuum, one has the freedom to choose a reference frame boosted to a momentum $P\neq 0$. As is well known from many-body theory and the exp[S] method, the vacuum state energy has a vol-

ume divergence, but the energy of a physical particle state has no such divergence. Thus, choosing a sector with $P \neq 0$ excludes the vacuum state, but may eventually allow more easily to compute the mass of a physical particle, compared to a computation in the rest frame where the vacuum is present.

D. Role of zero modes and left movers

Our approach being based on equal-time quantization is different from light-front quantization as well as from the infinite momentum frame. However, there are conceptional similarities between the different approaches: It is the goal to describe high energy deep-inelastic scattering and, in particular, structure and distribution functions. In the context of construction of the Hamiltonian it is very important to study the dynamical role of zero modes and left movers. In light-front physics, according to general folklore, the vacuum is trivial and positive momentum states decouple from the vacuum. There are examples in light-front physics, which show that zero modes play a dynamical role. This is discussed in the recent review article by Burkardt [11]. In the infinite momentum frame (IMF), left-moving fermions (with $x<0$, where x denotes the longitudinal parton momentum portion) are known to play a role, as has been discussed by Drell *et al.* [12]. In particular, in the second order self-energy IMF perturbation diagram left-moving fermions are essential, as has been shown by Brodsky *et al.* [13]. Another example is QCD in $1+1$ dimensions on a finite circle. Engelhardt and Schreiber [14] have shown that the gluon background field represented by zero modes couples nontrivially to the quarks, leading to an elimination of fermionic nonsinglet states. The role of zero modes in quantization using the ϵ coordinates, which are ''close'' to the light-cone coordinates, has been discussed by Lenz and Thies [15].

All these examples serve as warning that in the Breit-frame regularization a naive neglect of zero modes and left movers is dangerous. In this article we have introduced Breit-frame regularization at hand of the scalar model. Concerning left movers and zero modes in the scalar model, we can say the following.

(i) Zero modes. The following argument, in our opinion, shows for the case of the scalar model with spontaneous symmetry breaking that zero modes are better treated classically than by a Fock space expansion. In the scalar ϕ^4 model spontaneous symmertry breaking occurs. In the broken phase one has $\langle\phi\rangle=\phi_0\neq 0$, while in the symmetric phase $\langle\phi\rangle=0$. In particular, the zero mode $\phi^{(0)}=1/V\int d^3x\,\phi(x)$ is an order parameter. Spontaneous symmetry breaking implies that the zero mode field is peaked around $\langle\phi^{(0)}\rangle=\phi_o^{(0)}$. In the thermodynamic limit the peak is infinitely sharp, i.e., the fluctuation of the zero mode goes to zero. Thus, in the thermodynamic limit the behavior of the zero-mode degree of freedom is adequately described by a classical solution, which corresponds to fluctuation zero. In our calculation using a finite lattice volume we are not at the thermodynamic limit. Nevertheless, we consider the classical treatment of the zero mode physically more suitable and justified than a particle number truncation in the zero-mode sector. In this work we have considered the ϕ^4 model only in the symmetric phase, where $\langle\phi^{(0)}\rangle=\phi_o^{(0)}=0$.

(ii) Left movers. According to Brodsky *et al.* [13], left movers do not play a role in IMF perturbation theory for the scalar model.

(iii) Actually, we have taken into account left movers in our model: The Hamiltonian, given by Eq. (2), consists of three terms, the noninteracting part (first term), the interacting term with four creation or annihilation operators (second term), and the contraction term with two creation or annihilation operators (last term). The Breit-frame regularization is a quantum mechanical regularization which imposes all momenta occurring in creation or annihilation operators to lie in the Breit domain, Eq. (5). Therefore, in the last term, the sum $\Sigma_l 1/\omega(l)$ runs over all the momenta l from the symmetric lattice ($-\Lambda$ to $+\Lambda$) rather than over the Breit domain. This corresponds to taking into account tadpole terms. This has been found essential in order to reproduce the critical behavior of the mass spectrum.

We have presented our method for the explicit example of the scalar ϕ^4 theory in $3+1$ dimensions. However, we do *not* claim that the treatment adequate for the scalar model would be the same for any other model. In particular, the situation is different when considering gauge theories such as QCD, as there is no spontaneous symmetry breaking (Elitzur's theorem). Therefore, zero modes have then to be treated quantum mechanically. We are fully aware of the fact that application of the Breit-frame regularization to any other model beyond the scalar model requires a careful study of the dynamical role of left movers and zero modes.

III. MASS SPECTRUM AND CRITICAL BEHAVIOR OF ϕ^4_{3+1} THEORY

First, we need to convince ourselves that the method allows the correct computation of physical observables. We have chosen the scalar ϕ^4_{3+1} theory because it is a quite well-understood theory which has a second order phase transition, thus allowing us to test this method near a critical point. The Hamiltonian of the ϕ^4 theory is given by Eq. (2), constrained by the Breit condition (5). It is expressed in terms of free field creation and annihilation operators corresponding to the lattice momenta in the Breit frame. Because the Hamiltonian and the momentum operators commute, we compute the energy spectrum E_n in a Hilbert space sector of a given momentum $\vec{P}$. Since we are not in the rest frame we use the mass-shell condition $M_n:=\sqrt{E_n^2-\vec{P}^2}$ in order to obtain the physical mass spectrum.

It is known [1] that the critical line between the symmetric and the broken phase lies entirely in the region where the bare parton mass squared m_0^2 is negative. Hence, we cannot build the Fock space in terms of partons with those masses. As a remedy, we have split the bare mass squared $m_0^2=m_{\rm kin}^2+m_{\rm int}^2$ into a positive kinetic part $m_{\rm kin}^2$ and an interaction part $m_{\rm int}^2$. The Fock states are built from positive bare masses $m_{\rm kin}$. In numerical calculations close to the critical point shown in Fig. 1 we have chosen, for simplicity, a small positive value. We found that the lower-lying physical mass spectrum is not very sensitive to the value of $m_{\rm kin}$ (this is not the case for higher-lying masses). A better choice of $m_{\rm kin}$ in our view would be to take the renormalized mass m_R. Although m_R is unknown initially, it can be computed by mak-

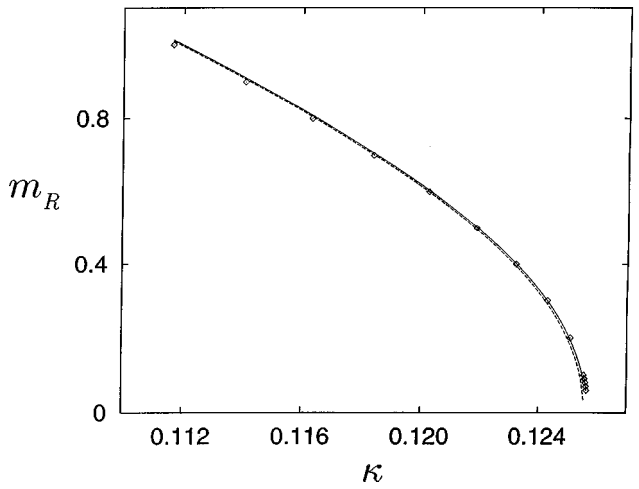

FIG. 1. The ground state mass m_R in lattice units ($a \equiv 1$) versus κ for $\lambda = 0.003\,457\,39$ ($\overline{\lambda} = 0.01$ in Ref. [1]), λ and κ are given by Eq. (6). The points correspond to the results of Ref. [1]. Our results correspond to $\Lambda/\Delta p = 3$ (dashed line) and $\Lambda/\Delta p = 4$ (solid line).

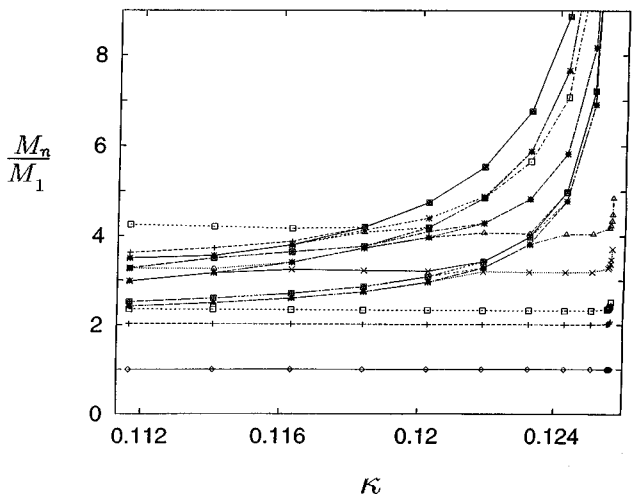

FIG. 2. The lowest lying mass spectrum versus κ. The ground state mass is set to one, λ as in Fig. 1; λ and κ are given by Eq. (6); $\Lambda/\Delta p = 4$.

ing an initial guess and then iteratively improving the answer.

We have diagonalized the Hamiltonian on two lattices: $\Lambda/\Delta p = 3$ and $\Lambda/\Delta p = 4$. This would correspond to symmetric lattices $[-\Lambda, +\Lambda]$ of size 7^3 and 9^3 nodes, respectively. This results in a very small Hilbert space of only 6 and 21 states, respectively. We want to check that the renormalization group predictions are reproduced correctly. Brézin *et al.* [16] have computed those via perturbation theory. Lüscher and Weisz [1] later have solved the renormalization group equations analytically near the critical line starting from boundary conditions (initial data for renormalization group differential equations) which have been obtained from a hopping parameter expansion to high order using the lattice action. In order to compare our numerical results to those of Lüscher and Weisz [1], we express the bare parameters m_0 and g_0 in terms of the parameters λ and κ:

$$m_0^2 = (1 - 2\lambda)/\kappa - 8, \quad g_0 = 6\frac{\lambda}{\kappa^2}. \tag{6}$$

We define the renormalized mass by $m_R = M_1$. The equivalence of our mass definition with that of Lüscher and Weisz is discussed in the appendix. Figure 1 displays the renormalized mass m_R versus κ. One observes that our results, computed on very small lattices, are quite close to the results of Lüscher and Weisz [1]. Masses M computed on the lattice must obey $a < 1/M < L$, where L is the length of the lattice and a denotes the lattice spacing of a space-time lattice. It is related to Λ by $\Lambda = \pi/a$. It can be shown from perturbation theory [16,1] that the physical masses close to the critical point obey the scaling law

$$M \sim C\tau^{1/2}|\ln\tau|^{-1/6}, \tag{7}$$

where $\tau := 1 - \kappa/\kappa_{\rm crit}$ and C is a constant (integration constant of renormalization group equations). Since the results of Ref. [1] are based on the solution of the renormalization group equations, this scaling law fits its results. One should note, however, that two different regularizations (this work and that of Ref. [1]) in general correspond to two different critical lines corresponding in general to different values of $\kappa_{\rm crit}$. In Table I we have displayed our results for the critical points $\kappa_{\rm crit}$ as a function of λ and compared our results with those of Ref. [1]. Again, our results are very close to those of Lüscher and Weisz. These results cover a domain of the bare parameter space extending quite far away from the Gaussian fixed point at $\kappa = 1/8$, $\lambda = 0$.

Another way to test continuum physics is to look at the mass ratios M_n/M_1 from the spectrum on the lattice and check whether they become independent of the cutoff Λ or else independent of the coupling constant $g_0(\Lambda)$ (i.e., they scale). These mass ratios M_n/M_1 are shown in Fig. 2. As can be seen, for a number of states $M_n/M_1 \to$ const in a wide range of κ values, i.e., they scale. However, for some states M_n/M_1 diverge, i.e., there is no scaling. The physical reason behind this is the following: The ϕ^4_{3+1} model describes a gas of partons repelling each other [1]. The spectrum of Fig. 2 shows states dominated by the 1-, 2-, 3-, 4-particle Fock space sectors plus a spectrum of excited (scattering) states.

TABLE I. The critical points $\kappa_{\rm crit}$ versus λ; λ and κ are given by Eq. (6). $\kappa_{\rm crit}^{\rm LW}$ is taken from Ref. [1]. $\alpha := \kappa_{\rm crit}^{\rm KS}/\kappa_{\rm crit}^{\rm LW}$ denotes the ratio between the results of this work and Ref. [1]. In this work, $\kappa_{\rm crit}$ has been determined under the condition that the renormalized mass m_R becomes imaginary. $\Lambda/\Delta p = 4$.

λ	0.0005	0.001	0.005	0.01	0.05	0.1
$\kappa_{\rm crit}^{\rm LW}$	0.125101	0.125202	0.125991	0.126968	0.132368	0.13601
α	0.99997	0.99993	0.99972	0.9993	1.0073	1.0275

The picture of repulsive two-particle-exchange force is confirmed by observation that the mass of the lowest-lying n-body state is larger than n times the mass of the one-body state. The states which scale are just those lowest-lying n-body states. The higher-lying part of the spectrum consists of states with more nodes in the wave function than lattice points, having also a wider range and contributions from higher Fock-state sectors. Because of the fact that in the calculation corresponding to Fig. 2, the parameters Δp, Λ, and the parton number cutoff are all kept fixed, we cannot properly describe these higher-lying states. Consequently, they do not show scaling. When we go to bigger lattices ($\Delta p \to 0$) we then observe (not displayed here) more states which show scaling.

IV. STRUCTURE FUNCTIONS

A. Why structure functions in the Breit frame?

Hadron structure is probed by deep-inelastic scattering (DIS). Over recent years a great deal of experimental data has been gathered from high energy collider experiments. While perturbative quantum chromodynamics (QCD) describes successfully the large Q^2 dependence of DIS structure functions, it cannot predict the correct dependence on the Bjorken variable x_B. Thus, much effort has been devoted to compute quark or gluon distribution functions and proton structure functions from QCD with *nonperturbative* methods. E.g., Martinelli and Sachrajda [17] have computed the first two moments of the pion structure function via Monte Carlo lattice simulations. Schierholz and co-workers [18] have recently computed moments of proton and neutron structure functions. These calculations are notoriously difficult. A particular problem is the determination of small x_B behavior from a few moments (see Ref. [19] for details on the present status of lattice calculations of structure functions). This situation calls for alternative techniques.

Let us briefly outline the reasons for the choice of our method: (i) Structure functions are computed from wave functions. Wave functions are defined in Minkowski space. The Hamiltonian approach offers the advantage of allowing direct computation of Minkowski space observables. E.g., scattering wave functions for glueball-like states in compact QED_{2+1} have been computed in a Hamiltonian formulation on a momentum lattice [20] (for a review of Hamiltonian lattice methods see [21,6,7]). (ii) The usefulness of a momentum lattice to compute physics close to a critical point has been demonstrated in Refs. [22–27]. (iii) The reason for our choice of the Breit frame has been explained above. However, Hamiltonian methods are known to lead to numerical problems because of the huge number of degrees of freedom involved [28]. To the best of the authors' knowledge nobody has succeeded before to observe scaling behavior indicating continuum physics in a (3+1)-dimensional Hamiltonian lattice formulation.

The Breit frame has a distinct property: Only in this frame the photon momentum transfer Q can be interpreted as resolution ability of the photon. The quark and gluon distribution functions of a proton or a neutron which are measured by DIS show a peak for small x_B even for a moderate resolution Q [29]. This indicates a huge number of partons in the proton, because a system of n identical partons would be peaked at $x_B = 1/n$ for symmetry reasons. The physical reason is that the strong forces which bind the proton can easily create gluons or quark-antiquark pairs. Contrary to a typical nonrelativistic problem, particle number is not conserved. Consequently, because of the enormous number of degrees of freedom which are usually associated with a relativistic many-body system, it is almost impossible to calculate quark or gluon distribution functions or mass spectra in a Hamiltonian QCD approach [30]. Our regularization, given by Eq. (5), however, enables us to treat a large number of partons with a *reasonable* numerical effort.

The Q *dependence* of the distribution functions is also a many-particle effect. An intuitive explanation for this dependence is that more partons can be seen inside the proton, if the resolution Q is increased. Partons, however, which are heavy with respect to the forces between them, can be described in a simple constituent model because many-particle effects are negligible. Hence, their distribution functions are neither peaked at $x_B = 0$ nor do they depend considerably on the resolution Q. Examples are heavy quarkonia, electromagnetically bound particles (such as atoms), or the ϕ^4_{3+1} theory which we are investigating below. The renormalized coupling constant of ϕ^4_{3+1} theory is weak everywhere in the critical region and the forces between ''partons'' are even repulsive [1].

B. Relation between structure functions and distribution functions in the Breit frame for QCD: Unpolarized structure functions

In this section we compute analytically the relation between hadronic tensor, structure functions, and distribution functions. Because we work in a fast moving frame and *not* in the infinite momentum frame, we can explicitly take into account a nonzero parton mass. The cross section for deep-inelastic lepton-hadron (electron-proton) scattering has the form (following Jaffe's notation [31])

$$d^2\sigma \propto l^{\mu\nu} W_{\mu\nu}, \tag{8}$$

where $l_{\mu\nu}$ denotes the leptonic tensor and $W_{\mu\nu}$ stands for the hadronic tensor. The hadronic tensor can be split into a symmetric part, which corresponds to unpolarized structure functions and an antisymmetric part, corresponding to the polarized structure functions. The symmetric part can be parametrized in terms of the structure functions F_1 and F_2:

$$W^{\mu\nu}_{\mathrm{sym}} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) F_1 + \left[\left(P^\mu - \frac{\nu}{q^2} q^\mu \right) \left(P^\nu - \frac{\nu}{q^2} q^\nu \right) \right] \frac{F_2}{\nu}, \tag{9}$$

where q^μ is the photon momentum, P^μ is the proton momentum, and $\nu = q \cdot P$. Now, we choose the Breit frame as the reference frame: In the Breit frame the proton momentum is $P^\mu = (E,0,0,P_3)$ with $E^2 = P_3^2 + M^2$, M being the proton rest mass. The photon momentum is $q^\mu = (0,0,0,-Q)$; Q is defined to be $q^2 = -Q^2$. As a result, we find that all components of $W^{\mu\nu}_{\mathrm{sym}}$ vanish, except for

$$W^{00}_{\rm sym}=-F_1+\frac{E^2}{P_3Q}F_2,$$

$$W^{11}_{\rm sym}=W^{22}_{\rm sym}=F_1. \tag{10}$$

The hadronic tensor is defined [31] as

$$4\pi W^{\mu\nu}=\sum_X (2\pi)^4\delta(P+q-P_X)\langle PS|J^\mu(0)|X\rangle \times\langle X|J^\nu(0)|PS\rangle, \tag{11}$$

where X denotes the unobserved fragments of the proton and P is the proton momentum. We have normalized the proton state to $\langle P'|P\rangle=2E(2\pi)^3\delta^3(P'-P)$. S is the proton spin (Pauli-Lubanski vector) normalized to $S^2=-M^2$; $J^\mu(x)=\overline{\psi}(x)\gamma^\mu\psi(x)$ denotes the fermionic (quark) current. The hadronic tensor can be expressed as a current commutator:

$$4\pi W^{\mu\nu}=\int d^4y\exp[-iq\cdot y]\langle P|[J^\mu(y),J^\nu(0)]|P\rangle. \tag{12}$$

In deep-inelastic scattering it is customary to use the impulse approximation in the axial gauge. The partons lie on the mass shell; thus, one can expand the field $\psi(x)$

$$\psi(x)=\sum_s\int\frac{d^3k}{(2\pi)^{3/2}}[2\omega(\vec{k})]^{-1/2}[u_s(\vec{k})e^{-ik\cdot x}b_s(\vec{k}) +v_s(\vec{k})e^{ik\cdot x}d_s^\dagger(\vec{k})]. \tag{13}$$

The spinors are normalized to $\overline{u_s}u_{s'}=2m\delta_{s,s'}$, $\overline{v_s}v_{s'}=-2m\delta_{s,s'}$, where m is the parton rest mass. The parton spin is normalized to $s^2=-m^2$. The creation and annihilation operators obey $[b_s(\vec{k}),b_{s'}^\dagger(\vec{k}')]_+=\delta_{s,s'}\delta(\vec{k}-\vec{k}')$ and $[d_s(\vec{k}),d_{s'}^\dagger(\vec{k}')]_+=\delta_{s,s'}\delta(\vec{k}-\vec{k}')$.

In the computation of the matrix element of the current commutator the following leptonic tensor occurs:

$$\begin{aligned}l^{\mu\nu}_{\overline{u}uuu}(k,s,k',s')&=\overline{u(k,s)}\gamma^\mu u(k',s')\,\overline{u(k',s')}\gamma^\nu u(k,s)\\&=k^\mu k'^\nu+k'^\mu k^\nu+g^{\mu\nu}(m^2-k\cdot k')\\&\quad-im\epsilon^{\mu\nu\alpha\beta}(k-k')_\alpha(s+s')_\beta,\end{aligned} \tag{14}$$

and summing over s' yields

$$\begin{aligned}l^{\mu\nu}_{\overline{u}uuu}(k,k',s)&=2[k^\mu k'^\nu+k'^\mu k^\nu+g^{\mu\nu}(m^2-k\cdot k')\\&\quad-im\epsilon^{\mu\nu\alpha\beta}(k-k')_\alpha s_\beta],\end{aligned} \tag{15}$$

being the standard result [31]. Because of the current commutator, there are four fermion fields involved, which gives 16 combinations of fermion and antifermion creation and annihilation oprators. A straightforward but lengthy calculation gives the term (for other terms see below)

$$W^{\mu\nu}_{b^\dagger bb^\dagger b}=\frac{E}{4}\int d^3k\frac{\delta(q^0+k^0-k'^0)}{\omega(\vec{k})\omega(\vec{k}+\vec{q})} \times\sum_s l^{\mu\nu}_{\overline{u}uuu}(k,k+q,s)\Pi_s^b(P,\vec{k}). \tag{16}$$

Here, we have switched to the following normalization of the proton state $\langle P'|P\rangle=\delta^3(P'-P)$. The calculation includes the matrix element $\langle PS|b_s^\dagger(\vec{k})b_{s'}(\vec{k'})|PS\rangle$ which allows one to split off $\delta(\vec{k}-\vec{k}')$ due to the conservation of total three-momentum as well as to split off $\delta_{s,s'}$ due to the conservation of spin quantum numbers in the helicity basis. Thus, we have defined $\Pi_s^b(PS,\vec{k})$ by

$$\langle PS|b_{s'}^\dagger(\vec{k'})b_s(\vec{k})PS\rangle=\Pi_s^b(PS,\vec{k})\,\delta(\vec{k}-\vec{k'})\,\delta_{s,s'}, \tag{17}$$

which is the the expectation value in the proton state of the quark number operator corresponding to momentum $\vec{k}$ and spin s. Note that Π^b has the same dimension as $\langle PS|PS\rangle$. We now go into the Breit frame. In particular, we employ the Breit condition, Eqs. (4) and (5). In the Breit frame one has $q^0=0$, moreover $\vec{k}$ lies on the mass shell, $(k^0)^2=\vec{k}^2+m^2$, due to the impulse approximation. Also $\vec{k}'=\vec{k}+\vec{q}$ is on shell. Thus, we obtain

$$W^{\mu\nu}_{b^\dagger bb^\dagger b}=\frac{E}{4}\int d^2k_\perp\sum_s\frac{l^{\mu\nu}_{\overline{u}uuu}(k_Q,k'_Q,s)}{Q\omega(\vec{k}_Q)}\Pi_s^b(P,\vec{k}_Q), \tag{18}$$

where we have defined $\vec{k}_Q=(\vec{k}_\perp,Q/2,)$ and $\vec{k'}_Q=(\vec{k}_\perp,-Q/2)$ (parallel and perpendicular denotes the orientation of components with respect to the space component of the proton momentum).

A second term, which contributes to the $W^{\mu\nu}$ tensor, is

$$W^{\mu\nu}_{dd^\dagger dd^\dagger}=\frac{E}{4}\int d^2k_\perp\sum_s\frac{l^{\mu\nu}_{\overline{v}vvv}(k'_Q,k_Q,s)}{Q\omega(\vec{k'}_Q)}\Pi_s^d(P,\vec{k}_Q). \tag{19}$$

$\Pi_s^d(PS,\vec{k})$ is defined in analogy to Eq. (17), but for the antiquark number operator. The leptonic tensor $l^{\mu\nu}_{\overline{v}vvv}$, corresponding to the v spinor, is defined by

$$l^{\mu\nu}_{\overline{v}vvv}(k,s,k',s')=\overline{v(k,s)}\gamma^\mu v(k',s')\,\overline{v(k',s')}\gamma^\nu v(k,s), \tag{20}$$

yielding, after summation over the spin s'

$$\begin{aligned}l^{\mu\nu}_{\overline{v}vvv}(k,k',s)&=2[k^\mu k'^\nu+k'^\mu k^\nu+g^{\mu\nu}(m^2-k\cdot k')\\&\quad+im\epsilon^{\mu\nu\alpha\beta}(k-k')_\alpha s_\beta].\end{aligned} \tag{21}$$

All other terms give vanishing contributions to $W^{\mu\nu}$ due to the fact that all parton momenta lie in the Breit sphere.

Thus, the symmetric part of $W^{\mu\nu}$, corresponding to the unpolarized structure functions, yields the following result in the Breit frame. The only nonzero components are those with $\mu=\nu=0,1,2$:

$$W^{\mu\mu}_{\rm sym}=\frac{E}{4}\int d^2k_\perp \frac{l^{\mu\mu}_{\rm unpol}(k_Q,k'_Q)}{Q\omega(\vec{k}_Q)}\sum_s [\Pi^b_s(P,\vec{k}_Q)+\Pi^d_s(P,\vec{k}_Q)], \tag{22}$$

where $l^{\mu\nu}_{\rm unpol}$ denotes the symmetric part of $l^{\mu\nu}_{\underline{v}\underline{v}\underline{v}\underline{v}}$ and $l^{\mu\nu}_{\underline{v}\underline{v}\underline{v}\underline{v}}$,

$$l^{\mu\nu}_{\rm unpol}(k,k')=2[k^\mu k'^\nu+k'^\mu k^\nu+g^{\mu\nu}(m^2-k\cdot k')]. \tag{23}$$

In particular, one has

$$\begin{aligned} l^{00}_{\rm unpol}(k_Q,k'_Q)&=4[m^2+(\vec{k}_\perp)^2],\\ l^{11}_{\rm unpol}(k_Q,k'_Q)&=4(k_1)^2+Q^2,\\ l^{22}_{\rm unpol}(k_Q,k'_Q)&=4(k_2)^2+Q^2. \end{aligned} \tag{24}$$

Let us now consider the Bjorken limit of these expressions. The Bjorken limit is defined by $Q\to\infty$ and $x_B=$const. In the Breit frame, this implies for the proton momentum P_3 that $P_3\to\infty$ and $Q/P_3=2x_B=$const. Thus, we compute the kinematical factors $E=\sqrt{M^2+P_3^2}\sim Q/2x_B$ and

$$\frac{l^{00}_{\rm unpol}(k_Q,k'_Q)}{Q\omega(\vec{k}_Q)}=\frac{4[m^2+(\vec{k}_\perp)^2]}{Q\sqrt{Q^2/4+(\vec{k}_\perp)^2+m^2}}\sim Q^{-2}\to 0. \tag{25}$$

This implies

$$W^{00}_{\rm sym}\to 0. \tag{26}$$

On the other hand, Eq. (10) yields in the Bjorken limit

$$W^{00}=-F_1+\frac{E_P^2}{P_3Q}F_2\to -F_1+\frac{1}{2x_B}F_2. \tag{27}$$

Thus, the last two equations imply, in the Bjorken limit, that the Callan-Gross relation holds

$$2x_BF_1=F_2, \tag{28}$$

which is the standard result as in the parton model. Similarly, we compute

$$\begin{aligned} \frac{l^{11}_{\rm unpol}(k_Q,k'_Q)}{Q\omega(\vec{k}_Q)}&=\frac{4(k_1)^2+Q^2}{Q\sqrt{Q^2/4+(\vec{k}_\perp)^2+m^2}}\to 2,\\ \frac{l^{22}_{\rm unpol}(k_Q,k'_Q)}{Q\omega(\vec{k}_Q)}&=\to 2. \end{aligned} \tag{29}$$

Thus, we obtain the following result for $W^{00}_{\rm sym}$, $W^{11}_{\rm sym}$, and $W^{22}_{\rm sym}$, in the Bjorken limit

$$\begin{aligned} W^{00}_{\rm sym}&=0,\\ W^{11}_{\rm sym}=W^{22}_{\rm sym}&\to\frac{P_3}{2}\int d^2k_\perp\sum_s [\Pi^b_s(P,\vec{k}_Q)+\Pi^d_s(P,\vec{k}_Q)]. \end{aligned} \tag{30}$$

Our regularization scheme allows direct computation of the parton distribution function. We define

$$f(p_3,P_3,\mu)=\int d^2k_\perp\sum_s [\Pi^b_s(P,\vec{k}_Q)+\Pi^d_s(P,\vec{k}_Q)], \tag{31}$$

where p_3 is the longitudinal parton momentum, P_3 is the longitudinal proton momentum, and μ is a fixed but arbitrary scale parameter with dimension of mass (e.g., $\Lambda_{\rm QCD}$). $f(p_3,P_3,\mu)$ is the probability of finding a parton with longitudinal momentum p_3 in a bound state (proton) with longitudinal momentum P_3, where momenta are measured in terms of the scale μ. The $W^{\mu\nu}$ tensor is dimensionless. Thus, we have

$$W^{22}_{\rm sym}=\frac{P_3}{2}f. \tag{32}$$

Thus, $G(p_3,P_3,\mu)=P_3f(p_3,P_3,\mu)$ is a dimensionless function which scales

$$G(\lambda p_3,\lambda P_3,\lambda\mu)=G(p_3,P_3,\mu). \tag{33}$$

We have shown in the Bjorken limit that $W^{22}_{\rm sym}=F_1$, thus

$$F_1(x,Q,\mu)=\frac{P_3}{2}f(p_3,P_3,\mu). \tag{34}$$

Expressing p_3 and P_3 in terms of Q and x yields $p_3=Q/2$ and $P_3=Q/2x$ and hence

$$F_1(x,Q,\mu)=\frac{Q}{4x}f(Q/2,Q/2x,\mu). \tag{35}$$

Making a scale transformation by multiplying all variables of dimension mass by λ, where λ is chosen to obey $\lambda P_3=1$, yields

$$\begin{aligned} F_1(x,Q,\mu)&=\tfrac{1}{2}f(x,1,2x\mu/Q),\\ F_2(x,Q,\mu)&=xf(x,1,2x\mu/Q). \end{aligned} \tag{36}$$

Here, $f(x,1,2x\mu/Q)$ denotes the probability of finding a parton with a longitudinal momentum fraction x for a total longitudinal momentum $=1$, where the scale is given by $2x\mu/Q$. Note that in Eq. (30) the sum runs over all spin values. For a spin 1/2 parton this is equivalent to a sum over the helicity quantum numbers $+$ and $-$. If we take into account e_i, the electric charge of a quark with flavor i relative to the charge of the electron, and redefine Eq. (31) by $f=\int d^2k_\perp\Sigma_s\Pi^b_s$ and $\overline{f}=\int d^2k_\perp\Sigma_s\Pi^d_s$ we obtain, from Eq. (36),

$$\begin{aligned} F_1(x,Q)=\sum_i e_i^2\tfrac{1}{2}[&f^{(i)}_+(x,1,2x\mu/Q)+f^{(i)}_-(x,1,2x\mu/Q)\\ &+\overline{f}^{(i)}_+(x,1,2x\mu/Q)+\overline{f}^{(i)}_-(x,1,2x\mu/Q)],\\ F_2(x,Q)=\sum_i e_i^2x[&f^{(i)}_+(x,1,2x\mu/Q)+f^{(i)}_-(x,1,2x\mu/Q)\\ &+\overline{f}^{(i)}_+(x,1,2x\mu/Q)+\overline{f}^{(i)}_-(x,1,2x\mu/Q)]. \end{aligned} \tag{37}$$

The standard expression from the parton model [10,31] is given by

$$F_1(x)=\sum_i e_i^2\tfrac{1}{2}[q_+^{(i)}(x)+q_-^{(i)}(x)+\overline{q_+^{(i)}}(x)+\overline{q_-^{(i)}}(x)],$$

$$F_2(x)=\sum_i e_i^2 x[q_+^{(i)}(x)+q_-^{(i)}(x)+\overline{q_+^{(i)}}(x)+\overline{q_-^{(i)}}(x)]. \tag{38}$$

In the Bjorken limit our result, Eqs. (37), agrees with the standard result, Eqs. (38). As can be seen, the quark distribution function $q(x)$ occurring in Eqs. (38) does not have any Q dependence. It corresponds to the naive parton model, which has no Q dependence. However, perturbative QCD introduces a Q dependence via logarithmic corrections (violation of scaling). Thus, one arrives at $q(x,Q,\mu)$ which is a quark distribution function from a ''renormalization group-improved parton model.'' $q(x,Q,\mu)$ is interpreted as the probability to find a parton with momentum fraction x in a hadron with momentum $=\infty$, where the resolution (by the photon) Q is finite, and momenta are measured in terms of a mass scale μ. Note that our distribution function $f(x,1,2x\mu/Q)$ has a different interpretation than $q(x,Q,\mu)$: $f(x,1,2x\mu/Q)$ corresponds to the Breit frame where the hadron moves fast but with *finite* momentum, while $q(x,Q,\mu)$ corresponds to the *infinite* momentum frame. However, in the Bjorken limit both coincide.

C. Relation between structure functions and distribution functions in the Breit frame for QCD: Polarized structure functions

The antisymmetric part of the $W^{\mu\nu}$ tensor describing the spin-dependent part can be parametrized in terms of the spin structure functions g_1 and g_2 [31]:

$$W_{\rm as}^{\mu\nu}=-i\epsilon^{\mu\nu\sigma\rho}q_\sigma\left[\frac{S_\rho}{\nu}(g_1+g_2)-\frac{q\cdot SP_\rho}{\nu^2}g_2\right]. \tag{39}$$

Here, S denotes the proton spin, with $P\cdot S=0$ and $S^2=-M^2$. The proton spin can be polarized in two ways: $\vec{S}\|\vec{P}$ (longitudinal, helicity) or $\vec{S}\perp\vec{P}$ (transverse). In order to extract both spin structure functions from the tensor $W_{as}^{\mu\nu}$ one needs both polarizations. Let us consider first longitudinal polarization. Then we find, in the Breit frame,

$$W_{\rm as}^{12}=i\left[g_1-\left(\frac{M}{P_3}\right)^2 g_2\right],$$

$$W_{\rm as}^{21}=-W_{\rm as}^{12}. \tag{40}$$

All other elements of $W_{\rm as}^{\mu\nu}$ vanish. Now, let us consider transverse polarization. We then find, in the Breit frame,

$$W_{\rm as}^{02}=-i\frac{M}{P_3}(g_1+g_2),$$

$$W_{\rm as}^{20}=-W_{\rm as}^{02}. \tag{41}$$

Once again, all other elements of $W_{\rm as}^{\mu\nu}$ vanish.

In the following let us consider the case where the proton, as well as the partons, are polarized longitudinally (helicity). We want to compute in the Breit frame the 12 component of the hadronic tensor $W_{\rm as}^{\mu\nu}$. We obtain

$$W_{\rm as}^{12}=\frac{E}{4}\int d^2k_\perp\sum_s\frac{l_{\rm pol}^{12}(k_Q,k_Q',s)}{Q\omega(\vec{k}_Q)}[\Pi_s^b(P,\vec{k}_Q)+\Pi_s^d(P,\vec{k}_Q)], \tag{42}$$

where we have defined

$$l_{\rm pol}^{12}(k_Q,k_Q',s)=l^{12}_{\overline{uuuu}}(k_Q,k_q',s). \tag{43}$$

In the Breit frame the antisymmetric part of the leptonic tensor $l^{\mu\nu}_{\overline{uuuu}}(k,k',s)$, Eq. (15), and of $l^{\mu\nu}_{\overline{vvvv}}(k,k',s)$, Eq. (20), are given by

$$l^{12}_{\overline{uuuu}}(k,k',s)=i2Qs_0,$$

$$l^{12}_{\overline{vvvv}}(k,k',s)=-i2Qs_0. \tag{44}$$

For the parton spin s the following holds: $s\cdot k=0$, $s^2=-m^2$. In the helicity basis, one has $\vec{s}\|\vec{k}$; thus one defines the parton helicity h by

$$h_s=\frac{\vec{s}\cdot\vec{k}}{|\vec{k}|\cdot\omega(\vec{k})}. \tag{45}$$

In the Bjorken limit one obtains

$$\frac{l_{\rm pol}^{12}}{Q\omega(\vec{k_Q})}\to i2h_s, \tag{46}$$

and, hence,

$$W_{\rm as}^{12}\to i\frac{P_3}{2}\int d^2k_\perp\sum_s h_s[\Pi_s^b(P,\vec{k}_Q)+\Pi_s^d(P,\vec{k}_Q)]. \tag{47}$$

Equation (40) implies, in the Bjorken limit,

$$W_{\rm as}^{12}\to ig_1. \tag{48}$$

From this and Eq. (47), after doing the same scale change as in the unpolarized case, we arrive at

$$g_1(x,Q)=\frac{1}{2}\sum_i e_i^2[h_+f_+^{(i)}(x,1,2x\mu/Q)+h_-f_-^{(i)}(x,1,2x\mu/Q)+h_+\overline{f_+^{(i)}}(x,1,2x\mu/Q)+h_-\overline{f_-^{(i)}}(x,1,2x\mu/Q)]. \tag{49}$$

This is in agreement with the standard result of the parton model [31]:

$$g_1(x)=\frac{1}{2}\sum_i e_i^2[h_+q_+^{(i)}(x)+h_-q_-^{(i)}(x)+h_+\overline{q_+^{(i)}}(x)+h_-\overline{q_-^{(i)}}(x)]. \tag{50}$$

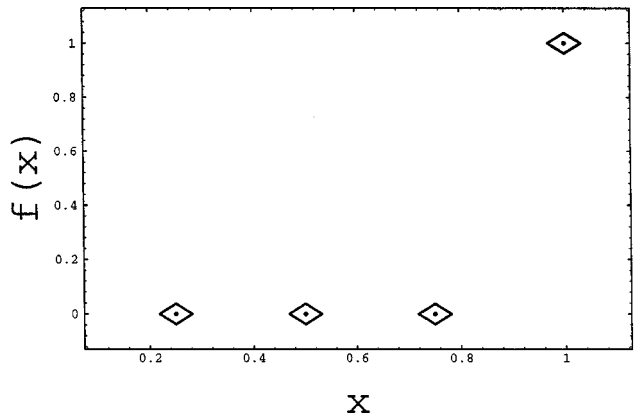

FIG. 3. The distribution function $f(x_B)$ of ϕ^4_{3+1} versus the momentum fraction x_B. $\lambda = 0.003\,457\,39$ (as in Fig. 1); $\Lambda/\Delta p = 4$.

In summary of this section, we have computed analytically using Breit-frame regularization the relation between the hadronic tensor, structure functions, and parton distribution functions. The main results are given by Eqs. (22) and (23) and Eqs. (42) and (43). The results are ''renormalization group improved'' compared to the naive parton model, taking into account parton mass m and scale parameter μ.

D. Numerical results for distribution functions of the scalar model

In this section we want to show how distribution functions can be computed numerically using the Hamiltonian approach and Breit-frame regularization. We apply the method to the scalar model in $3+1$ dimensions. This model has been extensively studied and represents, for finite cutoff, a nontrivial effective theory. We compute distribution functions for the ϕ^4 model. Let us consider the ϕ^4_{3+1} model. The first excited state consists of only one parton (three-particle contributions have been found to be extremely small, i.e., within the error margin). This is because our particular choice of regularization (Breit frame), which makes all parton momenta positive, implies that for the Hamiltonian the terms which conserve particle number are dominant. Consequently, the distribution function is peaked at $x_B = 1$, as shown in Fig. 3. We did not find a noticeable dependence on the resolution Q, i.e., many-particle effects are absent. We have not made any effort to obtain a finer x_B resolution because this state does not display the interesting structure of a bound state. We have also observed that higher excited states display a dominant 2-, 3-, 4-particle content (with very small mixtures between different sectors). One should note, however, that the simplicity of the first excited state is due to the fact that the positivity of the longitudinal parton momenta prevents the creation of partons directly from the vacuum. Had we worked in the rest frame ($\vec{P} = 0$), the ''valence parton'' of the first excited state would be surrounded by a large cloud of partons with opposite momenta $\vec{p}$ and even the vacuum, lying in the $\vec{P} = 0$ sector, would be made up of such a cloud.

E. Scaling in the Bjorken limit

It is physically very important to study the scaling behavior in the Bjorken limit of parton distribution functions. The most interesting case to do this is QCD, where perturbative QCD calculations based on asymptotic freedom predict logarithmic deviations from Bjorken scaling. It would be most important to have numerical results on this, and it is our goal of future work to go in this direction. In this work we have considered the scalar model, because it is a relatively simple model, however, having a nontrivial critical behavior, and it has been widely studied. This makes it a suitable benchmark model to test a new method. Thus, the close agreement found for the mass spectrum in the critical regime with lattice results is one of the most important results of this work.

Concerning the scaling behavior of distribution functions in the scalar model, we would like to make the following remarks.

(i) In order to test the scaling behavior, it would be necessary to compare numerical results obtained from our method with analytical perturbation calculations, based on asymptotic freedom. One of the few models having asymptotic freedom, except non-Abelian gauge theories, is the scalar ϕ^3 theory in six dimensions [32]. However, it is unbounded from below and does not give a physical mass spectrum.

(ii) Despite the defect of unboundedness from below, we have done numerical studies of the scalar ϕ^3 theory in $3+1$ dimensions. While the ϕ^4 theory has no bound states [1], the ϕ^3 theory has some bound states with nontrivial distribution functions. In QCD the scale dependence occurs in the running coupling constant $\alpha_s \sim 1/\ln(Q^2/\Lambda^2)$. In the ϕ^3 theory, there is no critical line with a second order phase transition, and there is no physical mass spectrum. Therefore, we have looked at the distribution function as a function x_B and of the bare coupling g_0. For $g_0 = 0$ we have found a one-particle distribution function which, for increasing g_0, smoothly goes over to a distribution function with many-parton signature. In particular, it yields a substantial increase for small x_B, similar to the behavior seen in QCD.

(iii) As mentioned above, the ϕ^4 theory has no bound states. This property of the model has also been seen in parton distribution functions, which shows the one-particle structure (Fig. 3). We have looked at its scaling behavior. But instead of changing the momenta p, P, and Q^2, such that $Q^2 \to \infty$ and $x_B = \text{const}$, we have kept the momentum cutoff, which corresponds to $Q^2 = \text{const}$, but have varied the model parameters λ, κ along the critical line, such that renormalized mass m_{ren} and renormalized coupling constant g_{ren} go to zero. As a result one finds for the parton distribution function that in addition to the dominant contribution from the one-particle state, there are very small contributions from three-particle states. The three-particle contribution relative to the one-particle contribution is in the order of 10^{-5}, i.e., of the size of the numerical error. We have chosen not to present this in the figure.

V. APPLICATION TO GAUGE THEORIES

Given the fact that the most important physical models are gauge theories, we want to discuss the treatment of gauge theories in the Hamiltonian formulation with Breit-frame regularization. In the previous sections we have given arguments and numerical results showing the usefulness of a momentum lattice regularization in connection with the Breit

frame. The usefulness of a momentum lattice corresponding to the rest frame has previously been investigated and demonstrated by several workers: Kuti and co-workers [22] have investigated the one-component scalar ϕ^4 model and the O(4) symmetric scalar model and estimated a bound on the Higgs boson mass. Kröger and co-workers [23] have solved the Langevin equation on a momentum lattice for the scalar ϕ^4_{3+1} model and extracted critical behavior. Glueball scattering in compact QED_{2+1} (QCD-like model) has been computed on a momentum lattice in Ref. [20]. Properties of nuclear matter have been computed by Brockmann and Frank [24]. Kogut and Lagaë [25] have studied the phase diagram of quenched QED on a momentum lattice. Espriu and Traresset [26] have studied the renormalization group flow by use of a momentum lattice. Finally, Koutsoumbras [27] has computed the gluon propagator of finite temperature QCD from a momentum lattice. Thus, momentum lattice regularization has proven to be useful when numerically studying physics near a critical point.

When one treats gauge theories on a momentum lattice the following problem occurs: If one takes the gauge fields $A_\mu(k_i)$ as variables (so-called noncompact formulation), where k_i denotes a momentum lattice, then the gauge action is not manifestly gauge invariant. As a consequence, one has observed nonlocal counterterms when computing, from lattice perturbation theory, the axial anomaly and the one-loop vacuum polarization. This has been seen by Karsten and Smit [33] by computing the triangle diagram using the SLAC derivative in the action and by Kröger and co-workers [34] using an action defined on a momentum lattice with a momentum cutoff Λ. As Wilson has pointed out, it is desirable to conserve gauge symmetry manifestly in a regularized gauge theory. E.g., there is numerical evidence [35] that a lattice action which is not manifestly gauge invariant yields no area law for the Wilson loop in pure SU(2) gauge theory.

The space-time lattice Hamiltonian, corresponding to the Wilson action and being manifestly gauge invariant, has been constructed by Kogut and Susskind [36]. Here, we are confronted with the following problem: How to introduce a momentum lattice as a regulator while manifestly conserving gauge invariance? We suggest doing this as follows: We take closed Wilson loops as variables (for pure gauge theory without color charges). The Hilbert space is then built from these loops. Gauge invariance corresponds to satisfying Gauss's law

$$G|\phi>=0. \tag{51}$$

For a fixed lattice site i, one has $G_i=\Sigma_{\{i,j\}\ni i} l^a_{ij}$, i.e., the sum over generators of gauge transformations (where the temporal gauge is fixed). States corresponding to closed loops obey this law, while states corresponding to open strings fail to obey it. The physical states are color singlet states, and thus open string states are unphysical. Nevertheless, we will make use of them as an intermediate step in constructing a Hilbert space of states obeying the Breit condition.

In order to introduce a regularization, we start from a conventional space-time lattice (regular, hypercube) with lattice spacing a. Next, closed loops as well as open strings are defined as curves connecting adjacent lattice sites (straight line between neighboring lattice sites). E.g., a loop state is given by

$$|\phi(x_1)\rangle=|U_\mu(x_1)U_\nu(x_1+a\hat{\mu})\cdots U_\omega(x_N)\rangle, \tag{52}$$

where $x_N+a\hat{\omega}=x_1$. In order to introduce a momentum lattice we make a discrete Fourier transformation

$$|\tilde{\phi}(k_i)\rangle=a\sum_{x_i}\exp[-ix_ik_i]|\phi(x_i)\rangle, \tag{53}$$

where each component of k_i runs over the Brillouin zone $-\pi/a$ to $+\pi/a$. One can define the lattice momentum operator P_μ via the lattice translation $T_\mu(a)$, which translates each configuration on the lattice by an increment a in the direction μ. It is given by

$$T_\mu(a)=\exp[-iaP_\mu]. \tag{54}$$

The eigenvalues of P are k_i, which are the possible momenta of the loop state.

In order to construct states with well-defined momentum, obeying the Breit condition, as well as satisfying gauge invariance, we suggest to proceed as follows: We construct a Hilbert space built from link states. Using discrete Fourier transformation we associate a discrete lattice momentum to each link, say $\tilde{U}_\mu(k_i)$. Then we construct multiple link states

$$|\psi(k_1,\ldots,k_N)\rangle=|\tilde{U}_\mu(k_1)\tilde{U}_\nu(k_2)\cdots\tilde{U}_\omega(k_N)\rangle. \tag{55}$$

This state corresponds to momentum $k_{\mathrm{tot}}=k_1+\cdots+k_N$. We then impose the Breit condition [see Eq. (4)]

$$(\vec{k}_i-\vec{P}/2)^2\leqslant(\vec{P}/2)^2, \tag{56}$$

where the parton momenta are given by the lattice momenta of the links. Thus, as in the scalar model, positivity of parton momenta in all three components and a given total momentum gives a bound $|\vec{P}|/\Delta p$ to the total number of links and thus gives a strong bound on the dimension of the effective Hilbert space. Eventually, we implement gauge symmetry by requiring Gauss's law, Eq. (51), to be respected. Hence the Breit condition and Gauss's law define our basis of Hilbert states.

VI. *S* MATRIX

The Hamiltonian in the Breit-frame regularization has been shown above to be a suitable tool in the scalar model for computation of the mass spectrum and physics at the critical line, as well as distribution functions. In this section we want to suggest that it is also a valuable tool for scattering phenomena and in particular for the nonperturbative computation of the S matrix. When considering the nonperturbative computation of scattering observables, standard Euclidean lattice field theory is faced with the following problem: The scattering matrix elements are directly related to Minkowski n-point functions. On the lattice one can compute Euclidean n-point functions. In principle, there is an analytic continuation between these two types of n-point functions. However, when the Euclidean n-point function is

only known at some lattice points within the uncertainty of statistical errors, it is very difficult (almost impossible) to get reliable numerical results from an analytic continuation. A way out of this dilemma has been proposed by Lüscher [37]. The idea is that continuum scattering phases can be extracted from the finite-size behavior of a mass spectrum on a finite lattice. This requires mass calculations via standard Euclidean lattice techniques, but also requires quite precise data in order to resolve finite-size effects.

An alternative way to compute nonperturbatively an S matrix has been suggested by Kröger [21]. The idea is the following. The S matrix, as has been introduced by Heisenberg [38] and Møller [39], is defined as

$$S=\langle\psi^{(-)}|\psi^{(+)}\rangle, \tag{57}$$

which is the probability amplitude to find an outgoing scattering state in an incoming scattering state. The scattering states are characterized by two conditions: (i) they are eigenstates of the Hamiltonian, and (ii) for $t\to\pm\infty$ they approach an asymptotic state. The asymptotic state describes two noninteracting particles (in the case of two-particle scattering). The so-called Møller operator maps the asymptotic states $|\phi^{\mathrm{as}}\rangle$ onto the scattering states $|\phi^{(\pm)}\rangle$:

$$|\phi^{(\pm)}\rangle=\Omega^{(\pm)}|\phi^{\mathrm{as}}\rangle=s-\lim_{t\to\mp\infty}\exp[iHt]\exp[-iH^0t]\phi^{\mathrm{as}}\rangle. \tag{58}$$

These equations define scattering states and the S matrix. They can be carried over to quantum field theory with some care.

A. Asymptotic one- and two-particle states

One problem in constructing the S matrix is the construction of asymptotic one-particle states, asymptotic two-particle states, etc. In constructive quantum field theory this is resolved by Haag-Ruelle theory [40], which indicates how to construct asymptotic one-particle states through the application of suitable local field operators on the physical vacuum:

$$|1\rangle_{\mathrm{phys}}=a^\dagger(f)|0\rangle_{\mathrm{phys}}. \tag{59}$$

A two-particle state is given by

$$|2\rangle_{\mathrm{phys}}=a^\dagger(f_1)a^\dagger(f_2)|0\rangle_{\mathrm{phys}}. \tag{60}$$

Here $a^\dagger$ is the creation operator of a one-particle state with wave function f created from the physical vacuum. The existence of such an operator has been proven by Haag and Ruelle [40]. An explicit form of this operator for the case of glueball states in pure gauge theory has been given by Lüscher [41]. However, Haag-Ruelle theory says nothing about how to find the physical vacuum. In the Hamiltonian approach in connection with the Breit-frame regularization, as advocated here, we avoid constructing the physical vacuum. Thus we follow an alternate route to the Haag-Ruelle theory. We directly construct a one-particle state with momentum $\vec{p}$ directly by calculating an eigenvector of the regularized Hamiltonian H,

$$H|\vec{p}\rangle=E(\vec{p})|\vec{p}\rangle. \tag{61}$$

The property of being a one-particle state is verified by computing its mass (see Sec. III). If, e.g., its mass is the lowest mass in the mass spectrum, the state $|\vec{p}\rangle$ is a one-particle state. Let $|f\rangle$ denote such a one-particle state with a momentum distribution given by a wave function f. In the language of Haag-Ruelle theory, the explicit construction of the state $|f\rangle$ means that we have found a creation operator $A^\dagger(f)$ with

$$|1\rangle=A^\dagger(f)|0\rangle_{\mathrm{free}}. \tag{62}$$

I.e., it creates a one-particle state from the vacuum of the regularized free Hamiltonian. There is a theorem by Haag [42] which says that in the continuum limit of relativistic quantum field theory, the physical Hilbert states of the interacting field theory (Hamiltonian) have nothing to do with those of the free field theory (free Hamiltonian). In particular, there is no unitary transformation between the physical vacuum to the free vacuum. However, this theorem does not apply when we consider the *regularized* field theory (Hamiltonian). Then there is a unitary transformation U, mapping the (regularized) free vacuum onto the (regularized) physical vacuum,

$$|0\rangle_{\mathrm{phys}}^{\mathrm{reg}}=U|0\rangle_{\mathrm{free}}^{\mathrm{reg}}. \tag{63}$$

This relates the Haag-Ruelle creation operator $a^\dagger(f)$ (of the regularized field theory) to the creation operator $A^\dagger(f)$ through

$$a^\dagger(f)=UA^\dagger(f)U^{-1}. \tag{64}$$

Finally, using $A^\dagger(f)$ from Eq. (42), we can construct, in analogy to Eq. (40), asymptotic noninteracting two-particle states given by

$$|2\rangle=A^\dagger(f_1)A^\dagger(f_2)|0\rangle_{\mathrm{free}}. \tag{65}$$

B. Møller wave operators and S matrix

Let us denote by $|\phi^{\mathrm{as}}(\vec{p}_1,\vec{p}_2)\rangle$ the asymptotic two-particle state, corresponding to two noninteracting particles with momentum $\vec{p}_1$ and $\vec{p}_2$, respectively. Then the Møller wave operator is given by

$$\begin{aligned}|\phi_{\mathrm{scatt}}^{(\pm)}(T)\rangle&=\Omega^{(\pm)}(T)|\phi^{\mathrm{as}}(\vec{p}_1,\vec{p}_2)\rangle\\&=\exp[\mp iHT]\exp\{\pm i[E(\vec{p}_1)+E(\vec{p}_2)]T\}\\&\quad\times|\phi^{\mathrm{as}}(\vec{p}_1,\vec{p}_2)\rangle.\end{aligned} \tag{66}$$

Here, $E(\vec{p})$ denotes the energy-momentum dispersion relation of the one-particle state of mass m. H denotes the regularized Hamiltonian. The time parameter t, which goes to infinity in the continuum limit, has to be chosen to take a positive finite value T in the regularized theory (see below). In a similar way, one can construct the S matrix

$$S_{\mathrm{fi,in}}(T)=\langle\phi_{\mathrm{fi}}^{\mathrm{as}}|\exp\{i[E(\vec{p'}_1)+E(\vec{p'}_2)]T\}\times\exp[-i2HT]\exp\{i[E(\vec{p}_1)+E(\vec{p}_2)]T\}|\phi_{\mathrm{in}}^{\mathrm{as}}\rangle. \tag{67}$$

From the numerical point of view, the computation of the S-matrix element proceeds most simply by diagonalizing the regularized Hamiltonian

$$H|\eta_\nu\rangle=\epsilon_\nu|\eta_\nu\rangle,\quad \nu=1,2,\ldots,$$

$$\exp[iHT]=\sum_\nu |\eta_\nu\rangle\exp[i\epsilon_\nu T]\langle\eta_\nu|. \tag{68}$$

How should one choose the scattering time parameter T? When applying this time-dependent Hamiltonian method to nonrelativistic quantum mechanics as well as to field theory models [21], the following general observations have emerged from numerical calculations: The matrix element $S_{\mathrm{fi,in}}(t)$ considered as a function of t has the following behavior. At $t=0$ it takes the value $\langle\phi_{\mathrm{fi}}^{\mathrm{as}}|\phi_{\mathrm{in}}^{\mathrm{as}}\rangle$ (in the case of elastic scattering). When increasing t it deviates from the starting value and eventually reaches a plateau region. With a further increase in t, it leaves the plateau region and after a while exhibits an (irregular) oscillatory behavior. The plateau region is the region of physical interest. Its existence can be shown analytically for nonrelativistic potential scattering (see Ref. [21] and references therein). The location and size of this plateau region depends on the model and dimension. In particular, it depends on the dimension of the regularized Hamiltonian. When increasing this dimension, i.e., when exploring a larger Hilbert space, the size of the plateau region becomes larger. In the continuum limit, when the S matrix converges, the size of the plateau should become infinitely large. The time parameter T should be chosen from this plateau region, either by determining where the matrix element $S_{\mathrm{fi,in}}(t)$ has the smallest variation with changing t, or by the following criterion of conservation of energy: In the continuum limit, energy conservation in a scattering reaction means that

$$\langle\psi^{(\pm)}|H|\psi^{(\pm)}\rangle=E_{\mathrm{as}}, \tag{69}$$

where E_{as} denotes the energy of the asymptotic noninteracting two-particle state. Thus we define the function

$$\Delta E(t)=|\langle\Omega^{(\pm)}(t)\phi^{\mathrm{as}}(\vec{p}_1,\vec{p}_2)|H\times|\Omega^{(\pm)}(t)\phi^{\mathrm{as}}(\vec{p}_1,\vec{p}_2)\rangle-E_{\mathrm{as}}|/E_{\mathrm{as}}, \tag{70}$$

where $\Omega^{(\pm)}$ is given by Eq. (58). This function is a measure of violation of energy conservation in a scattering reaction computed with the regularized Hamiltonian at some finite time t. In the continuum limit this should be zero. Thus we can choose the time parameter T such that $\Delta E(t)$ is minimum. Numerical experience has shown that the value of T determined as the position of minimal variation of the S-matrix element and its corresponding value, taken as the position of the minimum of ΔE, agree quite well. This is an indication of consistency.

In order to get the physical S matrix one has to carry out renormalization and take into account the vacuum structure. Renormalization means that first one has to determine the counterterms in the Hamiltonian. E.g., for the scalar ϕ^4 model, one has to renormalize the wave function, the mass, and the coupling constant. Then one computes physical observables like, e.g., masses or scattering cross sections and tunes the bare parameters of the model, such that the physical observables remain fixed. Finally, the vacuum structure needs some careful treatment. The computation of the S matrix, as described above, yields the full S matrix, which includes the connected part (which is the part observed in scattering experiments) but also all disconnected parts. The factorization of n-point Green's functions into connected pieces is known as vacuum structure [43]. This allows the extraction of the connected part of the S matrix.

The time-dependent Hamiltonian method, as described above, but using of the rest-frame regularization instead of the Breit-frame regularization, has been applied to glueball scattering in compact U(1) gauge theory (compact QED) in $2+1$ dimensions [20]. To conclude this section we wish to address the question: What advantage does it bring to use the Breit-frame regularization for scattering calculations in the time-dependent Hamiltonian formulation? First, as mentioned above, the Breit-frame regularization avoids the calculation of the vacuum state when constructing asymptotic noninteracting two-particle states. Second, this regularization reduces the number of effective degrees of freedom by the same mechanism as was shown to be useful for the calculation of the mass spectrum. However, one must pay attention to the following limitation: Because we take into account parton momenta inside the sphere given by the Breit condition, Eq. (4), the momenta of the asymptotic particles, i.e., $\vec{p}_1',\vec{p}_2',\vec{p}_1,\vec{p}_2$ should lie well inside the Breit sphere. This limits the scattering reactions which can be treated. E.g., head-on collisions are not included. However, this constraint is not very stringent because a suitable Lorentz boost can be applied to map the momenta into the Breit sphere.

VII. FINITE DENSITY THERMODYNAMICS

The computation of thermodynamic observables at finite temperature and finite density is an important problem in the physics of neutron stars, high energy heavy ion collisions, as well as the question of phase transitions from the hadronic phase to a quark-gluon plasma in QCD. However, when treating finite temperature QCD in the standard Lagrangian lattice approach, there is a well-known problem when a nonzero chemical potential is included to describe finite density effects. The fermionic determinant then becomes complex yielding a complex lattice action. This has led to great difficulties when solving the model numerically via Monte Carlo methods [44]. In order to study the infrared dynamics of Yang-Mills and Yang-Mills-Higgs theories at finite temperature, which cannot be addressed by Euclidean methods, Moore [45] has suggested an improved Hamiltonian for Minkowski Yang-Mills theory.

In this section we want to discuss how finite temperature and finite density thermodynamics can be treated in a Hamiltonian formulation with the Breit frame regularization. The point is that the Hamiltonian formulation also allows the

treatment of non-Hermitian Hamiltonians (complex actions). Consider the following partition function:

$$Z=\mathrm{Trexp}\left[-\frac{1}{k_BT}(H_0+\mu N)\right], \tag{71}$$

where H_0 stands for a Hermitian Hamiltonian, μ denotes the chemical potential, and N stands for a particle number operator. Let us suppose now, for the sake of argument, that the term μN is non-Hermitian. What then is the advantage of using a Hamiltonian formulation? In a Hamiltonian formulation this partition function can be computed nonperturbatively via diagonalization of $H_0+\mu N$ in the same way as $\exp[iHt]$ has been computed in calculating the S matrix (Sec. VI).

What is the advantage of using the Breit-frame regularization? Let us consider the following scenario: One wishes to study hot nuclear matter, respectively a quark-gluon plasma in a state with total momentum $\vec{P}\neq 0$. In order to investigate this experimentally, one can perform a high energy heavy ion collision experiment with large momentum $\vec{P}$. However, it is still an open question whether thermodynamical equilibrium can be reached during the short collision time, before the decay into fragments, which would justify the use of the Boltzmann-Gibbs partition function. Putting aside, for the moment, the question of experimental realization, it is nevertheless physically interesting to ask the following question: What are the properties of matter at finite temperature and finite density at thermodynamical equilibrium in a sector of momentum $\vec{P}\neq 0$? Thus we consider the partition function at momentum $\vec{P}$:

$$Z(\vec{P})=\mathrm{Tr}\left\{\exp\left[-\frac{1}{k_BT}(H_0+\mu N)\right]\right\}\Big|_{\vec{P}}. \tag{72}$$

The evaluation of this function now can be done in the Breit-frame regularization, which will reduce the effective number of degrees of freedom, i.e., the dimension of the effective Hilbert space for the same reason as it did in the computation of structure functions.

VIII. SUMMARY

In conclusion, we have suggested a Hamiltonian method and a momentum regularization corresponding to the Breit frame. We have shown that this method allows one to extract continuum physics by presenting numerical results for the ϕ^4_{3+1} theory in the symmetric phase close to the critical line. We find a close agreement with the solution of the renormalization group equations by Lüscher and Weisz. We have seen scaling behavior of several low-lying masses near the critical point. Using the Breit frame, we have computed analytically for DIS in QCD the relation between the hadronic tensor, the structure functions, and the quark distribution functions. In the Bjorken limit we find the conventional relations between F_1, F_2, g_1, and the quark distribution functions. We have presented numerical results for parton distribution functions for the ϕ^4 model. We have proposed how the Breit-frame regularization can be applied to gauge theories, while keeping gauge symmetry manifestly conserved. We have suggested that this regularization might be useful also for the computation of scattering reactions (S matrix), as well as finite temperature and finite density thermodynamics. We are optimistic that the method can be applied to numerically compute structure functions in QCD; this work is in progress.

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APPENDIX

We wish to show that the renormalized mass defined in this work, in a Hamiltonian formulation, agrees with the standard definition in Euclidean lattice field theoery and, in particular, with the definition used by Lüscher and Weisz [1]. We have defined the renormalized mass by $m_R=M_1$, which is the lowest energy eigenvalue of the Hamiltonian at $\vec{p}=0$. The mass is measured relative to the vacuum, i.e., the vacuum energy is subtracted. This mass corresponds to the mass as it is computed in Euclidean lattice field theory from the exponential decay of the two-point function. Subtraction of the vacuum energy corresponds, in the two-point function, to consider the connected two-point function. The connected Euclidean two-point function is given by $\langle\phi(x)\phi(0)\rangle_c$, where x denotes the lattice site. Let $x\equiv(\vec{x},\tau)$, where τ is the Euclidean time. Then

$$\phi(\vec{x},\tau)=\exp(H\tau)\,\phi(\vec{x},0)\exp(-H\tau) \tag{A1}$$

and

$$\langle\phi(x)\phi(0)\rangle_c=\sum_n |\langle 0|\phi(\vec{x},\tau)|n\rangle|^2 e^{-E_n\tau}$$

$$\underset{\tau\to\infty}{\sim} |\langle 0|\phi(\vec{x},\tau)|1\rangle|^2 e^{-E_1\tau}. \tag{A2}$$

Here, $|1\rangle$ denotes the lowest lying state above the vacuum and E_1 its energy. In order to obtain the mass one projects onto momentum zero:

$$\sum_{\vec{x}} \langle\phi(\vec{x},\tau)\phi(0)\rangle_c \sim \mathrm{const}\times e^{-M_1\tau}. \tag{A3}$$

Thus the exponential fall-off behavior in Euclidean time of the connected two-point function determines the mass M_1. The connected two-point function is the inverse of the two-

point vertex function $\Gamma^{(2,0)}$, by which Lüscher and Weisz define the mass. They define the renormalized mass m_R by

$$\Gamma^{(2,0)}(p,-p) = -Z_R^{-1}[m_R^2 + p^2 + O(p^4)], \qquad \text{(A4)}$$

and the physical mass m by the pole of the renormalized propagator. The relation between m_R and m is given by

$$m = m_R[1 - 0.001\,287(g_R/16\pi^2)^2 + O(g_R^2)]. \qquad \text{(A5)}$$

Thus for the curve shown in Fig. 1, the relative difference between m_R and m is less than 5×10^{-6}, i.e., indistinguishable by eye.

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