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Nonrelativistic scattering problem by a global monopole

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We study the nonrelativistic quantum scattering problem of a charged or massive particle by the global monopole background metric. In addition to the purely gravitational effects, we consider the electrostatic or gravitational self-interaction. [S0556-2821(97)04014-9]

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I. INTRODUCTION

Global monopoles are heavy objects formed in the phase transition of a system composed of a self-coupling scalar field triplet Φ^a whose original global O(3) symmetry is spontaneously broken to U(1). The scalar matter field plays the role of an order parameter which outside the monopole core acquires a nonvanishing value. Coupling this matter field with the Einstein equations, Barriola and Vilenkin [1] have shown that the effect produced by this object in the geometry can be approximately represented by a solid angle deficit in $(3+1)$-dimensional space-time. The metric of this manifold can be expressed by the line element

$$ds^2=-dt^2+\frac{dr^2}{\alpha^2}+r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{1}$$

where the parameter $\alpha^2=1-8\pi G v^2$ is smaller than 1 and depends on the energy scale v where the global symmetry is spontaneously broken. The area of a sphere of unit radius in this manifold is not 4π but $4\pi\alpha^2$, and the surface $\theta=\pi/2$ presents the geometry of a cone with deficit angle $\Delta=8\pi^2 G v^2$.

Although this manifold presents no gravitational fields, some global effects of this geometry can be measured, for example, by the scattering cross section for massless bosonic [2] and fermionic [3] particles propagating in it.

Charged global monopoles were recently found in a system containing the global O(3) scalar triplet coupled to a local U(1) complex scalar field. For a given parameter range of the theory, the complex scalar field acquires a nonvanishing value in the core of the global monopole, thus giving it a charge. This implies that there appears an Abelian Coulomb potential [4]. The interaction of the charged monopole with a charged particle, as far as we know, has not yet been studied.

In this present paper we shall consider the quantum motion of a charged or massive particle in the background space-time metric descibed by Eq. (1); however, we shall also take into account the effect of the self-interaction potential in our analysis. In the Appendix we show that similarly to what happens in the conical space-time produced by a cosmic string [5], a charged or massive test particle, when placed in the space-time of a global monopole, becomes subjected to a self-interaction potential given by

$$U=\frac{K}{r}, \tag{2}$$

where r is the distance from the particle to the monopole, considered as a point, and

$$K=\frac{q^2S(\alpha)}{2}>0 \tag{3}$$

for the electrostatic effect or

$$K=-\frac{GM^2S(\alpha)}{2}<0 \tag{4}$$

for the induced gravitational one, q and M being the charge and mass of the particle and G the Newton's gravitational constant. The numerical factor $S(\alpha)$ is shown to be a finite positive number for $\alpha<1$ and a negative one for $\alpha>1$. These self-interaction terms are a consequence of the distortion on the particle fields caused by the solid angle deficit in this geometry.

This paper is organized as follows: In Sec. II, we obtain exact solutions for bound and scattering states in a nonrelativistic quantum-mechanical treatment. Using previous results, in Sec. III we return to the calculation of the scattering amplitude, which was already estimated by Mazur and Papavassiliou [2]. We extend their analysis by including the correction on the phase shifts due to the new interaction potential. In Sec. IV we summarize our main results. Finally, in the Appendix we derive the self-interaction potentials.

II. QUANTUM-MECHANICS ANALYSIS

This section is devoted to the nonrelativistic quantum analysis of bound and scattering states of a charged or massive particle by a global monopole. In order to do that we shall use the Schrödinger equation in spherical polar coordinates written in a covariant form in this geometry and in the presence of the self-interaction term. Then, we have

$$\left[-\frac{\hbar^2}{2M}\frac{\alpha^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\hbar^2}{2Mr^2}\mathbf{L}^2+\frac{K}{r}\right]\Psi(\mathbf{r})=E\Psi(\mathbf{r}), \tag{5}$$

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where **L** is the usual orbital angular momentum operator. This equation is similar to the Schrödinger one for a charged particle interacting with a Coulomb potential; however, the presence of the parameter α^2 in Eq. (5) makes the analysis a little more complicated, as we shall see below.

We study first the bound states, which, of course, makes sense only for the attractive ($K<0$) gravitational self-interaction.

Our procedure to obtain bound states is the standard one. We shall assume for the eigenfunction the form

$$\Psi(\mathbf{r})=R_l(r)Y_l^m(\theta,\varphi), \tag{6}$$

which substituting into Eq. (5) results in

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_l}{dr}\right)-\frac{l(l+1)}{\alpha^2 r^2}R_l+\frac{2M}{\hbar^2\alpha^2}\left(E-\frac{K}{r}\right)R_l=0. \tag{7}$$

Defining $\lambda(\lambda+1)=l(l+1)\alpha^{-2}$ and choosing the positive solution $\lambda_l=-1/2+\sqrt{\alpha^2+4l(l+1)}/2\alpha$, we can obtain for $R_l(r)$ solutions which obey appropriate boundary conditions at the origin. They are given in terms of confluent hypergeomtric function by

$$R_l(r)=r^{\lambda_l}\exp(-\kappa r)\,{}_1F_1(-\gamma+\lambda_l+1,2(\lambda_l+1);2\kappa r), \tag{8}$$

where

$$\kappa=\sqrt{-\frac{2ME}{\hbar^2\alpha^2}},\quad \gamma=-K\sqrt{-\frac{M}{2\hbar^2\alpha^2 E}}. \tag{9}$$

In order to have bound states we shall assume $E<0$, and choose appropriate parameters to terminate the series in Eq. (8). Admitting for the hypergeometric function a polynomial of degree n, we must impose

$$\gamma-\lambda_l-1=n. \tag{10}$$

With this condition we get discrete values for the energy given by

$$E_{n,l}=-\frac{MK^2}{2\hbar^2\alpha^2}\frac{1}{(n+\lambda_l+1)^2}. \tag{11}$$

From the expressions above we can see that (i) because $\alpha\neq1$, only for a few special values of this parameter, $n+\lambda_l+1$ happens to be an integer number, and so this geometry reduces the degree of the degeneracy of this Coulomb-like problem, (ii) the energy depends linearly on K^2, and so it is proportional to the square of the Newton's gravitational constant G^2, (iii) from the radial function $R_l(r)$, we can infer a gravitational Bohr radius $a_B=2\hbar^2\alpha/GM^3S(\alpha)$ for this system. [For the $\alpha>1$ case, the electrostatic Bohr radius would be $a_B=2\hbar^2\alpha/Mq^2S(\alpha)$.]

The scattering state for this system can be obtained by the sum of partial wave functions assuming $E>0$. For our spherically symmetric scatterer, only $m=0$ spherical harmonic components will be important. If we consider the incident propagating vector **k** pointing along the z axis, the wave function will be given by

$$\Psi(\mathbf{r})=\sum_{l=0}^{\infty}(2l+1)i^l R_l(r)P_l(\cos\theta). \tag{12}$$

Again substituting Eq. (12) into Eq. (5) and taking into account the long distance behavior for the solutions, the radial function can be expressed as

$$R_l(r)=r^{\lambda_l}\exp(ikr)\,{}_1F_1(\lambda_l+1+i\eta,2(\lambda_l+1);-2ikr), \tag{13}$$

where

$$k=\sqrt{\frac{2ME}{\hbar^2\alpha^2}},\quad \eta=K\sqrt{\frac{M}{2E\hbar^2\alpha^2}}. \tag{14}$$

Following the standard procedure [6] we can take the asymptotic form of the confluent hypergeometric function and give the long distance behavior for the radial function:

$$R_l(r)\simeq\frac{\Gamma(2\lambda_l+2)\exp(\pi\eta/2)}{(2k)^{\lambda_l}|\Gamma(\lambda_l+1+i\eta)|}\times\frac{\cos[kr-\pi/2(\lambda_l+1)-\eta\ln(2kr)+\gamma_l]}{kr}, \tag{15}$$

where $\gamma_l=\arg\Gamma(\lambda_l+1+i\eta)$.

Before concluding this section we would like to emphasize that from Eq. (15) it is possible to measure the effects of the global monopole metric space on the wave function of the particle. It is also possible to detect the effects on the energy of a bound state, Eq. (11). For both cases the effects can be observed in two different ways: (i) from the modification on the effective angular quantum number λ_l due to the geometry of the manifold itself and (ii) from the presence of a self-interaction term as another indirect consequence of this nontrivial topology.

In the next section we shall study the influence of the presence of both effects on the scattering wave function, particularly on the phase shift produced by this scatterer.

III. SCATTERING AMPLITUDE

Recently Mazur and Papavassiliou [2] and also Ren [3] have calculated the gravitational scattering amplitude $f(\theta)$ for a massless bosonic and a massive fermionic particle, respectively, by a global monopole. The approximation procedure adopted in [2] for small scattering angles, i.e., close to $\theta_0=\pi(1-\alpha^{-1})$, takes into account that the solid angle deficit parameter α^2 is close to 1. In fact, for a typical grand unifed theory the parameter v, associated with the scale where the global symmetry is spontaneously broken, is of the order of 10^{16} GeV, and $\Delta=1-\alpha^2=8\pi Gv^2\simeq10^{-5}$. As will be explained below in the calculation of $f(\theta)$, a series expansion in powers of Δ is developed, and for small scattering angles only large polar quantum numbers l (or $l+1/2$) will be relevant. In this section we return to the calculation of the scattering amplitude for a nonrelativistic scattering problem of a charged or massive particle by the global monopole, taking into consideration the nontrivial topology of this manifold as well as the self-interaction potential that should also be present. As we shall see, the latter contribution for $f(\theta)$ is of the same order of magnitude as the others.

From the asymptotic behavior of the scattering wave function it is possible to obtain the phase shifts δ_l, the most relevant parameter for our calculation. Using our result, Eq. (15), we can write

$$\delta_l = \frac{\pi}{2}(l - \lambda_l) + \gamma_l . \tag{16}$$

[The extra term ln(2kr) in Eq. (15) is due to the long range of the self-interaction term. As this asymptotic contribution does not depend on the partial wave component and varies slowly with r, it cannot be considered as a contribution for δ_l [7].]

The scattering amplitude can be obtained knowing the phase shifts and is given by the standard relation

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty} (2l+1)[\exp(2i\delta_l) - 1]P_l(\cos\,\theta). \tag{17}$$

Moreover, from this amplitude the total elastic cross section can be obtained by

$$\sigma = \frac{4\pi}{k}\,\text{Im}\,f(0). \tag{18}$$

Our above result for the phase shifts, Eq. (16), is an exact one, however, because it is not a simple function of l. Its direct substitution into Eq. (17) will provide us only with a formal expression for $f(\theta)$. An estimative value for the scattering amplitude can be obtained if we consider small scattering angles $\theta = O(\Delta)$. In this case, because only large angular quantum numbers are important, the scattering amplitude can be easily evaluated. It is useful to change the variable l to $z_l = l + 1/2$. Let us consider first

$$\delta_l^{(1)} = \frac{\pi}{2}(l - \lambda_l) \simeq \frac{\pi}{2}\left[z_l(1-\alpha^{-1}) + \frac{a^2}{2\alpha z_l} + O\left(\frac{a^4}{z_l^3}\right)\right], \tag{19}$$

where $a^2 = (1-\alpha^2)/4 = \Delta/4$, which agrees with a previous expansion [2].

The calculation of the self-interaction contribution follows the same procedure; however, there is an extra analysis. Because the parameter η depends on the factor $S(\alpha)$ [see Eqs. (14), (3), and (4)], which is small as shown in the Appendix, we also can develop a series in powers of η,

$$\delta_l^{(2)} = \gamma_l = \arg\Gamma(\lambda_l + 1 + i\eta), \tag{20}$$

which admits the expansion [8]

$$\delta_l^{(2)} = \eta\Psi(\lambda_l+1) + \sum_{n=0}^{\infty}\left[\frac{\eta}{\lambda_l+1+n} - \arctan\frac{\eta}{\lambda_l+1+n}\right], \tag{21}$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$ is the psi function. Taking $\arctan x \cong x - x^3/3 + \cdots$ in the equation above we get

$$\delta_l^{(2)} \cong \eta\Psi(\lambda_l+1) + O(\eta^3). \tag{22}$$

Taking now the asymptotic expansion of the psi function, $\Psi(z) = \ln z - 1/2z^2 - \sum_{n=1}^{\infty} B_{2n}/2nz^{2n}$, B_n being the Bernoulli numbers, we obtain, after some calculation,

$$\delta_l^{(2)} \cong \eta\left[\ln\left(\frac{z_l}{\alpha}\right) - \frac{1}{24z_l^2} + O\left(\frac{1}{z_l^4}\right)\right] + O(\eta^3). \tag{23}$$

Substituting Eqs. (19) and (23) into Eq. (16), we get

$$\delta_l \cong \frac{\pi}{2}\left[z_l(1-\alpha^{-1}) + \frac{a^2}{2\alpha z_l}\right] + \eta\left[\ln\left(\frac{z_l}{\alpha}\right) - \frac{1}{24z_l^2}\right] + \cdots. \tag{24}$$

In the expression above the leading term in z_l is the linear one followed by the logarithmic and so on. Comparing our result with the similar one obtained in Ref. [2], one notices the appearence of the extra logarithmic term. So our expression for the scattering amplitude takes into account the contributions from pure gravitational scattering plus a Coulomb one, as an indirect consequence of this manifold on the deformation of the electrical or gravitational field of the test particle. Considering the expansion in z_l we can write

$$\exp(2i\delta_l) \cong \alpha^{-2i\eta} e^{[i\pi z_l(1-\alpha^{-1})]} \times \left[1 + \frac{i\pi a^2}{2\alpha z_l} + 2i\eta \ln z_l + O\left(\frac{1}{z_l^2}\right)\right]. \tag{25}$$

Substituting Eq. (25) into Eq. (17) we get

$$f(\theta) = f_s(\theta) + f_0(\theta), \tag{26}$$

where

$$f_0(\theta) = \frac{i}{k}\delta(1-\cos\,\theta), \quad f_s(\theta) = \frac{\alpha^{-2i\eta}}{ik}F_s(\theta), \tag{27}$$

with

$$F_s(\theta) = \sum_{l=0}^{\infty} z_l \exp(i\omega z_l)\left[1 + \frac{i\pi a^2}{2\alpha z_l} + 2i\eta \ln z_l + \cdots\right] \times P_l(\cos\,\theta), \tag{28}$$

and $\omega = \pi(1-\alpha^{-1})$.

The two first contributions to the scattering amplitude in Eq. (28) have already been obtained in Ref. [2], and shown to be singular for the scattering angle $\theta = \omega$. However, there is also an extra singular contribution to $f(\theta)$ given by the logarithmic term which comes from the self-interaction potential:

$$\overline{F_s}(\theta) = 2i\eta\sum_{l=0}^{\infty} z_l \ln z e^{i\omega z_l} P_l(\cos\,\theta). \tag{29}$$

Using the identity $\int_0^\infty dx(a\,\sin bx - b\sin ax)/x^2 = ab\ln(a/b)$, we can obtain an integral representation for $\overline{F_s}(\theta)$ where the singularity is explicitly exhibited. After some straightfoward calculation we get

$$\overline{F_s}(\theta) = \frac{2\eta\,\sin\,\omega}{[2(\cos\,\omega - \cos\,\theta)]^{3/2}}\Theta_\omega(\theta), \tag{30}$$

where the factor $\Theta_\omega(\theta)$ is

$$\Theta_\omega(\theta)=\int_0^\infty dx f_\omega(x,\theta), \tag{31}$$

with

$$f_\omega(x,\theta)=\frac{1}{x^2}\Bigg\{\sin x-\frac{(\cos\,\omega-\cos\,\theta)}{\sin\,\omega}\times\Bigg[\left(\frac{\cos\,\omega-\cos\,\theta}{\cos(x+\omega)-\cos\,\theta}\right)^{1/2}-\left(\frac{\cos\,\omega-\cos\,\theta}{\cos(x-\omega)-\cos\,\theta}\right)^{1/2}\Bigg]\Bigg\}. \tag{32}$$

Unfortunately we could not obtain an explicit expression for $\overline{F_s}(\theta)$; however, as one can see, $f_\omega(x,\theta)$ is well defined at $x=0$. In fact, near this point $f_\omega\sim-3\sin\omega/4(\cos\omega-\cos\theta)$, and when $x\to\infty$, $f_\omega\sim 1/x^2$. Moreover, the factor $\Theta_\omega(\theta)$ is finite for $\theta\neq\omega$.

IV. CONCLUDING REMARKS

Although a global monopole exerts no gravitational forces on the matter, its effects at the quantum level can be detected by measuring important observables related with the movement of a particle. Of particular interest in recent years was the study of gravitational scattering of bosonic or fermionic particles by this object. The metric space-time produced by the monopole couples with the angular momentum operator associated with the particle, modifying the effective polar quantum number, producing, in turn, a nonvanishing phase shift.

In this paper we returned to this subject considering at this time the complete quantum treatment of the movement of a charged or massive particle in the global monopole background metric. Besides the purely geometric effect of this manifold, we considered the effect produced by the electrostatic or gravitational self-interaction potential on the movement of the test particle, showing how this term modifies the self-energy and the scattering amplitude.

APPENDIX

The self-interaction potentials, for a particle at rest outside the monopole are given by [5]

$$U_{\rm grav}(r_0)=-\frac{GM^2}{2}G_R(r_0,r_0), \tag{A1}$$

and

$$U_{\rm elec}(r_0)=\frac{q^2}{2}G_R(r_0,r_0) \tag{A2}$$

where the $G_R(\mathbf{r},\mathbf{r}')$ is the renormalized Green function defined by

$$G_R(\mathbf{r},\mathbf{r}')=G_\alpha(\mathbf{r},\mathbf{r}')-G_1(\mathbf{r},\mathbf{r}'), \tag{A3}$$

$G_\alpha(\mathbf{r},\mathbf{r}')$ being a solution of the Poisson equation below, which when written in a covariant form, in this geometry reads

$$\nabla^2 G_\alpha(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}')=-4\pi\alpha\frac{\delta(r-r')\delta(\cos\,\theta-\cos\,\theta')\delta(\varphi-\varphi')}{r^2}. \tag{A4}$$

The solution for $G_\alpha(\mathbf{r},\mathbf{r}')$ can be obtained following the standard procedure, which leads the expression

$$G_\alpha(\mathbf{r},\mathbf{r}')=\frac{1}{\alpha}\sum_{l=0}^\infty\frac{2l+1}{2\lambda_l+1}\frac{r_<^{\lambda_l}}{r_>^{\lambda_l+1}}P_l(\cos\,\gamma), \tag{A5}$$

where γ satisfies the well known relation between the original angles (θ,φ) and (θ',φ') and λ_l was given in Sec. II.

The self-interaction potentials above depend on the evaluation of $G_R(\mathbf{r},\mathbf{r_0})$ at the coincidence limit $\mathbf{r}\to\mathbf{r_0}$, whose value is

$$G_R(r_0,r_0)=\frac{1}{r_0}S(\alpha), \tag{A6}$$

which is finite for $r_0\neq 0$, and $S(\alpha)$ is a finite numerical factor given by the sum

$$S(\alpha)=\sum_{l=0}^\infty\left[\frac{2l+1}{\sqrt{\alpha^2+4l(l+1)}}-1\right]. \tag{A7}$$

Developing a series expansion in the parameter $\Delta=1-\alpha^2$, after some steps [9] we get

$$S(\alpha)=\frac{1}{2}\sum_{n=1}^\infty\frac{(\pi^2\Delta)^n}{(n!)^2}|B_{2n}|(1-2^{-2n}), \tag{A8}$$

where B_n are the Bernoulli numbers. The first term in the expression above is $S(\alpha)=\pi(1-\alpha^2)/16$. We can see that $S(\alpha)$ is positive for $\alpha<1$ and negative for $\alpha>1$.

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