# Fermionic string representation for the three-dimensional Ising model

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## I. INTRODUCTION

It is well known that the partition functional of the three-dimensional (3D) Ising gauge model can be rigorously described on a regular lattice in $R^3$ by a sum over self-intersecting surfaces [1] on this lattice manifold (here after denoted by $\mathbb{Z}^3$):

$$Z[\beta]=(\cosh\beta)^N \sum_{\{S\}\subset\mathbb{Z}^3} \left\{ \exp\left[-A(S)\left(\ln\frac{1}{\tanh\beta}\right)\right] \times\Phi[\widetilde{C}(S)]\right\}, \tag{1}$$

where the sum in the above written equation is defined over the set of all closed two-dimensional surfaces $S\subset\mathbb{Z}^3$ with a weight given by the (lattice) area of $S$; $N$ is the number of the plaquettes, $\beta=J/kT$ denotes the ratio of the Ising hope parameter and the temperature. The presence of the Ising model functional $\Phi[\widetilde{C}(S)]$ inside Eq. (1) is a further weight given by the famous sign factor defined on the manifold of the lines of self-intersection $\widetilde{C}(S)$ of a given surface $S$ on the sum Eq. (1). Its explicit expression is given by

$$\Phi[\widetilde{C}(S)]=(-1)^{l[\widetilde{C}(S)]}=\exp\{i\pi l[\widetilde{C}(S)]\}, \tag{2}$$

where $l[\widetilde{C}(S)]$ denotes the total length of $\widetilde{C}(S)\subset S$.

It has been argued elsewhere [2] that the dependence of the 3D Ising model partition functional Eq. (1) on the area of the lattice closed surfaces $S$ is a strong indication that, near its critical point, some formal continuum string theory representation should be possible.

In this Brief Report we address the problem of writing a geometric string path integral involving only the string world-sheet geometry as in our previous work [3], which upon fermionization possesses formally on the lattice the same partition functional given by Eq. (1) after a ''replica'' limit. This study is presented in Sec. I. In the same section we show the usefulness of our proposed string framework for the 3D Ising model by writing in the lattice the associated partitional functional in the presence of an externed magnetic field.

## II. THE PROPOSED STRING THEORY

In our previous study, we proposed on formal mathematical grounds the following geometrical path-integral as a continuum limit of the sum Eq. (1) without the sign factor [3]:

$$Z(\alpha')=\int d_\mu^{\rm cov}[g_{ab},X^i]\,\delta_{\rm cov}^{(F)}(g_{ab}-\partial_a X^i\partial_b X_i) \times\exp\left[-\frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}d^2\xi \times\left(\frac{1}{2}\sqrt{g}g^{ab}\partial_a X^i\partial_b X_i\right)(\xi)\right]. \tag{3}$$

The above written string path integral is the same as that considered by Polyakov [2], but with a fundamental difference: we have used a covariant functional restricting the intrinsic metric field $g_{ab}(\xi)$ to be the string world-sheet-induced metric. As a result, the physical quantum theory obtained after integrating the $g_{ab}(\xi)$ field depends only on the string vector position [after considering $(2\pi\alpha' l^{-1}=1)$] and the metric piece $h_{ab}^{(J)}(\xi)$ related to the metric module space associated with the nontrivial topology of $S$ [2];

$$Z=\int d_\mu^{(\rm Weyl)}[h_{ab}^{(J)}(\xi)]\int D^{\rm cov}_{\sqrt{h^{(J)}}}[X^i(\xi)] \times\exp\left[\left(-\frac{1}{2}+\mu_0^2\,\frac{(26-3)}{48\pi}\right) \times\int_{-\infty}^{+\infty}d^2\xi(\sqrt{h^{(J)}}h_{ab}^{(J)}\partial^a X^i\partial^b X_i)(\xi)\right] \times\exp\left[-\left(\frac{26-3}{48\pi}\right)\int_{-\infty}^{+\infty}d^2\xi\{\sqrt{h^{(J)}}h_{mn}^{(J)}\partial_m \times[\ln(h_{ab}^{(J)}\partial^a X^i\partial^b X_i)]\partial_n[\ln(h_{a'b'}^{(J)}\partial^{a'}X^i\partial^{b'}X_i)]\}(\xi)\right]. \tag{4}$$

At this point we proceed by analogy by searching a continuum functional defined on the physical geometrical string degrees of freedom leading formally on the lattice to the sign factor $\Phi[\widetilde{C}(S)]$. Our purpose is to consider a new intrinsic field $\Omega(\xi)$ taking values on the SO(3) group with a similar role of the intrinsic metric field in Eqs. (1)–(3). We have, thus, to consider in Eq. (1) besides the terms already written there, a further path integral over the $\Omega(\xi)$ field with a weight given by a $\sigma$ model action added with a Wess-Zumino functional $\Gamma_{\rm WZ}(\Omega)$ and the following SO(3)-invariant $\delta$ functional:

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$$\delta^{(F)}_{\mathrm{Haar}}(\Omega_{ij}(\xi)-\hat{C}_{ij}(\xi,[X^i],[g_{ab}])). \tag{5}$$

Here $\hat{C}_{ij}$ denotes the (covariant) Cartan matrix relating the orthonormal basis $e_1=(1,0,0)$, $e_2=(0,1,0)$, and $e_3=(0,0,1)$ to the orthonormal basis defined by the tangent vectors $\{v_1(\xi),v_2(\xi)\}$ and the normal vector $\{v_3(\xi)\}$ on the string surface at the point $\{X^i(\xi)\}$ [4]:

$$e_i=\hat{C}_{ij}(\xi,[X^i],[g_{ab}])v_j(\xi), \tag{6}$$

where

$$v_1^{(i)}(\xi)=\partial_1 X^{(i)}(\xi)/(\partial_1 X^a g^{11}\partial_1 X_a)^{1/2}(\xi), \tag{7}$$

$$v_2^{(i)}(\xi)=\partial_2 X^{(i)}_{(\xi)}/(\partial_2 X^a g^{22}\partial_2 X_a)^{1/2}(\xi), \tag{8}$$

$$v_2^{(i)}(\xi)=\left(\frac{v_1(\xi)\wedge v_2(\xi)}{|v_1(\xi)\wedge v_2(\xi)|}\right)^{(i)}. \tag{9}$$

The geometrical string path integral to be considered now is given by

$$\begin{aligned}\overline{Z(\alpha')}=&\int d^{\mathrm{cov}}\mu[g_{ab};X^i]D^{\mathrm{cov}}_{\mathrm{Haar}}[\Omega]\,\delta^{(F)}_{\mathrm{cov}}(g_{ab}\\&-\partial_a X^i\partial_b X_i)\,\delta^{(F)}_{\mathrm{Haar}}(\Omega(\xi)-\hat{C}(\xi,[X^i],[g_{ab}]))\\&\times\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}d^2\xi(\sqrt{g}g^{ab}\partial_a X^i\partial_b X_i)(\xi)\right\}\\&\times\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}d^2\xi(\sqrt{g}\ \mathrm{Tr}(\Omega^{-1}\partial_a\Omega)^2)(\xi)\right\}\\&\times\exp\{4\pi i\Gamma_{\mathrm{WZ}}[\Omega]\},\end{aligned} \tag{10}$$

where the quantum measure defining the $\sigma$-quantum model is the invariant SO(3) measure associated with the invariant metric

$$dS^2=\int_{-\infty}^{+\infty}d^2\xi[\sqrt{g}\ \mathrm{Tr}(\Omega^{-1}\delta\Omega)^2](\xi). \tag{11}$$

It is an important step in our study to consider the fermionic version of the above displayed $\sigma$-model path integral as a result of the presence of the Wess-Zumino functional in Eq. (10):

$$\begin{aligned}\overline{Z(\alpha')}=&\int d^{\mathrm{cov}}\mu[g_{ab};X^i]d^{\mathrm{cov}}[\psi_A;\overline{\psi_A}]\,\delta^{(F)}_{\mathrm{cov}}(g_{ab}-\partial_a X^i\partial_b X_i)\\&\times\exp\left(-\frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}d^2\xi(\sqrt{g}g^{ab}\partial_a X^i\partial_b X_i)(\xi)\right)\\&\times\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}d^2\xi\right.\\&\left.\times\left[\sum_{A=1}^{3}(\sqrt{g}\,\overline{\psi_A}(\gamma^a\nabla_a)\psi_A)(\xi)\right]\right\}.\end{aligned} \tag{12}$$

Here, the Dirac curved space-time matrices satisfy the usual (Euclidean) anticommuting relationship $\{\gamma^a(\xi),\gamma^b(\xi)\}_+=g^{ab}(\xi)=e^a_{b'}(\xi)e^{b,b'}(\xi)$ and the spin connection is given by the following expression involving the surface Cartan matrix:

$$\omega_a(\xi)=e_a^{a'}(\xi)\gamma_{a'}(\xi)(\hat{C}^{-1}\partial_a\hat{C})(\xi). \tag{13}$$

Let us now give a formal argument that the string theory Eq. (12) represents the 3D Ising model at a replica limit on the geometrical fermionic degrees of freedom. In order to implement such an argument, we introduce $N$ copies of the fermionic fields $\{(\psi_A^{(m)};\overline{\psi_A^{(m)}})_i 1\leq m\leq N\}$ in the fermionic action Eq. (12). After integrating out these fermion fields, writing the fermionic functional determinant by the Grassmanian proper-time technique implemented on the surface loop space (see [5], Appendix B) and using the well-known replica limit on the fermion species, we have the following loop space path integral for the fermionic effective action in Eq. (12):

$$\begin{aligned}&\lim_{N\to 0}(\det{}^N(\gamma^a\nabla_a)-1)/N\\&\quad=\frac{1}{2}\int_0^\infty dT\ \exp(-l(C_a)T)\int_{-\infty}^{+\infty}d^2\xi\sqrt{g(\xi)}\\&\quad\times\mathrm{Tr}_{\mathrm{Dirac}}\left\{\int_{C_a(0)=C_a(T)=\xi_a}D[C_a(t)]D[\pi_a(t)]\right.\\&\quad\times\exp\left(i\int_0^T dt\,\pi_a(t)dC_a(t)\right)\\&\quad\times\mathbb{P}_{\mathrm{Dirac}}\left\{\exp\left(i\int_0^T dt(\gamma^a\pi_a)(t)\right)\right\}\\&\quad\times\mathrm{Tr}_{\mathrm{SO(3)}}\left\{\exp\left[i\int_0^T dt\left((\hat{C}^{-1}\partial_a\hat{C})\right.\right.\right.\\&\quad\left.\left.\left.\left.\times(t)\ \frac{dC^a(t)}{dt}\right)\right]\right\}\right\},\end{aligned} \tag{14}$$

where $\{l_a(t)\}$ belongs to the manifold of closed bosonic trajectories on the string surface and $\{\pi_a(t)\}$ the Grassmanian degrees of freedom associated to the 2D Dirac indexes. If one considers formally the above replica limit on the lattice, one can see that the Wilson loop defined by the Cartan matrix in Eq. (14) coincides exactly with the sign factor as Sedrakyan and Kavalov showed by using topological-homotopical techniques.

As a consequence, we have the following string representation at the critical point for the 3D-Ising model with $\beta = \mathrm{arctanh}_0(e^{-1/2\pi\alpha'})$,

$$Z_{\text{critical point}}[\beta] = \int d^{\mathrm{cov}}\mu[g_{ab};X^i]\,\delta^{(F)}_{\mathrm{cov}}(g_{ab}-\partial_a X^i \partial_b X_i) \times \exp\left(-\frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty} d^2\xi \times (\sqrt{g}g^{ab}\partial_a X^i \partial_b X_i)(\xi)\right) \times \lim_{N\to 0}\left\{\frac{\det^N(\gamma^a \nabla_a)-1}{N}\right\}. \tag{15}$$

This is our main result.

It is worth remarking that all of the above results are of a formal mathematical nature and real checks will be to compute (at least numerically) physical quantities. However, one can use Eq. (15) to suggest some new formulas on the lattice. Let us show the usefulness of Eqs. (12)–(15) by coupling the proposed Ising string theory to an external magnetic field $\vec{H}(\xi)$ by means of the well-known string electromagnetic flux action (see Ref. [5]):

$$\exp\left\{-\frac{1}{2}e\int_{-\infty}^{+\infty} d^2\xi \sqrt{g(\xi)} H^i[X^j(\xi)]\partial_a X_i(\xi) \times \left(\sum_{A=1}^{3} \overline{\psi_A}(\gamma^a)\psi_A\right)(\xi)\right.. \tag{16}$$

By considering the replica limit of the resulting string path integral as in Eq. (14), we obtain as a candidate for the partition Ising model in the presence of the external magnetic field the following sum over closed surfaces on the lattice:

$$Z[\beta, e\vec{H}] = (\cosh\beta)^N \sum_{\{S\}\subset \mathbb{Z}^3}\left\{\exp\left[-\left(\ln\frac{1}{\tanh\beta}\right)A(S)\right] \times \Phi[\widetilde{C}(S)] \times W[C(S)]\right\}, \tag{17}$$

where we note the appearance of the usual Wilson loop defined by the 2D-closed loops $\{l^a(t)\}$ on the surface $S$ and the external magnetic field:

$$W[C(S)] = \prod_{\{C(S)\subset S\}}\left\{\exp\left(ie\int_{C(S)} \widetilde{H}^a[C^b(t)]\,\frac{dC_a(t)}{dt}\right)\right\}, \tag{18}$$

where $\widetilde{H}^a[C^b(t)]$ is the restriction of the surface magnetic flux $H^i(X^j(\xi))\partial_a X_i(\xi)$ to the 2D loop $\{l^a(t),\ a=1,2\}$ which are obtained from the string surface parametrization by supposing an implicit relation of the form $[\xi=(\xi_1,\xi_2)]$

$$\xi_2 = \beta(\xi_1) \Rightarrow X^i(\xi_1,\beta(\xi_1)) = X^i(C_1(\xi),C_2(\xi)). \tag{19}$$

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