Chiral zero modes on the domain-wall model in 4+1 dimensions

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We investigate an original domain-wall model in 4+1 dimensions numerically in the presence of U(1) dynamical gauge field only in an extra dimension, corresponding to a weak coupling limit of four-dimensional physical gauge coupling. Using a quenched approximation we carry out a numerical simulation for this model at $\beta_s(=1/g_s^2)=$ 0.29 (''symmetric'' phase) and 0.5 (''broken'' phase), where g_s is the gauge coupling constant of the extra dimension. In the broken phase, we found that there exists a critical value of a domain-wall mass m_0^c which separates a region with a fermionic zero mode on the domain wall from the one without it in the same case of (2+1)-dimensional model. On the other hand, in the symmetric phase, our numerical data suggest that the chiral zero modes disappear in the infinite limit of four-dimensional volume. From these results it seems difficult to construct the U(1) lattice chiral gauge theory via an original domain-wall formulation. [S0556-2821(97)04412-3]

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I. INTRODUCTION

Although perturbative aspects of electroweak interaction are well described by the standard model, its nonperturbative phenomenon has to be investigated beyond the perturbation theory. At present, the most powerful nonperturbative technique is the lattice field theory. However, it is nontrivial to define the standard model on a lattice since it is a type of chiral gauge theories, construction of which is one of the long-standing problems of lattice field theory: Because of the fermion doubling problems, a naively D-dimensional discretized lattice fermion field yields 2^D fermion particles, half of one chirality and half of the other, so that the theory becomes nonchiral [1]. Several lattice approaches have been proposed, but so far none of them have been proven to work successfully [2].

Kaplan has proposed a new construction of lattice chiral gauge theories via domain-wall models [3]. Starting from a vectorlike gauge theory in $2k+1$ dimensions with a fermion mass term being a shape of a domain wall in the (extra) $(2k+1)$th dimension, he showed in the weak gauge coupling limit that a massless chiral state arises as a zero mode bound to the $2k$-dimensional domain wall while all the doublers have large masses of the lattice cutoff scale. It has been also shown that the model works well for smooth background gauge fields [4–7].

Two simplified variants of the original Kaplan's domain-wall model have been proposed: an ''overlap formula'' [8,9] and a ''waveguide model'' [10,11]. Gauge fields appearing in these variants are $2k$ dimensional and are independent of the extra $(2k+1)$th coordinate, while those in the original model are $2k+1$ dimensional and depend on the extra $(2k+1)$th coordinate. These variants work successfully for smooth background gauge fields [12–18,21], as the original one does. Nonperturbative investigations for these variants seem easier than that for the original model due to the simpler structure of gauge fields. Indeed some positive results are obtained for the vector Schwinger model by the overlap formula [22].

However, it has been reported [10,11] that the waveguide model in the weak gauge coupling limit cannot produce chiral zero modes needed to construct chiral gauge theories. In this limit, if gauge invariance were maintained, pure gauge field configurations equivalent to the unity by gauge transformation would dominate and gauge fields would become smooth. In the setup of the waveguide model, however, $2k$-dimensional gauge fields are nonzero only in the layers near domain wall (waveguide), so that the gauge invariance is broken in the edge of the waveguide. Therefore, even in the weak gauge coupling limit, gauge fields are no more smooth and become very ''rough,'' due to the gauge degrees of freedom appearing to be dynamical in this edge. As a result of the rough gauge dynamics, a new chiral zero mode with the opposite chirality to the original zero mode on the domain wall appears in the edge, so that the fermionic spectrum inside the waveguide becomes vectorlike. It has been claimed [10,11] that this ''rough gauge problem'' also exists in the overlap formula since the gauge invariance is broken by the boundary condition at the infinity of the extra dimension [17,18]. Furthermore, an equivalence between the waveguide model and the overlap formula has been pointed out for the special case [19]. Although the claimed equivalence has been challenged in Refs. [20,21], it is still crucial for the success of the overlap formula to solve the ''rough gauge problem'' and to show the existence of a chiral zero mode in the weak gauge coupling limit. Recently, two results have been obtained in this aspect, one is positive [23], the other is negative [24], so more work may be necessary for the definite conclusion.

In the original model there are two inverse gauge couplings $\beta=1/g^2$ and $\beta_s=1/g_s^2$, where g is the coupling constant in (physical) $2k$ dimensions and g_s is the one in the (extra) $(2k+1)$th dimension. Very little is known about this model except $\beta_s=0$ case [10,25,26] (known as ''the layered phase'') where the spectrum seems vectorlike and except the case of (2+1)-dimensional U(1) model [27]. Since perturbation theory for the physical gauge coupling g is expected to hold, the fermion spectrum of the model can be determined in the limit that $g\to 0$. In this weak coupling limit, all gauge fields in the physical dimensions can be gauged away, while the gauge field in the extra dimension is still dynamical and its dynamics is controlled by β_s. Instead of the gauge degrees of freedom in the edge of the waveguide, $(2k+1)$th component of gauge fields represents roughness of $2k$-dimensional gauge fields. An important question is whether the chiral zero mode on the domain wall survives in the presence of this rough dynamics. The dynamics of the gauge field in this limit is equivalent to $2k$-dimensional scalar model with $2L_s$ independent copies where $2L_s$ is the number of sites in the extra dimension. In general at large β_s such a system is in a ''broken'' phase where some global symmetry is spontaneously broken, while at small β_s the system is in a ''symmetric'' phase. Therefore, there exists a critical point β_s^c, and it is likely that the phase transition at $\beta_s=\beta_s^c$ is continuous (second or higher order). The ''gauge field'' becomes rougher and rougher at smaller β_s. Indeed we know that the zero mode disappears at $\beta_s=0$ [25,26], while the zero mode exists at $\beta_s=\infty$ (free case). So far we do not know the fate of the chiral zero mode in the intermediate range of the coupling β_s. In the symmetric phase, if chiral zero modes exist, these modes, one at the domain wall and the other at the antidomain wall, decouple each other even in the presence of the extra dimensional gauge fields. In the broken phase, on the other hand, there are arguments that massless gauge fields, whose couplings to zero modes are vectorlike, always exist both in Abelian case [10] and in non-Abelian case [27]. Although there is a possibility that the broken phase leads to a (some) chiral gauge theory [27] in the continuum limit ($\beta_s=\beta_s^c$), this depends on whether the zero modes exist or not in the symmetric phase. Then there are the following three possibilities: (a) The chiral zero mode always exists except $\beta_s=0$. In this case we may likely construct a lattice chiral gauge theory in both symmetric ($\forall\beta_s<\beta_s^c$) and broken ($\beta_s\searrow\beta_s^c$) phases. This is the best case for the domain-wall model. (b) The chiral zero mode exists only in the broken phase ($\beta_s>\beta_s^c$). It is likely, however, that the continuum limit taken at $\beta_s=\beta_s^c$ from above leads to a vector gauge theory. (c) No chiral zero mode survives except $\beta_s=\infty$. The original model cannot describe lattice chiral gauge theories at all. Therefore, it is very important to determine which possibility is indeed realized in the domain-wall model.

Instead of (4+1)-dimensional models, we have recently investigated a (2+1)-dimensional U(1) model [27]. Using a quenched approximation we have carried out a numerical simulation to see whether chiral zero modes exist or not in this model. In the weak coupling limit of the physical gauge coupling the (2+1)-dimensional U(1) gauge system is reduced to the two-dimensional U(1) spin system.[1] Strictly speaking, there is no order parameter in the two-dimensional U(1) spin system. On a large but finite lattice, however, the behavior of the model is similar to the one of a four-dimensional scalar model: On a finite lattice we regard the Kosterlitz-Thouless phase as the ''symmetric'' phase and the spin-wave phase as the ''broken'' phase. In the ''broken'' phase we have numerically found that there exists a critical value of a domain-wall mass m_0^c which separates a region with a fermionic zero mode on the domain wall from one without it. In the ''symmetric'' phase the critical values of the domain-wall mass seem to also exist but are very close to its upper bound $m_0=1$, so that the region with a fermionic zero mode is very narrow. Because of the difficulty observed in the numerical simulation near $m_0=1$ we cannot exclude a possibility that the existence of the zero mode is an artifact of finite lattice size effects. Further simulation we have made on larger lattice sizes cannot give a definite conclusion. Since, as mentioned before, the phases of the model in the infinite volume limit is different from the ones of the four-dimensional model, we did not attempt to increase lattice sizes further, for example, 100^2 to see the fate of zero mode in the symmetric phase. Instead, we have decided to investigate the (4+1)-dimensional U(1) model directly, to obtain the definite conclusion on the existence of zero modes in the symmetric phase.

In this paper, in order to know the fate of the chiral zero mode, we have carried out a numerical simulation of a domain-wall model in 4+1 dimensions with a quenched U(1) gauge field in the $\beta=\infty$ limit. In Sec. II, we have defined our domain-wall model with dynamical gauge fields. We have calculated a fermion propagator by using a kind of mean-field approximation, to show that there is a critical value of the domain-wall mass parameter above which the zero mode exists. The value of the critical mass may depend on β_s, which controls the dynamics of the gauge field. In Sec. III, we have calculated the fermion spectrum numerically using quenched approximation at $\beta_s=0.29,0.5$, and at various values of domain-wall masses. We have found that in the broken phase ($\beta_s=0.5$) there exists the range of a domain-wall mass parameter in which the chiral zero mode survives on the domain wall. In the symmetric phase ($\beta_s=0.29$), however, from data on several lattice sizes we have found a numerical evidence that the chiral zero mode disappears in the infinite volume limit of four-dimensional Euclidean space-time. Our conclusions and some discussions are given in Sec. IV.

II. DOMAIN-WALL MODEL

A. Definition of the model

We consider a vector gauge theory in $D=2k+1$ dimensions with a domain-wall mass term, which has a kinklike mass term in the coordinate of an extra dimension. This domain-wall model is originally proposed by Kaplan [3], and a fermionic part of the action is reformulated by Narayanan-Neuberger [8], in terms of a $2k$-dimensional theory. The model is defined by the action

$$S=S_G+S_F, \tag{1}$$

where S_G is the action of a dynamical gauge field and S_F is the fermionic action. S_G is given by

$$S_G=\beta\sum_{n,\mu>\nu}\sum_{s}\{1-\mathrm{Re\,Tr}[U_{\mu\nu}(n,s)]\} \tag{2}$$

$$+\beta_s\sum_{n,\mu}\sum_{s}\{1-\mathrm{Re\,Tr}[U_{\mu D}(n,s)]\},$$

[1]This is explained in the next section.

where μ,ν run from 1 to $2k$, n is a $2k$-dimensional lattice point, and s is a coordinate of an extra dimension. $U_{\mu\nu}(n,s)$ is a $2k$-dimensional plaquette and $U_{\mu D}(n,s)$ is a plaquette containing two link variables in the extra direction. β is the inverse gauge coupling for the plaquette $U_{\mu\nu}$ and β_s is the one for the plaquette $U_{\mu D}$. In general, $\beta\neq\beta_s$. The fermion action S_F on the Euclidean lattice, in terms of the $2k$-dimensional notation, is given by

$$S_F=\frac{1}{2}\sum_{n\mu}\sum_{s}\overline{\psi_s}(n)\gamma_\mu[U_{s,\mu}(n)\psi_s(n+\mu)-U^\dagger_{s,\mu}(n-\mu)\psi_s(n-\mu)]+\sum_{n}\sum_{s,t}\overline{\psi_s}(n)[M_0P_R+M_0^\dagger P_L]\psi_t(n)$$
$$+\frac{1}{2}\sum_{n\mu}\sum_{s}\overline{\psi_s}(n)[U_{s,\mu}(n)\psi_s(n+\mu)+U^\dagger_{s,\mu}(n-\mu)\psi_s(n-\mu)-2\psi_s(n)], \tag{3}$$

where s,t are extra coordinates, $P_{R/L}=\frac{1}{2}(1\pm\gamma_{2k+1})$,

$$(M_0)_{s,t}=U_{s,D}(n)\delta_{s+1,t}-a(s)\delta_{s,t},$$
$$(M_0^\dagger)_{s,t}=U^\dagger_{s-1,D}(n)\delta_{s-1,t}-a(s)\delta_{s,t}. \tag{4}$$

Here $U_{s,\mu}(n),U_{s,D}(n)$ $(D=2k+1)$ are link variables connecting a site (n,s) to $(n+\mu,s)$ or $(n,s+1)$, respectively. Because of a periodic boundary condition in the extra dimension, s,t run from $-L_s$ to L_s-1, and $a(s)$ is given by

$$a(s)=1-m_0\ \mathrm{sgn}[(s+\tfrac{1}{2})\ \mathrm{sgn}(L_s-s-\tfrac{1}{2})]$$
$$=\begin{cases}1-m_0 & \left(-\dfrac{1}{2}<s<L_s-\dfrac{1}{2}\right)\\ 1+m_0 & \left(-L_s-\dfrac{1}{2}<s<-\dfrac{1}{2}\right),\end{cases} \tag{5}$$

where m_0 is the height of the domain-wall mass. It is easy to check that the above fermionic action is identical to the one in $2k+1$ dimensions, proposed by Kaplan [3,8].

In weak coupling limit of both β and β_s, it has been shown[2] that at $0<m_0<1$ a desired chiral zero mode appears on a domain wall ($s=0$ plane) without unwanted doublers. Because of the periodic boundary condition in the extra dimension, however, a zero mode of the opposite chirality to the one on the domain wall appears on the antidomain wall ($s=L_s-1$). Overlap between two zero modes decreases exponentially at large L_s. A free fermion propagator is easily calculated and an effective action of a (2+1)- and (4+1)-dimensional model including the gauge anomaly and the Chern-Simons term can be obtained for smooth background gauge fields [4,5,7].

The original Kaplan's domain-wall models in the 4+1 dimensions, however, have not been investigated yet *nonperturbatively*, except $\beta_s=0$ [10,25,26] and (2+1)-dimensional case [27]. The main question is whether the chiral zero mode survives in the presence of rough gauge fields mentioned in the Introduction. To answer this question we will analyze the fate of the chiral zero mode in the weak coupling limit for β. In this limit, the gauge field action S_G is reduced to

$$S_G=\beta_s\sum_{s}\sum_{n,\mu}\{1-\mathrm{Re\ Tr}[V(n,s)V^\dagger(n+\mu,s)]\}, \tag{6}$$

where the link variable $U_{s,D}(n)$ in the extra direction is regarded as a site variable $V(n,s)[=U_{s,D}(n)]$. This action is identical to the one of a $2k$-dimensional spin model and s is regarded as an independent flavor. The action equation (6) is invariant under

$$V(n,s)\rightarrow g(s)V(n,s)g^\dagger(s+1)\quad [g(s)\in G], \tag{7}$$

where G is the gauge group of the original model. Therefore, the total symmetry of the model is G^{2L_s}, where $2L_s$, the size of the extra dimension, is regarded as the number of independent flavors. We use this (reduced) model for our numerical investigation.

B. Mean-field approximation for fermion propagators

When the dynamical gauge fields are added even on the extra dimension only, it is difficult to calculate the fermion propagator analytically. Instead of calculating the fermion propagator *exactly*, we use a mean-field approximation to see an effect of the dynamical gauge field qualitatively. The mean-field approximation we adopt is that the link variables are replaced as

$$V(n,s)[=U_{s,D}(n)]\rightarrow z, \tag{8}$$

where z is a (n,s)-independent constant. From Eq. (3) the fermion action in a $2k$-dimensional momentum space becomes

$$S_F\rightarrow\sum_{s,t,p}\overline{\psi_s}(-p)\left(\sum_{\mu} i\gamma_\mu\sin(p_\mu)\delta_{s,t}\right.$$
$$\left.+[M(z)P_R+M^\dagger(z)P_L]_{s,t}\right)\psi_t(p), \tag{9}$$

[2] In fact, the solutions of a chiral zero mode exist at $0<m_0<2$ in the free case. A smoothly damping solution is obtained at $0<m_0<1$, while an oscillating damping one is obtained at $1<m_0<2$. Since the two solutions are transformed to each other by simple oscillation factor, one may restrict the region of m_0 to $0<m_0<1$ without loss of a generality. At $m_0=1$, a solution becomes singular.

$$(M(z))_{s,t}=(M_0(z))_{s,t}+\frac{\nabla(p)}{2}\delta_{s,t},$$

$$(M^\dagger(z))_{s,t}=(M_0^\dagger(z))_{s,t}+\frac{\nabla(p)}{2}\delta_{s,t},$$

$$\nabla(p)\equiv\sum_{\mu=1}^{D-1}2(\cos p_\mu-1), \tag{10}$$

$$(M_0(z))_{s,t}=z\,\delta_{s+1,t}-a(s)\,\delta_{s,t},$$

$$(M_0^\dagger(z))_{s,t}=z\,\delta_{s-1,t}-a(s)\,\delta_{s,t}. \tag{11}$$

Following Refs. [4,5,8], especially Ref. [5], it is easy to obtain a mean-field fermion propagator on a finite lattice with the periodic boundary condition

$$G(p)_{s,t}=\left[\left\{\left(-i\sum_\mu \gamma_\mu\overline{p_\mu}+M(z)\right)G_L(p)\right\}_{s,t}P_L+\left\{\left(-i\sum_\mu \gamma_\mu\overline{p_\mu}+M^\dagger(z)\right)G_R(p)\right\}_{s,t}P_R\right], \tag{12}$$

$$G_L(p)=\frac{1}{\overline{p^2}+M^\dagger(z)M(z)},\quad G_R(p)=\frac{1}{\overline{p^2}+M(z)M^\dagger(z)}, \tag{13}$$

with $\overline{p_\mu}\equiv\sin(p_\mu)$. For large L_s where we neglect terms of $O(e^{-cL_s})$ with $c>0$, G_L and G_R are given by

$$[G_L(p)]_{s,t}=\begin{cases}Be^{-\alpha_+|s-t|}+(A_L-B)e^{-\alpha_+(s+t)}+(A_R-B)e^{-\alpha_+(2L_s-s-t)} & (s,t\geqslant 0),\\ A_Le^{-\alpha_+s+\alpha_-t}+A_Re^{-\alpha_+(L_s-s)-\alpha_-(L_s+t)} & (s\geqslant 0,t\leqslant 0),\\ A_Le^{\alpha_-s-\alpha_+t}+A_Re^{-\alpha_-(L_s+s)-\alpha_+(L_s-t)} & (s\leqslant 0,t\geqslant 0),\\ Ce^{-\alpha_-|s-t|}+(A_L-C)e^{\alpha_-(s+t)}+(A_R-C)e^{-\alpha_-(2L_s+s+t)} & (s,t\leqslant 0),\end{cases} \tag{14}$$

$$[G_R(p)]_{s,t}=\begin{cases}Be^{-\alpha_+|s-t|}+(A_R-B)e^{-\alpha_+(s+t+2)}+(A_L-B)e^{-\alpha_+(2L_s-s-t-2)} & (s,t\geqslant -1),\\ A_Re^{-\alpha_+(s+1)+\alpha_-(t+1)}+A_Le^{-\alpha_+(L_s-s-1)-\alpha_-(L_s+t+1)} & (s\geqslant -1,t\leqslant -1),\\ A_Re^{\alpha_-(s+1)-\alpha_+(t+1)}+A_Le^{-\alpha_-(L_s+s+1)-\alpha_+(L_s-t-1)} & (s\leqslant -1,t\geqslant -1),\\ Ce^{-\alpha_-|s-t|}+(A_R-C)e^{\alpha_-(s+t+2)}+(A_L-C)e^{-\alpha_-(2L_s+s+t+2)} & (s,t\leqslant -1),\end{cases} \tag{15}$$

where

$$a_\pm=z\left(1-\frac{\nabla(p)}{2}\mp m_0\right)=zb_\pm, \tag{16}$$

$$\alpha_\pm=\text{arccosh}\left[\frac{\overline{p^2}+z^2+b_\pm^2}{2zb_\pm}\right], \tag{17}$$

$$A_L=\frac{1}{a_+e^{\alpha_+}-a_-e^{-\alpha_-}},\quad A_R=\frac{1}{a_-e^{\alpha_-}-a_+e^{-\alpha_+}}, \tag{18}$$

$$B=\frac{1}{2a_+\sinh\alpha_+},\quad C=\frac{1}{2a_-\sinh\alpha_-}. \tag{19}$$

Like a free fermion theory, the terms of A_R, B, and C have no singularity for all z's as $p\to 0$. A behavior of A_L is, however, different. As $p\to 0$, A_L behaves as

$$A_L\to\begin{cases}\dfrac{1}{[(1-m_0)^2-z^2]+O(p^2)} & (0<m_0<1-z),\\[2ex] \dfrac{4m_0^2-[(z^2-1)-m_0^2]^2}{4m_0z^2p^2} & (1-z<m_0<1).\end{cases} \tag{20}$$

A critical value of the domain-wall mass that separates a region with a zero mode and a region without any zero modes is $m_0^c=1-z$. Since A_L term dominates[3] for $1-z<m_0<1$ in the G_L [Eq. (14)] and G_R [Eq. (15)], a right-handed zero mode appears in the $s=0$ plane, and a left-handed zero mode in the $s=L_s-1$ plane. For $0<m_0<1-z$ the right- and left-handed fermions are massive in all s planes. Since the terms of $A_L(A_R)$ and $B(C)$ are almost of the same values in this region of m_0, a translational invariant term dominates in G_L and G_R in the positive (negative) s layer, so that the spectrum becomes vectorlike.

If $z\to 1$, the model becomes a free theory. The propagator obtained in this section agrees with the one obtained in Ref. [4]. In the opposite limit that $z\to 0$, since there is no hopping term to the neighboring layers, this model becomes the one analyzed in Ref. [25] in the case of the strong coupling limit $\beta_s=0$, and in Ref. [26], in the case that z is identified to the vacuum expectation value of the link variables. This consideration suggests that the region where the zero modes exist becomes smaller and smaller as $z(1-z<m_0<1)$ approaches zero.

What corresponds to z? Boundary conditions which z satisfies are $z=1$ at $\beta_s=\infty$ and $z=0$ at $\beta_s=0$. As explained

[3]To be exact, the zero mode exists for $1-z<m_0<1+z$ when the domain-wall mass height is $0<m_0<2$ in which doublers decouple from the fermion spectrum. As mentioned in the footnote 2, however, since one can restrict the region of m_0 to $0<m_0<1$ without loss of a generality, it is sufficient that the range of m_0 with a zero mode is $1-z<m_0<1$.

before, the gauge field action of our model is identical to that of the U(1) spin system in four-dimensions. So the most naive candidate [26] is

$$z=\langle V(n,s)\rangle, \tag{21}$$

which is not invariant under the symmetry (7). In our simulations on $L^3\times L_4\times 2L_s$ lattices, as an order parameter, we take a vacuum expectation value of link variable calculated with rotational technique:

$$v=\left\langle \frac{1}{2L_s}\sum_s \left|\frac{1}{L_4L^3}\sum_n V(n,s)\right|\right\rangle. \tag{22}$$

Although v defined in Eq. (22) is always nonzero on finite lattices, it becomes zero in the symmetric phase but stays nonzero in the broken phase in the infinite volume limit. Figure 1(a) shows that, even on finite lattices, v behaves as if it was an order parameter: it is very small and decreases as the volume increases in the would-be symmetric phase. If the identification that $z=v$ is true, zero modes disappear in the symmetric phase, where $v=0$. The other choice, which is invariant under Eq. (7), is the vacuum expectation value of plaquette:

$$z^2=\langle \mathrm{TrRe}\{V(n,s)V^\dagger(n+\mu,s)\}\rangle. \tag{23}$$

In this identification the zero modes always exist in both phases, since $\langle \mathrm{Tr\,Re}\{V(n,s)V^\dagger(n+\mu,s)\}\rangle$ is nonzero for all β_s except $\beta_s=0$ and is insensitive to which phase we are in, as shown in Fig. 1(b).

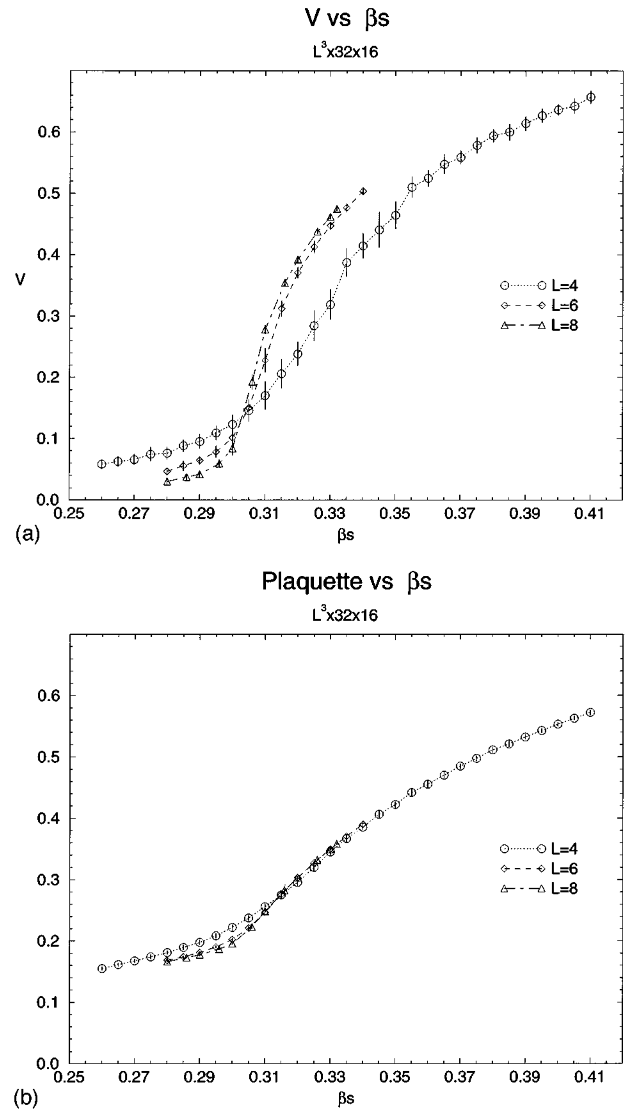

FIG. 1. (a) Vacuum expectation value of link variables v on $L^3\times 32\times 16$ lattices with $L=4$ (circles), 6 (diamonds), 8 (triangles) as a function of β_s. (b) Vacuum expectation value of plaquette w on $L^3\times 32\times 16$ lattices with $L=4$ (circles), 6 (diamonds), 8 (triangles) as a function of β_s.

III. NUMERICAL STUDY OF (4+1)-DIMENSIONAL U(1) MODEL

A. Method of numerical calculations

In this section we numerically study the domain-wall model in 4+1 dimensions with a U(1) dynamical gauge field in the extra dimension. As seen from Eq. (6), the gauge field action can be identified with a four-dimensional U(1) spin model (with $2L_s$ copies).

Our numerical simulation has been carried out by the quenched approximation. Configurations of U(1) dynamical gauge field are generated and fermion propagators are calculated on these configurations. The obtained fermion propagators are gauge noninvariant in general under the symmetry (7). The fermion propagator $G(p)_{s,t}$ becomes ''invariant'' if and only if $s=t$. Thus, we take the s-s layer as propagating plane ($\approx$ ''physical space''), and investigate the behavior of the fermion propagator in this layer.

To study the fermion spectrum, we first extract G_L and G_R, defined in Eq. (12), from the fermion propagator. Assuming Eq. (13), we then obtain corresponding ''fermion masses'' from $G_L^{-1}(p)$ and $G_R^{-1}(p)$ by fitting them linearly in $\overline{p^2}$ as follows:

$$G_L^{-1}=\overline{p^2}+M^\dagger M\rightarrow m_f^2(\mathrm{right}) \quad (p\rightarrow 0), \tag{24}$$

$$G_R^{-1}=\overline{p^2}+MM^\dagger\rightarrow m_f^2(\mathrm{left}) \quad (p\rightarrow 0). \tag{25}$$

We take the following setup for four-dimensional momenta. A periodic boundary condition is taken for the first-, second-, and third-directions and the momenta in these directions are fixed on $p_1,p_2,p_3=0$. An antiperiodic boundary condition is taken for the fourth direction and the momentum in this direction is variable, $p_4=(2n+1)\pi/L_4$, $n=-L_4/2,\ldots,L_4/2-1$, where L_4 is the number of site of fourth direction. (Our numerical simulations have been performed always for $L_4=32$.)

B. Simulation parameters

From Fig. 1(a) it is inferred that the system is in the symmetric phase at $\beta_s=0.29$ and in the broken phase at $\beta_s=0.5$. Our simulation is performed in the quenched approximation on $L^3\times 32\times 2L_s$ lattices with $L=4$, 6, 8 and $L_s=8$ at $\beta_s=$ 0.29 (symmetric phase) and 0.5 (broken phase), where L is a lattice size of first, second, and third

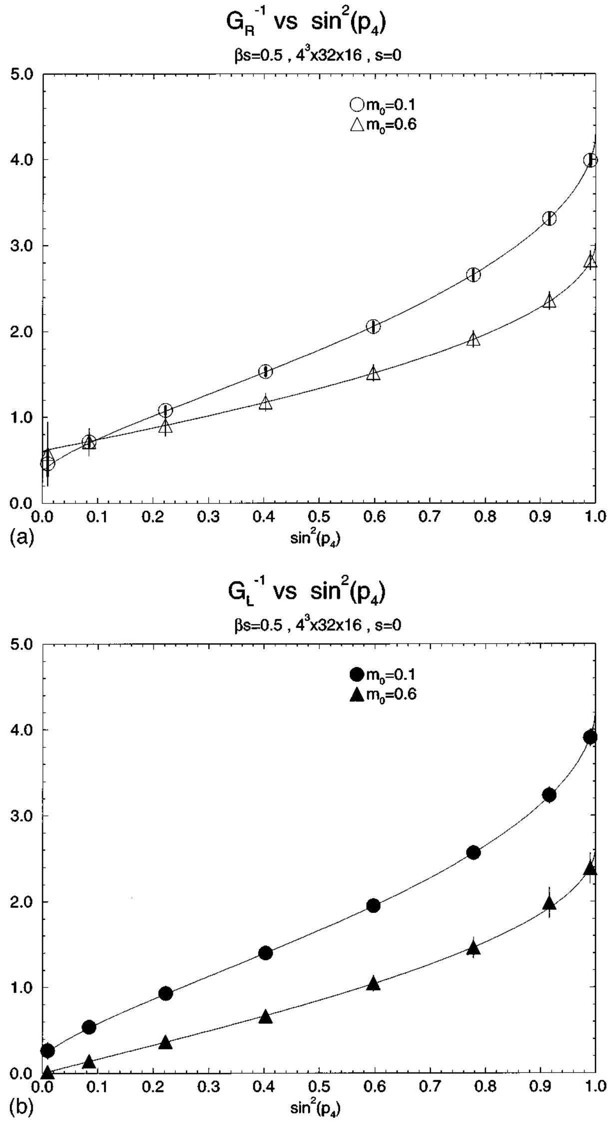

FIG. 2. (a) $[G_R]_{0,0}^{-1}$ as a function of $\sin^2(p_4)$ with $p_1, p_2, p_3=0$ at $\beta_s=0.5$ on a $4^3\times 32\times 16$ lattice, at $m_0=0.1$ (open circles) and 0.6 (open triangles). (b) $[G_L]_{0,0}^{-1}$ as a function of $\sin^2(p_4)$ with $p_1, p_2, p_3=0$ at $\beta_s=0.5$ on a $4^3\times 32\times 16$ lattice, at $m_0=0.1$ (solid circles) and 0.6 (solid triangles). Solid lines of both figures stand for the ones obtained from the mean-field propagator with the fitted parameter z.

directions. The coordinate s in the extra dimension runs $-8\leqslant s\leqslant 7$. Gauge configurations are generated by the five-hit Metropolis algorithm. For the thermalization first 5000 sweeps are discarded.

The fermion propagators are calculated by the conjugate gradient method on 50 configurations separated by 100 sweeps. We take the domain-wall mass $m_0=$ 0.7, 0.8, 0.9, 0.95, 0.99 at $\beta_s=$ 0.29 and $m_0=$ 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.9 at $\beta_s=$ 0.5. As mentioned before, the boundary conditions in first, second, third, and fifth directions are periodic and the one in fourth direction is antiperiodic. Wilson parameter r has been set to $r=1$. The fermion propagators have been investigated mainly at $s=0, -1$. These s's are the layers where we put sources. The layer at $s=0$ is the domain wall. Errors are all estimated by the jackknife method with unit bin size.

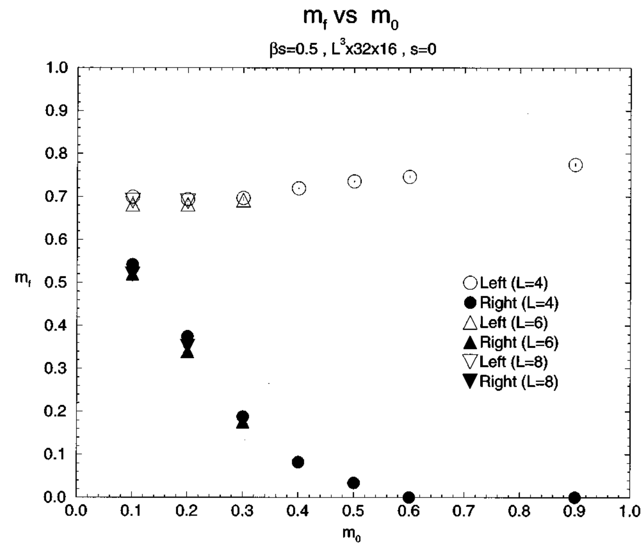

FIG. 3. m_f vs m_0 at $\beta_s=0.5$ on a $L^3\times 32\times 16$ lattices with $L=4$ (circles), 6 (up triangles), and 8 (down triangles) in the case of putting a source on the domain wall at $s=0$, for the right-handed fermion (solid symbols) and the left-handed fermion (open symbols).

C. Fermion spectrum in the broken phase

The system is in broken phase at $\beta_s=$ 0.5. We first consider the fermion spectrum on the layer at $s=0$. Let us show Fig. 2, which is a plot of the G_L^{-1} and G_R^{-1} as a function of $\overline{p_4^2}\equiv\sin^2(p_4)$ at m_0 =0.1 and 0.6. (Note we always set $p_1, p_2, p_3=0$.) In the limit $p_4\rightarrow 0$, G_R^{-1} remains nonzero at both m_0's, while G_L^{-1} vanishes at $m_0=0.6$. We obtain the value of m_f^2, which can be regarded as the mass square in four-dimensional world, by the linear fit in $\overline{p_4^2}$ near $\overline{p_4^2}=0$, and plot m_f as a function of m_0 in Fig. 3. The mass of

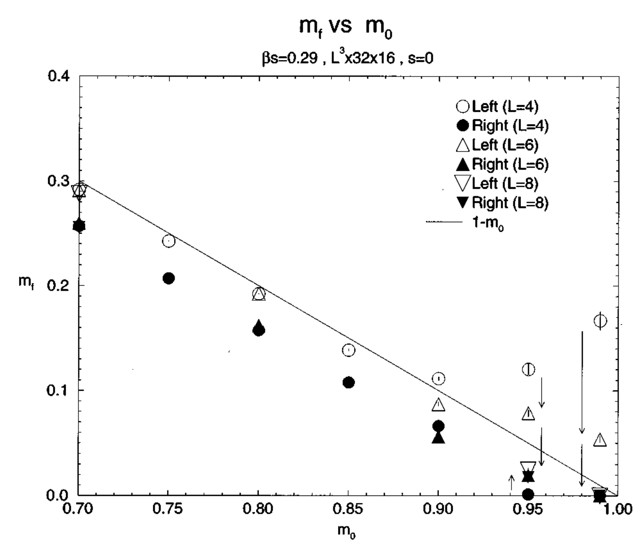

FIG. 4. m_f vs m_0 at $\beta_s=0.29$ on $L^3\times 32\times 16$ lattices with $L=4$ (circles), 6 (up triangles), and 8 (down triangles) in the case of putting a source on the domain wall at $s=0$, for the right-handed fermion (solid symbols) and the left-handed fermion (open symbols). Solid line corresponds to $1-m_0$.

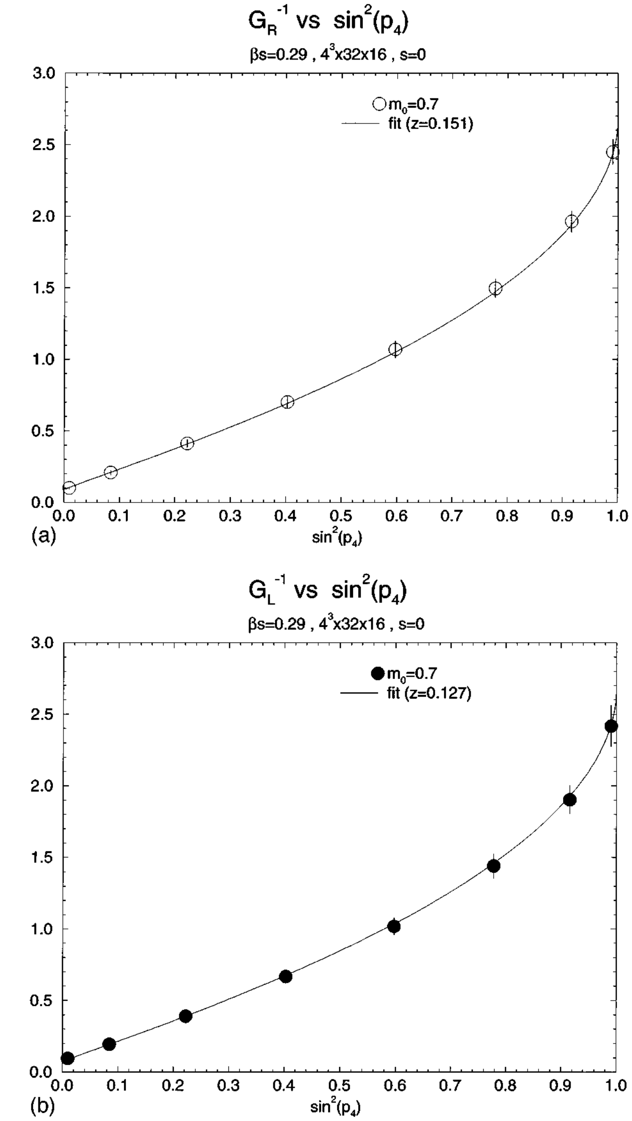

FIG. 5. (a) $[G_R]_{0,0}^{-1}$ as a function of $\sin^2(p_4)$ with $p_1,p_2,p_3=0$ at $\beta_s=0.29$ on a $4^3\times32\times16$ lattice, at $m_0=0.7$. (b) $[G_L]_{0,0}^{-1}$ as a function of $\sin^2(p_4)$ with $p_1,p_2,p_3=0$ at $\beta_s=0.29$ on a $4^3\times32\times16$ lattice, at $m_0=0.7$. Solid lines of both figures stand for the ones obtained from the mean-field propagator with the fitted parameter z.

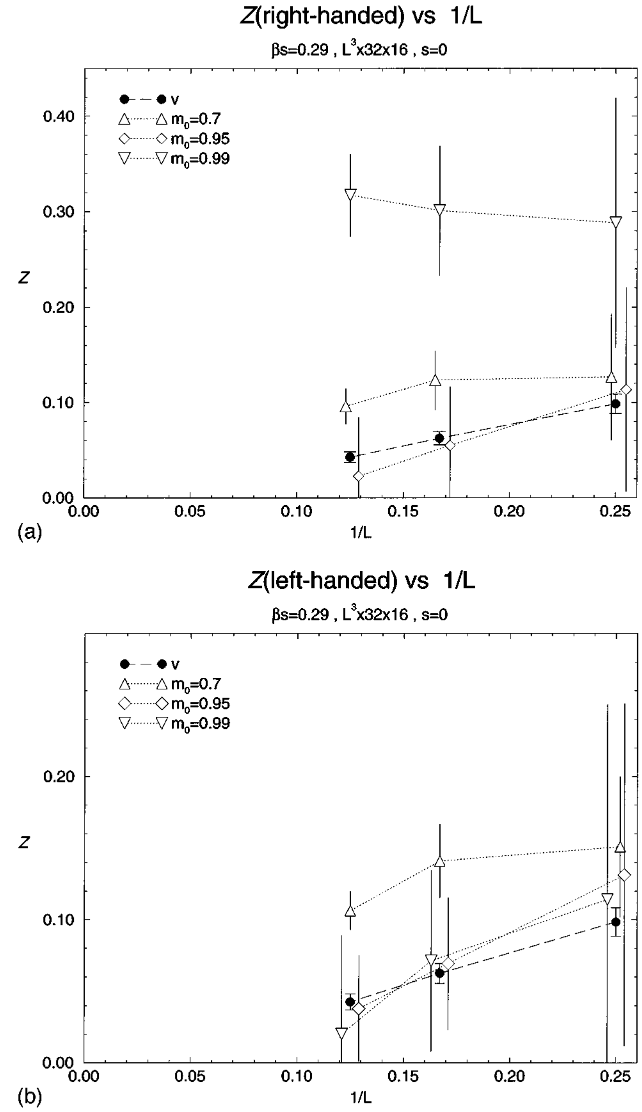

FIG. 6. (a) z (right handed) and v vs $1/L$ at $\beta_s=0.29$ on $L^3\times32\times16$ lattices with $L=4$, 6, and 8 at $m_0=0.7$ (up triangles), 0.95 (diamonds), and 0.99 (down triangles), in the case of putting a source on the domain wall at $s=0$. (b) z (left handed) and v vs $1/L$ at $\beta_s=0.29$ on $L^3\times32\times16$ lattices with $L=4$, 6, and 8 at $m_0=0.7$ (up triangles), 0.95 (diamonds), and 0.99 (down triangles), in the case of putting a source on the domain wall at $s=0$. Solid circles stand for the vacuum expectation value of link variable (the order parameter).

right-handed fermion, obtained from G_L^{-1}, becomes very small (<0.1) at m_0 larger than 0.35, so we conclude that the critical value is $m_0^c\sim0.35$. Whenever the domain-wall mass is larger than this value, this model produces the right-handed chiral zero mode on the domain wall at $s=0$. From the results above we conclude that the domain-wall model with the dynamical gauge field on the extra dimension (i.e., the weak coupling limit of the original model) can maintain the chiral zero mode on the domain wall, at least deep in the broken phase.

Also in Fig. 2, solid lines stand for the inverse propagators obtained from the mean-field propagator with appropriately tuned parameter z, and the lines show that the behavior of the fermion propagator is well described by the mean-field propagator.

An important question here is what a fermion spectrum is in the scaling limit. If the fermion spectrum stays chiral in the limit, it should stay chiral also in the symmetric phase, since the phase transition is continuous. This means that, in order to determine the fermion spectrum in the scaling limit even from the broken phase, we have to know the spectrum in the symmetric phase. Therefore, from the knowledge of the fermion spectrum obtained in the broken phase so far, we cannot draw any conclusions on the fermion spectrum, chiral or vectorlike, in the scaling limit.

D. Fermion spectrum in symmetric phase

The system is in the symmetric phase at $\beta_s=0.29$. The fermion propagator is analyzed in the same way as in the broken phase.

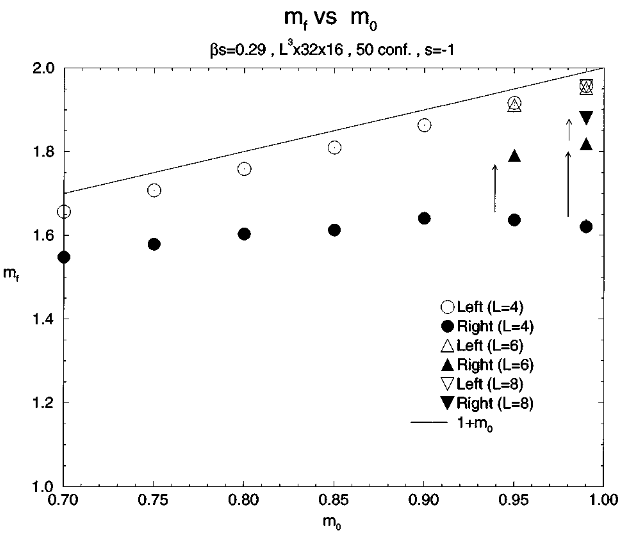

FIG. 7. m_f vs m_0 at $\beta_s=0.29$ on $L^3\times32\times16$ lattices with $L=4$ (circles), 6 (up triangles), and 8 (down triangles) in the case of putting a source at $s=-1$, for the right-handed fermion (solid symbols) and the left-handed fermion (open symbols). Solid line corresponds to $1+m_0$.

In Fig. 4, we have plotted mass m_f of the right- and left-handed modes at $s=0$ as a function of m_0. On a $4^3\times32\times16$ lattice, it seems that the right-handed fermion becomes massless at $m_0>0.95$, while the left-handed fermion stays massive at all m_0, so that the fermion spectrum on the domain wall is chiral. However, as the lattice sizes become larger, for example $6^3\times32\times16$ and $8^3\times32\times16$, mass differences between the right- and the left-handed modes become smaller. This suggests that the fermion spectrum becomes vectorlike in the infinite volume limit. From this data alone, however, we cannot exclude a possibility that the critical mass m_0^c is very close to 1.0, since the fermion mass near $m_0=1.0$ is very small. In order to make a definite conclusion on the absence of chiral zero modes in the symmetric phase, we try to fit G_L^{-1} and G_R^{-1} at given m_0 using the form of mean-field propagator, Eqs. (14) and (15), with the fitting parameter z. We show the quality of this fit in Figs. 5(a) and 5(b). These figures show that the fermion propagator is well described by the mean-field propagator if the fitting parameter z is chosen such that χ^2 is minimized. We then have plotted z obtained by the fit as a function of $1/L$ in Figs. 6(a) and 6(b), where L's take 4, 6, 8 on the $L^3\times32\times16$ lattices. The values of z's are almost independent of m_0 at each $1/L$ except the right-handed modes at $m_0=0.99$. The solid circles represent the order parameter v defined in Eq. (22). The behaviors of z's at different m_0's are almost identical to one another and are very similar to that of v except the right-handed ones at $m_0=0.99$. We think that the observed deviation of z (right handed) at $m_0=0.99$ from those at other m_0's is not a real effect but a statistical fluctuation, since the χ^2 for the fit at $m_0=0.99$ is almost flat in the region between $z=0$ and $z\sim0.3$. Ignoring this data at $m_0=0.99$ we see that z can be identified with v and, therefore, z becomes zero as the lattice size goes to infinity since the order parameter v defined in Eq. (22) becomes zero in the infinite volume limit. This analysis leads us to a conclusion that the fermion spectrum in the symmetric phase is *vectorlike*.

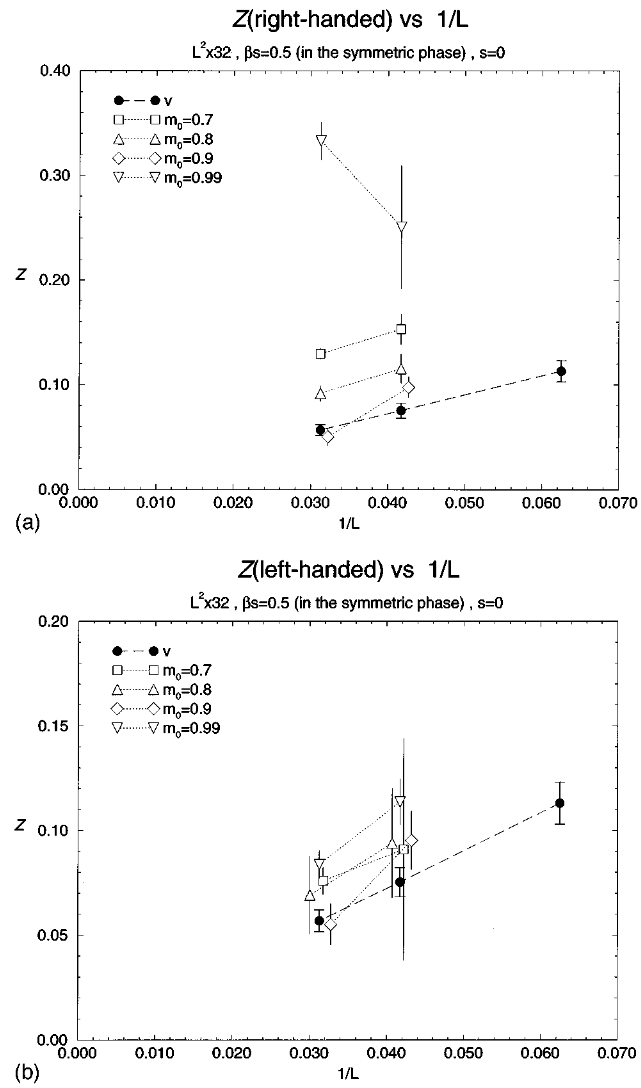

FIG. 8. (a) z (right handed) and v vs $1/L$ at $\beta_s=0.5$ in the symmetric phase on $L^2\times32$ lattices with $L=24$ and 32 at $m_0=0.7$ (squares), 0.8 (up triangles), 0.9 (diamonds), and 0.99 (down triangles), in the case of putting a source on the domain wall at $s=0$. (b) z (left handed) and v vs $1/L$ at $\beta_s=0.5$ in the symmetric phase on $L^2\times32$ lattices with $L=24$ and 32 at $m_0=0.7$ (squares), 0.8 (up triangles), 0.9 (diamonds), and 0.99 (down triangles), in the case of putting a source on the domain wall at $s=0$. Solid circles stand for the vacuum expectation value of link variable (the order parameter).

Figure 4 also shows that the fermion masses tend to approach the line $m_f=1-m_0$, which is the value of the mass for free Wilson fermion at a positive s layer, as the lattice volume increases. Furthermore, in Fig. 7 as a function of m_0, we have also plotted m_f at $s=-1$ (a negative s layer). As the volume increases the mass difference between right-handed and left-handed fermions becomes smaller and both masses seem to approach to the line $m_f=1+m_0$, the value of the mass for free Wilson fermion at a negative s layer. Since the Dirac and Wilson terms in the extra direction of the fermion action (3) are absent in the case of $\beta_s=0$ or, equivalently, the case of the mean-field parameter z being zero effectively, the fermion action is equivalent to the four-dimensional ones with the copies of the number of sites in the extra dimension and then two lines of m_f's at a positive and a negative layer represent a free Wilson fermion mass at each layer. The phase where this phenomenon occurs is known as the layered phase [26] and in this phase the chiral zero modes disappear because of the no-go theorem at each layer. Therefore, this data also supports our conclusion on the absence of chiral zero modes in the symmetric phase.

In summary our results of m_f at both positive s and negative s layer suggest that chiral zero modes in the symmetric phase disappear in the infinite volume limit. We conclude that the original Kaplan's model fails to describe lattice chiral gauge theories in the symmetric phase.

IV. CONCLUSIONS AND DISCUSSIONS

Using the quenched approximation, we have performed the numerical study of the domain-wall model in 4+1 dimensions with the U(1) dynamical gauge field on the extra dimension. From this study we obtain the following results. In the broken phase of the gauge field, there exists the critical value of the domain-wall mass separating the region with a chiral zero mode and the region without it in the same case of (2+1)-dimensional model. At the domain-wall mass larger than its critical value, a zero mode with one chirality exists on the domain wall. In the symmetric phase, on the other hand, our data on $L^3\times 32\times 16$ with $L=4$, 6, 8 lattices suggest the absence of chiral zero modes in the infinite volume limit, though the chiral zero mode seems to exist on finite lattices. We also found that fermion propagators obtained through numerical simulations on finite lattices are well described by the mean-field propagator with $z\simeq v$. Since the existence of chiral zero modes in the symmetric phase is essential for the success of the original domain-wall model, our results for the (4+1)-dimensional model indicate that the original domain-wall model cannot work as lattice chiral gauge theories.

In 2+1 dimensions we could not conclude whether or not the fermionic zero mode exists because of the difficulty for the simulation near $m_0=1$ and the absence of the order parameter in the infinite volume limit for the two-dimensional U(1) model. We try here to extract the parameter z of the (2+1)-dimensional model, by the method used for the (4+1)-dimensional model. In Figs. 8(a) and 8(b), the parameter z is plotted as a function of $1/L$. The behavior of z is almost identical to that of order parameter v in Eq. (22) but not to the square root of plaquette in Eq. (23). This new analysis shows that zero modes are absent in the symmetric phase in the (2+1)-dimensional U(1) model as well as in the (4+1)-dimensional model. Therefore, we conclude that U(1) chiral gauge theories cannot be constructed via an original domain-wall model, regardless of its dimensionality.

One of the remaining questions is whether the above negative conclusion also holds for other gauge groups such as SU(2). The answer is not so straightforward: For example, the two-dimensional SU(2) spin model has a symmetric phase only. We wonder whether chiral zero modes are absent in the symmetric phase even if the gauge field of the (2+1)-dimensional domain-wall model becomes ''smooth'' for large but finite β_s. To answer this question we are currently investigating the (2+1)-dimensional SU(2) domain-wall model.

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