Renormalons beyond one loop

Taekoon Lee
Department of Physics, Purdue University, West Lafayette, Indiana 47907
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Higher order renormalons beyond the chain of one-loop bubbles are discussed. A perturbation method for the infrared renormalon residue is found. The large order behavior of the current-current correlation function due to the first infrared renormalon is determined in both QED and QCD to the first three orders. [S0556-2821(97)01014-X]

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I. INTRODUCTION

Perturbation theory in field theories is generally plagued by the rapidly growing coefficients, which cause the series in weak coupling to be asymptotic. Classical solutions, instantons, cause the perturbative coefficients to grow as $n!$ for large n, where n is the order of perturbation, and so do certain subsets of Feynman diagrams, renormalons.[1] Some of the properties of the renormalons are discussed here.

A chain of the one-loop bubble diagrams in a photon propagator in massless QED is an example of the renormalons (Fig. 1). An exchange of the one-loop Gell-Mann–Low (GL) effective charge gives a contribution of $n!$ for an ultraviolet renormalon, in which the momentum flowing in the propagator is large compared to the renormalization scale $[k^2/\mu^2 \sim \exp(n)]$, and $(-1)^n n!$ for an infrared (IR) renormalon, in which a soft momentum $[k^2/\mu^2 \sim \exp(-n/2)]$ flows in the propagator.

The actual form of the large order behavior due to an infrared renormalon is generally given by

$$K n! n^{\nu} b_0^{-n}[1+O(1/n)], \tag{1}$$

where ν and b_0 are renormalon-specific, known constants. The coefficient K is an all-order quantity [1]. It depends not only on the one-loop renormalon mentioned above but also on an infinite set of higher order renormalons, and so determining it is nontrivial.

However, it should be emphasized that K is calculable, at least perturbatively. For example, if we had calculated the series to a very high order, then Eq. (1) implies that we could extract the coefficient to an accuracy of $O(1/n)$. Therefore, there must be a convergent sequence K_N for K, with K being its limit, associated with the perturbation of the amplitude in consideration. The main purpose of this paper is to present such a sequence for the first IR renormalon in the Borel plane.

The precise calculation of the large order behavior is important, besides its theoretical interest, because it could play an essential role in an effort to reconstruct the true amplitudes from the perturbation theory. The large order behavior due to the IR renormalons in non-Abelian gauge theory arises from the imaginary part of the nonperturbative effects, vacuum condensations, and so a precise calculation of the large order behavior gives detailed information on the imaginary part of the nonperturbative amplitudes, which could be essential in understanding the full amplitude. For a recent consideration in this direction one may refer to [2].

This paper is organized as follows. In Sec. II, we discuss in QED the higher order renormalons beyond the chain of one-loop bubbles, and show in detail how the large order behavior gets contribution from the higher order renormalons. In Secs. III–V, a systematic method of summing those higher order renormalons is discussed, and the renormalon residue to the first three orders is given. In Sec. VI, we discuss the calculation of the large order behavior in QCD using the analytic property in Borel plane, and give the large order behavior to the first three orders. In Sec. VII, the scheme dependence of the large order behavior is discussed.

II. HIGHER ORDER RENORMALONS

We first review how a chain of the one-loop bubble diagrams gives rise to factorial growing coefficients, and then show that the large order behavior of perturbation theory is an all-order property by giving an estimate of the higher order renormalons.

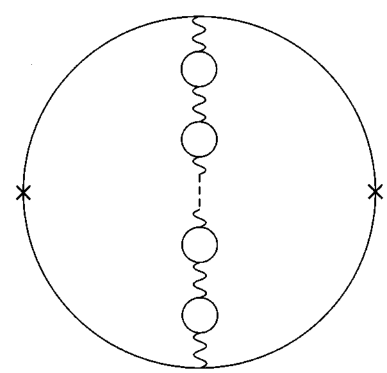

FIG. 1. One-loop renormalon.

The Green's function we consider is the electromagnetic current correlation function in QED in the Euclidean regime

[1]Renormalons also denote the singularities in the Borel plane.

$$i\int e^{iqx}\langle j_\mu(x)j_\nu(0)\rangle d^4x=(q_\mu q_\nu-q^2g_{\mu\nu})\frac{\Pi\left(\alpha(\mu),\frac{\mu^2}{Q^2}\right)}{\alpha(\mu)}, \tag{2}$$

where $Q^2=-q^2>0$, and

$$j^\mu(x)=\overline{\psi}\gamma^\mu\psi(x). \tag{3}$$

For large order behavior, it is more convenient to consider the renormalization scheme and scale-invariant quantity D defined by

$$\begin{aligned}D(\alpha(\mu),\mu^2/Q^2)&=Q^2\frac{\partial}{\partial Q^2}\left[\frac{\Pi\left(\alpha(\mu),\frac{\mu^2}{Q^2}\right)}{\alpha(\mu)}\right]\\&\quad-Q^2\frac{\partial}{\partial Q^2}\left[\frac{\Pi\left(\alpha(\mu),\frac{\mu^2}{Q^2}\right)}{\alpha(\mu)}\right]\Bigg|_{\alpha(\mu)=0}\\&=\sum_{n=0}^{\infty}a_n(\mu^2/Q^2)[\alpha(\mu)]^{n+1}.\end{aligned} \tag{4}$$

A single exchange of the GL effective charge gives rise to the first IR renormalon singularity in the Borel plane. It is also generally assumed that the leading residue of the first IR singularity can be completely determined by a single exchange of the GL effective charge, which implies equivalently that the large order behavior due to the IR renormalon can be determined by a single exchange of the GL effective charge (Fig. 1). For the other IR renormalon in the Borel plane, it is similarly believed that their residues can be saturated by the multiple exchanges of the GL effective charges. The analysis for IR renormalon in non-Abelian gauge theory using operator product expansion supports this assumption [3,4]. With this assumption for the large order behavior for the first IR renormalon, D may be written as

$$D(\alpha(\mu),\mu^2/Q^2)=\int_0 f(k^2)\mathbf{a}(k^2)dk^2, \tag{5}$$

where $\mathbf{a}(k^2)$ denotes the GL effective charge and

$$f(k^2)=\frac{-e_f^2k^2}{8\pi^3Q^4}\quad\text{for}\quad k^2\to 0, \tag{6}$$

with e_f denoting the charge of the fermion ψ. This infrared limit of $f(k^2)$ can be easily read off from the coefficient of $F^2_{\mu\nu}$ term in the operator product expansion of the current product in Eq. (2).

To see the $n!$ growth of the perturbative coefficient from the chain of the one-loop bubbles, we may substitute $\mathbf{a}(k^2)$ in Eq. (5) with its one-loop form

$$\mathbf{a}(k^2)=\frac{\alpha(\mu)}{1-\beta_0\alpha(\mu)\ln\left(\frac{k^2}{\mu^2}\right)}=\sum_0^\infty\left[\beta_0\ln\left(\frac{k^2}{\mu^2}\right)\right]^n\alpha(\mu)^{n+1}, \tag{7}$$

where β_0 is the first coefficient of the β function, to obtain

$$\begin{aligned}a_n&=-\frac{e_f^2\mu^4\beta_0^n}{8\pi^3Q^4}\int_0 t\ln(t)^n dt\\&=-\frac{e_f^2\mu^4}{16\pi^3Q^4}\left(-\frac{\beta_0}{2}\right)^n n![1+O(1/n)],\end{aligned} \tag{8}$$

where

$$t=\frac{k^2}{\mu^2}. \tag{9}$$

For large n the leading contribution to the integral comes from the kinetic region $k^2\sim\mu^2\exp(-n/2)$, and thus the leading large order behavior is independent of the upper bound of the integral.

Let us now consider the effect of higher order renormalons on the large order behavior of perturbation. First, we introduce some definitions. In the following the vacuum polarization diagrams are assumed to include two external photon propagators. An irreducible renormalon is defined by replacing all photon propagators in an irreducible vacuum polarization diagram with chains of the one-loop bubbles. Similarly, reducible renormalons are defined by replacing all photon lines in reducible vacuum polarization diagrams with chains of the one-loop bubbles. Thus for every vacuum polarization diagram there are corresponding renormalons.

We assign an order p, and the number of reduced photon propagators q, to each irreducible vacuum polarization diagram by

$$\begin{aligned}p&=n_A-n_L-1,\\q&=n_A-n_1,\end{aligned} \tag{10}$$

where n_A is the number of photon propagators and n_L,n_1 denote the number of irreducible vacuum polarization subdiagrams and the number of the one-loop bubbles, respectively. A reduced photon propagator is simply a chain of an unspecified number of one-loop bubbles. The same p and q of an irreducible vacuum polarization diagram are defined as the order and the number of the reduced photon lines of the corresponding renormalon. For example, the order and the number of reduced photon lines of the one-loop renormalon in Fig. 1 is $p=0,q=1$. Other higher order renormalons may be similarly characterized by the p and q. For reducible renormalons, the highest order of the irreducible subrenormalons of a reducible renormalon is defined as the order of the reducible renormalon. Some examples of the higher order renormalons are given in Fig. 2.

Before we discuss the effect on large order behavior of higher order renormalons in general, let us take some specific examples of low order renormalons, and see their contribution to the large order behavior. This exercise is very instructive and gives an insight to more complex renormalons.

It was first noticed by Grunberg [1], and diagrammatically by Mueller [5], that the large order behavior of perturbation is an all-order property (see also [6]). The following argument is motivated by Mueller's observation. Let us consider

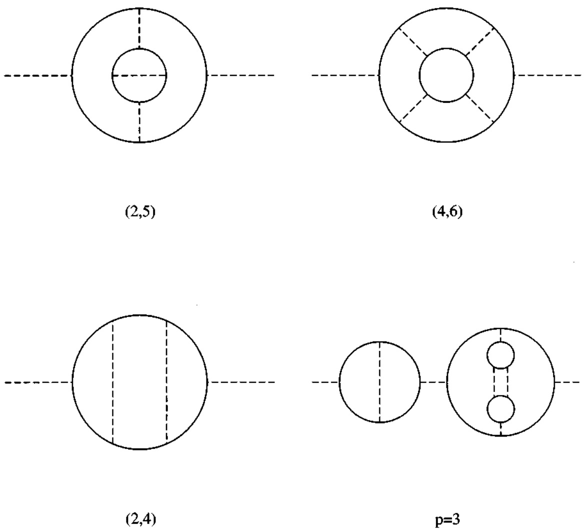

FIG. 2. Examples of higher order renormalons. Dashed lines denote chains of one-loop bubbles, and (p,q) denote the order and number of the reduced photon propagators, respectively.

the order-one renormalon in Fig. 3. The coefficient $\tilde{\mathbf{a}}_n(t)$ of $\alpha(\mu)^{n+1}$ due to this renormalon in the perturbation of the GL effective charge in $\alpha(\mu)$ is given by

$$\tilde{\mathbf{a}}_n(t)=\sum_{r=1}^{n-1}[-\Pi_1(t)]^{n-r-1}[-\Pi_{r+1}^{(r-1)}(t)](n-r), \tag{11}$$

where Π_r is the r-loop vacuum polarization function

$$\Pi(\alpha(\mu),t)=\sum_{r=1}^{\infty}\Pi_r(t)\alpha(\mu)^r. \tag{12}$$

The powers of Π_1 in Eq. (11) obviously come from the one-loop bubbles in the external reduced photon propagators, and the factor $(n-r)$ accounts for the $(n-r)$ possible location of the fermion loop with the internal reduced photon line. $\Pi_{r+1}^{(m)}$ denotes the terms proportional to β_0^m in Π_{r+1}.

The general form of $\Pi_{r+1}^{(m)}$ can be deduced by considering the following renormalization group equation for $\Pi(\alpha(\mu),t)$,

$$\mu^2\frac{d}{d\mu^2}\left[\frac{1}{\alpha(\mu)}+\frac{1}{\alpha(\mu)}\Pi(\alpha(\mu),t)\right]=0, \tag{13}$$

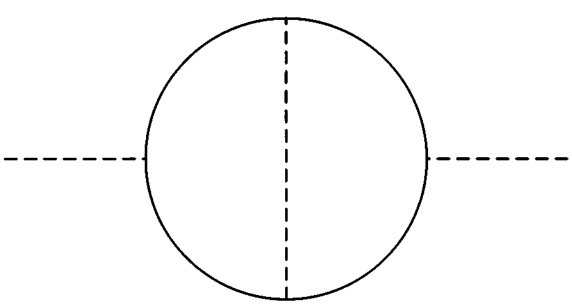

FIG. 3. Order-one renormalon. Dashed lines denote chains of the one-loop bubbles.

which comes from the renormalization scale invariance of the GL effective charge $\mathbf{a}(k^2)$:

$$\mathbf{a}(\alpha(\mu),t)=\frac{\alpha(\mu)}{1+\Pi(\alpha(\mu),t)}. \tag{14}$$

Putting the perturbative form of Π in Eq. (12) into Eq. (13), we get the recursion equation

$$t\frac{d}{dt}\Pi_{n+1}=-\beta_n+\sum_{m=2}^{n}(m-1)\beta_{n-m}\Pi_m \quad \text{for } n=2,3,\ldots, \tag{15}$$

with

$$\Pi_1=-\beta_0\ln t+p_1,$$

$$\Pi_2=-\beta_1\ln t+p_2. \tag{16}$$

Here β_m are the coefficients of the β function defined by

$$\beta(\alpha(\mu))=\mu^2\frac{d\alpha(\mu)}{d\mu^2}=\sum_{m=0}^{\infty}\beta_m\alpha(\mu)^{m+2}, \tag{17}$$

and p_i are constants.

Solving the recursion equation we have

$$\Pi_{r+1}^{(r-1)}=(\beta_0\ln(t))^{r-1}\left[\frac{-\beta_1}{r}\ln(t)+p_2\right]. \tag{18}$$

Substitution of this into Eq. (11) gives

$$\tilde{\mathbf{a}}_n(t)=\sum_{r=1}^{n-1}(\beta_0\ln(t))^{n-2}\left[\frac{\beta_1}{r}\ln(t)-p_2\right](n-r). \tag{19}$$

Here we kept only the $\ln(t)$ term in $\Pi_1(t)$ for simplicity, and the effect of the constant term in $\Pi_1(t)$ will be discussed shortly. Since a factor of $(\ln t)^n$ in the integrand in Eq. (5) would give rise to

$$\frac{(-1)^n n!}{2^{n+1}}, \tag{20}$$

$\tilde{\mathbf{a}}_n(t)$ gives the following large order behavior:

$$a_n(\mu^2/Q^2)=-\frac{e_f^2\mu^4}{16\pi^3Q^4}\left(-\frac{\beta_0}{2}\right)^n n! \times\left[\frac{-2\beta_1}{\beta_0^2}(\ln n+\gamma_E-1)-2\frac{p_2}{\beta_0^2}\right][1+O(1/n)], \tag{21}$$

which is comparable to the one-loop renormalon contribution. Here γ_E is the Euler constant.

Going back to Eq. (11), expanding the factor

$$(-\Pi_1)^{n-r-1}=(\beta_0\ln t-p_1)^{n-r-1}, \tag{22}$$

it is easy to see that every term in the expansion proportional to

$$(p_1)^i \quad \text{for } i\ll n \tag{23}$$

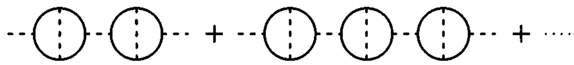

FIG. 4. Chains of the order-one renormalons.

also contributes to the large order behavior. Thus the inclusion of the constant term in $\Pi_1(t)$ would modify Eq. (21) into a form

$$a_n \sim \left(-\frac{\beta_0}{2}\right)^n n! \left[\frac{\beta_1}{\beta_0^2}[(\ln n)h(p_1)+h'(p_1)]+\frac{p_2}{\beta_0^2}h''(p_1)\right] \times [1+O(1/n)], \tag{24}$$

where

$$h(p_1)=h_0+h_1\left(\frac{p_1}{\beta_0}\right)+h_2\left(\frac{p_1}{\beta_0}\right)^2+\cdots. \tag{25}$$

Here h_i are calculable constants. Note that this series runs to an infinite order in the limit $n\to\infty$. $h'(p_1),h''(p_1)$ are similarly defined in a series form.

Using a similar method it is now easy to estimate the large order behavior from other renormalons. For example, it is straightforward to check that the chains of the order-one irreducible renormalon in Fig. 4 along with the one-loop renormalon contribution in Eq. (8) exponentiate the ln(n) term in Eq. (21) to give the n^ν factor in the large order behavior with

$$\nu=\frac{-2\beta_1}{\beta_0^2}, \tag{26}$$

which agrees with the well-known result for the first IR renormalon [4].

Now to discuss the higher order renormalons in general, consider an irreducible renormalon of order p and reduced photon propagator number q. The large order behavior due to this renormalon in the infrared regime is given by

$$a_n \sim \int_0 t \sum_{r=q-2}^{n-1} (\beta_0 \ln t)^{n-r-1}[\Pi_{r+1}^{(r-q+2)}(t)](n-r)dt. \tag{27}$$

Here we picked up only the ln(t) term in $\Pi_1(t)$ as before. The general solution of the recursion equation (15) may be organized in terms of p and q as

$$\Pi_{r+1}=\sum_{q'=1}^{r} \beta_0^{r-q'} \sum_{p=1}^{q'} \sum_{k=0}^{1} (\ln t)^{r-p-k+1} \times \sum_{\{m_i,\overline{m_i}\}} C^{rpqk}_{\{m_i,\overline{m_i}\}} \prod_{i=2}^{p+1} \beta_{i-1}^{m_i} p_i^{\overline{m_i}}, \tag{28}$$

where

$$q'=q-2 \tag{29}$$

and $m_i,\overline{m_i}$ are non-negative integers that satisfy

$$\sum_{i\geqslant 2} m_i+\overline{m_i}=q'-p+1,$$

$$\sum_{i\geqslant 2} \overline{m_i}=k,$$

$$\sum_{i\geqslant 2} (m_i+\overline{m_i})(i-1)=p. \tag{30}$$

Here p_i is the constant term of $\Pi_i(t)$. The $\Pi_{r+1}^{(r-q+2)}(t)$ of order p renormalon is then given by

$$\Pi_{r+1}^{(r-q+2)}(t)=\beta_0^{r-q+2}\sum_{k=0}^{1} (\ln t)^{r-p-k+1} \times \sum_{\{m_i,\overline{m_i}\}} C^{rpqk}_{\{m_i,\overline{m_i}\}} \prod_{i=2}^{p+1} \beta_{i-1}^{m_i} p_i^{\overline{m_i}}. \tag{31}$$

By solving the recursion equation (15) explicitly for the several low order diagrams, and considering the form of the large order behavior in Eq. (1), it is not difficult to convince oneself that asymptotically

$$C^{rpqk}_{\{m_i,\overline{m_i}\}} \sim r^{p+k-2} g(\ln r) \quad \text{for } r\to\infty. \tag{32}$$

Here the function $g(x)$ is a polynomial of degree at most m_2-1 for $m_2\geqslant 2$, and causes the n^ν term in the large order behavior.

Substituting Eq. (31) into Eq. (27) and ignoring the logarithmic dependence in g, we get

$$a_n \sim \frac{n!\left(-\frac{\beta_0}{2}\right)^n}{(-\beta_0)^{q-1}} \sum_{k=0}^{1} J(p,k) \sum_{\{m_i,\overline{m_i}\}} \prod_{i=2}^{p+1} \beta_{i-1}^{m_i} p_i^{\overline{m_i}}, \tag{33}$$

where $J(p,k)$ is a function of p and k. This shows that the irreducible renormalons of all order contribute to the large order behavior. Further note that the large order behavior due to an irreducible renormalon of order p involves only the coefficients of the vacuum polarization function and the β function to $(p+1)$-loop order.

These are also true for the reducible renormalons. As we chain more irreducible subrenormalons into a reducible renormalon, more powers of ln$t$ in $\tilde{\mathbf{a}}_n(t)$ due to this renormalon are being lost, resulting in suppressed integral in t, but this suppression is exactly compensated by the larger combinatoric factor caused by the more possible locations in putting the subrenormalons, giving a large order behavior comparable to those from the irreducible renormalons. Therefore, all reducible renormalons also contribute to the leading large order behavior. Also, since $\tilde{\mathbf{a}}_n(t)$ for a reducible renormalon with m irreducible subrenormalons is proportional to

$$\Pi_{r_1+1}^{(r_1-q_1+2)}\Pi_{r_2+1}^{(r_2-q_2+2)}\cdots\Pi_{r_m+1}^{(r_m-q_m+2)}, \tag{34}$$

with each $\Pi_{r_i+1}^{(r_i-q_i+2)}$ coming from the subrenormalons, it is obvious that the large order behavior from this reducible renormalon depends on the vacuum polarization function and the β function to $(p+1)$-loop order.

The inclusion of the constant term in $\Pi_1(t)$ in Eq. (27) would modify the large order behavior in Eq. (33) in a similar fashion as in the example of the order-one irreducible renormalon in Eq. (24). With p_1 included, each term in Eq. (33) will be multiplied by a series in p_1 in the form of Eq. (25), with the coefficients h_i now depending on each particular term.

Though this analysis of diagrams is very helpful in understanding general higher order renormalons qualitatively, it seems difficult, or at least inconvenient, to systematically calculate the higher order renormalons using this technique. We need a more straightforward approach for systematic evaluation of the higher order renormalons. In the following sections such an approach is discussed.

III. BOREL TRANSFORM OF GL EFFECTIVE CHARGE

The problem of determining the renormalon residues of $D(\alpha(\mu),\mu^2/Q^2)$ eventually reduces to finding the Borel transform of the GL effective charge. We give in this section a perturbation method for the Borel transform of $\mathbf{a}(k^2)$. We are going to introduce a scheme and scale-independent coupling, and then write the GL effective charge in terms of the coupling in a form that is particularly convenient for Borel transform.

The Borel transform of $D(\alpha(\mu),\mu^2/Q^2)$ is defined by

$$D(\alpha(\mu),\mu^2/Q^2)=\int_0^\infty \exp\left(-\frac{b}{\alpha(\mu)}\right)\tilde{D}(b)\,db. \quad (35)$$

With the perturbative series of $D(\alpha(\mu),\mu^2/Q^2)$ in Eq. (4),

$$\tilde{D}(b)=\sum_{n=0}^{\infty}\frac{a_n}{n!}b^n. \quad (36)$$

The large order behavior of the form in Eq. (1) causes a singularity (renormalon) in $\tilde{D}(b)$ and, conversely, the singularity in $\tilde{D}(b)$ determines the large order behavior. Thus by studying $\tilde{D}(b)$ near the renormalon singularities we can determine the large order behavior of $D(\alpha(\mu),\mu^2/Q^2)$.

Let us first consider the renormalization group equation for the GL effective charge $\mathbf{a}(k^2)$,

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha)\frac{\partial}{\partial\alpha}\right)\mathbf{a}(\alpha(\mu),t)=0. \quad (37)$$

Solving the equation we may write the GL effective charge as

$$\mathbf{a}(k^2)=\frac{1}{\dfrac{1}{A(k^2)}+C(\mathbf{a}(k^2))}, \quad (38)$$

where $C(\mathbf{a})$ is a scheme-independent function. The effective coupling A is defined by

$$A(\alpha(\mu),t)=\frac{1}{-\beta_0\left(\ln t+\int^{\alpha(\mu)}\dfrac{1}{\beta(\alpha)}d\alpha-\dfrac{p_1}{\beta_0}\right)}, \quad (39)$$

where p_1 is defined in Eq. (16) and the integral of $1/\beta(\alpha)$ is defined in the perturbative form

$$\int^{\alpha(\mu)}\frac{1}{\beta(\alpha)}d\alpha=-\frac{1}{\beta_0\alpha(\mu)}-\frac{\beta_1}{\beta_0^2}\ln\alpha(\mu)+\cdots. \quad (40)$$

Since

$$A(k^2)=\alpha(k^2)\left[1-\frac{\beta_1}{\beta_0}\alpha(k^2)\ln\alpha(k^2)+\cdots\right], \quad (41)$$

A can be as good an expansion parameter as α. $A(k^2)$ is also independent of renormalization scheme. The scheme independence of $A(k^2)$ can be easily understood by considering the difference of the integral in Eq. (39) between two schemes. Since the difference is μ independent, it cannot depend on $\alpha(\mu)$ and thus must be a constant. It is in fact given by

$$\int^{\alpha'(\mu)}\frac{dx}{\beta'(x)}-\int^{\alpha(\mu)}\frac{dx}{\beta(x)}=\gamma_1, \quad (42)$$

where γ_1 is the first coefficient of the relation between the couplings of the two schemes

$$\alpha'(\mu)=\alpha(\mu)[1+\gamma_1\alpha(\mu)+\gamma_2\alpha(\mu)^2+\cdots]. \quad (43)$$

The difference in Eq. (42) is then canceled by the scheme dependence of p_1

$$p_1'=p_1+\gamma_1, \quad (44)$$

leading to the scheme independence of $A(k^2)$. Note that a similar effective charge was considered in relation to the renormalization scheme-invariant perturbation by Maximov and Vovk [7].

Now $C(\mathbf{a}(k^2))$ in Eq. (38) may be expanded in an asymptotic series as

$$C(\mathbf{a}(k^2))=c_1\ln(\mathbf{a}(k^2))+\sum_2^{\infty} c_i(\mathbf{a}(k^2))^{i-1}. \quad (45)$$

The coefficients c_i can be determined by solving perturbatively $\mathbf{a}(k^2)$ in Eq. (38) with the help of Eqs. (39) and (40) in terms of $\alpha(\mu)$, and comparing it with the perturbative expansion of the GL effective charge in $\alpha(\mu)$ using Eq. (14). Then it is not difficult to see that c_i are given by the renormalization scheme-invariant combinations of the coefficients of the vacuum polarization function and β function. This expansion turns out to be a critical step for the Borel transform of the effective charge. Introduce $\mathbf{a}^{(N)}(k^2)$:

$$\mathbf{a}^{(N)}(k^2)=\frac{1}{\dfrac{1}{A(k^2)}+c_1\ln(\mathbf{a}^{(N)}(k^2))+\sum_2^{N} c_i\mathbf{a}^{(N)}(k^2)^{i-1}}. \quad (46)$$

Solving $\mathbf{a}^{(N)}(k^2)$ in Eq. (46) recursively in $\alpha(\mu)$, it can be seen that this equation generates all the higher order renormalon diagrams to order N.

Let us now consider a modified Borel transform of $\mathbf{a}^{(N)}\times(k^2)$ defined by

$$\mathbf{a}^{(N)}(\alpha(\mu),t)=\int_0^\infty \exp\left(-\frac{b}{A(t)}\right)\widetilde{\mathbf{a}}^{(N)}(b)db. \tag{47}$$

Note that this Borel transform of GL effective charge was introduced by Grunberg [1]. Substituting Eq. (47) into Eq. (5), we can write $D^{(N)}(\alpha(\mu),\mu^2/Q^2)$, which is defined by replacing $\mathbf{a}(k^2)$ in Eq. (5) with $\mathbf{a}^{(N)}(k^2)$, as

$$D^{(N)}(\alpha(\mu),\mu^2/Q^2)=\int \exp\left[b\beta_0\int^{\alpha(\mu)}\frac{d\alpha}{\beta(\alpha)}\right]\times\{e^{-bp_1}\widetilde{f}(b)\widetilde{\mathbf{a}}^{(N)}(b)\}db, \tag{48}$$

where

$$\widetilde{f}(b)=\int_0 f(t)\exp(b\beta_0\ln t)dt. \tag{49}$$

The first IR renormalon singularity arises from the IR divergence in the integral in Eq. (49). Substituting Eq. (6) into Eq. (49),

$$\begin{aligned}\widetilde{f}(b)&=\int_0^M f(t)e^{b\beta_0\ln t}dt\\&=-\frac{e_f^2\mu^4}{8\pi^3Q^4}\int_0^M te^{b\beta_0\ln t}dt\\&=-\frac{e_f^2\mu^4}{8\pi^3Q^4}\frac{1}{2+b\beta_0}[1+(2+b\beta_0)\ln M+\cdots],\end{aligned} \tag{50}$$

where M is an arbitrary UV cutoff. Notice that the leading renormalon singularity is cutoff independent.

The scheme dependence of the Borel transform of $D^{(N)}(\alpha(\mu),\mu^2/Q^2)$ is now isolated in

$$e^{-bp_1}\quad\text{and}\quad \exp\left(b\beta_0\int^{\alpha(\mu)}\frac{d\alpha}{\beta(\alpha)}\right). \tag{51}$$

To find the Borel transform explicitly, let us take a renormalization scheme in which the β function is given by the simple form

$$\beta(\alpha)=\frac{\beta_0\alpha^2}{1-\lambda\alpha}, \tag{52}$$

where

$$\lambda=\frac{\beta_1}{\beta_0}. \tag{53}$$

With this β function, Eq. (48) defines the modified Borel transform by Brown, Yaffe, and Zhai (BYZ) [8]:

$$D^{(N)}(\alpha,\mu^2/Q^2)=\int_0^\infty \exp\left[-b\left\{\frac{1}{\alpha}+\lambda\ln\left(\frac{\alpha}{b}\right)\right\}\right]\widetilde{f}(b)\times e^{-bp_1}e^{-\lambda b\ln b}\widetilde{\mathbf{a}}^{(N)}(b)db. \tag{54}$$

From this we can read off the modified Borel transform of $D^{(N)}(\alpha(\mu),\mu^2/Q^2)$

$$\widetilde{D}^{(N)}_{\rm BYZ}(b)=e^{-bp_1}\widetilde{f}(b)e^{-\lambda b\ln b}\widetilde{\mathbf{a}}^{(N)}(b). \tag{55}$$

Then using the relation between the ordinary Borel transform and the modified one [8], we have the ordinary Borel transform

$$\widetilde{D}^{(N)}(b)=-\frac{e_f^2\mu^4}{16\pi^3Q^4}e^{-b_0p_1}e^{-\lambda b_0\ln b_0}\widetilde{\mathbf{a}}^{(N)}(b_0)\times\frac{(-2\lambda/\beta_0)!}{\left(1+\frac{1}{2}b\beta_0\right)^{1-2\lambda/\beta_0}}[1+O(2+b\beta_0)]. \tag{56}$$

The real problem here is to find the $\widetilde{\mathbf{a}}^{(N)}(b)$. However, with the help of the expansion in Eq. (46) it is now easy to calculate the Borel transform of $\mathbf{a}^{(N)}(k^2)$. The inverse of Eq. (47) is

$$\widetilde{\mathbf{a}}^{(N)}(b)=\frac{1}{2\pi i}\int_C e^{bx}\mathbf{a}^{(N)}(x)dx, \tag{57}$$

with the contour wrapping around the negative real axis. Here

$$x=\frac{1}{A(k^2)} \tag{58}$$

and

$$\mathbf{a}^{(N)}(x)=\frac{1}{x+c_1\ln \mathbf{a}^{(N)}(x)+\Sigma_2^N c_i[\mathbf{a}^{(N)}(x)]^{i-1}}. \tag{59}$$

Putting

$$y=\frac{1}{\mathbf{a}^{(N)}(x)}, \tag{60}$$

we can write Eq. (59) as

$$x=y+c_1\ln y-\sum_{i=2}^N\frac{c_i}{y^{i-1}}, \tag{61}$$

and Eq. (57) as

$$\widetilde{\mathbf{a}}^{(N)}(b)=\frac{1}{2\pi i}\int e^{by}y^{bc_1-1}\exp\left(-b\sum_{i=2}^N\frac{c_i}{y^{i-1}}\right)\sum_{i=0}^N\overline{\frac{c_i}{y^i}}dy, \tag{62}$$

where

$$\overline{c}_i = \begin{cases} 1 & \text{for } i=0, \\ c_1 & \text{for } i=1, \\ (i-1)c_i & \text{for } i \geqslant 2. \end{cases} \tag{63}$$

The exponential term in the integrand may be expanded as

$$\exp\left(-b\sum_{2}^{N}\frac{c_i}{y^{i-1}}\right) = \sum_{k=0}^{\infty}\frac{(-b)^k}{k!}\sum_{l=k}^{k(N-1)}\frac{h_{Nkl}}{y^l}, \tag{64}$$

where

$$h_{Nkl} = k!\sum_{\{n_i\}}\frac{\prod_{i=1}^{N-1} c_{i+1}^{n_i}}{\prod_{i=1}^{N-1} n_i!}, \tag{65}$$

with the set $\{n_i\}$ of non-negative integers satisfying

$$\sum_{i=1}^{N-1} n_i i = l, \quad \sum_{i=1}^{N-1} n_i = k. \tag{66}$$

Substituting Eq. (64) into Eq. (62), we finally have

$$\begin{aligned} b^{bc_1}\tilde{\mathbf{a}}^{(N)}(b) &= \frac{b^{bc_1}}{2\pi i}\int e^{by}\sum_{k=0}^{\infty}\frac{(-b)^k}{k!} \\ &\quad \times \sum_{l=k}^{k(N-1)}\frac{h_{Nkl}}{y^l}\sum_{i=0}^{N}\overline{c}_i y^{bc_1-i-1}dy \\ &= \sum_{k=0}^{\infty}\sum_{l=k}^{k(N-1)}\sum_{i=0}^{N}\frac{(-1)^k h_{Nkl}\overline{c}_i}{k!\Gamma(l+i+1-bc_1)}b^{k+l+i}. \end{aligned} \tag{67}$$

This completes the Borel transform of the GL effective charge.

IV. RENORMALON RESIDUE

To find the leading renormalon residue of $\tilde{D}^{(N)}(b)$, we have to evaluate $b^{bc_1}\tilde{\mathbf{a}}^{(N)}(b)$ at the first IR renormalon position, $b_0 = -2/\beta_0$. If we directly substitute b in Eq. (67) with b_0, the resulting large order behavior would sum all the contribution from the renormalons to order N, but unfortunately this large order behavior does not have a finite limit for $N \to \infty$ [14]. The reason for this is that $\tilde{\mathbf{a}}(b)$ is singular at the UV and IR renormalon positions, and its radius of convergence when it is expanded as in Eq. (67) is given by the position at $b = 1/\beta_0$ of the first UV renormalon, which is the closest renormalon to the origin in the Borel plane. Therefore, we cannot substitute b with b_0 in Eq. (67) to correctly evaluate the Borel transform at the first IR renormalon.

This problem can be avoided by introducing an analytic transform of the Borel plane so that the closest renormalon to the origin in the new complex plane is the first IR renormalon [9]. Because the singularity of $\tilde{\mathbf{a}}(b)$ at the IR renormalon is such that it is finite but has divergent derivative [1], we can then express the residue as a convergent series.

For this purpose, we can take any analytic transform that puts the IR renormalon as the closest singularity to the origin, but here we consider a simple form

$$z = \frac{\beta_0 b}{1-\beta_0 b}, \tag{68}$$

with its inverse

$$b = \frac{1}{\beta_0}\left(\frac{z}{1+z}\right). \tag{69}$$

In the z plane, the closest singularity to the origin is the first IR renormalon at

$$z_0 = -\tfrac{2}{3}, \tag{70}$$

and all the UV renormalons are pushed beyond $z = -1$ on the real axis. It is interesting to note that the freedom in choosing this analytic transform is similar to that for the renormalization scheme. As the renormalization scheme can be optimized for a particular process, the analytic transform could be chosen to optimize the perturbative evaluation of the residue.

Now to find $b^{bc_1}\tilde{\mathbf{a}}^{(N)}(b)$ at the first IR renormalon, we have to substitute b in Eq. (67) with that in Eq. (69) and expand it in Taylor series at $z = 0$ to order N, and evaluate it at $z = z_0$. Thus the Borel transform of GL effective charge at the first IR renormalon is given by

$$\kappa_N = b^{bc_1}\tilde{\mathbf{a}}^{(N)}(b)|_{b=b_0} = \sum_{M=0}^{N} q_M z_0^M, \tag{71}$$

with

$$\begin{aligned} q_M &= (M-1)!\,\beta_0^{-M}\sum_{k=0}^{N}\sum_{l=k}^{k(N-1)} \\ &\quad \times \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\frac{(-1)^{k+m}h_{Nkl}\overline{c}_i c_1^j}{k!m!(M-m-1)!} \\ &\quad \times \gamma_j^{\{l+i\}}\beta_0^m\delta_{k+l+i+j+m,M}. \end{aligned} \tag{72}$$

Here $\gamma_j^{\{n\}}$ is defined by

$$\frac{1}{\Gamma(n+1-x)} = \sum_{j=0}\gamma_j^{\{n\}}x^j. \tag{73}$$

Note that the coefficient q_N is completely determined by $\tilde{\mathbf{a}}^{(N)}(b)$ in Eq. (67), and is not modified by the Borel transform $\tilde{\mathbf{a}}^{(N+m)}(b)$ of the higher order effective charge. The IR residue is then

$$\kappa = \sum_{M=0}^{\infty} q_M z_0^M. \tag{74}$$

It should be emphasized that, though this series is being evaluated at its radius of convergence, it is convergent because of the finiteness of $\tilde{\mathbf{a}}(b(z))$ at $z = z_0$.

V. LARGE ORDER BEHAVIOR

The Borel transform $\widetilde{D}(b)$ can now be used in determining the leading large order behavior of $D(\alpha)$. Note that the leading large order behavior is determined by the leading Borel singularity, and the $1/n$ correction in the large order behavior corresponds to the $O(b-b_0)$ correction in the Borel transform. First, we give the large order behavior in a renormalization scheme in which the β function is given by Eq. (52), and then in Sec. VII discuss a class of schemes in which all schemes share a common large order behavior except for a trivial scheme-dependent term. Using the known result for the vacuum polarization function and the β function to three loop and four loop, respectively, we will also give numerical values for the large order behavior.

Let us go back to the Borel transform of D in Eq. (56). Expanding $\widetilde{D}^{(N)}(b)$ at the origin, and using the definition of the Borel transform in Eq. (36), the leading large order behavior of $D^{(N)}(\alpha(\mu),\mu^2/Q^2)$ is given by

$$a_n^{(N)}=-\frac{e_f^2\mu^4}{16\pi^3Q^4}e^{-b_0p_1}e^{-\lambda b_0\ln b_0}\widetilde{\mathbf{a}}^{(N)}(b_0)n!\,n^{\lambda b_0}b_0^{-n}$$

$$\text{with}\quad b_0=\frac{-2}{\beta_0}. \tag{75}$$

Then the sequence for the large order behavior mentioned in Sec. I may be defined as

$$K_N=-\frac{e_f^2\mu^4}{16\pi^3Q^4}e^{-b_0p_1}\kappa_N, \tag{76}$$

with κ_N defined in Eq. (71).

To evaluate the numerical values for κ_N, we have to find the coefficients c_i in Eq. (46) explicitly. The β function and the vacuum polarization function in the modified minimal subtraction ($\overline{\text{MS}}$) scheme to three loop and four loop, respectively, are given by [10]

$$\beta_0=\frac{1}{2\pi}\left(\frac{2}{3}N_f\right),$$

$$\beta_1=\frac{1}{2\pi^2}\left(\frac{N_f}{2}\right),$$

$$\beta_2=\frac{1}{2\pi^3}\left(-\frac{1}{16}N_f-\frac{11}{72}N_f^2\right),$$

$$\beta_3=\frac{1}{2\pi^4}\left[-\frac{23}{64}N_f+\left(\frac{95}{432}-\frac{13}{18}\zeta(3)\right)N_f^2-\frac{77}{1944}N_f^3\right], \tag{77}$$

and

$$p_1=\frac{5}{9\pi}N_f,$$

$$p_2=\frac{N_f}{\pi^2}\left[\frac{55}{48}-\zeta(3)\right],$$

$$p_3=\frac{1}{\pi^3}\left[\left(-\frac{143}{288}-\frac{37}{24}\zeta(3)+\frac{5}{2}\zeta(5)\right)N_f+\left(-\frac{3701}{2592}+\frac{19}{18}\zeta(3)\right)N_f^2\right], \tag{78}$$

where N_f is the number of fermion flavors. Solving $\mathbf{a}(k^2)$ in Eq. (38) in terms of $\alpha(\mu)$ and comparing it with Eq. (14) we find

$$c_1=-\frac{\beta_1}{\beta_0},$$

$$c_2=-\frac{\beta_2}{\beta_0}+\frac{\beta_1^2}{\beta_0^2}-\frac{p_1\beta_1}{\beta_0}+p_2,$$

$$c_3=-\frac{\beta_3}{2\beta_0}+\frac{\beta_1\beta_2}{\beta_0^2}-\frac{\beta_1^3}{2\beta_0^3}-\frac{p_1\beta_2}{\beta_0}+\frac{p_1\beta_1^2}{\beta_0^2}-\frac{p_1^2\beta_1}{2\beta_0}-\frac{p_2\beta_1}{\beta_0}+p_1p_2+p_3. \tag{79}$$

Note that the use of the vacuum polarization function and the β function in $\overline{\text{MS}}$ is allowed, because c_i are independent of renormalization scheme.

The κ_N obtained by substituting Eq. (79) into Eq. (71) is given in Table I for several flavor numbers. In the table, we see that for a reliable estimation of the large order behavior, a higher order calculation beyond the current one is required for the vacuum polarization function and the β function.

TABLE I. The first three elements of the sequence for the first IR renormalon residue in QED. κ_0 denotes the residue from the one-loop renormalon.

	$N_f=1$	$N_f=2$	$N_f=3$	$N_f=4$	$N_f=5$	$N_f=100$
κ_0	1	1	1	1	1	1
κ_1	1.63	1.32	1.21	1.16	1.13	1.00
κ_2	0.71	1.31	1.31	1.27	1.24	1.02
κ_3	−1.53	1.25	1.41	1.39	1.34	1.02

Not surprisingly, the numbers in the table also suggest that the large N_f limit is the one-loop renormalon. This is indeed the case. To see this, note that the following coefficients scale as

$$\beta_i\sim N_f^i \quad \text{for}\quad i>0,$$

$$c_i\sim N_f^{i-1} \tag{80}$$

for large N_f. Then scaling the variable y in Eq. (62) by b_0y, it is straightforward to see that

$$\lim_{N_f\to\infty}\kappa_N=1. \tag{81}$$

VI. RESIDUE IN QCD

In QCD, there is unfortunately no satisfactory definition of renormalization scheme and scale-invariant effective charge that may be used in the diagrammatic study of renormalon. However, as long as such an effective charge is defined, the formalism developed in QED may be used without modification.

Often in renormalon calculation in QCD, $\Pi(t)$ in Eq. (14) that defines the effective charge is considered in certain limit [11], for example, as in the $1/N_f$ approximation combined with ''naive-non-Abelianization'' [12], and the pinch technique [13]. The pinch technique appears to be promising, though presently there is no all-order definition for the effective charge in this scheme. In pinch technique, $\Pi(t)$ at one-loop level is defined by collecting the gluon vacuum polarization, and the vacuum polarizationlike term in the vertex and box diagrams.

However, if we are only interested in the calculation of the residue, the definition of the effective charge is not required. Indeed the calculation is cunningly simple; it only requires the strength of the renormalon singularity and the perturbative calculation of $D(\alpha)$.

Consider the Borel transform of the current correlation function in QCD. The renormalon singularity of $\widetilde{D}(b)$ in QCD

$$\widetilde{D}(b)\approx\frac{\hat{D}}{(1-b/b_0)^{1+\lambda b_0}} \tag{82}$$

gives the large order behavior

$$a_n\approx\frac{\hat{D}}{(\lambda b_0)!}n!n^{\lambda b_0}b_0^{-n}. \tag{83}$$

To calculate the residue $\hat{D}$, consider a function

$$R(b)=\widetilde{D}(b)(1-b/b_0)^{1+\lambda b_0}. \tag{84}$$

Then because of Eq. (82), we have

$$\hat{D}=R(b_0). \tag{85}$$

To avoid the first UV renormalon, we introduce a new variable z, as we did in QED, which is defined by

$$z=-\frac{\beta_0 b}{1-\beta_0 b}, \tag{86}$$

with its inverse

$$b=\frac{-1}{\beta_0}\left(\frac{z}{1-z}\right). \tag{87}$$

In the z plane, the IR renormalon at

$$z_0=\tfrac{2}{3} \tag{88}$$

is the closest singularity to the origin, and so the radius of convergence of the Taylor series of $\widetilde{D}(b(z))$ at $z=0$ is given by the first IR renormalon.

Now $\hat{D}$ can be expressed in a convergent series form

TABLE II. The first three elements of the sequence for the large order behavior in QCD.

	$N_f=1$	$N_f=2$	$N_f=3$	$N_f=4$	$N_f=5$
K_1	0.881	0.904	0.946	1.018	1.132
K_2	0.521	0.546	0.592	0.674	0.813
K_3	0.592	0.549	0.494	0.411	0.307

$$\begin{aligned}\hat{D}&=\widetilde{D}(b)(1-b/b_0)^{1+\lambda b_0}\big|_{b=b_0}\\&=\left(\sum_{n=0}^{\infty}\frac{a_n}{n!}(b(z))^n\right)[1-b(z)/b_0]^{1+\lambda b_0}\big|_{z=z_0}=\sum_{n=0}^{\infty}r_n z_0^n,\end{aligned} \tag{89}$$

where it is straightforward to find r_n in terms of the perturbative coefficients a_n. Note that the series is convergent even if $R(b(z))$ is not analytic at $z=z_0$, because then the radius of convergence of the series is given by $z=z_0$, and $R(b(z_0))$ is finite.

Using the perturbative calculation of the current correlation function, and $D(\alpha)$, to three loop [8], we have

$$R(b(z))=\frac{3\Sigma_f Q_f^2}{16\pi^3}[1.333-0.748z-0.311z^2+O(z^3)] \tag{90}$$

for $N_f=3$. This is in the renormalization scheme in which the one-loop renormalization point is same as that of $\overline{\text{MS}}$ scheme, and the β function is given in the form in Eqs. (52) and (53). Evaluating this series at the renormalon position at $z=z_0$, we have

$$\begin{aligned}K_1&=\frac{1.333}{(\lambda b_0)!}=0.946,\\K_2&=\frac{(1.323-0.748z_0)}{(\lambda b_0)!}=0.592,\\K_3&=\frac{(1.323-0.748z_0-0.311z_0^2)}{(\lambda b_0)!}=0.494.\end{aligned} \tag{91}$$

For several other flavor numbers we give K_n in Table II.

VII. SCHEME DEPENDENCE OF LARGE ORDER BEHAVIOR

In Sec. V, we determined the large order behavior in renormalization schemes in which the β function is given by the simple form in Eq. (52). With this β function, the scheme dependence of the large order behavior arises only through the factor

$$e^{-b_0 p_1} \tag{92}$$

in Eq. (76). In fact, this result is more general. All renormalization schemes for which the coefficients of the β function do not grow faster than a_n share the same large-large order behavior except for the scheme dependence in Eq. (92).

To see this, let us consider two renormalization schemes (say, unprimed and primed) in which the relation between the couplings is given by Eq. (43). We now assume that the large order behavior by the first UV renormalon is extracted out so that the leading large order behavior of a scheme-independent subamplitude D' of D is given by the first IR renormalon. Then using

$$D'=\sum_{n=0}^{\infty} a_n\alpha(\mu)^{n+1}=\sum_{n=0}^{\infty} a_n'\alpha'(\mu)^{n+1}, \tag{93}$$

we get the relation between a_n and a_n'

$$a_n=a_n'\left(1+(n+1)\frac{a_{n-1}'}{a_n'}\gamma_1+\frac{n(n+1)}{2}\frac{a_{n-2}'}{a_n'}\gamma_1^2+\cdots+\frac{a_0'\gamma_n}{a_n'}\right). \tag{94}$$

With the large order behavior of a_n' in the form Eq. (1), this equation becomes

$$a_n'=a_n e^{-b_0\gamma_1}[1+O(1/n)], \tag{95}$$

provided

$$\lim_{n\to\infty}\frac{n\gamma_n}{a_n'}=\text{ const.} \tag{96}$$

Thus if γ_n does not grow faster than

$$(n-1)!n^{\nu}b_0^{-n}, \tag{97}$$

the scheme dependence of large order behavior is given by the simple relation in Eq. (95). In fact the large order behavior we found in Sec. V exactly transforms according to Eq. (95) under scheme changes.

We may now translate the limit in Eq. (96) on γ_n to that of β_n. Let the β function in the unprimed scheme be given by the simple form in Eq. (52). Then Eq. (43) gives the $\beta'(\alpha')$ in the form

$$\begin{aligned}\beta'(\alpha')&=\sum_{n=0}^{\infty}\beta_n'\alpha'^{n+2}=\beta(\alpha)\sum_{n=0}^{\infty}(n+1)\gamma_n\alpha^n\\&=\frac{\beta_0\alpha^2}{1-\lambda\alpha}\sum_n(n+1)\gamma_n\alpha^n\\&=\beta_0\sum_{n=0}^{\infty}\overline{\beta_n'\alpha}^{n+2},\end{aligned} \tag{98}$$

where

$$\overline{\beta_n'}=\sum_{k=0}^{n}\lambda^{n-k}(k+1)\gamma_k\sim n\gamma_n[1+O(1/n)]. \tag{99}$$

Inverting Eq. (43) to express α in terms of α', and substituting it in Eq. (98), we have

$$\beta_n'=\overline{\beta'}_n[1+O(1/n)]=n\gamma_n[1+O(1/n)]. \tag{100}$$

Then the restriction on γ_n in Eq. (97) implies that any renormalization scheme in which β_n does not grow faster than

$$n!n^{\nu}b_0^{-n} \tag{101}$$

has the same large order behavior [except for the factor in Eq. (92)] as in a scheme for which the β function is given by Eq. (52).

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