Generalized two-dimensional chiral QED: Anomaly and exotic statistics

Fuad M. Saradzhev
Institute of Physics, Academy of Sciences of Azerbaijan, Huseyn Javid prospect 33, 370143 Baku, Azerbaijan
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We study the influence of the anomaly on the physical quantum picture of the generalized chiral Schwinger model defined on S^1. We show that the anomaly (i) results in the background linearly rising electric field and (ii) makes the spectrum of the physical Hamiltonian nonrelativistic without a massive boson. The physical matter fields acquire exotic statistics. We construct explicitly the algebra of the Poincaré generators and show that it differs from the Poincaré one. We exhibit the role of the vacuum Berry phase in the failure of the Poincaré algebra to close. We prove that, in spite of the background electric field, such phenomenon as the total screening of external charges characteristic for the standard Schwinger model takes place in the generalized chiral Schwinger model, too. [S0556-2821(97)02014-6]

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I. INTRODUCTION

Two-dimensional QED with massless fermions, i.e., the Schwinger model (SM), demonstrates such phenomena as the dynamical mass generation and the total screening of the charge [1]. Although the Lagrangian of the SM contains only massless fields, a massive boson field emerges out of the interplay of the dynamics that govern the original fields. This mass generation is due to the complete compensation of any external charge inserted into the vacuum.

In the chiral Schwinger model (CSM) [2,3] the right and left chiral components of the fermionic field have different charges. The left-right asymmetric matter content leads to an anomaly. At the quantum level, the local gauge symmetry is not realized by a unitary action of the gauge symmetry group on Hilbert space. The Hilbert space furnishes a projective representation of the symmetry group [4–6].

In this paper, we aim to study the influence of the anomaly on the physical quantum picture of the CSM. Do the dynamical mass generation and the total screening of charges take place also in the CSM? Are there any new physical effects caused just by the left-right asymmetry? These are the questions which we want to answer.

To get the physical quantum picture of the CSM we need first to construct a self-consistent quantum theory of the model and then solve all the quantum constraints. In the quantization procedure, the anomaly manifests itself through a special Schwinger term in the commutator algebra of the Gauss law generators. This term changes the nature of the Gauss law constraint: instead of being a first-class constraint, it turns into a second-class one. As a consequence, the physical quantum states cannot be defined as annihilated by the Gauss law generator.

There are different approaches to overcome this problem and to consistently quantize the CSM. The fact that the second-class constraint appears only after quantization means that the number of degrees of freedom of the quantum theory is larger than that of the classical theory. To keep the Gauss law constraint first class, Faddeev and Shatashvili proposed adding an auxiliary field in such a way that the dynamical content of the model does not change [7]. At the same time, after quantization it is the auxiliary field that furnishes the additional ''irrelevant'' quantum degrees of freedom. The auxiliary field is described by the Wess-Zumino (WZ) term. When this term is added to the Lagrangian of the original model, a new, anomaly-free model is obtained. Subsequent canonical quantization of the new model is achieved by the Dirac procedure.

For the CSM, the corresponding WZ term is not defined uniquely. It contains the so-called Jackiw-Rajaraman parameter $a>1$. This parameter reflects an ambiguity in the bosonization procedure and in the construction of the WZ term. The spectrum of the new, anomaly-free model turns out to be relativistic and contains a relativistic boson. However, the mass of the boson also depends on the Jackiw-Rajaraman parameter [2,3]. This mass corresponds therefore to the ''irrelevant'' quantum degrees of freedom. The quantum theory with such a parameter in the spectrum is not physical, i.e., that final version of the quantum theory which we would like to get. The latter should not contain any nonphysical parameters, otherwise one cannot say anything about a physical quantum picture.

In another approach also formulated by Faddeev [8], the auxiliary field is not added, so the quantum Gauss law constraint remains second class. The standard Gauss law is assumed to be regained as a statement valid in matrix elements between some states of the total Hilbert space, and it is the states that are called physical. The theory is regularized in such a way that the quantum Hamiltonian commutes with the nonmodified, i.e., second-class quantum Gauss law constraint. The spectrum turns out to be nonrelativistic [9,10].

Here, we follow the approach given in our previous work [11]. A peculiarity of the CSM is that its anomalous behavior is trivial in the sense that the second-class constraint which appears after quantization can be turned into first class by a simple redefinition of the canonical variables. This allows us to formulate a modified Gauss law to constrain physical states. The physical states are gauge-invariant up to a phase, the phase being one-cocycle of the gauge symmetry group algebra. In [12–14], the modification of the Gauss law constraint is obtained by making use of the adiabatic approach.

Contrary to [11], where the CSM is defined on R^1, we suppose here that the space is a circle of length L, $-L/2 \leq x < L/2$, so the space-time manifold is a cylinder $S^1 \times R^1$. The gauge field then acquires a global physical degree of freedom represented by the nonintegrable phase of the Wilson integral on S^1. We show that this brings into the physical quantum picture new features of principle.

Another way of making two-dimensional gauge field dynamics nontrivial is by fixing the spatial asymptotics of the gauge field [15,16]. If we assume that the gauge field defined on R^1 diminishes rather rapidly at spatial infinities, then it again acquires a global physical degree of freedom. We will see that the physical quantum picture for the model defined on S^1 is equivalent to that obtained in [15,16].

We consider the general version of the CSM with a U(1) gauge field coupled with different charges to both chiral components of a fermionic field. We show that the charges are not arbitrary, but satisfy a quantization condition. The SM where these charges are equal is a special case of the generalized CSM. This will allow us at each step of our consideration to see the distinction between the two models.

We work in the temporal gauge $A_0=0$ in the framework of the canonical quantization scheme and the Dirac's quantization method for the constrained systems [17]. We use the system of units where $c=1$. In Sec. II, we quantize our model in two steps. First, the matter fields are quantized, while A_1 is handled as a classical background field. The gauge field A_1 is quantized afterwards, using the functional Schrödinger representation. We derive the anomalous commutators with nonvanishing Schwinger terms which indicate that our model is anomalous.

In Sec. III, we show that the Schwinger term in the commutator of the Gauss law generators is removed by a redefinition of these generators and formulate the modified quantum Gauss law constraint. We prove that this constraint can be also obtained by using the adiabatic approximation and the notion of quantum holonomy.

In Sec. IV, we construct the physical quantum Hamiltonian consistent with the modified quantum Gauss law constraint, i.e., invariant under the modified gauge transformations both topologically trivial and nontrivial. We introduce the modified topologically nontrivial gauge transformation operator and define θ states which are its eigenstates. We consider in detail the case of the SM and demonstrate its equivalence to the free field theory of a massive scalar field. For the generalized CSM, we define the exotic statistics matter field and reformulate the quantum theory in terms of this field.

In Sec. V, we construct two other Poincaré generators, i.e., the momentum and the boost. We act in the same way as before with the Hamiltonian, namely we define the physical generators as those which are invariant under both topologically trivial and nontrivial gauge transformations. We show that the algebra of the constructed generators is not a Poincaré one and that the failure of the Poincaré algebra to close is connected to the nonvanishing vacuum Berry curvature.

In Sec. VI, we study the charge screening. We introduce external charges and calculate (i) the energy of the ground state of the physical Hamiltonian with external charges and (ii) the current density induced by these charges. Section VII contains our conclusions and a discussion.

II. QUANTIZATION PROCEDURE

A. Classical theory

The Lagrangian density of the generalized CSM is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi} i\hbar \gamma^\mu \partial_\mu \psi + e_+ \overline{\psi_+} \gamma^\mu \psi_+ A_\mu + e_- \overline{\psi_-} \gamma^\mu \psi_- A_\mu , \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $(\mu,\nu) = \overline{0,1}$, $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, $\gamma^0\gamma^1 = \gamma^5 = \sigma_3$, σ_i $(i=\overline{1,3})$ are Pauli matrices. The field ψ is two-component Dirac spinor $\overline{\psi} = \psi^\dagger \gamma^0$ and $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$.

In the temporal gauge $A_0=0$, the Hamiltonian density is

$$\mathcal{H} = \mathcal{H}_{\mathrm{EM}} + \mathcal{H}_F , \tag{2}$$

where $\mathcal{H}_{\mathrm{EM}} = \frac{1}{2}E^2$, with E momentum canonically conjugate to A_1, and

$$\mathcal{H}_F = \mathcal{H}_+ + \mathcal{H}_- ,$$

$$\mathcal{H}_\pm \equiv \psi_\pm^\dagger d_\pm \psi_\pm = \mp \psi_\pm^\dagger (i\hbar \partial_1 + e_\pm A_1)\psi_\pm .$$

On the circle boundary conditions for the fields must be specified. We impose the periodic ones

$$A_1\left(-\frac{L}{2}\right) = A_1\left(\frac{L}{2}\right),$$

$$\psi_\pm\left(-\frac{L}{2}\right) = \psi_\pm\left(\frac{L}{2}\right). \tag{3}$$

We require also that $\mathcal{H}$ and the classical fermionic currents $j_\pm \equiv \psi_\pm^\dagger \psi_\pm$ be periodic.

The Lagrangian and Hamiltonian densities are invariant under local time-independent gauge transformations

$$A_1 \to A_1 + \partial_1 \lambda, \quad \psi_\pm \to \exp\left\{\frac{i}{\hbar} e_\pm \lambda\right\} \psi_\pm ,$$

generated by

$$G = \partial_1 E + e_+ j_+ + e_- j_- ,$$

λ being a gauge function, as well as under global gauge transformations of the right-handed and left-handed Dirac fields which are generated by

$$Q_\pm = e_\pm \int_{-L/2}^{L/2} dx\, j_\pm(x).$$

Due to the gauge invariance, the Hamiltonian density is not uniquely determined. On the constrained submanifold $G \approx 0$ of the full phase space, the Hamiltonian density

$$\tilde{\mathcal{H}} = \mathcal{H} + v_H \cdot G, \tag{4}$$

where v_H is an arbitrary Lagrange multiplier which can be any function of the field variables and their momenta, reduces to the Hamiltonian density $\mathcal{H}$. In this sense, our theory cannot distinguish between $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, and so both Hamiltonian densities are physically equivalent to each other.

For arbitrary e_+, e_- the gauge transformations do not respect the boundary conditions (3). The gauge transformations compatible with the boundary conditions must be either of the form

$$\lambda\left(\frac{L}{2}\right)=\lambda\left(-\frac{L}{2}\right)+\hbar\frac{2\pi}{e_+}n, \quad n\in\mathcal{Z} \tag{5}$$

with $e_+\neq 0$ and

$$\frac{e_-}{e_+}=N, \quad N\in\mathcal{Z} \tag{6}$$

or of the form

$$\lambda\left(\frac{L}{2}\right)=\lambda\left(-\frac{L}{2}\right)+\hbar\frac{2\pi}{e_-}n, \quad n\in\mathcal{Z},$$

with $e_-\neq 0$ and

$$\frac{e_+}{e_-}=\overline{N}, \quad \overline{N}\in\mathcal{Z}. \tag{7}$$

Equations (6) or (7) imply the charge quantization condition for our system. Without loss of generality, we choose the condition (6). For $N=1$, $e_-=e_+$ and we have the standard Schwinger model. For $N=0$, we get the model in which only the right-handed component of the Dirac field is coupled to the gauge field.

From Eq. (5) we see that the gauge transformations under consideration are divided into topological classes characterized by the integer n. If $\lambda(L/2)=\lambda(-L/2)$, then the gauge transformation is topologically trivial and belongs to the $n=0$ class. If $n\neq 0$ it is nontrivial and has winding number n.

Given Eq. (5), the nonintegrable phase

$$\Gamma(A)=\exp\left\{\frac{i}{\hbar}e_+\int_{-L/2}^{L/2}dxA_1(x,t)\right\}$$

is a unique gauge-invariant quantity that can be constructed from the gauge field [18–21]. By a topologically trivial transformation we can make A_1 independent of x,

$$A_1(x,t)=b(t),$$

i.e., obeying the Coulomb gauge $\partial_1 A_1=0$, then

$$\Gamma(A)=\exp\left\{\frac{i}{\hbar}e_+Lb(t)\right\}.$$

In contrast with $\Gamma(A)$, the line integral

$$b(t)=\frac{1}{L}\int_{-L/2}^{L/2}dxA_1(x,t)$$

is invariant only under the topologically trivial gauge transformations. The gauge transformations from the nth topological class shift b by $\hbar(2\pi/e_+L)n$. By a nontrivial gauge transformation of the form $g_n=\exp\{i(2\pi/L)\hbar nx\}$, we can then bring b into the interval $[0,\hbar(2\pi/e_+L)]$. The configurations $b=0$ and $b=\hbar(2\pi/e_+L)$ are gauge equivalent, since they are connected by the gauge transformation from the first topological class. The gauge-field configuration space is therefore a circle with length $\hbar(2\pi/e_+L)$.

B. Quantization and anomaly

The eigenfunctions and the eigenvalues of the first quantized fermionic Hamiltonians are

$$d_\pm\langle x|n;\pm\rangle=\pm\varepsilon_{n,\pm}\langle x|n;\pm\rangle,$$

where

$$\langle x|n;\pm\rangle=\frac{1}{\sqrt{L}}\exp\left\{\frac{i}{\hbar}e_\pm\int_{-L/2}^{x}dzA_1(z)+\frac{i}{\hbar}\varepsilon_{n,\pm}x\right\},$$

$$\varepsilon_{n,\pm}=\frac{2\pi}{L}\left(n\hbar-\frac{e_\pm bL}{2\pi}\right).$$

We see that the spectrum of the eigenvalues depends on b. For $e_+bL/2\pi\hbar=$integer, the spectrum contains the zero energy level. As b increases from 0 to $\hbar(2\pi/e_+L)$, the energies of $\varepsilon_{n,+}$ decrease by $\hbar(2\pi/L)$, while the energies of $(-\varepsilon_{n,-})$ increase by $\hbar(2\pi/L)N$. Some of energy levels change sign. However, the spectra at the configurations $b=0$ and $b=\hbar(2\pi/e_+L)$ are the same, namely, the integers, as it must be since these gauge-field configurations are gauge equivalent. In what follows, we will use separately the integer and fractional parts of $e_+bL/2\pi\hbar$ (and $e_-bL/2\pi\hbar$), denoting them as $[e_\pm bL/2\pi\hbar]$ and $\{e_\pm bL/2\pi\hbar\}$, correspondingly.

Now we introduce the second quantized right-handed and left-handed Dirac fields. For the moment, we will assume that $d_\pm$ do not have zero eigenvalues. At time $t=0$, in terms of the eigenfunctions of the first quantized fermionic Hamiltonians the second quantized (ζ-function regulated) fields have the expansion [22]

$$\psi_+^s(x)=\sum_{n\in\mathcal{Z}}a_n\langle x|n;+\rangle|\lambda\varepsilon_{n,+}|^{-s/2},$$

$$\psi_-^s(x)=\sum_{n\in\mathcal{Z}}b_n\langle x|n;-\rangle|\lambda\varepsilon_{n,-}|^{-s/2}. \tag{8}$$

Here λ is an arbitrary constant with dimension of length which is necessary to make $\lambda\varepsilon_{n,\pm}$ dimensionless, while $a_n, a_n^\dagger$ and $b_n, b_n^\dagger$ are, correspondingly, right-handed and left-handed fermionic annihilation and creation operators which fulfil the commutation relations

$$[a_n,a_m^\dagger]_+=[b_n,b_n^\dagger]_+=\delta_{m,n}.$$

For $\psi_\pm^s(x)$, the equal time anticommutators are

$$[\psi_\pm^s(x),\psi_\pm^{\dagger s}(y)]_+=\zeta_\pm(s,x,y), \tag{9}$$

with all other anticommutators vanishing, where

$$\zeta_\pm(s,x,y)\equiv\sum_{n\in\mathcal{Z}}\langle x|n;\pm\rangle\langle n;\pm|y\rangle|\lambda\varepsilon_{n,\pm}|^{-s},$$

s being large and positive. In the limit where the regulator is removed, i.e., $s=0$, $\zeta_{\pm}(s=0,x,y)=\delta(x-y)$ and Eq. (9) takes the standard form.

The vacuum state of the second quantized fermionic Hamiltonian

$$|\text{vac};A\rangle=|\text{vac};A;+\rangle\otimes|\text{vac};A;-\rangle$$

is defined such that all negative energy levels are filled and the others are empty:

$$a_n|\text{vac};A;+\rangle=0 \quad \text{for} \quad n>\left[\frac{e_+bL}{2\pi\hbar}\right],$$

$$a_n^{\dagger}|\text{vac};A;+\rangle=0 \quad \text{for} \quad n\leqslant\left[\frac{e_+bL}{2\pi\hbar}\right], \tag{10}$$

and

$$b_n|\text{vac};A;-\rangle=0 \quad \text{for} \quad n\leqslant\left[\frac{e_-bL}{2\pi\hbar}\right],$$

$$b_n^{\dagger}|\text{vac};A;-\rangle=0 \quad \text{for} \quad n>\left[\frac{e_-bL}{2\pi\hbar}\right]. \tag{11}$$

Excited states are constructed by operating creation operators on the Fock vacuum.

In the ζ-function regularization scheme, we define the action of the functional derivative on first quantized fermionic kets and bras by

$$\frac{\delta}{\delta A_1(x)}|n;\pm\rangle=\lim_{s\to 0}\sum_{m\in\mathcal{Z}}|m;\pm\rangle \times\left\langle m;\pm\left|\frac{\delta}{\delta A_1(x)}\right|n;\pm\right\rangle|\lambda\varepsilon_{m,\pm}|^{-s/2},$$

$$\left\langle n;\pm\left|\frac{\overleftarrow{\delta}}{\delta A_1(x)}=\lim_{s\to 0}\sum_{m\in\mathcal{Z}}\left\langle n;\pm\right|\frac{\overleftarrow{\delta}}{\delta A_1(x)}\right|m;\pm\right\rangle \times\langle m;\pm||\lambda\varepsilon_{m,\pm}|^{-s/2}.$$

From Eq. (8) we get the action of $\delta/\delta A_1(x)$ on the operators a_n, $a_n^{\dagger}$ in the form

$$\frac{\delta}{\delta A_1(x)}a_n=-\lim_{s\to 0}\sum_{m\in\mathcal{Z}}\left\langle n;+\left|\frac{\delta}{\delta A_1(x)}\right|m;+\right\rangle \times a_m|\lambda\varepsilon_{m,+}|^{-s/2},$$

$$\frac{\delta}{\delta A_1(x)}a_n^{\dagger}=\lim_{s\to 0}\sum_{m\in\mathcal{Z}}\left\langle m;+\left|\frac{\delta}{\delta A_1(x)}\right|n;+\right\rangle \times a_m^{\dagger}|\lambda\varepsilon_{m,+}|^{-s/2}.$$

The action of $\delta/\delta A_1(x)$ on $b_n,b_n^{\dagger}$ can be written analogously.

Next we define the quantum fermionic currents and fermionic parts of the second-quantized Hamiltonian as

$$\hat{j}^s_{\pm}(x)=\frac{1}{2}[\psi_{\pm}^{\dagger s}(x),\psi_{\pm}^{s}(x)]_-$$

and

$$\hat{H}^s_{\pm}=\int_{-L/2}^{L/2}dx\hat{\mathcal{H}}^s_{\pm}(x)=\frac{1}{2}\int_{-L/2}^{L/2}dx(\psi_{\pm}^{\dagger s}d_{\pm}\psi_{\pm}^{s}-\psi_{\pm}^{s}d_{\pm}^{*}\psi_{\pm}^{\dagger s}).$$

Substituting Eq. (8) into these expressions, we obtain

$$\hat{j}^s_{\pm}(x)=\sum_{n\in\mathcal{Z}}\frac{1}{L}\exp\left\{i\frac{2\pi}{L}nx\right\}\rho^s_{\pm}(n),$$

where

$$\rho^s_{+}(n)\equiv\sum_{k\in\mathcal{Z}}\frac{1}{2}[a_k^{\dagger},a_{k+n}]_-|\lambda\varepsilon_{k,+}|^{-s/2}|\lambda\varepsilon_{k+n,+}|^{-s/2},$$

$$\rho^s_{-}(n)\equiv\sum_{k\in\mathcal{Z}}\frac{1}{2}[b_k^{\dagger},b_{k+n}]_-|\lambda\varepsilon_{k,-}|^{-s/2}|\lambda\varepsilon_{k+n,-}|^{-s/2}$$

are momentum space charge density (or current) operators, and

$$\hat{\mathcal{H}}^s_{\pm}(x)=\sum_{n\in\mathcal{Z}}\frac{1}{L}\exp\left\{i\frac{2\pi}{L}nx\right\}\mathcal{H}^s_{\pm}(n),$$

$$\mathcal{H}^s_{\pm}(n)\equiv\mathcal{H}^s_{0,\pm}(n)\mp e_{\pm}b\rho^s_{\pm}(n), \tag{12}$$

where

$$\mathcal{H}^s_{0,+}(n)\equiv\hbar\frac{\pi}{L}\sum_{k\in\mathcal{Z}}(2k+n)\frac{1}{2}[a_k^{\dagger},a_{k+n}]_-|\lambda\varepsilon_{k,+}|^{-s/2} \times|\lambda\varepsilon_{k+n,+}|^{-s/2},$$

$$\mathcal{H}^s_{0,-}(n)\equiv\hbar\frac{\pi}{L}\sum_{k\in\mathcal{Z}}(2k+n)\frac{1}{2}[b_{k+n},b_k^{\dagger}]_-|\lambda\varepsilon_{k,-}|^{-s/2} \times|\lambda\varepsilon_{k+n,-}|^{-s/2}.$$

The charges corresponding to the currents $\hat{j}^s_{\pm}(x)$ are

$$\hat{Q}^s_{\pm}=e_{\pm}\int_{-L/2}^{L/2}dx\hat{j}^s_{\pm}(x)=e_{\pm}\rho^s_{\pm}(0).$$

With Eqs. (10) and (11), we have, for the vacuum expectation values,

$$\langle\text{vac};A;\pm|\hat{j}_{\pm}(x)|\text{vac};A;\pm\rangle=-\frac{1}{2}\eta_{\pm},$$

$$\langle\text{vac},A|\hat{H}_F|\text{vac},A\rangle=-\frac{1}{2}(\xi_{+}+\xi_{-}),$$

where

$$\eta_{\pm} \equiv \pm \lim_{s\to 0} \frac{1}{L}\sum_{k\in\mathcal{Z}} \mathrm{sgn}(\varepsilon_{k,\pm})|\lambda\varepsilon_{k,\pm}|^{-s},$$

$$\xi_{\pm} \equiv \lim_{s\to 0} \frac{1}{\lambda}\sum_{k\in\mathcal{Z}} |\lambda\varepsilon_{k,\pm}|^{-s+1}.$$

Taking the sums, we obtain

$$\eta_{\pm} = \pm\frac{2}{L}\left(\left\{\frac{e_{\pm}bL}{2\pi\hbar}\right\} - \frac{1}{2}\right),$$

$$\xi_{\pm} = -\hbar\frac{2\pi}{L}\left[\left(\left\{\frac{e_{\pm}bL}{2\pi\hbar}\right\} - \frac{1}{2}\right)^2 - \frac{1}{12}\right].$$

The quantum fermionic currents, charges, and Hamiltonians can therefore be written as

$$\hat{j}_{\pm}(x) = :\hat{j}_{\pm}(x): - \frac{1}{2}\eta_{\pm},$$

$$\hat{Q}_{\pm} = e_{\pm}:\rho_{\pm}(0): - \frac{L}{2}e_{\pm}\eta_{\pm},$$

$$\hat{H}_{\pm} = \hat{H}_{0,\pm} \mp e_{\pm}b:\rho_{\pm}(0): - \frac{1}{2}\xi_{\pm}, \tag{13}$$

where double dots indicate normal ordering with respect to $|\mathrm{vac},A\rangle$,

$$\hat{H}_{0,+} = \hbar\frac{2\pi}{L}\lim_{s\to 0}\Bigg\{\sum_{k>[e_+bL/2\pi\hbar]} k a_k^\dagger a_k|\lambda\varepsilon_{k,+}|^{-s} - \sum_{k\le[e_+bL/2\pi\hbar]} k a_k a_k^\dagger|\lambda\varepsilon_{k,+}|^{-s}\Bigg\},$$

$$\hat{H}_{0,-} = \hbar\frac{2\pi}{L}\lim_{s\to 0}\Bigg\{\sum_{k>[e_-bL/2\pi\hbar]} k b_k b_k^\dagger|\lambda\varepsilon_{k,-}|^{-s} - \sum_{k\le[e_-bL/2\pi\hbar]} k b_k^\dagger b_k|\lambda\varepsilon_{k,-}|^{-s}\Bigg\},$$

and

$$:\rho_+(0): = \lim_{s\to 0}\Bigg\{\sum_{k>[e_+bL/2\pi\hbar]} a_k^\dagger a_k|\lambda\varepsilon_{k,+}|^{-s} - \sum_{k\le[e_+bL/2\pi\hbar]} a_k a_k^\dagger|\lambda\varepsilon_{k,+}|^{-s}\Bigg\},$$

$$:\rho_-(0): = \lim_{s\to 0}\Bigg\{\sum_{k\le[e_-bL/2\pi\hbar]} b_k^\dagger b_k|\lambda\varepsilon_{k,-}|^{-s} - \sum_{k>[e_-bL/2\pi\hbar]} b_k b_k^\dagger|\lambda\varepsilon_{k,-}|^{-s}\Bigg\}.$$

The operators $:\hat{j}_{\pm}(x):$ and $:\hat{H}_{\pm}:$ are well defined when acting on finitely excited states which have only a finite number of excitations relative to the Fock vacuum.

To construct the quantum electromagnetic Hamiltonian, we quantize the gauge field using the functional Schrödinger representation. In this representation, when the vacuum and excited fermionic Fock states are functionals of A_1, the gauge field operators are represented as $\hat{A}_1(x)\to A_1(x)$, $\hat{E}(x)\to -i\hbar[\delta/\delta A_1(x)]$ and the inner product is evaluated by functional integration. We first introduce the Fourier expansion for the gauge field

$$A_1(x) = b + \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} e^{i(2\pi/L)px}\alpha_p.$$

Since $A_1(x)$ is a real function, α_p satisfies

$$\alpha_p = \alpha_{-p}^*.$$

The Fourier expansion for the canonical momentum conjugate to $A_1(x)$ is then

$$\hat{E}(x) = \frac{1}{L}\hat{\pi}_b - \frac{i}{L}\hbar\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} e^{-i(2\pi/L)px}\frac{d}{d\alpha_p},$$

where $\hat{\pi}_b \equiv -i\hbar(d/db)$. The electromagnetic part of the Hamiltonian density is

$$\hat{\mathcal{H}}_{\mathrm{EM}}(x) = \sum_{p\in\mathcal{Z}}\frac{1}{L}\exp\left\{i\frac{2\pi}{L}px\right\}\mathcal{H}_{\mathrm{EM}}(p),$$

where

$$\mathcal{H}_{\mathrm{EM}}(p) \equiv -\frac{1}{L}\hbar^2\frac{d}{d\alpha_{-p}}\frac{d}{db} - \frac{1}{2L}\hbar^2\sum_{\substack{q\in\mathcal{Z}\\ q\neq(0;p)}}\frac{d}{d\alpha_{-p+q}}\frac{d}{d\alpha_{-q}} \quad (p\neq 0), \tag{14}$$

so the corresponding quantum Hamiltonian becomes

$$\hat{H}_{\mathrm{EM}} = \mathcal{H}_{\mathrm{EM}}(p=0) = \frac{1}{2L}\hat{\pi}_b^2 - \frac{1}{L}\hbar^2\sum_{q>0}\frac{d}{d\alpha_q}\frac{d}{d\alpha_{-q}}.$$

The total quantum Hamiltonian is

$$\hat{H} = \hat{H}_{0,+} + \hat{H}_{0,-} + \hat{H}_{\mathrm{EM}} - e_+b:\rho_+(0): + e_-b:\rho_-(0): - \frac{1}{2}(\xi_+ + \xi_-).$$

If we multiply two operators that are finite linear combinations of the fermionic creation and annihilation operators, the ζ-function regulated operator product agrees with the naive product. However, if the operators involve infinite summations their naive product is not generally well defined. We then define the operator product by mutiplying the regulated operators with s large and positive and analytically continue the result to $s=0$. In this way we obtain the relations

$$[\rho_{\pm}(m),\rho_{\pm}(n)]_- = \pm m\,\delta_{m,-n}, \tag{15}$$

$$[\mathcal{H}_{0,\pm}(n),\mathcal{H}_{0,\pm}(m)]_- = \pm\hbar\frac{2\pi}{L}(n-m)\mathcal{H}_{0,\pm}(n+m),$$

$$[\hat{H}_{0,\pm},\rho_\pm(m)]_- = \mp\hbar\frac{2\pi}{L}m\rho_\pm(m),$$

and

$$\frac{d}{db}\rho_\pm(m)=0,$$

$$\frac{d}{d\alpha_{\pm p}}\rho_+(m) = -\frac{e_+L}{2\pi\hbar}\delta_{p,\pm m},$$

$$\frac{d}{d\alpha_{\pm p}}\rho_-(m) = \frac{e_-L}{2\pi\hbar}\delta_{p,\pm m} \quad (p>0). \tag{16}$$

The quantum Gauss operator is

$$\hat{G}=\hat{G}_0+\frac{2\pi}{L^2}\sum_{p>0}\{\hat{G}_+(p)e^{i(2\pi/L)px}-\hat{G}_-(p)e^{-i(2\pi/L)px}\},$$

where

$$\hat{G}_0 \equiv \frac{1}{L}e_+\rho_N(0),$$

$$\hat{G}_\pm(p)\equiv\hbar p\frac{d}{d\alpha_{\mp p}}\pm\frac{e_+L}{2\pi}\rho_N(\pm p),$$

and $\rho_N=\rho_+ + N\rho_-$ is momentum space total charge density operator.

Using Eqs. (15) and (16), we easily get that $\rho_+(\pm p)$ [and $\rho_-(\pm p)$] are gauge invariant. For example, for $\rho_+(\pm p)$ we have

$$[\hat{G}_+(p),\rho_+(\pm q)]_-=0,$$

$$[\hat{G}_-(p),\rho_+(\pm q)]_-=0,$$

$(p>0,q>0)$. The operators $\hat{G}_\pm(p)$ do not commute with either themselves,

$$[\hat{G}_+(p),\hat{G}_-(q)]_- = (1-N^2)\frac{e_+^2L^2}{4\pi^2}p\,\delta_{p,q},$$

or the Hamiltonian,

$$[\hat{H},\hat{G}_\pm(p)]_- = \pm(1-N^2)\hbar\frac{e_+^2L}{4\pi^2}\frac{d}{d\alpha_{\mp p}} \quad (p>0).$$

The last two commutators reflect an anomalous behavior of the generalized CSM. The appearance of the Schwinger term in the first commutator changes the nature of the Gauss law constraints: instead of being first-class constraints, they turn into second-class ones. The Schwinger term in the second commutator means that the total quantum Hamiltonian is not invariant under the topologically trivial gauge transformations generated by $\hat{G}_\pm(p)$.

For $N=1$, i.e., for the standard SM, both commutators vanish. Another case of vanishing Schwinger terms is axial electrodynamics where $N=-1$ and the fermionic fields $\psi_\pm$ are of opposite charge.

III. QUANTUM CONSTRAINTS

A. Quantum symmetry

In nonanomalous gauge theories, Gauss law is considered to be valid for physical states only. This identifies physical states as those which are gauge invariant. The problem with the anomalous behavior of the generalized CSM, in terms of states in Hilbert space, is apparent: owing to the Schwinger terms we cannot require that states be annihilated by the Gauss law generators $\hat{G}_\pm(p)$.

Let us represent the action of the topologically trivial gauge transformations by the operators

$$U_0(\tau)=\exp\left\{\frac{i}{\hbar}\hat{G}_0\tau_0+\frac{i}{\hbar}\sum_{p>0}(\hat{G}_+\tau_+ + \hat{G}_-\tau_-)\right\} \tag{17}$$

with $\tau_0,\tau_\pm(p)$ smooth, then

$$U_0^{-1}(\tau)\alpha_{\pm p}U_0(\tau)=\alpha_\pm - ip\tau_\mp(p),$$

$$U_0^{-1}(\tau)\frac{d}{d\alpha_{\pm p}}U_0(\tau)=\frac{d}{d\alpha_{\pm p}}\mp\frac{i}{\hbar^2}(1-N^2)\left(\frac{e_+L}{2\pi}\right)^2\tau_\pm(p),$$

$$(p>0).$$

The composition law for the operators U_0 is

$$U_0(\tau^{(1)})U_0(\tau^{(2)})=\exp\{2\pi i\omega_2(\tau^{(1)},\tau^{(2)})\}U_0(\tau^{(1)}+\tau^{(2)}),$$

where

$$\omega_2(\tau^{(1)},\tau^{(2)})\equiv -\frac{i}{4\pi}(1-N^2)\left(\frac{e_+L}{2\pi\hbar}\right)^2 \times\sum_{p>0}p(\tau_-^{(1)}\tau_+^{(2)}-\tau_+^{(1)}\tau_-^{(2)})$$

is a two-cocycle of the gauge group algebra. Thus for $N \neq \pm 1$ we are dealing with a projective representation.

The two-cocycle $\omega_2(\tau^{(1)},\tau^{(2)})$ is trivial, since it can be removed by a simple redefinition of $U_0(\tau)$. Indeed, the modified operators

$$\tilde{U}_0(\tau)=\exp\{i2\pi\alpha_1(\gamma;\tau)\}U_0(\tau), \tag{18}$$

where

$$\alpha_1(\gamma,\tau)\equiv -\frac{1}{4\pi}(1-N^2)\left(\frac{e_+L}{2\pi\hbar}\right)^2\sum_{p>0}(\alpha_{-p}\tau_- - \alpha_p\tau_+)$$

is a one-cocycle, satisfy the ordinary composition law

$$\tilde{U}_0(\tau^{(1)})\tilde{U}_0(\tau^{(2)})=\tilde{U}_0(\tau^{(1)}+\tau^{(2)}),$$

i.e., the action of the topologically trivial gauge transformations represented by Eq. (18) is unitary.

The modified Gauss law generators corresponding to Eq. (18) are

$$\hat{\tilde{G}}_{\pm}(p)=\hat{G}_{\pm}(p)\pm\frac{1}{\hbar}(1-N^2)\frac{e_+^2L^2}{8\pi^2}\alpha_{\pm p}. \tag{19}$$

The generators $\hat{\tilde{G}}_{\pm}(p)$ commute:

$$[\hat{\tilde{G}}_{+}(p),\hat{\tilde{G}}_{-}(q)]_{-}=0.$$

This means that Gauss law can be maintained at the quantum level for $N\neq\pm1$, too. We define physical states as those which are annihilated by $\hat{\tilde{G}}_{\pm}(p)$ [11]:

$$\hat{\tilde{G}}_{\pm}(p)|\text{phys};A\rangle=0. \tag{20}$$

The zero component $\hat{G}_0$ is a sum of quantum generators of the global gauge transformations of the right-handed and left-handed fermionic fields, so the other quantum constraints are

$$:\rho_{\pm}(0):|\text{phys};A\rangle=0. \tag{21}$$

It follows from Eq. (20) that the physical states $|\text{phys};A\rangle$ respond to a gauge transformation from the zero topological class with the phase

$$U_0(\tau)|\text{phys};A\rangle=\exp\{-i2\pi\alpha_1(\gamma;\tau)\}|\text{phys};A\rangle. \tag{22}$$

Only for models without anomaly, i.e., for $N=\pm1$, does this equation translate into the statement that physical states are gauge invariant.

Equation (22) expresses in an exact form the nature of anomaly in the CSM. At the quantum level the gauge invariance is not broken, but realized projectively. The one-cocycle α_1 occurring in the projective representation contributes to the commutator of the Gauss law generators by a Schwinger term and therefore produces the anomaly.

B. Adiabatic approach

Let us show now that we can come to the quantum constraints (20) and (21) in a different way, using the adiabatic approximation [23,24]. In the adiabatic approach, the dynamical variables are divided into two sets, one which we call fast variables and the other which we call slow variables. In our case, we treat the fermions as fast variables and the gauge fields as slow variables.

Let $\mathcal{A}^1$ be a manifold of all static gauge field configurations $A_1(x)$. On $\mathcal{A}^1$ a time-dependent gauge field $A_1(x,t)$ corresponds to a path and a periodic gauge field to a closed loop.

We consider the fermionic part of the second-quantized Hamiltonian $:\hat{H}_F:$ which depends on t through the background gauge field A_1 and so changes very slowly with time. We consider next the periodic gauge field $A_1(x,t)(0\leq t<T)$. After a time T the periodic field $A_1(x,t)$ returns to its original value $A_1(x,0)=A_1(x,T)$, so that $:\hat{H}_F:(0)=:\hat{H}_F:(T)$.

At each instant t we define eigenstates for $:\hat{H}_F:(t)$ by

$$:\hat{H}_F:(t)|F,A(t)\rangle=\varepsilon_F(t)|F,A(t)\rangle.$$

The state $|F=0,A(t)\rangle\equiv|\text{vac},A(t)\rangle$ is a ground state of $:\hat{H}_F:(t)$,

$$:\hat{H}_F:(t)|\text{vac},A(t)\rangle=0.$$

The Fock states $|F,A(t)\rangle$ depend on t only through their implicit dependence on A_1. They are assumed to be periodic in time, $|F,A(T)\rangle=|F,A(0)\rangle$, orthonormalized,

$$\langle F',A(t)|F,A(t)\rangle=\delta_{F,F'},$$

and nondegenerate.

The time evolution of the wave function of our system (fermions in a background gauge field) is clearly governed by the Schrödinger equation

$$i\hbar\frac{\partial\psi(t)}{\partial t}=:\hat{H}_F:(t)\psi(t).$$

For each t, this wave function can be expanded in terms of the ''instantaneous'' eigenstates $|F,A(t)\rangle$.

Let us choose $\psi_F(0)=|F,A(0)\rangle$, i.e., the system is initially described by the eigenstate $|F,A(0)\rangle$. According to the adiabatic approximation, if at $t=0$ our system starts in a stationary state $|F,A(0)\rangle$ of $:\hat{H}_F:(0)$, then it will remain, at any other instant of time t, in the corresponding eigenstate $|F,A(t)\rangle$ of the instantaneous Hamiltonian $:\hat{H}_F:(t)$. In other words, in the adiabatic approximation transitions to other eigenstates are neglected.

At time $t=T$ our system will be described by the state

$$\psi_F(T)=\exp\{i\gamma_F^{\text{dyn}}+i\gamma_F^{\text{Berry}}\}\psi_F(0),$$

where

$$\gamma_F^{\text{dyn}}\equiv-\frac{1}{\hbar}\int_0^T dt\varepsilon_F(t),$$

while

$$\gamma_F^{\text{Berry}}\equiv\int_0^T dt\int_{-L/2}^{L/2}dx\dot{A}_1(x,t)\left\langle F,A(t)\left|i\frac{\delta}{\delta A_1(x,t)}\right|F,A(t)\right\rangle \tag{23}$$

is Berry's phase [24].

If we define the U(1) connection

$$\mathcal{A}_F(x,t)\equiv\left\langle F,A(t)\left|i\frac{\delta}{\delta A_1(x,t)}\right|F,A(t)\right\rangle, \tag{24}$$

then

$$\gamma_F^{\text{Berry}}=\int_0^T dt\int_{-L/2}^{L/2}dx\dot{A}_1(x,t)\mathcal{A}_F(x,t).$$

We see that upon parallel transport around a closed loop on $\mathcal{A}^1$ the Fock state $|F,A(t)\rangle$ acquires an additional phase which is integrated exponential of $\mathcal{A}_F(x,t)$. Whereas the dynamical phase γ_F^{dyn} provides information about the duration of the evolution, the Berry's phase reflects the nontrivial holonomy of the Fock states on $\mathcal{A}^1$.

However, a direct computation of the diagonal matrix elements of $\delta/\delta A_1(x,t)$ in Eq. (23) requires a globally single-valued basis for the eigenstates $|F,A(t)\rangle$ which is not available [25]. For that reason, to calculate $\gamma_F^{\rm Berry}$ it is more convenient to compute first the U(1) curvature tensor

$$\mathcal{F}_F(x,y,t) \equiv \frac{\delta}{\delta A_1(x,t)}\mathcal{A}_F(y,t) - \frac{\delta}{\delta A_1(y,t)}\mathcal{A}_F(x,t) \tag{25}$$

and then deduce $\mathcal{A}_F$.

The vacuum curvature tensor is evaluated as [25]

$$\begin{aligned}\mathcal{F}_{F=0} &= (1-N^2)\frac{e_+^2}{2\pi^2\hbar^2}\sum_{n>0}\frac{1}{n}\sin\left(\frac{2\pi}{L}n(x-y)\right)\\ &= (1-N^2)\frac{e_+^2}{2\pi\hbar^2}\left(\frac{1}{2}\epsilon(x-y) - \frac{1}{L}(x-y)\right).\end{aligned} \tag{26}$$

The corresponding U(1) connection is easily deduced as

$$\mathcal{A}_{F=0}(x,t) = -\frac{1}{2}\int_{-L/2}^{L/2} dy \mathcal{F}_{F=0}(x,y,t)A_1(y,t).$$

The Berry phase becomes

$$\begin{aligned}\gamma_{F=0}^{\rm Berry} = -\frac{1}{2}\int_0^T dt \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \dot{A}_1(x,t)\\ \times \mathcal{F}_{F=0}(x,y,t)A_1(y,t).\end{aligned}$$

In terms of the Fourier components, the connection $\mathcal{A}_{F=0}$ is rewritten as

$$\left\langle \text{vac},A(t)\left|\frac{d}{db(t)}\right|\text{vac},A(t)\right\rangle = 0,$$

$$\begin{aligned}\left\langle \text{vac},A(t)\left|\frac{d}{d\alpha_{\pm p}(t)}\right|\text{vac},A(t)\right\rangle &\equiv \mathcal{A}_\pm(p,t)\\ &= \pm(1-N^2)\frac{e_+^2L^2}{8\pi^2\hbar^2}\frac{1}{p}\alpha_{\mp p},\end{aligned}$$

so the nonvanishing curvature is

$$\mathcal{F}_{+-}(p) \equiv \frac{d}{d\alpha_{-p}}\mathcal{A}_+ - \frac{d}{d\alpha_p}\mathcal{A}_- = (1-N^2)\frac{e_+^2L^2}{4\pi^2\hbar^2}\frac{1}{p}.$$

A parallel transportation of the vacuum $|\text{vac},A(t)\rangle$ around a closed loop in (α_p,α_{-p}) space $(p>0)$ yields back the same vacuum state multiplied by the phase

$$\gamma_{F=0}^{\rm Berry} = (1-N^2)\frac{e_+^2L^2}{4\pi^2\hbar^2}\int_0^T dt \sum_{p>0}\frac{1}{p} i\alpha_p\dot{\alpha}_{-p}.$$

This phase is associated with the projective representation of the gauge group. For $N=\pm 1$, when the representation is unitary, the curvature $\mathcal{F}_{+-}$ and the Berry phase vanish.

As mentioned in the beginning of this section, the projective representation is trivial and the two-cocycle in the composition law of the gauge transformation operators can be removed by a redefinition of these operators. Analogously, if we redefine the momentum operators as

$$\frac{d}{d\alpha_{\pm p}} \to \frac{\widetilde{d}}{d\alpha_{\pm p}} \equiv \frac{d}{d\alpha_{\pm p}} \mp (1-N^2)\frac{e_+^2L^2}{8\pi^2\hbar^2}\frac{1}{p}\alpha_{\mp p}, \tag{27}$$

then the corresponding connection and curvature vanish

$$\widetilde{\mathcal{A}}_\pm \equiv \left\langle \text{vac},A(t)\left|\frac{\widetilde{d}}{d\alpha_{\pm p}}\right|\text{vac},A(t)\right\rangle = 0,$$

$$\widetilde{\mathcal{F}}_{+-} = \frac{\widetilde{d}}{d\alpha_{-p}}\widetilde{\mathcal{A}}_+ - \frac{\widetilde{d}}{d\alpha_p}\widetilde{\mathcal{A}}_- = 0.$$

However, the nonvanishing curvature $\mathcal{F}_{+-}(p)$ shows itself in the algebra of the modified momentum operators which are noncommuting:

$$\left[\frac{\widetilde{d}}{d\alpha_p},\frac{\widetilde{d}}{d\alpha_{-q}}\right]_- = \mathcal{F}_{+-}(p)\delta_{p,q}.$$

Following Eq. (27), we modify the Gauss law generators as

$$\hat{G}_\pm(p) \to \hat{\widetilde{G}}_\pm(p) = \hbar p\frac{\widetilde{d}}{d\alpha_{\mp p}} \pm \frac{e_+L}{2\pi}\rho_N(\pm p)$$

that coincides with Eq. (19). The modified Gauss law generators have vanishing vacuum expectation values,

$$\langle \text{vac},A(t)|\hat{\widetilde{G}}_\pm(p,t)|\text{vac},A(t)\rangle = 0.$$

This justifies the definition (20).

For the zero component $\hat{G}_0$, the vacuum expectation value

$$\langle \text{vac},A(t)|\hat{G}_0|\text{vac},A(t)\rangle = -\frac{1}{2}(e_+\eta_+ + e_-\eta_-)$$

can be also made equal to zero by the redefinition

$$\hat{G}_0 \to \hat{\widetilde{G}}_0 = \hat{G}_0 + \frac{1}{2}(e_+\eta_+ + e_-\eta_-) = \frac{1}{L}e_+ :\rho_N(0):$$

that leads to Eq. (21). Thus, both quantum constraints (20) and (21) can be realized in the framework of the adiabatic approximation.

IV. PHYSICAL QUANTUM CSM

A. Construction of physical Hamiltonian

(1) From the point of view of Dirac quantization, there are many physically equivalent classical theories of a system with first-class constraints. The origin of such an ambiguity lies in the gauge freedom. For the classical CSM, the gauge freedom is characterized by an arbitrary $v_H(x)$ in Eq. (4). If

we use the Fourier expansion for $v_H(x)$, then the general form of the classical Hamiltonian is rewritten as

$$\tilde{H}=H+\sum_{p>0}(v_{H,+}G_{+}+v_{H,-}G_{-}). \tag{28}$$

Any Hamiltonian $\tilde{H}$ with fixed nonzero $(v_{H,-},v_{H,+})$ gives rise to the same weak equations of motion as those deduced from H, although the strong form of these equations may be quite different. The physics is, however, described by the weak equations. Different $(v_{H,-},v_{H,+})$ lead to different mathematical descriptions of the same physical situation.

To construct the quantum theory of any system with first-class constraints, we usually quantize one of the corresponding classical theories. All the possible quantum theories constructed in this way are believed to be equivalent to each other.

In the case where gauge degrees of freedom are anomalous, the situation is different: the physical equivalence of quantum Hamiltonians is lost. For the CSM, the quantum Hamiltonian $\hat{\tilde{H}}$ does not reduce to $\hat{H}$ on the physical states:

$$\hat{\tilde{H}}|\text{phys};A\rangle \neq \hat{H}|\text{phys};A\rangle.$$

The quantum theory consistently describing the dynamics of the CSM should be definitely compatible with Eq. (20). The corresponding quantum Hamiltonian is then defined by the conditions

$$[\hat{\tilde{H}},\hat{\tilde{G}}_{\pm}(p)]_{-}=0 \quad (p>0), \tag{29}$$

which specify that $\hat{\tilde{H}}$ must be invariant under the modified topologically trivial gauge transformations generated by $\hat{\tilde{G}}_{\pm}(p)$.

We have in Eq. (29) a system of nonhomogeneous equations in the Lagrange multipliers $\hat{v}_{H,\pm}$ which become operators at the quantum level. The solution of these equations is

$$\hat{v}_{H,\pm}(p)=\frac{\hbar}{L}\frac{1}{p^2}\left\{p\frac{d}{d\alpha_{\pm p}}\mp(1-N^2)\left(\frac{e_+L}{4\pi\hbar}\right)^2\alpha_{\mp p}\right\}.$$

Substituting this expression for $\hat{v}_{H,\pm}(p)$ into the quantum counterpart of Eq. (28), on the physical states $|\text{phys};A\rangle$ we obtain

$$\frac{1}{2}\sum_{p>0}\{[\hat{v}_{H,+}(p),\hat{G}_{+}(p)]_{+}+[\hat{v}_{H,-}(p),\hat{G}_{-}(p)]_{+}\}$$

$$=\frac{1}{L^2}\hbar^2\sum_{p>0}\left(\frac{d}{d\alpha_p}\frac{d}{d\alpha_{-p}}-\frac{1}{2}\left[\frac{\tilde{d}}{d\alpha_p},\frac{\tilde{d}}{d\alpha_{-p}}\right]_{+}\right),$$

i.e., the last term in the right-hand side of Eq. (28) contributes only to the electromagnetic part of the Hamiltonian, changing $d/d\alpha_{\pm}$ to $\tilde{d}/d\alpha_{\pm}$:

$$\hat{H}_{\text{EM}}\rightarrow\hat{\tilde{H}}_{\text{EM}}=\tilde{\mathcal{H}}_{\text{EM}}(0)\equiv\frac{1}{2L}\hat{\pi}_b^2-\frac{1}{2L}\hbar^2\sum_{p>0}\left[\frac{\tilde{d}}{d\alpha_p},\frac{\tilde{d}}{d\alpha_{-p}}\right]_{+}.$$

In terms of the momentum space charge density operators, the gauge invariant electromagnetic Hamiltonian becomes

$$\hat{\tilde{H}}_{\text{EM}}=\frac{1}{2L}\hat{\pi}_b^2+V(\rho_N;\rho_N),$$

where

$$V(\rho_N;\rho_N)\equiv\frac{e_+^2L}{8\pi^2}\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}}\frac{1}{p^2}\rho_N(-p)\rho_N(p)$$

is the energy of the Coulomb current-current interaction.

In order to make the dependence on N for the Hamiltonian more obvious, let us represent ρ_N as

$$\rho_N=\frac{1}{2}(1+N)\rho+\frac{1}{2}(1-N)\sigma,$$

where

$$\rho\equiv\rho_1=\rho_{+}+\rho_{-},$$

$$\sigma\equiv\rho_{-1}=\rho_{+}-\rho_{-},$$

and

$$[\rho(p),\rho(q)]_{-}=[\sigma(p),\sigma(q)]_{-}=0,$$

$$[\sigma(p),\rho(q)]_{-}=2p\,\delta_{p,-q}.$$

Then the Coulomb interaction energy takes the form

$$V(\rho_N;\rho_N)=\frac{1}{4}(1+N)^2V(\rho;\rho)+\frac{1}{4}(1-N)^2V(\sigma;\sigma)$$
$$+\frac{1}{2}(1-N^2)V(\rho;\sigma). \tag{30}$$

For $N=1$, $\rho(p)$ and $\sigma(p)$ are respectively momentum space electric and axial charge density operators, the electromagnetic Hamiltonian depending only on ρ:

$$\hat{\tilde{H}}_{\text{EM}}=\frac{1}{2L}\hat{\pi}_b^2+V(\rho;\rho).$$

For $N=-1$, the momentum space electric charge density operator is $\sigma(p)$ and

$$\hat{\tilde{H}}_{\text{EM}}=\frac{1}{2L}\hat{\pi}_b^2+V(\sigma;\sigma).$$

For $N\neq\pm1$, i.e., for models with anomaly, the last term in Eq. (30) does not vanish and is of principal importance. This term means that ρ and σ are not decoupled as before for the cases without anomaly and that the electromagnetic Hamiltonian involves the noncommuting charge density operators.

(2) The topologically nontrivial gauge transformations change the integer part of $e_+bL/2\pi\hbar$:

$$\left[\frac{e_+bL}{2\pi\hbar}\right]\rightarrow\left[\frac{e_+bL}{2\pi\hbar}\right]+n,$$

$$\hat{\psi}_+ \to \exp\left\{ i\frac{2\pi n}{L} x \right\} \hat{\psi}_+ ,$$

and

$$\left[\frac{e_- bL}{2\pi\hbar}\right] \to \left[\frac{e_- bL}{2\pi\hbar}\right] + N\cdot n,$$

$$\hat{\psi}_- \to \exp\left\{ iN\frac{2\pi n}{L} x \right\} \hat{\psi}_- .$$

The action of the topologically nontrivial gauge transformations on the states can be represented by the operators

$$U_n = \exp\left\{ -\frac{i}{\hbar} n\cdot \hat{T}_b \right\} U_0 , \tag{31}$$

where

$$\hat{T}_b \equiv \hat{\pi}_{[e_+ bL/2\pi\hbar]} - \frac{2\pi}{L}\int_{-L/2}^{L/2} dx x [\hat{j}_+(x) + N\hat{j}_-(x)]$$

$$\equiv -i\hbar \frac{d}{d[e_+ bL/2\pi\hbar]} + i\hbar \sum_{\substack{n\in\mathcal{Z}\\ n\neq 0}} \frac{(-1)^n}{n}\rho_N(n)$$

and U_0 is given by Eq. (17).

To identify the gauge transformation as belonging to the nth topological class we use the index n in Eq. (31). The case $n=0$ corresponds to the topologically trivial gauge transformations.

The topologically nontrivial gauge transformation operators satisfy the same composition law as the topologically trivial ones. The modified operators are

$$\widetilde{U}_n = \exp\left\{ -\frac{i}{\hbar} n\cdot \hat{T}_b \right\} \widetilde{U}_0 .$$

On the physical states

$$\widetilde{U}_n |\text{phys};A\rangle = \left(\exp\left\{ -\frac{i}{\hbar}\hat{T}_b \right\} \right)^n |\text{phys};A\rangle .$$

Among all states $|\text{phys};A\rangle$ one may identify the eigenstates of the operators of the physical variables. The action of the topologically nontrivial gauge transformations on such states may, generally speaking, change only the phase of these states by a C number, since with any gauge transformations both topologically trivial and nontrivial, the operators of the physical variables and the observables cannot be changed. Using $|\text{phys};\theta\rangle$ to designate these physical states, we have

$$\exp\left\{ \mp\frac{i}{\hbar}\hat{T}_b \right\} |\text{phys};\theta\rangle = e^{\pm i\theta} |\text{phys};\theta\rangle .$$

The states $|\text{phys};\theta\rangle$ are easily constructed in the form

$$|\text{phys};\theta\rangle = \sum_{n\in\mathcal{Z}} e^{-in\theta} \left(\exp\left\{ -\frac{i}{\hbar}\hat{T}_b \right\} \right)^n |\text{phys};A\rangle$$

(so-called θ states [26,27]), where $|\text{phys};A\rangle$ is an arbitrary physical state from Eq. (20).

In one dimension the parameter θ is related to a constant background electric field. To show this, let us introduce states which are invariant even against the topologically nontrivial gauge transformations. Recalling that $[e_+ bL/2\pi\hbar]$ is shifted by n under a gauge transformation from the nth topological class, we obtain such states by the transition

$$|\text{phys};\theta\rangle \to |\text{phys}\rangle \equiv \exp\left\{ i\left[\frac{e_+ bL}{2\pi\hbar}\right]\theta \right\} |\text{phys};\theta\rangle . \tag{32}$$

The new states $|\text{phys}\rangle$ continue to be annihilated by $\hat{\widetilde{G}}_\pm(p)$, and are also invariant under the topologically nontrivial gauge transformations.

The electromagnetic part of the Hamiltonian transforms as

$$\hat{H}_{\text{EM}} \to \exp\left\{ i\left[\frac{e_+ bL}{2\pi\hbar}\right]\theta \right\} \hat{H}_{\text{EM}} \exp\left\{ -i\left[\frac{e_+ bL}{2\pi\hbar}\right]\theta \right\}$$

$$= \frac{1}{2L}(\hat{\pi}_b - L\mathcal{E}_\theta)^2 - \frac{1}{2L}\hbar^2 \sum_{p>0} \left[\frac{\widetilde{d}}{d\alpha_p}, \frac{\widetilde{d}}{d\alpha_{-p}} \right]_+ ,$$

i.e., in the new Hamiltonian the momentum $\hat{\pi}_b$ is supplemented by the electric field strength $\mathcal{E}_\theta \equiv (e_+/2\pi)\theta$.

(3) The Fourier components of the fermionic currents are transformed under the topologically nontrivial gauge transformations as

$$\rho_+(\pm p) \to \rho_+(\pm p) - (-1)^p n,$$

$$\rho_-(\pm p) \to \rho_-(\pm p) + (-1)^p Nn \quad (p>0),$$

being invariant under the topologically trivial ones.

The quantum Hamiltonian invariant under the topologically trivial gauge transformations is still not uniquely determined. We can add to it any linear combination of the operators $\rho_+(\pm p)$ and $\rho_-(\pm p)$:

$$\hat{\widetilde{H}} \to \hat{\widetilde{H}} + \beta_0 + \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} [\beta_+ \rho_+(p) + \beta_- \rho_-(p)], \tag{33}$$

where β_0, $\beta_\pm$ are arbitrary functions. The conditions (29) do not clearly fix these functions.

The Hamiltonian of the consistent quantum theory of the generalized CSM should be invariant under the topologically nontrivial gauge transformations as well. So in addition to Eq. (29) is the following condition:

$$[\hat{\widetilde{H}}, \hat{T}_b]_- = 0. \tag{34}$$

The condition (34) can be then rewritten as a system of linear equations in $(\beta_0, \beta_\pm)$. We can easily find a solution of these equations, which gives us $(\beta_0, \beta_\pm)$ as functions of $[e_+ bL/2\pi\hbar]$. The most general solution must involve constants depending on $\{e_\pm bL/2\pi\hbar\}$. However, these constants are irrelevant for our consideration and we neglect them.

Finding $(\beta_0, \beta_\pm)$ from Eq. (34) and substituting them into the expression (33), on the physical states we obtain

$$\hat{\tilde{H}}|\text{phys};A\rangle=\hat{H}_{\text{phys}}|\text{phys};A\rangle,$$

where

$$\hat{H}_{\text{phys}}=\hat{H}_F^{\text{phys}}+\hat{H}_{\text{EM}}^{\text{phys}},$$

$$\hat{H}_F^{\text{phys}}=\hat{H}_{0,+}+\hat{H}_{0,-}-\frac{1}{2}(\xi_+ +\xi_-)-\frac{\pi}{L}\hbar(1+N^2)\left(\left[\frac{e_+bL}{2\pi\hbar}\right]\right)^2 +\frac{2\pi}{L}\hbar\left[\frac{e_+bL}{2\pi\hbar}\right]\sum_{\substack{p\in\mathcal{Z}\\p\neq 0}}(-1)^p\rho_{-N}(p), \tag{35}$$

$$\hat{H}_{\text{EM}}^{\text{phys}}=\frac{1}{2L}\hat{\pi}_b^2+V(\rho_N;\rho_N)+\frac{e_+^2L}{4\pi^2}(1-N^2)\times\left[\frac{e_+bL}{2\pi\hbar}\right]\sum_{\substack{p\in\mathcal{Z}\\p\neq 0}}\frac{(-1)^p}{p^2}\rho_N(p)+\frac{e_+^2L}{24}(1-N^2)^2\left(\left[\frac{e_+bL}{2\pi\hbar}\right]\right)^2. \tag{36}$$

The free fermionic Hamiltonians $\hat{H}_{0,\pm}$ can be expressed in terms of $\rho_\pm(p)$, by making use of the bosonization procedure. Their bosonized version is

$$\hat{H}_{0,\pm}^s=\frac{2\pi}{L}\hbar\sum_{p>0}|\lambda\varepsilon_{p,\pm}|^{-s}\rho_\pm^s(-p)\rho_\pm^s(p).$$

Equations (35) and (36) give us a physical Hamiltonian invariant under both topologically trivial and nontrivial gauge transformations, $\hat{H}_F^{\text{phys}}$ and $\hat{H}_{\text{EM}}^{\text{phys}}$ being invariant separately. The last two terms in Eq. (35) make the free fermionic part of the Hamiltonian invariant, while the ones in Eq. (36) make the electromagnetic part invariant.

For $N=\pm 1$, the last two terms in Eq. (36) vanish. These terms are therefore caused by the anomaly and represent new types of interactions which are absent in the nonanomalous models.

The new interactions admit the following interpretation. Let us combine the last term in Eq. (36) with the kinetic part of the electromagnetic Hamiltonian, then

$$\frac{1}{2L}\hat{\pi}_b^2+\frac{e_+^2L}{24}(1-N^2)^2\left(\left[\frac{e_+bL}{2\pi\hbar}\right]\right)^2 =\frac{1}{2L^2}\int_{-L/2}^{L/2}dx[\hat{\pi}_b-L\mathcal{E}(x)]^2,$$

i.e., the momentum $\hat{\pi}_b$ is supplemented by the linearly rising electric field strength

$$\mathcal{E}(x)\equiv-\frac{e_+}{L}x(1-N^2)\left[\frac{e_+bL}{2\pi\hbar}\right].$$

As in four-dimensional models of a relativistic particle moving in an external field, we may define a generalized momentum operator in the form

$$\hat{\tilde{\pi}}_b(x)\equiv\hat{\pi}_b-L\mathcal{E}(x).$$

The commutation relations for $\hat{\tilde{\pi}}_b$ are

$$[\hat{\tilde{\pi}}_b(x),\hat{\tilde{\pi}}_b(y)]_-=i(1-N^2)\frac{e_+^2L}{2\pi}(x-y).$$

We see that due to the new interactions the physical degrees of freedom behave themselves as though moving in a background linearly rising electric field. This is an effective field not related directly to the original fields of our model. It may be considered as produced by a charge uniformly distributed on the circle with density

$$\rho_{\text{bgrd}}=-\frac{1}{L}(1-N^2)\left[\frac{e_+bL}{2\pi\hbar}\right].$$

This situation is similar to that in 2+1 or 3+1 dimensions. As known, in the non-Abelian models governed by Lagrangians with topological terms (the Pontryagin density in 3+1 dimensions or the Chern-Simons term in 2+1 dimensions) the non-Abelian gauge field is moving in a background $U(1)$ functional gauge potential expressed in terms of the non-Abelian gauge field components [5]. The peculiarity of the situation in our case is that there is no magnetic field related to the gauge field in 1+1 dimensions, so the background field is electric.

If we again make the transition to the physical states invariant under both the topologically trivial and nontrivial gauge transformations, then the density of the kinetic part of the physical electromagnetic Hamiltonian becomes

$$\frac{1}{2L^2}\hat{\tilde{\pi}}_b^2\rightarrow\frac{1}{2L^2}\{\hat{\pi}_b-L[\mathcal{E}_\theta+\mathcal{E}(x)]\}^2.$$

While the constant background electric field is general in one-dimensional gauge models defined on the circle, the linearly rising one is specific to models with left-right asymmetric matter content [15].

The next-to-last term in Eq. (36) means that the fermionic physical degrees of freedom and b are not decoupled in the physical Hamiltonian. This term represents the Coulomb-type background-matter interaction

$$\frac{e_+^2L}{4\pi^2}(1-N^2)\left[\frac{e_+bL}{2\pi\hbar}\right]\sum_{\substack{p\in\mathcal{Z}\\p\neq 0}}\frac{(-1)^p}{p^2}\rho_N(p) =-\frac{e_+^2L^2}{4\pi^2}\sum_{\substack{p\in\mathcal{Z}\\p\neq 0}}\frac{(-1)^p}{p^2}\rho_{\text{bgrd}}\rho_N(p).$$

It is just the background linearly rising electric field that couples b to the fermionic physical degrees of freedom in the Coulomb interaction.

As a consequence, the eigenstates of the physical Hamiltonian are not a direct product of the purely fermionic Fock states and wave functionals of b. This is a common feature of gauge theories with anomaly. That the Hilbert space in such theories is not a tensor product of the Hilbert space for a gauge field and the fixed Hilbert space for fermions was shown in [6,7].

The background charge interpretation is related to the definition of the Fock vacuum. The definition given in Eqs.

(10)–(11) depends on $[e_+bL/2\pi\hbar]$ and remains unchanged only locally on $\mathcal{A}^1$, in regions where $[e_+bL/2\pi\hbar]$ is fixed. The values of the gauge field in regions of different $[e_+bL/2\pi\hbar]$ are connected by the topologically nontrivial gauge transformations. If $[e_+bL/2\pi\hbar]$ changes, then there is a nontrivial spectral flow, i.e., some of the energy levels of the first quantized fermionic Hamiltonians cross zero and change sign. This means that the definition of the Fock vacuum changes.

The charge operators $\hat{Q}_\pm$ also change. Let $:\hat{Q}_\pm^{(0)}:$ be charge operators defined in the region where $[e_+bL/2\pi\hbar]=0$ and normal ordered with respect to the corresponding Fock vacuum. Then in regions with nonzero $[e_+bL/2\pi\hbar]$ the charge operators become $:\hat{Q}_\pm^{(0)}: \mp e_\pm[e_\pm bL/2\pi\hbar]$. For models without anomaly, the additional terms in the positive and negative chirality charges are opposite in sign, so the total charge is $:\hat{Q}_+^{(0)}:+:\hat{Q}_-^{(0)}:$ in all regions of different $[e_+bL/2\pi\hbar]$, i.e., defined globally on $\mathcal{A}^1$. For models with anomaly, the additional terms do not cancel each other and the total charge operator up to terms depending on $\{e_\pm bL/2\pi\hbar\}$ becomes $:\hat{Q}_+^{(0)}:+:\hat{Q}_-^{(0)}:+e_+L\rho_{\text{bgrd}}$. The background charge is therefore that part of the total charge which depends on $[e_+bL/2\pi\hbar]$ and changes in the transition between regions of different $[e_+bL/2\pi\hbar]$.

B. Exotization

We can formally decouple the matter and gauge field degrees of freedom by introducing the exotic statistics matter fields (the so-called exotization procedure [16]). Let us define the composite fields

$$\tilde{\psi}_\pm(x)=\exp\left\{\mp i\frac{\pi}{L}x\pm i\frac{2\pi}{e_\pm L}\left(\hat{\pi}_b\mp e_\pm x\left[\frac{e_\pm bL}{2\pi\hbar}\right]\right)\right\}\psi_\pm(x). \tag{37}$$

The fields $\tilde{\psi}_\pm(x)$ are invariant under the topologically nontrivial gauge transformations (we put $U_0=1$)

$$\exp\left\{\frac{i}{\hbar}n\hat{T}_b\right\}\tilde{\psi}_\pm\exp\left\{-\frac{i}{\hbar}n\hat{T}_b\right\}=\tilde{\psi}_\pm$$

and have the commutation relations

$$\tilde{\psi}_\pm^\dagger(x)\tilde{\psi}_\pm(y)+e^{\mp iF(x,y)}\tilde{\psi}_\pm(y)\tilde{\psi}_\pm^\dagger(x)=\delta(x-y),$$

$$\tilde{\psi}_\pm(x)\tilde{\psi}_\pm(y)+e^{\pm iF(x,y)}\tilde{\psi}_\pm(y)\tilde{\psi}_\pm(x)=0, \tag{38}$$

where $F(x,y)\equiv(2\pi/L)(x-y)$. The commutation relations (38) are indicative of exotic statistics of $\tilde{\psi}_\pm(x)$. These fields are neither fermionic nor bosonic. Only for $x=y$ Eqs. (38) become anticommutators: $\tilde{\psi}_\pm(x)$ [and $\tilde{\psi}_\pm^\dagger(x)$] anticommute with themselves, i.e., behave as fermionic fields.

Using Eq. (37) and the expansions (8), we obtain the Fourier expansions for the exotic fields $\tilde{\psi}_\pm(x)$:

$$\tilde{\psi}_+^s(x)=\sum_{n\in\mathcal{Z}}\tilde{a}_n\langle x|n;+\rangle|\lambda\varepsilon_{n,+}|^{-s/2},$$

$$\tilde{\psi}_-^s(x)=\sum_{n\in\mathcal{Z}}\tilde{b}_n\langle x|n;-\rangle|\lambda\varepsilon_{n,-}|^{-s/2},$$

where

$$\tilde{a}_n\equiv\exp\left\{i\frac{2\pi}{e_+L}\hat{\pi}_b\right\}a_{n+[e_+bL/2\pi\hbar]},$$

$$\tilde{b}_n\equiv\exp\left\{-i\frac{2\pi}{e_-L}\hat{\pi}_b\right\}b_{n+[e_-bL/2\pi\hbar]}.$$

The exotic creation and annihilation operators $\tilde{a}_n^\dagger,\tilde{a}_n$ and $\tilde{b}_n^\dagger,\tilde{b}_n$ fulfill the following commutation relations algebra:

$$\tilde{a}_n^\dagger\tilde{a}_m+\tilde{a}_{m-1}\tilde{a}_{n-1}^\dagger=\delta_{mn},$$

$$\tilde{a}_n\tilde{a}_m+\tilde{a}_{m+1}\tilde{a}_{n-1}=0,$$

and

$$\tilde{b}_n^\dagger\tilde{b}_m+\tilde{b}_{m+1}\tilde{b}_{n+1}^\dagger=\delta_{mn},$$

$$\tilde{b}_n\tilde{b}_m+\tilde{b}_{m-1}\tilde{b}_{n+1}=0.$$

We next introduce the new Fock vacuum $\overline{|\text{vac};A\rangle}=\overline{|\text{vac};A;+\rangle}\otimes\overline{|\text{vac};A;-\rangle}$ defined as

$$a_n\overline{|\text{vac};A;+\rangle}=0 \quad \text{for} \quad n>0,$$

$$a_n^\dagger\overline{|\text{vac};A;+\rangle}=0 \quad \text{for} \quad n\leqslant 0,$$

and

$$b_n\overline{|\text{vac};A;-\rangle}=0 \quad \text{for} \quad n\leqslant 0,$$

$$b_n^\dagger\overline{|\text{vac};A;-\rangle}=0 \quad \text{for} \quad n>0,$$

denoting the normal ordering with respect to $\overline{|\text{vac};A\rangle}$ by triple dots.

If we compare the old and the new definitions of the Fock vacuum, then we see a shift of the level that separates the filled levels and the empty ones. The new Fock vacuum is defined such that for all values of $[e_+bL/2\pi\hbar]$ only the levels with energy lower than (or equal to) the energy of the level $n=0$ are filled and the others are empty, i.e., the new definition does not depend on $[e_+bL/2\pi\hbar]$ and remains unchanged as the gauge configuration changes.

The exotic matter current operators are

$$\hat{\tilde{j}}_\pm^s(x)=\sum_{n\in\mathcal{Z}}\frac{1}{L}\exp\left\{i\frac{2\pi}{L}nx\right\}\tilde{\rho}_\pm^s(n),$$

$$\tilde{\rho}_+^s(n)=\sum_{k\in\mathcal{Z}}\tilde{a}_k^\dagger\tilde{a}_{k+n}|\lambda\varepsilon_{k,+}|^{-s/2}|\lambda\varepsilon_{k+n,+}|^{-s/2},$$

$$\tilde{\rho}_-^s(n)=\sum_{k\in\mathcal{Z}}\tilde{b}_k^\dagger\tilde{b}_{k-n}|\lambda\varepsilon_{k,-}|^{-s/2}|\lambda\varepsilon_{k-n,-}|^{-s/2}.$$

The new operators $\widetilde{\rho}_{\pm}(n)$ and the old ones $\rho_{\pm}(n)$ are connected in the following way:

$$\vdots\,\widetilde{\rho}_{\pm}(n)\,\vdots = :\rho_{\pm}(n): \pm \delta_{n,0}\left[\frac{e_{\pm}bL}{2\pi\hbar}\right].$$

The exotic matter charges are

$$\vdots\,\hat{\widetilde{Q}}_{\pm}\,\vdots = :\hat{Q}_{\pm}: \pm e_{\pm}\left[\frac{e_{\pm}bL}{2\pi\hbar}\right].$$

On the physical states (21) the exotic charges become

$$\vdots\,\hat{\widetilde{Q}}_{\pm}\,\vdots\,|\mathrm{phys};A\rangle = \pm e_{\pm}\left[\frac{e_{\pm}bL}{2\pi\hbar}\right]|\mathrm{phys};A\rangle. \qquad (39)$$

With Eq. (39), we decouple the matter and gauge-field degrees of freedom in the physical Hamiltonian (36). We obtain

$$\hat{H}_{\mathrm{phys}} = \frac{1}{2L}\hat{\pi}_b^2 - \frac{1}{2}(\xi_+ + \xi_-) + \hbar\frac{2\pi}{L}\sum_{p>0}[\rho_{\mathrm{tot},+}(-p)\rho_{\mathrm{tot},+}(p) + \rho_{\mathrm{tot},-}(p)\rho_{\mathrm{tot},-}(-p)] + V(\rho_N^{\mathrm{tot}};\rho_N^{\mathrm{tot}}),$$

where we have defined the operators

$$\rho_N^{\mathrm{tot}} \equiv \rho_{\mathrm{tot},+} + N\rho_{\mathrm{tot},-}\,,$$

$$\rho_{\mathrm{tot},\pm} \equiv \widetilde{\rho}_{\pm} + (-1)^p\frac{1}{e_{\pm}}\,\vdots\,\hat{\widetilde{Q}}_{\pm}\,\vdots\,.$$

These operators are invariant under both topologically trivial and nontrivial gauge transformations.

To diagonalize the exotic matter part of the physical Hamiltonian, we perform the Bogoliubov transformation over the operators $\rho_{\mathrm{tot},+}(\pm p)$ and $\rho_{\mathrm{tot},-}(\pm p),(p>0)$:

$$\begin{aligned}\rho_{\mathrm{tot},+}(\pm p) \to \overline{\rho}_{\mathrm{tot},+}(\pm p) &= \cosh t_p \rho_{\mathrm{tot},+}(\pm p) + \sinh t_p \rho_{\mathrm{tot},-}(\pm p),\\ \rho_{\mathrm{tot},-}(\pm p) \to \overline{\rho}_{\mathrm{tot},-}(\pm p) &= \sinh t_p \rho_{\mathrm{tot},+}(\pm p) + \cosh t_p \rho_{\mathrm{tot},-}(\pm p),\end{aligned} \qquad (40)$$

where

$$\cosh 2t_p = \frac{1}{E_p}\left(\frac{2\pi p}{L}\hbar + \frac{e_+^2 L}{8\pi^2 p}(1+N^2)\right),$$

$$\sinh 2t_p = \frac{1}{E_p}\frac{e_+^2 L}{4\pi^2 p}N,$$

and

$$E_p = \sqrt{E_p^2(N) + \left(\frac{e_+^2 L}{8\pi^2}\right)^2 (1-N^2)^2 \frac{1}{p^2}},$$

$$E_p^2(N) \equiv \left(\frac{2\pi p}{L}\right)^2\hbar^2 + \frac{e_+^2}{2\pi}\hbar(1+N^2).$$

The Bogoliubov transformed operators $\overline{\rho}_{\mathrm{tot},+}(\pm p), \overline{\rho}_{\mathrm{tot},-}(\pm p)$ satisfy the same commutation relations as the nontransformed ones:

$$[\overline{\rho}_{\mathrm{tot},+}(m), \overline{\rho}_{\mathrm{tot},+}(n)]_- = [\overline{\rho}_{\mathrm{tot},-}(n), \overline{\rho}_{\mathrm{tot},-}(m)]_- = m\delta_{m,-n}\,.$$

The generator of the Bogoliubov transformation (40) is

$$B_p \equiv \exp\left\{\frac{1}{p}t_p[\overline{\rho}_{\mathrm{tot},-}(p)\overline{\rho}_{\mathrm{tot},+}(-p) - \overline{\rho}_{\mathrm{tot},+}(p)\overline{\rho}_{\mathrm{tot},-}(-p)]\right\}.$$

The diagonalized form of the total physical Hamiltonian is

$$\hat{H}_{\mathrm{phys}} = \frac{1}{2L}\hat{\pi}_b^2 - \frac{1}{2}(\xi_+ + \xi_-) + \sum_{p>0}\frac{1}{p}E_p[\overline{\rho}_{\mathrm{tot},+}(-p)\overline{\rho}_{\mathrm{tot},+}(p) + \overline{\rho}_{\mathrm{tot},-}(p)\overline{\rho}_{\mathrm{tot},-}(-p)]. \qquad (41)$$

The physical Hamiltonian obtained is expressed in terms of the exotic matter and global gauge-field degrees of freedom. The exotic fields are composites of the fermionic matter and background electric fields.

For the $N=1$ model, $e_- = e_+ \equiv e$ and the linearly rising background electric field vanishes. The spectrum of the physical Hamiltonian becomes relativistic:

$$E_p = E_p(N=1) = \hbar\sqrt{\left(\frac{2\pi p}{L}\right)^2 + M^2},$$

where $M^2 \equiv (e^2/\pi)(1/\hbar)$.

If we introduce the creation and annihilation operators for b,

$$C^\dagger \equiv \frac{1}{\sqrt{2ML}}\left[-\hbar\frac{d}{db} + 2\sqrt{\pi\hbar}\left(\left\{\frac{ebL}{2\pi\hbar}\right\} - \frac{1}{2}\right)\right],$$

$$C \equiv \frac{1}{\sqrt{2ML}}\left[\hbar\frac{d}{db} + 2\sqrt{\pi\hbar}\left(\left\{\frac{ebL}{2\pi\hbar}\right\} - \frac{1}{2}\right)\right],$$

$$[C, C^\dagger] = 1,$$

then the global gauge-field part of the physical Hamiltonian becomes

$$\frac{1}{2L}\hat{\pi}_b^2 - \frac{1}{2}(\xi_+ + \xi_-) = M\left(C^\dagger C + \frac{\hbar}{2}\right).$$

The wave function of its lowest energy eigenstate is

$$f_0(b) = \left(\frac{ML}{\pi\hbar}\right)^{1/4}\exp\left[-\left(\frac{2\pi}{eL}\right)^2\frac{ML\hbar}{2}\left(\left\{\frac{ebL}{2\pi\hbar}\right\} - \frac{1}{2}\right)^2\right].$$

The total physical Hamiltonian takes the form

$$\hat{H}_{\mathrm{phys}} = MC^\dagger C + \sum_{p>0}\frac{1}{p}E_p[\overline{\rho}_{\mathrm{tot},+}(-p)\overline{\rho}_{\mathrm{tot},+}(p) + \overline{\rho}_{\mathrm{tot},-}(p)\overline{\rho}_{\mathrm{tot},-}(-p)]. \qquad (42)$$

This is just the Hamiltonian of a massive scalar boson with mass M.

The $N=-1$ model can be considered analogously. We get the same physical Hamiltonian (42). Thus, for both cases $N=\pm 1$, the quantum generalized CSM is equivalent to the free field theory of a massive scalar field.

For the $N\neq\pm 1$ models, the spectrum of the physical Hamiltonian is nonrelativistic and does not correspond to a massive boson. So the quantum theory of the models with anomaly is not equivalent to the theory of a free massive scalar field.

Since the matter and gauge-field degrees of freedom are decoupled in the physical Hamiltonian (41), its eigenstates can be represented as a direct product of the exotic matter Fock states and wave functionals of b. In particular, the ground state of the physical Hamiltonian is defined as

$$\left(\frac{1}{2L}\hat{\pi}_b^2-\frac{1}{2}(\xi_+ + \xi_-)\right)|\text{ground}\rangle=0,$$

$$\overline{\rho}_{\text{tot},+}(n)|\text{ground}\rangle=\overline{\rho}_{\text{tot},-}(-n)|\text{ground}\rangle=0, \quad n>0.$$

For the $N=1$ model, the ground state is

$$\begin{aligned}|\text{ground}\rangle&=f_0(b)\left(\prod_{n>0}U_n^\dagger\right)\overline{|\text{vac};A\rangle}\\&=f_0(b)\exp\Bigg\{-\sum_{n>0}\frac{1}{n}[\overline{\rho}_{\text{tot},-}(n)\overline{\rho}_{\text{tot},+}(-n)\\&\qquad-\overline{\rho}_{\text{tot},+}(n)\overline{\rho}_{\text{tot},-}(-n)]\Bigg\}\overline{|\text{vac};A\rangle}.\end{aligned}$$

All the excited states are constructed by acting the Bogoliubov transformed operators $\overline{\rho}_{\text{tot},+}(-n),\overline{\rho}_{\text{tot},-}(n),(n>0)$ and the global gauge-field degree of freedom creation operator $C^\dagger$ on the ground state.

Thus, the quantum generalized CSM can be formulated in two equivalent ways. In the first way, the matter fields are fermionic and coupled nontrivially to the global gauge-field degree of freedom. In the second way, the matter and gauge-field degrees of freedom are decoupled in the physical Hamiltonian, but the matter fields acquire exotic statistics.

V. POINCARÉ ALGEBRA

(1) The classical momentum and boost generators are given by

$$P=\int_{-L/2}^{L/2}dx(-i\hbar\psi_+^\dagger\partial_1\psi_+ - i\hbar\psi_-^\dagger\partial_1\psi_- - E\partial_1 A),$$

$$K=\int_{-L/2}^{L/2}dx\,x\mathcal{H}(x).$$

After a straightforward calculation we obtain

$$\{H,P\}=0,$$

$$\{P,K\}=-H, \quad \{H,K\}=-P,$$

i.e., at the classical level, these generators obey the Poincaré algebra.

At the quantum level, the momentum and boost generators become

$$\hat{P}=\hat{P}_+ + \hat{P}_- - \int_{-L/2}^{L/2}dx\hat{E}\partial_1 A_1,$$

$$\hat{P}_\pm\equiv\frac{1}{2}\hbar\int_{-L/2}^{L/2}dx[\psi_\pm^\dagger(-i\partial_1)\psi_\pm-\psi_\pm(i\partial_1)\psi_\pm^\dagger],$$

$$\hat{K}=\int_{-L/2}^{L/2}dx\,x[\hat{\mathcal{H}}_+(x)+\hat{\mathcal{H}}_-(x)+\hat{\mathcal{H}}_{\text{EM}}(x)].$$

Using the Fourier expansions for the fermionic and gauge fields, we rewrite the quantum generators as

$$\hat{P}=\hat{P}_+ - \hat{P}_- - \frac{e_+^2 L}{2\pi}(1-N^2)\sum_{p>0}\alpha_{-p}\alpha_p,$$

$$\hat{P}_\pm=\pm\hat{H}_{0,\pm}\mp\frac{1}{2}\xi_\pm-\frac{1}{2}e_\pm\eta_\pm bL,$$

$$\hat{K}=-i\frac{L}{2\pi}\sum_{\substack{p\in\mathcal{Z}\\p\neq 0}}\frac{(-1)^p}{p}[\mathcal{H}_+(p)+\mathcal{H}_-(p)+\mathcal{H}_{\text{EM}}(p)],$$

where $\mathcal{H}_\pm(p)$ and $\mathcal{H}_{\text{EM}}(p)$ are given, respectively, by Eqs. (12) and (14).

As the Hamiltonian, the quantum momentum and boost generators are not uniquely determined. We can use this arbitrariness in order to make them invariant under both topologically trivial and nontrivial gauge transformations. Acting in the same way as before in Sec. IV, we obtain the physical momentum and boost generators in the form

$$\hat{P}_{\text{phys}}=[\mathcal{H}_+^{\text{phys}}(0)-\mathcal{H}_-^{\text{phys}}(0)],$$

$$\hat{K}_{\text{phys}}=-i\frac{L}{2\pi}\sum_{\substack{p\in\mathcal{Z}\\p\neq 0}}\frac{(-1)^p}{p}[\mathcal{H}_+^{\text{phys}}(p)+\mathcal{H}_-^{\text{phys}}(p)+\mathcal{H}_{\text{EM}}^{\text{phys}}(p)],$$

where

$$\mathcal{H}_\pm^{\text{phys}}(p)\equiv\hbar\frac{\pi}{L}\sum_{\substack{q\in\mathcal{Z}\\q\neq(0;-p)}}\rho_{\text{tot},\pm}(p+q)\rho_{\text{tot},\pm}(-q),$$

$$\mathcal{H}_\pm^{\text{phys}}(0)=\mathcal{H}_\pm^{\text{phys}}(p=0),$$

and

$$\mathcal{H}_{\text{EM}}^{\text{phys}}(p)=\frac{\hbar}{p}\frac{e_+}{2\pi}\rho_{\text{tot}}^N(p)\frac{d}{db}+\frac{e_+^2 L}{8\pi^2}\times\sum_{\substack{q\in\mathcal{Z}\\q\neq(0;-p)}}\frac{1}{q(q+p)}\rho_{\text{tot}}^N(p+q)\rho_{\text{tot}}^N(-q).$$

(2) Let us now construct the algebra of the physical Hamiltonian, momentum, and boost generators. For $N \neq \pm 1$, the relativistic invariance is broken, so this algebra is not certainly a Poincaré one.

For the operators $\mathcal{H}_{\pm}^{\text{phys}}(p)$, $\mathcal{H}_{\text{EM}}^{\text{phys}}(p)$, and $\mathcal{H}_{\text{EM}}^{\text{phys}}(0)=\hat{H}_{\text{EM}}^{\text{phys}}$, we get the commutation relations

$$[\mathcal{H}_{\pm}^{\text{phys}}(n),\mathcal{H}_{\pm}^{\text{phys}}(m)]_{-}=\pm\hbar\frac{2\pi}{L}(n-m)\mathcal{H}_{\pm}^{\text{phys}}(n+m),$$

$$[\mathcal{H}_{+}^{\text{phys}}(0)-\mathcal{H}_{-}^{\text{phys}}(0),\mathcal{H}_{\text{EM}}^{\text{phys}}(p)]_{-}=-\hbar\frac{2\pi}{L}p\mathcal{H}_{\text{EM}}^{\text{phys}}(p),$$

and

$$[\mathcal{H}_{+}^{\text{phys}}(0)-\mathcal{H}_{-}^{\text{phys}}(0),\mathcal{H}_{\text{EM}}^{\text{phys}}(0)]_{-}=0,$$

$$[\mathcal{H}_{\pm}^{\text{phys}}(p),\mathcal{H}_{\text{EM}}^{\text{phys}}(0)]_{-}=[\mathcal{H}_{\pm}^{\text{phys}}(0),\mathcal{H}_{\text{EM}}^{\text{phys}}(p)]_{-}.$$

With these commutation relations, we easily obtain the algebra of the Poincaré generators:

$$[\hat{H}_{\text{phys}},\hat{P}_{\text{phys}}]_{-}=0,$$

$$[\hat{P}_{\text{phys}},\hat{K}_{\text{phys}}]_{-}=-i\hbar\hat{H}_{\text{phys}}+(\text{boundary terms})_1,$$

and

$$[\hat{H}_{\text{phys}},\hat{K}_{\text{phys}}]_{-}=-i\hbar\hat{P}_{\text{phys}}-i\frac{L}{2\pi}\sum_{\substack{p\in\mathcal{Z}\\p\neq 0}}\frac{(-1)^p}{p}[\mathcal{H}_{\text{EM}}^{\text{phys}}(0),\mathcal{H}_{\text{EM}}^{\text{phys}}(p)]_{-}+(\text{boundary terms})_2, \tag{43}$$

where

$$[\mathcal{H}_{\text{EM}}^{\text{phys}}(0),\mathcal{H}_{\text{EM}}^{\text{phys}}(p)]_{-}=-\hbar^3\frac{e_+}{2\pi L}\frac{1}{p}\mathcal{F}_{+-}(p)\rho_{\text{tot}}^{N}(p)\frac{d}{db}+\hbar^2\frac{e_+^2}{4\pi^2}\sum_{\substack{q\in\mathcal{Z}\\q\neq(0;-p)}}\mathcal{F}_{+-}(q)\frac{1}{q(p+q)}\rho_{\text{tot}}^{N}(p+q)\rho_{\text{tot}}^{N}(-q),$$

while the boundary terms are

$$(\text{boundary terms})_1\equiv i\hbar L\left[\hat{\mathcal{H}}_{+}^{\text{phys}}\left(\frac{L}{2}\right)+\hat{\mathcal{H}}_{-}^{\text{phys}}\left(\frac{L}{2}\right)+\hat{\mathcal{H}}_{\text{EM}}^{\text{phys}}\left(\frac{L}{2}\right)\right],$$

$$(\text{boundary terms})_2\equiv i\hbar L\left[\hat{\mathcal{H}}_{+}^{\text{phys}}\left(\frac{L}{2}\right)-\hat{\mathcal{H}}_{-}^{\text{phys}}\left(\frac{L}{2}\right)\right].$$

The algebra obtained differs from the Poincaré one in the boundary terms and in the commutator $[\mathcal{H}_{\text{EM}}^{\text{phys}}(0),\mathcal{H}_{\text{EM}}^{\text{phys}}(p)]_{-}$. The curvature $\mathcal{F}_{+-}$ associated with the projective representation of the gauge group makes this commutator nonvanishing for the models with anomaly. This is another point where the nonvanishing curvature $\mathcal{F}_{+-}$ shows itself (recall the commutator of the modified momentum operators).

For $N=\pm 1$, $\mathcal{F}_{+-}$ vanishes and up to the boundary terms we get the Poincaré algebra. In the limit $L\to\infty$, these boundary terms vanish on the physical states, because the energy density is assumed to diminish at spatial infinities faster than $1/L$. Otherwise, the total energy of the system would become infinite. Therefore, for the $N=\pm 1$ models on the line, the Poincaré algebra closes exactly. However, the boundary terms do not affect the spectrum of the physical Hamiltonian which is relativistic in the case of the circle too.

For $N\neq\pm 1$, in the limit $L\to\infty$ the first term in the commutator $[\mathcal{H}_{\text{EM}}^{\text{phys}}(0),\mathcal{H}_{\text{EM}}^{\text{phys}}(p)]_{-}$ disappears (since the gauge field has no global gauge-field degrees of freedom), while the second one survives. For the $N\neq\pm 1$ models the Poincaré algebra does not close even on the line. This means that such models are not relativistically invariant.

We can conclude that the nonclosure of the Poincaré algebra in Eq. (43) is essentially due to the projective representation of the local gauge symmetry. Working on the circle allowed us to construct explicitly the Poincaré algebra breaking term connected to the nonvanishing curvature $\mathcal{F}_{+-}$.

Let us note that the Poincaré algebra fails to close in the physical sector where the states satisfy the quantum Gauss law constraint (20) and the Poincaré generators are gauge invariant. The physical Hamiltonian and momentum generator commute, so the translational invariance is preserved. The origin of the breakdown of the relativistic invariance lies in the anomaly or, more exactly, in the fact that the local gauge symmetry is realized projectively.

VI. CHARGE SCREENING

Let us introduce a pair of external charges, namely, a positive charge with strength q at x_0 and a negative one with the same strength at y_0 [28]. The external current density is

$$j_{\text{ex},0}(x)=q[\delta(x-x_0)-\delta(x-y_0)]=\frac{q}{L}\sum_{p\in\mathcal{Z}}j_p^{\text{ex}}e^{-i(2\pi p/L)x},$$

where

$$j_p^{\text{ex}}\equiv e^{i(2\pi p/L)x_0}-e^{i(2\pi p/L)y_0}.$$

The total external charge is zero, so the external current density has vanishing zero mode, $j_0^{\text{ex}}=0$. The Lagrangian density of the generalized CSM changes as

$$\mathcal{L}\to\mathcal{L}+A_0 j_{\text{ex},0}.$$

The classical generalized CSM with the external charges added can be quantized in the same way as that without external charges. The quantum Gauss law operator becomes

$$\hat{G}_{\text{ex}}\equiv\hat{G}+j_{\text{ex},0}=\partial_1\hat{E}+e_+\hat{j}_++e_-\hat{j}_-+j_{\text{ex},0}.$$

Its Fourier expansion is

$$\hat{G}_{\text{ex}}=\hat{G}_0+\frac{2\pi}{L^2}\sum_{p>0}[\hat{G}_{+}^{\text{ex}}(p)e^{i(2\pi/L)px}-\hat{G}_{-}^{\text{ex}}(p)e^{-i(2\pi/L)px}],$$

where

$$\hat{G}_{+}^{\rm ex}(p) \equiv \hat{G}_{+}(p) + \frac{qL}{2\pi}(j_p^{\rm ex})^*,$$

$$\hat{G}_{-}^{\rm ex}(p) \equiv \hat{G}_{-}(p) - \frac{qL}{2\pi} j_p^{\rm ex}.$$

The physical states $|{\rm phys};A;{\rm ex}\rangle$ are defined as

$$\hat{\tilde{G}}_{\pm}^{\rm ex}(p)|{\rm phys};A;{\rm ex}\rangle \equiv \left(\hat{G}_{\pm}^{\rm ex}(p) \pm \frac{1}{\hbar}\frac{e_+^2 L^2}{8\pi^2} \times (1-N^2)\alpha_{\pm p} \right)|{\rm phys};A;{\rm ex}\rangle = 0.$$

The physical quantum Hamiltonian becomes

$$\hat{H}_{\rm phys} = \frac{1}{2L}\hat{\pi}_b^2 - \frac{1}{2}(\xi_+ + \xi_-) + \hbar \frac{2\pi}{L}\sum_{p>0} [\rho_{{\rm tot},+}(-p)\rho_{{\rm tot},+}(p) + \rho_{{\rm tot},-}(p)\rho_{{\rm tot},-}(-p)] + V_{\rm ex},$$

where

$$V_{\rm ex} \equiv \frac{e_+^2 L}{8\pi^2}\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} \frac{1}{p^2}\left(\rho_N^{\rm tot}(-p) + \frac{q}{e_+} j_p^{\rm ex}\right)\left(\rho_N^{\rm tot}(p) + \frac{q}{e_+}(j_p^{\rm ex})^*\right)$$

is the Coulomb energy in the presence of the external charges. Two new interactions contribute to the Coulomb energy owing to the external charges. One is the classical Coulomb interaction between the external charges and the other is the interaction between the total internal current and the external current.

After some calculations we rewrite the physical Hamiltonian as

$$\begin{aligned}\hat{H}_{\rm phys} = {} & \frac{1}{2L}\hat{\pi}_b^2 - \frac{1}{2}(\xi_+ + \xi_-) + \sum_{p>0}\frac{E_p}{p}\{[\overline{\rho}_{{\rm tot},+}(-p) \\ & + \kappa_{p,+} j_p^{\rm ex}][\overline{\rho}_{{\rm tot},+}(p) + \kappa_{p,+}(j_p^{\rm ex})^*] \\ & + [\overline{\rho}_{{\rm tot},-}(p) + \kappa_{p,-}(j_p^{\rm ex})^*][\overline{\rho}_{{\rm tot},-}(-p) + \kappa_{p,-} j_p^{\rm ex}]\} \\ & + \hbar^2 \frac{q^2}{L}\sum_{p>0}\frac{1}{E_p^2(N)} j_p^{\rm ex}(j_p^{\rm ex})^*, \end{aligned} \tag{44}$$

with

$$\kappa_{p,+} \equiv \hbar \frac{qe_+}{2\pi}\frac{1}{E_p^2(N)}(\cosh t_p + N\sinh t_p),$$

$$\kappa_{p,-} \equiv \hbar \frac{qe_+}{2\pi}\frac{1}{E_p^2(N)}(\sinh t_p + N\cosh t_p).$$

Comparing this Hamiltonian with the physical Hamiltonian without the external charges, we see that the external charges change the ground state. The ground state of the physical Hamiltonian (44) satisfies

$$[\overline{\rho}_{{\rm tot},+}(p) + \kappa_{p,+}(j_p^{\rm ex})^*]|{\rm ground};{\rm ex}\rangle = 0,$$

$$[\overline{\rho}_{{\rm tot},-}(-p) + \kappa_{p,-} j_p^{\rm ex}]|{\rm ground};{\rm ex}\rangle = 0, \quad p>0.$$

The last term in Eq. (44) is just the energy of the ground state

$$E_0 = \langle {\rm ground};{\rm ex}|\hat{H}_{\rm phys}|{\rm ground};{\rm ex}\rangle = \hbar^2\frac{q^2}{L}\sum_{p>0}\frac{1}{E_p^2(N)} j_p^{\rm ex}(j_p^{\rm ex})^*.$$

The energy E_0 depends only on the distance between the two external charges:

$$\begin{aligned} E_0 &= \hbar^2\frac{2q^2}{L}\sum_{p>0}\frac{1}{E_p^2(N)}\left\{1-\cos\left(\frac{2\pi p}{L}(x_0-y_0)\right)\right\} \\ &= \frac{q^2}{2M_N}\frac{\cosh(LM_N/2) - \cosh(LM_N/2 - M_N|x_0-y_0|)}{\sinh(LM_N/2)}, \end{aligned}$$

where $M_N^2 = (e^2/2\pi)(1/\hbar)(1+N^2)$. In the limit $L \gg 1$, we obtain

$$E_0 = \frac{q^2}{2M_N}(1 - e^{-M_N|x_0-y_0|}),$$

i.e., the ground-state energy has the form of the Yukawa-type potential. The long-range Coulomb force between widely separated external charges disappears. Since there is no long-range force, the external charges are screened. To show this we calculate currents induced by the charges. The induced current (or charge density) is

$$\langle {\rm ground};{\rm ex}|:\hat{j}_+(x) + N\hat{j}_-(x):|{\rm ground};{\rm ex}\rangle = -\frac{1}{2}\rho_{\rm bgrd} + \varphi(x,x_0) - \varphi(x,y_0),$$

where $(-\frac{1}{2}\rho_{\rm bgrd})$ is a current induced by the background charge, while $\varphi(x,x_0)$ and $\varphi(x,y_0)$ are currents induced by the external charges at points x_0 and y_0, correspondingly,

$$\begin{aligned}\varphi(x,y) &\equiv -\sum_{n>0}\hbar\frac{e_+ q}{\pi L}(1+N^2)\frac{1}{E_n^2(N)}\cos\left(\frac{2\pi n}{L}(x-y)\right) \\ &= -\frac{qM_N}{2e_+}\frac{\cosh(LM_N/2 - M_N|x-y|)}{\sinh(LM_N/2)}.\end{aligned}$$

The current induced by the two external charges is a sum of the currents induced by each charge.

In the limit $L \gg 1$,

$$\varphi(x,x_0)\simeq -\frac{qM_N}{2e_+}e^{-M_N|x-x_0|}$$

and damps exponentially as x goes far from x_0. Screening occurs only globally. The induced charge density distribution is spread within the range of the order M_N^{-1}. So if we are far away from the external charges, we cannot find them.

The screening of each external charge occurs independently of the other charges. That is, the charge density induced around any of the two external charges does not depend on the location of the other one. Next, the external charges are screened independently of the fact whether the background charge vanishes or not.

The screening mechanism works therefore in the generalized CSM, too. When a charge is placed in the system, the accompanying external current polarizes the vacuum producing the complete compensation of the charge. The background linearly rising electric field characteristic for models with $N\neq\pm1$ does not influence this mechanism.

VII. DISCUSSION

We have shown that the anomaly influences essentially the physical quantum picture of the generalized CSM.

(i) For the models with $N\neq\pm1$ and defined on S^1, when the gauge field has a global physical degree of freedom, the left-right asymmetric matter content results in the background linearly rising electric field. This is a new physical effect just caused by the anomaly and absent in the models without the anomaly, i.e., for $N=1$ (the standard Schwinger model) and $N=-1$ (axial electrodynamics).

This effect distinguishes the generalized CSM on S^1 from the model defined on R^1 as well. In the latter case, the gauge field has neither local nor global physical degrees of freedom and the background field disappears.

(ii) The anomaly leads also to the breakdown of the relativistic invariance. For the quantum theory of both the $N=\pm1$ and $N\neq\pm1$ models we have presented the exotic statistics matter formulation. In this formulation the physical Hamiltonian is written in a compact diagonalized form. For the models with anomaly, the spectrum of the physical Hamiltonian turns out to be nonrelativistic and does not contain a massive boson.

We have constructed the physical quantum Poincaré generators and shown that their algebra is not a Poincaré one. We have demonstrated a relation between the anomaly, Berry phase, and breakdown of the relativistic invariance. Namely, the curvature $\mathcal{F}_{+-}$ related to the Berry phase does not vanish because of the left-right asymmetric matter content. At the same time, just the nonvanishing $\mathcal{F}_{+-}$ makes the algebra of the Poincaré generators different from the Poincaré one.

The Poincaré algebra fails to close on the physical states in the chiral two-dimensional QCD (QCD_2) as well [29]. The origin of the breakdown of the relativistic invariance is the same in both models and lies in the anomaly. It would be of interest to study the question of whether the relativistic invariance is broken for other models with the projective realization of a local gauge symmetry, especially in higher dimensions.

(iii) The total screening of charges characteristic for the SM takes place in the generalized CSM too. External charges are screened globally even in the background linearly rising electric field. The current density induced by the external charges damps exponentially far away from them independently of the background charge.

Because of Schwinger [1], the total screening of external charges implies the existence of a massive particle. The breakdown of the relativistic invariance does not mean, in principle, that in the anomalous models the dynamical mass generation mechanism fails and that the massive particle cannot exist. Using the fact that the physical Hamiltonian and momentum commute, we may try to prove the existence of the simultaneous eigenstates of the relativistic massive particle energy-momentum relation and then to identify these states with massive physical particles. For the $N=0$ model defined on R^1, such a massive eigenstate is constructed in [11]. The existence of the massive eigenstates for the $N\neq\pm1$ models defined on S^1 will be investigated. We intend to report on that in a future publication.

[1] J. Schwinger, Phys. Rev. D **128**, 2425 (1962); in *Theoretical Physics, Trieste Lectures*, 1962 (IAEA, Vienna, 1963).

[2] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. **54**, 1219 (1985).

[3] R. Rajaraman, Phys. Lett. **154B**, 305 (1985).

[4] E. Wigner, Ann. Math. **40**, 149 (1939).

[5] R. Jackiw, in *Relativity, Groups and Topology II*, Proceedings of the Les Houches Summer School, 1983 (North-Holland, Amsterdam, 1984).

[6] P. Nelson and L. Alvarez-Gaume, Commun. Math. Phys. **99**, 103 (1985).

[7] L. D. Faddeev and S. L. Shatashvili, Phys. Lett. **167B**, 225 (1986).

[8] L. D. Faddeev, Phys. Lett. **14B**, 81 (1984); Nuffield Workshop on Supersymmetry and Supergravity, 1985.

[9] I. G. Halliday, E. Rabinovici, and A. Schwimmer, Nucl. Phys. **B268**, 413 (1986).

[10] M. B. Paranjape, Nucl. Phys. **B307**, 649 (1988).

[11] F. M. Saradzhev, Int. J. Mod. Phys. A **6**, 3823 (1991); **8**, 2915 (1993); **8**, 2937 (1993).

[12] A. Niemi and G. Semenoff, Phys. Rev. Lett. **55**, 927 (1985); **56**, 1019 (1986).

[13] A. Niemi and G. Semenoff, Phys. Lett. B **175**, 439 (1986).

[14] G. Semenoff, in *Super Field Theory*, edited by H. Lee *et al.* (Plenum, New York, 1987).

[15] F. M. Saradzhev, Phys. Lett. B **278**, 449 (1992).

[16] F. M. Saradzhev, Phys. Lett. B **324**, 192 (1994).

[17] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).

[18] N. S. Manton, Ann. Phys. (N.Y.) **159**, 220 (1985).

[19] S. Rajeev, Phys. Lett. B **212**, 203 (1988).

[20] J. E. Hetrick and Y. Hosotani, Phys. Rev. D **38**, 2621 (1988); Phys. Lett. B **230**, 88 (1989).

[21] F. M. Saradzhev, Sov. Phys. Lebedev. Inst. Rep., No. 9, 57 (1988); Int. J. Mod. Phys. A **9**, 3179 (1994); A. O. Barut and F. M. Saradzhev, Ann. Phys. (N.Y.) **234**, 220 (1994).
[22] A. Niemi and G. Semenoff, Phys. Rep. **135**, 99 (1986).
[23] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968).
[24] M. V. Berry, Proc. R. Soc. London **A392**, 45 (1984).
[25] F. M. Saradzhev, J. Phys. A **30**, 2219 (1997).
[26] R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
[27] C. G. Callan, Jr., R. Dashen, and D. J. Gross, Phys. Lett. **63B**, 334 (1976).
[28] S. Iso and H. Murayama, Prog. Theor. Phys. **84**, 142 (1990).
[29] F. M. Saradzhev, Phys. Lett. B **372**, 283 (1996).