Coarse grainings and irreversibility in quantum field theory

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In this paper we are interested in studying coarse graining in field theories using the language of quantum open systems. Motivated by the ideas of Hu and Calzetta on correlation histories we employ the Zwanzig projection technique to obtain evolution equations for relevant observables in self-interacting scalar field theories. Our coarse-graining operation consists in concentrating solely on the evolution of the correlation functions of degree less than n, a treatment which corresponds to the familiar truncation of the BBKGY hierarchy at the nth level. We derive the equations governing the evolution of mean-field and two-point functions thus identifying the terms corresponding to dissipation and noise. We discuss possible applications of our formalism, the emergence of classical behavior, and the connection to the decoherent histories framework. [S0556-2821(97)05114-X]

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I. INTRODUCTION

A. Motivation

Quantum field theory (QFT) has a rich structure, which manifests itself in the possibility of describing the same field system from diverse points of view. Hence, depending on the problem of interest one could focus, for instance, on the Hamiltonian; the statistical, or the particle aspects of the quantum field. This potentiality for description within different frameworks, inherent in quantum field theory, is the cause of its large domain of applications, but is also a source of interesting questions.

The more important one is to identify the level of observation in a field theory or, putting it another way, what an actual observer measures in a quantum field. The answer to this question is not easy and it is clear that the level of observation cannot be fixed uniquely. Unlike nonrelativistic quantum mechanics where one is essentially measuring phase-space quantities for a particle system spatially localized, in quantum field theory local measurements contain only a very small portion of information about the state of the field. A local observer will, for instance, be able to record only the mean field, and the higher order correlation functions are inaccessible to him. Therefore, for most of the possible configurations, one might lose all sense of predictability for the field observables.

This is closely connected with the problem of the classical limit of field theories. In the context of the decoherent histories approach to quantum mechanics, the classical domain corresponds to a set of coarse-grained and noninterfering histories, from which one can obtain almost deterministic equations for a class of observables [1]. In our case there is a large number of such classes. At low energies one might consider the particlelike behavior of the fields and obtain in the classical limit a theory of interacting nonrelativistic particles. Or one could concentrate on phase-space histories, to see the extent to which QFT behaves as a Hamiltonian system. Or even consider histories of quantities such as energy and momentum density and obtain a classical hydrodynamics description.

The above issues are also of value for early Universe cosmology. The transition from quantum to classical is of great importance in models of inflation since many of its predictions are based on the fact that the long wavelength modes of the inflaton exhibit classical behavior. When considering the nonequilibrium dynamics of fields (mainly for the study of phase transitions), the first point needed to be settled is what are the variables we should concentrate, that contain the relevant information for the problem in hand.

The notion of natural coarse graining in field theories is also important in the context of field theories in curved spacetime. For it is only one quantity that actually governs the back-reaction dynamics of spacetime: the expectation value of the field energy-momentum tensor (essentially constructed from the two-point correlation functions in the case of free fields).

To address these problems a number of techniques from nonequilibrium statistical mechanics has been employed with varying degree of success: the Feynman-Vernon influence functional technique [2–4] and the close time-path formalism [5,6]. It is the aim of this paper to exhibit the use of another powerful technique of statistical mechanics in a field theoretic context: the Zwanzig projection method (for a review see [7–9]). The great advantage of this method lies in its wide range of possible applications: for any choice of coarse graining it can be applied once we are able to identify the coarse-graining operation with an indempotent map on the space of states. Our choice of coarse graining is motivated by the ideas of Hu and Calzetta [10] on the truncation of the Schwinger-Dyson hierarchy of n-point functions.

But before discussing the approach we adopt in this paper, we find it meaningful to give a short discussion on possible choices for coarse graining.

B. Coarse grainings

There are two important constraints one might impose on our possible choices for coarse graining: naturality and Lorentz covariance. To see what we mean by naturality, let us

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consider cases of typical coarse grainings in standard nonequilibrium statistical mechanics. A typical situation is to separate relevant and irrelevant observables according to the order of magnitude of some physical parameter characterizing them. Hence we can for instance average out the effect of ''fast'' variables (evolving within very short timescales) or trace out the contribution of particles with the smaller masses (as is the case in quantum Brownian motion). Such a separation of scales, while quite common in nonrelativistic many-particle systems is rare in relativistic quantum field theory. If possible, it would involve a fine-tuning of the the coupling constants and masses of the field systems as well as the imposition of a particular initial condition. In a generic system it is unlikely that such ''autocratic'' coarse grainings can emerge naturally [10].

The requirement of Lorentz covariance, though it can be relaxed in a number of situations (for instance, when nonrelativistic matter is present [11]), is of great importance both for the cosmological applications and the emergence of classical behavior for the field variables. For when we try to study a field system from first principles, there is no natural way a non-Lorentz-invariant quantity can be introduced in our schemes. Hence, for instance, a coarse graining taking the form of a high momentum cutoff for the field modes should not be considered as fundamental but rather as emerging from the full dynamics of the theory under particular circumstances.

In addition to those two *a priori* criteria for our choice of coarse graining, there is an equally important one that can be considered only *a posteriori*, that is, after we have identified the dynamics of the relevant variables. This is the requirement of persistent predictability for the evolution equations. In the language of the decoherent histories approach it states that histories of relevant observables ought to form a quasiclassical domain. This means that the evolution equations have to be approximately dynamically autonomous [9] (even though we cannot expect to obtain Markovian behavior). This again implies that the noise due to the irrelevant part of the field, though sufficient to decohere the histories of relevant observables, is weak enough to allow a degree of predictability [2]. In general, this is expected to be possible only for a small class of initial states of our system (or the Universe for cosmological applications). This is a fact that we will verify in our analysis.

C. Truncation of the Schwinger-Dyson hierarchy

The coarse-graining operation we shall examine is one proposed by Hu and Calzeta [10] (see also [12]). They remarked on the similarity of the chain of Dyson equations linking each Green function to others of higher order with the Bugolinbov-Burn-Green-Kirkwood-Yvon (BBKGY) hierarchy of correlation functions in classical statistical mechanics. Since the set of expectation values of field products contains all information about the state of the field, a truncation in the chain of Green function will form a natural coarse-graining operation and the lower order n-point functions will be our relevant observables. The authors then proceed to compute the effective equations of motion from a master effective action using a generalization of the close time-path formalism. The important feature of these equations is the presence of correlation noise, which under particular conditions may guarantee decoherence of the ''correlation histories.''

The truncation of the Schwinger-Dyson hierarchy does satisfy the conditions of Lorentz covariance and naturality for the choice of the coarse-graining operation. First, this choice of coarse graining is closer to actual measurements of the quantum field since any finite measurement device cannot obtain information about arbitrarily high orders of correlation. Actually, a local observer might be expected to monitor only the mean-field values. Second, being an intrinsically justifiable division between relevant and irrelevant observables, it can be applied to a wide variety of systems, without the need to recourse to special arguments for each particular case. Third, it seems promising when trying to consider evolution of hydrodynamic quantities since quantities such as energy and momentum density can be obtained through the knowledge of low order correlation functions. In particular, when dealing with the back-reaction problem in curved spacetime, truncation of the hierarchy at the level $n=2$ might give interesting results since the energy-momentum tensor determining back reaction can be determined through the knowledge of two-point functions.

As far as the third requirement of predictability is concerned, we need to have a detailed calculation of the dynamical evolution of the relevant observables. Still, it is important to note that the classical behavior of the two-point correlations observed at later stages gives us at least a hint for the possibility of an initial condition such that the dynamics of observables obtained from a truncation at the level $n=2$ are approximately autonomous.

D. The Zwanzig method

To obtain the evolution equations for the relevant observables, we are going to utilize, as mentioned earlier, the Zwanzig projection technique. There is a number of reasons for believing that this provides an important calculational tool when dealing with the above issues.

(1) It allows us to use a canonical formalism, hence gaining intuition by comparison with well-studied systems in nonrelativistic quantum statistical mechanics. Our results are still covariant, though not manifestly, since we have restricted ourselves to an invariant choice of coarse graining.

(2) To perform a perturbation expansion for the equations of motion it is sufficient to construct perturbatively the field propagator $e^{-i\hat{H}t}$. This is best carried out in the Fock representation [13], which turns out to be particularly useful for implementing our choice of coarse graining.

(3) We are allowed a certain degree of flexibility since the choice of the projector onto the level of description is not unique [7]. Hence, depending on the details of our problem (mainly the initial condition), we can choose a projector so as to reduce the strength of the noise terms.

(4) It provides a straightforward relation between the initial state of the irrelevant variables and the noise terms in the evolution equations.

(5) It does not depend on the particular dynamics of the full system, that is one can apply it even when the field evolution is nonunitary, non-Markovian, or nonautonomous. Therefore, it might be used in conjunction with other meth-

ods (in particular, the influence functional technique) in order to reduce the amount of calculations needed for a particular problem.

(6) The Zwanzig method is essentially algebraic, in the sense that it depends solely on the properties of the space of observables and not on any particular realization in some Hilbert space. This means that, at least in principle, one can employ it in systems where quantum variables are coupled to classical ones as is the case of the field theory in curved spacetime.

E. This paper

It is the aim of this paper to apply the above ideas in the simplest of field systems, first to exhibit the technique and to understand the insight it can offer in particular for the case of quantum to classical transition. Hence, we concentrate on a single self-interacting scalar field in Minkowski spacetime and consider coarse grainings corresponding to truncation at the levels $n=1$ and $n=2$.

We mainly focus on two issues: the derivation of the effective equation for the relevant variables and the estimation of strength of the noise term, which determines the degree of predictability of our preferred set of variables.

The paper is then organized as follows: In Sec. II we give a brief review of the Zwanzig projection formalism, and construct the indempotent operators that implement the coarse-graining operation in the space of observables. In Sec. III, we derive the mean-field dynamics in a $\lambda\phi^4$ scalar field theory and give a general discussion on the relation of correlation noise with the initial condition. In Sec. IV we perform the same analysis for a $g\phi^3$ theory for the case of truncation at the level of two-point functions. Finally, in Sec. V, we give a discussion of our results, on the possibility of obtaining Markovian behavior, and on future applications of the formalism.

We have found it more convenient to implement the coarse graining on the normal-ordered form of the observables. The expressions we obtain are simplified significantly if we use an index notation to denote products of creation and annihilation operators $\hat{a}(\mathbf{x})$ and $\hat{a}^\dagger(\mathbf{x})$. The conventions of this notation are found in Appendix A. Finally, some useful formulas concerning the Fock representation and the normal form of operators are to be found in Appendix B.

II. THE METHOD

A. The Zwanzig technique

We will give a brief summary of the Zwanzig projection formalism, following the conventions of Zeh [8]. The main idea in the Zwanzig formalism is the representation of the coarse-graining operator by an indempotent mapping $\mathbf{P}$ in the space of states

$$\rho \rightarrow \rho_{\mathrm{rel}} = \mathbf{P}\rho, \quad \mathbf{P}^2 = \mathbf{P}. \tag{2.1}$$

The irrelevant part of the state is then given by

$$\rho_{\mathrm{irr}} = (\mathbf{1}-\mathbf{P})\rho. \tag{2.2}$$

$\mathbf{P}$ is essentially a projection operator in the space of states and determines through the trace functional a conjugate projector $\mathbf{P}^*$ on the space of observables. The projector needs not be self-adjoint ($\mathbf{P}=\mathbf{P}^*$) but for convenience we shall assume so.

The projector $\mathbf{P}$ determines the level of description for our system. We should remark that the choice of $\mathbf{P}$ projecting to a particular class of observables is not unique; there can be different inequivalent choices. Strictly speaking, $\mathbf{P}$ should be considered as an operation on the states of the system and only in this sense it is unique.

To obtain the evolution equation for the relevant observables one starts from the full dynamics of our system. The formalism is not restricted to unitary dynamics; it can be applied equally well when the dynamics are nonunitary, or non-Markovian, or nonlocal in time. In our case, we shall restrict ourselves to unitary evolution given through the von Neumann equation

$$i\,\frac{\partial \rho}{\partial t} = \mathbf{L}\rho \equiv [H,\rho], \tag{2.3}$$

from which we obtain the following system of coupled differential equations for ρ_{rel} and ρ_{irr}:

$$i\,\frac{\partial \rho_{\mathrm{rel}}}{\partial t} = \mathbf{PL}\rho_{\mathrm{rel}} + \mathbf{PL}\rho_{\mathrm{irr}}, \tag{2.4}$$

$$i\,\frac{\partial \rho_{\mathrm{irr}}}{\partial t} = (\mathbf{1}-\mathbf{P})\mathbf{L}\rho_{\mathrm{rel}} + (\mathbf{1}-\mathbf{P})\mathbf{L}\rho_{\mathrm{irr}}. \tag{2.5}$$

We can solve Eq. (2.5) by treating the ρ_{rel} term as an external force:

$$\rho_{\mathrm{irr}}(t) = e^{-i(\mathbf{1}-\mathbf{P})\mathbf{L}t}\rho_{\mathrm{irr}}(0) - i\int_0^t d\tau e^{-i(\mathbf{1}-\mathbf{P})\mathbf{L}\tau}(\mathbf{1}-\mathbf{P}) \times \mathbf{L}\rho_{\mathrm{rel}}(t-\tau). \tag{2.6}$$

Here we have denoted by $e^{-i(\mathbf{1}-\mathbf{P})\mathbf{L}t} = (\mathbf{1}-\mathbf{P})e^{-i\mathbf{L}t}(\mathbf{1}-\mathbf{P})$ the evolution operator of the equation

$$i\,\frac{\partial \rho}{\partial t} = (\mathbf{1}-\mathbf{P})\mathbf{L}\rho. \tag{2.7}$$

The exponential is used to denote the solution of the equation. It is not an actual operator exponential unless $\mathbf{L}$ is bounded [14]. Substituting Eq. (2.6) into Eq. (2.4) we get the Zwanzig premaster equation

$$i\,\frac{\partial \rho_{\mathrm{rel}}(t)}{\partial t} = \mathbf{PL}\rho_{\mathrm{rel}}(t) + \mathbf{PL}e^{-i(\mathbf{1}-\mathbf{P})\mathbf{L}t}\rho_{\mathrm{irr}}(0) - i\int_0^t d\tau \mathbf{G}(\tau)\rho_{\mathrm{rel}}(t-\tau). \tag{2.8}$$

Here, $\mathbf{G}$ stands for the kernel:

$$\mathbf{G}(\tau) = \mathbf{PL}e^{-i(\mathbf{1}-\mathbf{P})\mathbf{L}\tau}(\mathbf{1}-\mathbf{P})\mathbf{LP}. \tag{2.9}$$

Given then a relevant observable A, i.e., one such that $\mathbf{P}A = A$, we obtain for the evolution of its expectation value $\langle A\rangle$

$$i\frac{\partial}{\partial t}\langle A\rangle(t)-\langle\mathbf{PL}A\rangle(t)+i\int_0^t d\tau\langle\mathbf{PL}(\mathbf{1}-\mathbf{P})e^{i\mathbf{L}\tau}(\mathbf{1}-\mathbf{P})\mathbf{LP}A\rangle(t-\tau)=F_A(t). \tag{2.10}$$

$F_A(t)$ is ''driving force'' term, essentially stochastic in nature, since it depends on the irrelevant components of the initial state that are inaccessible from our level of description. It reads

$$F_A(t)=-\mathrm{Tr}(\rho(0)[(\mathbf{1}-\mathbf{P})e^{i(\mathbf{1}-\mathbf{P})\mathbf{L}t}\mathbf{L}A]). \tag{2.11}$$

Note, that in general the evolution of the relevant observables is nonlocal in time.

B. The coarse-graining operator

Equation (2.11) provides the starting point for a detailed calculation of the evolution equations for the relevant observables. The only input one needs is the particular form of the coarse-graining operator **P**.

We want **P** to correspond as closely as possible to the notion of the truncation of the hierarchy of correlation functions at some order n. To see how one can proceed in the construction, let us examine first the case for $n=1$. Here, the relevant variables are the values of the field $\hat{\phi}(x)$ at each given instant of time. Recall that the field operator can be written in terms of creation and annihilation operators. Then consider any density matrix written in normal-ordered form

$$\rho=\sum_{r,s}\hat{a}^\dagger_{a_1}\ldots\hat{a}^\dagger_{a_r}\rho^{a_1\ldots a_r}{}_{b_1\ldots b_s}\hat{a}^{b_1}\ldots\hat{a}^{b_s}. \tag{2.12}$$

We remark that the contributions to the expectation value of ϕ arise solely from the terms in the summation characterized by $r=s+1$ or $r=s-1$. That is, only terms differing in the number of $\hat{a}$'s and $a^\dagger$'s by 1 are the contributing ones.

Requiring that **P** projects any operator into a linear combination of $\hat{a}$'s and $a^\dagger$'s (this corresponds to considering field and momentum expectation values for relevant observables as is natural in a canonical treatment) and taking the above remark into consideration, we arrive at a natural choice for the projector. Write any observable into its normal-ordered form

$$\hat{A}=\sum_{r,s}\hat{a}^\dagger_{a_1}\ldots\hat{a}^\dagger_{a_r}A^{a_1\ldots a_r}{}_{b_1\ldots b_s}\hat{a}^{b_1}\ldots\hat{a}^{b_s} \tag{2.13}$$

and implement the action of **P** in each term in the series as follows: if $|r-s|\neq 1$ then the action of **P** yields zero. If $r=s+1$ then

$$\mathbf{P}(\hat{a}^\dagger_{a_1}\ldots\hat{a}^\dagger_{a_s+1}A^{a_1\ldots a_{s+1}}{}_{b_1\ldots b_s}\hat{a}^{b_1}\ldots\hat{a}^{b_s})=\hat{a}^\dagger_a K^a, \tag{2.14}$$

where K^a is obtained by summing over all possible contractions of the $s+1$ upper indices with the s lower ones.

Let us give one simple example to illustrate this. Consider a term of the form $A=\hat{a}^\dagger_a\hat{a}^\dagger_b A^{ab}{}_c\hat{a}^c$. The action of **P** reads

$$\mathbf{P}\hat{A}=\hat{a}^\dagger_a A^{ab}{}_b+\hat{a}^\dagger_b A^{ab}{}_a. \tag{2.15}$$

We proceed similarly for the case $r=s-1$.

The generalization for higher order products of operators follows along the same lines. Consider, for instance, a level of description fixed at one- and two-point correlation functions. We then have **P** projecting onto linear combinations of operators of the form $\hat{a}$, $\hat{a}^\dagger$, $\hat{a}\hat{a}$, $\hat{a}^\dagger\hat{a}$, and $\hat{a}^\dagger\hat{a}^\dagger$. When acting on any normal-ordered operator **P** will yield a nonzero expression if $|r-s|\in\{0,1,2\}$. For example, consider a term $\hat{a}^\dagger_a\hat{a}^\dagger_a A^{ab}{}_{cd}\hat{a}^c\hat{a}^d$. Action with **P** will yield

$$\hat{a}^\dagger_a A^{ab}{}_{cb}\hat{a}^c+\hat{a}^\dagger_a A^{ab}{}_{bd}\hat{a}^d+\hat{a}^\dagger_b A^{ab}{}_{ca}\hat{a}^c+\hat{a}^\dagger_a A^{ab}{}_{ad}\hat{a}^d. \tag{2.16}$$

C. Perturbation expansion

Having identified **P** we are only left with the calculation of the terms appearing in Eq. (2.11). In the following we shall assume that the Hamiltonian is of the form $\hat{H}=\hat{H}_0+\hat{V}$. We should note that evolution according to the free Hamiltonian does not change the level of description (since $\mathbf{L_0P}=\mathbf{PL_0}$ where $L_0\rho=[\hat{H}_0,\rho]$) and, therefore, the expression of the nonlocal term simplifies

$$i\int_0^t d\tau\langle\mathbf{PV}(\mathbf{1}-\mathbf{P})e^{i\mathbf{L}\tau}(\mathbf{1}-\mathbf{P})\mathbf{VP}A\rangle(t-\tau), \tag{2.17}$$

where $\mathbf{V}\rho=[\hat{V},\rho]$. From this expression we can readily see that in a perturbative expansion local in time terms will be at least of second order in the coupling constant. This is easily understood since this term comes from correlations, that start as relevant at time 0, become irrelevant due to interaction at time τ, propagate as irrelevant, and become relevant again at time t. Hence in the perturbative expansion diagrams containing at least two vertices are having nonzero contribution. On the other hand, the noise term, containing the evolution of correlations starting and propagating as irrelevant and due to an interaction at time t becoming relevant, can be of the first order to the coupling constant thus being dominant in lowest part of the perturbation series. This means that unless we consider some particular initial condition the effect of the noise might destroy any sense of predictability for our selected variables.

Another important observation is that the potential appears in the nonlocal term only in the combination **PV**. This part of the potential essentially scatters relevant information only to a particular sector of irrelevant states (these are sometimes called ''doorway states'' [8]). For example, in the $g\phi^3$ theory with truncation at the level of $n=2$, we shall examine in the following sections, the doorway states are the ones supporting third order correlations. Further propagation is needed to reach states with higher order correlations.

When considering the lowest order term in the perturbation expansion the expression of the nonlocal terms is significantly simplified. To see this, note that these can be writen in the form

$$i\int_0^t d\tau\{\mathbf{P}[\hat{V},(\mathbf{1}-\mathbf{P})(e^{-i\hat{H}_0\tau}[\hat{V},\rho_{\mathrm{rel}}(t-\tau)]e^{i\hat{H}_0\tau})],A\}, \tag{2.18}$$

where (,) refers to the Hilbert-Schmidt inner product. Now, since $\|e^{-iH_0\tau}\rho(t-\tau)e^{iH_0\tau}-\rho(t)\|=O(g)$ we can easily verify that within second order in the coupling constant we get

$$i\langle \mathbf{P}([(1-\mathbf{P})([\hat{A},\hat{V}]),W])\rangle(t), \tag{2.19}$$

where

$$\hat{W}(t)=\int_0^t d\tau e^{-i\hat{H}_0\tau}\hat{V}e^{i\hat{H}_0\tau}. \tag{2.20}$$

Hence to the lowest order in the perturbative expansion the nonunitary term becomes local in time. This is due to the fact that the free propagation can not remove correlations from the doorway states into the more deeply lying states of the irrelevant sector. Evolution within the sector of doorway states makes the correlations lose fast the memory of the initial condition (within a time interval proportional to the coupling constant) and hence when they reappear in the relevant channel they do not impose a time correlation in the relevant dynamics.

We are going to carry our calculation in the lowest order of perturbation theory. We should remark that apart from the technical complication the computation of higher order corrections is not difficult. It is sufficient to have a perturbation expansion in the propagator $e^{-i\hat{H}t}$. This is best carried in the Fock representation [13], which is a desirable feature given the connection of our coarse-graining projector with the normal-ordered form of the observables.

III. EVOLUTION OF THE MEAN FIELD IN A $\lambda\phi^4$ THEORY

Let us apply now the above construction to the case of a $\lambda\phi^4$ theory for truncation at the level $n=1$. The operator for the potential is given by Eqs. (B14)–(B19), while the operator $\hat{W}$ is easily computed:

$$\hat{W}=\frac{\lambda}{4!}(W_{abcd}\hat{a}^a\hat{a}^b\hat{a}^c\hat{a}^d+4\hat{a}_a^\dagger W^a{}_{bcd}\hat{a}^b\hat{a}^c\hat{a}^d+6\hat{a}_a^\dagger\hat{a}_b^\dagger W^{ab}{}_{cd}\hat{a}^c\hat{a}^d+4\hat{a}_a^\dagger\hat{a}_b^\dagger\hat{a}_c^\dagger W^{abc}{}_d\hat{a}^d+\hat{a}_a^\dagger\hat{a}_b^\dagger\hat{a}_c^\dagger\hat{a}_d^\dagger W^{abcd}), \tag{3.1}$$

with

$$W_{abcd}\rightsquigarrow\int\prod_{i=1}^4\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}e^{-i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{e^{-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4})t}-1}{-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3})+\omega_{\mathbf{k}_4}}, \tag{3.2}$$

$$W^a{}_{bcd}\rightsquigarrow\int\prod_{i=1}^4\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}e^{-i(-\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{e^{-i(-\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4})t}-1}{-i(-\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3})+\omega_{\mathbf{k}_4}}, \tag{3.3}$$

$$W^{ab}{}_{cd}\rightsquigarrow\int\prod_{i=1}^4\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}e^{-i(-\mathbf{k}_1\mathbf{x}_1-\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{e^{-i(-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4})t}-1}{-i(-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3})+\omega_{\mathbf{k}_4}}, \tag{3.4}$$

$$W^{abc}{}_d\rightsquigarrow\int\prod_{i=1}^4\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}e^{-i(-\mathbf{k}_1\mathbf{x}_1-\mathbf{k}_2\mathbf{x}_2-\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{e^{-i(-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4})t}-1}{-i(-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3})+\omega_{\mathbf{k}_4}}, \tag{3.5}$$

$$W^{abcd}\rightsquigarrow\int\prod_{i=1}^4\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}e^{i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{e^{i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4})t}-1}{i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3})+\omega_{\mathbf{k}_4}}. \tag{3.6}$$

Having the expression for $\hat{W}$ one can use in a straightforward way Eq. (2.19) to compute the dissipative terms in the evolution equation. Let us perform the calculations step by step.

First we compute the commutator $[\hat{a}^a,\hat{V}]$. It reads

$$[\hat{a}^a,\hat{V}]=\frac{\lambda}{4!}(4V^a{}_{bcd}\hat{a}^b\hat{a}^c\hat{a}^d+12\hat{a}_b^\dagger V^{ab}{}_{cd}\hat{a}^c\hat{a}^d+12\hat{a}_b^\dagger\hat{a}_c^\dagger V^{abc}{}_d\hat{a}^d+\hat{a}_b^\dagger\hat{a}_c^\dagger\hat{a}_d^\dagger V^{abcd}). \tag{3.7}$$

Acting the projector $\mathbf{P}$ on this we obtain

$$\mathbf{P}[\hat{a}^a,\hat{V}]=\frac{\lambda}{4!}(24V^{ac}{}_{cb}\hat{a}^c+24\hat{a}_b^\dagger V^{abc}{}_c). \tag{3.8}$$

Hence we can easily read the operator $(\mathbf{1}-\mathbf{P})[\hat{a}^a,\hat{V}]$.

One then needs to compute its commutator with the operator $\hat{W}$. This is indeed the difficult part of the calculations. We will get 24 terms, out of which only 12 will survive after the action on them of $\mathbf{P}$. There is no need to reproduce the whole of the calculations here, but for purposes of exposition we shall present the computations involved in one term.

A. An example

We consider the term

$$16W^e{}_{fgh}V^{abcd}[\hat{a}_b^\dagger\hat{a}_c^\dagger\hat{a}_d^\dagger,\hat{a}_e^\dagger\hat{a}^f\hat{a}^g\hat{a}^h]. \tag{3.9}$$

After computing the commutator we will obtain

$$-16[9\hat{a}_e^\dagger\hat{a}_c^\dagger\hat{a}_d^\dagger W^e{}_{fgb}V^{abcd}\hat{a}^f\hat{a}^g+18\hat{a}_e^\dagger\hat{a}_f^\dagger W^e{}_{fbc}V^{abcd}\hat{a}^f+6\hat{a}_e^\dagger W^e{}_{bcd}V^{abcd}]. \tag{3.10}$$

The action of **P** on Eq. (3.10) will yield

$$\begin{aligned}&-16[9(4\hat{a}_c^\dagger W^e{}_{ebd}V^{abcd}+2\hat{a}_e^\dagger W^e{}_{bcd}V^{abcd})+12\hat{a}_d^\dagger W^e{}_{ebc}V^{abcd}+12\hat{a}_e^\dagger W^e{}_{bcd}V^{abcd}+6\hat{a}_e^\dagger W^e{}_{bcd}V^{abcd}]\\&=-16\times 12(4\hat{a}_d^\dagger W^e{}_{ebc}+3\hat{a}_e^\dagger W^e{}_{bcd}V^{abcd}).\end{aligned} \tag{3.11}$$

B. The evolution equations

The final result reads

$$\begin{aligned}&\lambda^2\left[\frac{3}{2}(W^{cbd}{}_dV^a{}_{bce}-W^d{}_{dbc}V^{abc}{}_e)+(-W_{bcde}V^{abcd}-W^d{}_{ebc}V^{abc}{}_d+W^{cd}{}_{be}V^{ab}{}_{cd}+W^{bcd}{}_eV^a{}_{bcd})\right]\\&+\lambda^2\hat{a}_e^\dagger\left[\frac{3}{2}(W^{cbd}{}_dV^{ae}{}_{cb}-W^d{}_{dbc}V^{abce})+(W^{bcde}V^a{}_{bcd}+W^{cde}{}_bV^{ab}{}_{cd}-W^{de}{}_{bc}V^{abc}{}_d-W^e{}_{bcd}V^{abcd})\right].\end{aligned} \tag{3.12}$$

Note, the symmetry between the terms contracting $\hat{a}^e$ and $\hat{a}_e^\dagger$.

We can, therefore, write down the evolution equation for $a(\mathbf{x})=\langle\hat{a}(\mathbf{x})\rangle$ and $a^*(\mathbf{x})=\langle\hat{a}^\dagger(\mathbf{x})\rangle$:

$$\begin{aligned}&i\frac{\partial}{\partial t}a(\mathbf{x})-\int d\mathbf{x}'h(\mathbf{x},\mathbf{x}')a(\mathbf{x}')-\lambda\int dx[V(\mathbf{x},\mathbf{x}')a(\mathbf{x}')+V(\mathbf{x},-\mathbf{x}')a^*(\mathbf{x}')]\\&-i\lambda^2\int dx'[A(\mathbf{x},\mathbf{x}')a(\mathbf{x}')+A^*(\mathbf{x},-\mathbf{x}')a^*(\mathbf{x}')]+\lambda^2\int dx'[B(\mathbf{x},\mathbf{x}')a(\mathbf{x}')+B(\mathbf{x},-\mathbf{x}')a^*(\mathbf{x}')]=F_{a(\mathbf{x})}(t),\end{aligned} \tag{3.13}$$

where $h(\mathbf{x},\mathbf{x}')$ is given by Eq. (B2), $V(\mathbf{x},\mathbf{x}')$ (essentially, $V^{ac}{}_{cb}$) reads

$$V(\mathbf{x},\mathbf{x}')=\int\frac{dk_1}{(2\omega_{\mathbf{k}_1})^{1/2}}\frac{d\mathbf{k}_2}{(2\omega_{\mathbf{k}_2})^{1/2}}\frac{1}{2\omega_{(\mathbf{k}_1+\mathbf{k}_2)/2}}e^{-i\mathbf{k}_1\mathbf{x}+i\mathbf{k}_2\mathbf{x}'}, \tag{3.14}$$

while

$$A(\mathbf{x},\mathbf{x}')=\int\prod_{i=1}^4\frac{dk_i}{2\omega_{\mathbf{k}_i}}e^{-i\mathbf{k}_1(\mathbf{x}-\mathbf{x}')}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\Delta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4;t), \tag{3.15}$$

$$B(\mathbf{x},\mathbf{x}')=3\int\frac{dk_1}{(2\omega_{\mathbf{k}_1})^{1/2}}\frac{d\mathbf{k}_2}{(2\omega_{\mathbf{k}_2})^{1/2}}e^{i\mathbf{k}_1\mathbf{x}-i\mathbf{k}_2\mathbf{x}'}\left(\int\frac{d\mathbf{k}_1}{2\omega_{\mathbf{k}_3}}\frac{d\mathbf{k}_1}{2\omega_{\mathbf{k}_4}}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)E(\mathbf{k}_3,\mathbf{k}_4;t)\right). \tag{3.16}$$

Δ and E contain the time dependence of the kernels A and B and read

$$\Delta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4;t)=\int_0^t d\tau e^{-i\omega_{\mathbf{k}_1}\tau}(-e^{-i(\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4})\tau}-e^{-i(\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}-\omega_{\mathbf{k}_4})\tau}+e^{-i(\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3}-\omega_{\mathbf{k}_4})\tau}+e^{i(\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4})\tau}), \tag{3.17}$$

$$E(\mathbf{k},\mathbf{k}';t)=\frac{\cos(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t-1}{\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}}. \tag{3.18}$$

Note that for times $t\ll m^{-1}$ we have $\Delta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4;t)\approx t$.

A more transparent form is given when calculating the expectation values of creation and annihilation operators in momentum space

$$\begin{aligned}&\frac{\partial}{\partial t}a(\mathbf{k})+i\omega_{\mathbf{k}}a(\mathbf{k})+i\int dk'[V(\mathbf{k},\mathbf{k}')-\lambda B(\mathbf{k},\mathbf{k}')][a(\mathbf{k}')+a^*(\mathbf{k}')]\\&-\lambda^2[A(\mathbf{k};t)a(\mathbf{k})+A^*(\mathbf{k};t)a^*(\mathbf{k})]=-iF_{a(\mathbf{k})}(t),\end{aligned} \tag{3.19}$$

$$\frac{\partial}{\partial t}a^*(\mathbf{k})-i\omega_{\mathbf{k}}a^*(\mathbf{k})-i\int dk'[V(\mathbf{k},\mathbf{k}')-\lambda B(\mathbf{k},\mathbf{k}')][a(\mathbf{k}')+a^*(\mathbf{k}')]-\lambda^2[A(\mathbf{k},t)a(\mathbf{k})+A^*(\mathbf{k};t)a^*(\mathbf{k})]=-iF_{a^*(\mathbf{k})}(t), \tag{3.20}$$

with

$$A(\mathbf{k})=\frac{1}{2\omega_{\mathbf{k}}}\int \frac{dk_1}{2\omega_{\mathbf{k}_1}}\frac{dk_2}{2\omega_{\mathbf{k}_2}}\frac{1}{2\omega_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2}}\Delta(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2;t), \tag{3.21}$$

$$B(\mathbf{k},\mathbf{k}')=\frac{3}{16\omega_{\mathbf{k}+\mathbf{k}'}\omega_{\mathbf{k}'}}\int \frac{dk_1}{\omega_{\mathbf{k}_1}\omega_{\mathbf{k}+\mathbf{k}'+\mathbf{k}_1}}E(\mathbf{k}_1,\mathbf{k}+\mathbf{k}'+\mathbf{k}_1;t), \tag{3.22}$$

$$V(\mathbf{k},\mathbf{k}')=\frac{1}{4\omega_{\mathbf{k}'}\omega_{(\mathbf{k}+\mathbf{k}')/2}}. \tag{3.23}$$

C. Renormalization

The function $A(\mathbf{k};t)$ is actually divergent. We can perform a Taylor expansion of A around $k=0$ and verify that the term $A(0;t)$ is divergent, the terms containing first derivatives vanish while the ones containing the second order derivatives are finite. Hence, as could be expected, it is the zero modes of the field that give a divergent contribution. This can be removed by a redefinition:

$$A_{\mathrm{ren}}(\mathbf{k};t)=A(\mathbf{k};t)-A(0;t) \tag{3.24}$$

and by absorbing $A(0;t)$ in a field renormalization. To see this, note that

$$\frac{\partial}{\partial t}\begin{pmatrix} a(\mathbf{k}) \\ a^*(\mathbf{k})\end{pmatrix}=\text{finite terms} +\lambda^2\begin{pmatrix} A(0;t) & A^*(0;t) \\ A(0;t) & A^*(0;t)\end{pmatrix}\begin{pmatrix} a(\mathbf{k}) \\ a^*(\mathbf{k})\end{pmatrix}. \tag{3.25}$$

Hence, the divergencies can be absorbed through a redefinition of the Heisenberg picture operators $\hat{a}(k,t)$, $\hat{a}^\dagger(\mathbf{k},t)$;

$$\begin{pmatrix} \hat{a}(\mathbf{k},t) \\ \hat{a}^\dagger(\mathbf{k},t)\end{pmatrix}\to\exp\left[\lambda^2\int_0^t d\tau\begin{pmatrix} A(0;t) & A^*(0;t) \\ A(0;t) & A^*(0;t)\end{pmatrix}\right]\begin{pmatrix} \hat{a}(\mathbf{k},t) \\ \hat{a}^\dagger(\mathbf{k},t)\end{pmatrix}. \tag{3.26}$$

It is easy to interpret the terms in (3.19) and (3.20). The term V contains the lowest order contribution from the potential to our coarse-grained dynamics. Its form is better understood by observing that the mean-field theory approximation amounts to substituting four-point vertices (say with incoming momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ and outcoming $\mathbf{k}_3$ and $\mathbf{k}_4$) with free propagation of a mode with momentum the average of the incoming (or the outcoming) modes' momenta $(\mathbf{k}_1+\mathbf{k}_2)/2$.

The term B is higher order, time-dependent correction to the contribution of the potential, while the term A corresponds to dissipation. This is easily verified when we take the time reverse of Eqs. (3.19) and (3.20). The terms containing A are the only noninvariant terms.

D. The noise terms

Most important, from the point of view of the classical behavior and predictability of the mean field, is the noise term. As we said it is at least of first order to the coupling constant and in principle can dominate both the potential and the dissipation terms.

Starting from Eq. (2.11) it is straightforward to calculate the leading (first order to λ) contribution to the noise. It reads (we switch back to the index notation)

$$F_{a^a}(t)=\frac{\lambda}{4!}\,\mathrm{Tr}[\rho(0)\hat{A}(t)], \tag{3.27}$$

where

$$\begin{aligned}\hat{A}(t)=&\,4V^a{}_{bcd}\hat{a}^b(t)\hat{a}^c(t)\hat{a}^d(t)+12\hat{a}^\dagger_b(t)V^{ab}{}_{cd}\hat{a}^c(t)\hat{a}^d(t)\\ &+12\hat{a}^\dagger_b(t)\hat{a}^\dagger_c(t)V^{abc}{}_d\hat{a}^d(t)+4\hat{a}^\dagger_b(t)\hat{a}^\dagger_c(t)\hat{a}^\dagger_d(t)V^{abcd}\\ &-24V^{ac}{}_{cb}\hat{a}^b(t)-24\hat{a}^\dagger_bV^{abc}{}_c,\end{aligned} \tag{3.28}$$

where with $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ we denote the Heisenberg picture operators evolving according to the free Hamiltonian.

In order for our coarse-grained description to satisfy the predictability criterion, the noise term should be sufficiently weak (though strong enough to cause decoherence of the mean-field histories). This, as we see, cannot be true for a generic initial state of the system. We can nevertheless observe that the noise terms vanish when the initial state is the vacuum $\rho_{\mathrm{vac}}=|0\rangle\langle 0|$. This means that for states $\rho(0)$ sufficiently close to the vacuum the noise term becomes smaller and smaller. This means that for any state $\rho(0)$ such that $\|\rho(0)-\rho_{\mathrm{vac}}\|_{HS}<\epsilon$ the noise term will be of order $O(\epsilon)$.

Consider for instance that the initial state of the system is some coherent state $|\alpha(\mathbf{x})\rangle$, determined by a square-integrable function $\alpha(\mathbf{x})$. Coherent states are eigenstates of the annihilation operators, hence the trace in Eq. (2.11) is easily performed. Now, if we assume that $\|\alpha(\mathbf{x})\|<\epsilon$ it is easy to establish that $\|\rho(0)-\rho_{\mathrm{vac}}\|_{HS}=O(\epsilon)$. Hence to leading order in ϵ the noise term reads

$$F_{a^a}(t)=-\lambda\epsilon(V^{ac}{}_{cb}\zeta^b(t)+\zeta^*_b(t)V^{abc}{}_c), \tag{3.29}$$

where we wrote $\alpha(t)=\epsilon\zeta(t)$. This is an example of an initial condition that renders the noise term sufficiently weak to allow for predictability. This particular condition, we believe, is realistic when considering cosmological scenarios.

Finally, we should remark that it is straightforward to obtain evolution equations for the mean field and momentum by using the equations

$$\hat{a}(\mathbf{k})=\int dx e^{i\mathbf{kx}}(\omega_{\mathbf{k}}\hat{\phi}(\mathbf{x})+i\hat{\pi}(\mathbf{x})), \tag{3.30}$$

$$\hat{a}^{\dagger}(\mathbf{k})=\int dx e^{-i\mathbf{kx}}(\omega_{\mathbf{k}}\hat{\phi}(\mathbf{x})-i\hat{\pi}(\mathbf{x})). \tag{3.31}$$

IV. TWO-POINT FUNCTIONS IN $g\phi^3$ THEORY

In this section we are going to give the results for the truncation of the hierarchy at the level $n=2$ for a $g\phi^3$ scalar field theory.

A. The mean-field equations

For completeness we will give very briefly the results of the mean-field analysis for the $g\phi^3$ case. The expectation value of the operator $\hat{a}(\mathbf{k})$ evolves according to an equation similar to Eq. (3.19)

$$\frac{\partial}{\partial t}a(\mathbf{k})+i\omega_{\mathbf{k}}a(\mathbf{k})-ig^2\int B(\mathbf{k},\mathbf{k}')[a(\mathbf{k}')+a^*(\mathbf{k}')]-g^2[A(\mathbf{k};t)a(\mathbf{k})+A^*(\mathbf{k};t)a^*(\mathbf{k})]=F_{a(\mathbf{k})}(t), \tag{4.1}$$

where the functions A and B are given by

$$A(\mathbf{k};t)=\frac{3}{16\omega_{\mathbf{k}}}\int\frac{dk_1}{\omega_{\mathbf{k}_3+\mathbf{k}}\omega_{\mathbf{k}_3}}\Delta(\mathbf{k},\mathbf{k}+\mathbf{k}_3,\mathbf{k}_3;t), \tag{4.2}$$

$$B(\mathbf{k},\mathbf{k}';t)=\frac{1}{2\omega_{\mathbf{k}'}\omega_{\mathbf{k}+\mathbf{k}'}}\frac{\cos\omega_{\mathbf{k}+\mathbf{k}'}t-1}{\omega_{\mathbf{k}+\mathbf{k}'}}, \tag{4.3}$$

with

$$\Delta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t)=-2i\int_0^t e^{-i\omega_{\mathbf{k}_1}\tau}\sin(\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}). \tag{4.4}$$

As is well known, the potential does not contribute in the lowest order equation for the mean-field theory, and the quantities A and B again characterize dissipation and time-dependent correction to the potential.

B. Evolution equations for two-point functions

Let us now give the results for the case of truncation at the $n=2$ level. We prefer to give them in terms of the functions $G(\mathbf{k})$ and $Z(\mathbf{k})$ defined by

$$\langle\hat{a}(\mathbf{x})\hat{a}(\mathbf{x}')\rangle=\int\frac{dk}{2\omega_{\mathbf{k}}}e^{-i\mathbf{k}(\mathbf{x}+\mathbf{x}')}G(\mathbf{k}), \tag{4.5}$$

$$\langle\hat{a}^{\dagger}(\mathbf{x})\hat{a}(\mathbf{x}')\rangle=\int\frac{dk}{2\omega_{\mathbf{k}}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}Z(\mathbf{k}). \tag{4.6}$$

We will skip all calculations and present straightforwardly the results, since the way to proceed is exactly as previously and the only difficulty is a computational one. Thus, we get for a final result

$$\frac{\partial}{\partial t}a(\mathbf{k})+i\omega_{\mathbf{k}}a(\mathbf{k})=-iF_{a(\mathbf{k})}, \tag{4.7}$$

$$\frac{\partial}{\partial t}Z(\mathbf{k})-g\left(\frac{2}{\omega_{\mathbf{k}}}+\frac{1}{(\omega_{\mathbf{k}}\omega_{\mathbf{k}/2})^{1/2}}\right)[a(\mathbf{k})+a^*(\mathbf{k})]+ig^2\int dk'[r(\mathbf{k},\mathbf{k}';t)G(\mathbf{k}')+r^*(\mathbf{k},\mathbf{k}';t)G^*(\mathbf{k}')+s(\mathbf{k},\mathbf{k}';t)Z(\mathbf{k}')]+ig^2[D_1(\mathbf{k};t)G(\mathbf{k})+D_2(\mathbf{k};t)G^*(\mathbf{k})+D_3(\mathbf{k};t)Z(\mathbf{k})]=-iF_{Z(\mathbf{k})}(t), \tag{4.8}$$

$$\frac{\partial}{\partial t}G(\mathbf{k})+2i\omega_{\mathbf{k}}G(\mathbf{k})-g\left(\frac{2}{\omega_{\mathbf{k}}}+\frac{1}{(\omega_{\mathbf{k}}\omega_{\mathbf{k}/2})^{1/2}}\right)[a(\mathbf{k})+a^*(\mathbf{k})]+ig^2\int dk'[K_1(\mathbf{k},\mathbf{k}';t)G(\mathbf{k}')+K_2(\mathbf{k},\mathbf{k}';t)G^*(\mathbf{k}')+K_3(\mathbf{k},\mathbf{k}';t)Z(\mathbf{k}')]+ig^2[L_1(\mathbf{k};t)G(\mathbf{k})+L_2(\mathbf{k};t)Z(\mathbf{k})]=-iF_{G(\mathbf{k})}(t). \tag{4.9}$$

The form of the functions appearing in these equations can be found in Appendix C.

Now, in the equations for the mean field the noise can be shown to vanish, since

$$\mathbf{P}[V,a^a]=[V,a^a]. \tag{4.10}$$

Hence, the mean field evolves freely. In the second-order correlation functions the noise terms read

$$F_{a^a a^b}=g\ \mathrm{Tr}(\rho(0)\hat{A}^{ab}(t)), \tag{4.11}$$

$$F_{a_a^{\dagger}a^b}=g\ \mathrm{Tr}(\rho(0)\hat{A}_a^b(t)), \tag{4.12}$$

where

$$A^{ab}=-2\hat{a}_c^{\dagger}(t)V^{abc}-V_c^{ac}\hat{a}^b(t)-V_d^{ac}\delta_c^b\hat{a}^e(t), \tag{4.13}$$

$$A_a^b=V_{ca}^c\hat{a}^b-V_{ac}^b\hat{a}^c+\hat{a}_c^{\dagger}(t)V_a^{bc}-\hat{a}_a^{\dagger}(t)bc_c, \tag{4.14}$$

in terms of the Heisenberg picture operators evolving according to the free Hamiltonian.

This noise, terms more or less has the same features as the similar in Eq. (3.28).

V. CONCLUSIONS AND REMARKS

The techniques we have employed in this paper have given us a picture for the evolution of relevant variables, when the coarse-graining operation consists in the truncation of the Schwinger-Dyson hierarchy of n-point functions.

One of the great difficulties in such considerations is the complicated expressions we get for our equations in the end. It seems that it is very difficult to find a regime in a field theory where the dynamics would be Markovian. This essentially means that noise should be with good approximation ''white'' and in the autonomous part of the dynamics one should have no time-dependent coefficients. It seems unlikely that we can obtain Markovian evolution for a generic state of the system. In any case we should expect it when the field is in a state of partial (local) equilibrium [8]. This regime can still be studied using our techniques, but it might be that a different choice of coarse-graining projector might be more of use. The Kawasaki-Gunton and the Mori projector [7] might prove more convenient when dealing with this regime.

Another avenue to explore towards obtaining Markovian equations is to consider nonunitary dynamics for the evolution of the total system. This might come from a contact with a heat bath or through the interaction with other ignored degrees of freedom (a supermassive field or gravitons for the case of cosmology).

As far as the noise is concerned, we should stress that the Zwanzig method allows one to derive the noise term in the evolution equations solely from the knowledge of the initial state of the system. The comparison of its strength with the size of the terms entering the evolution equations offers a good criterion (though rather heuristic) for the classicalization of the variables under study. Remember, that noise should be strong enough to decohere but weak enough to allow for predictability and not covering up the effects of the potential. Only a particular class of initial states offers this possibility.

Finally, we should make some remarks concerning the classical domain in generic field theories. The techniques developed in this paper do provide a useful tool for dealing with the emergence of classical behavior. Still, it is my belief, that concrete understanding of the quantum to classical transition requires, in addition, employment of the conceptual technical tools of the decoherent histories approach to quantum mechanics. To obtain a complete and rigorous characterization of the classical domain (for instance [15–17]), one needs to construct the decoherence functional for coarse-grained correlation histories in a manageable computational form. This is currently under investigation.

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APPENDIX A: THE INDEX NOTATION

In the paper we have heavily used an index notation connected with the normal-ordered form of an operator, which we describe in detail here.

For reasons of symmetry in our expressions we prefer to work using the creation and annihilation operators in the configuration space instead of the momentum as is usual. Hence, we write $\hat{a}(x)$ and $\hat{a}^\dagger(x)$. They are related to the standard operators in momentum space by

$$\hat{a}(\mathbf{x})=\int \frac{dk}{(2\omega_{\mathbf{k}})^{1/2}}\, e^{-i\mathbf{kx}}\hat{a}(\mathbf{k}). \tag{A1}$$

We denote $\hat{a}(\mathbf{x})$ by $\hat{a}^a$ (index up) and $\hat{a}^\dagger(x)$ by $\hat{a}_a^\dagger$ (index down). To any function or distribution assign an abstract index to each of its arguments. The index is lower or upper according to whether the corresponding argument is integrated out with an $\hat{a}$ or an $\hat{a}^\dagger$, respectively. Hence the operator

$$\int dx_1 dx_2 dx_3 K(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_3)\hat{a}^\dagger(\mathbf{x}_1)\hat{a}^\dagger(\mathbf{x}_2)\hat{a}(\mathbf{x}_3) \tag{A2}$$

will be represented as

$$\hat{a}_a^\dagger \hat{a}_b^\dagger K^{ab}{}_c \hat{a}_c\,. \tag{A3}$$

We can easily verify that lowering a single index corresponds to changing the argument in the distribution from $\mathbf{x}$ to $-\mathbf{x}$, and inversion of all indices amounts to complex conjugation.

APPENDIX B: USEFUL FORMULAS

Here we list a number of expressions we make use of in the paper.

The free Hamiltonian can be written

$$\hat{H}_0=\frac{1}{2}\int dx dx'\,\hat{a}^\dagger(\mathbf{x})h(\mathbf{x},\mathbf{x}')\hat{a}(\mathbf{x}'), \tag{B1}$$

with

$$h(\mathbf{x},\mathbf{x}')=\int dk e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\omega_k\,. \tag{B2}$$

The evolution operator $\hat{U}_0(t)=e^{-it\hat{H}_0}$ reads

$$\hat{U}_0(t)=:\exp\left[\int dx\hat{a}^\dagger(\mathbf{x})[\Delta(\mathbf{x}-\mathbf{x}';t)-\delta(\mathbf{x}-\mathbf{x}')]\hat{a}(\mathbf{x}')\right]: \tag{B3}$$

where

$$\Delta(\mathbf{x}-\mathbf{x}';t)=\int dk e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}e^{-i\omega_{\mathbf{k}}t}. \tag{B4}$$

A coherent state is characterized by the square integrable function $\alpha(\mathbf{x})$ and is an eigenstate of the annihilation operator $\hat{a}(\mathbf{x})$. Under evolution of the free Hamiltonian we have

$$\hat{U}_0(t)|\alpha(\mathbf{x})\rangle=|\alpha(\mathbf{x},t)\rangle, \tag{B5}$$

where

$$\alpha(\mathbf{x},t)=\int dx'\,\Delta(\mathbf{x}-\mathbf{x}';t)\,\alpha(\mathbf{x}'). \tag{B6}$$

The operator

$$\hat{V}=:\int dx\,\frac{g}{3!}\,\hat{\phi}^3: \tag{B7}$$

reads in the index notation

$$V(\alpha^*,\alpha)=\frac{g}{3!}\,(V_{abc}\hat{a}^a\hat{a}^b\hat{a}^c+3\hat{a}_a^\dagger V^a{}_{bc}\hat{a}^b\hat{a}^c+3\hat{a}_a^\dagger\hat{a}_b^\dagger V^{ab}{}_c\hat{a}^c+\hat{a}_a^\dagger\hat{a}_b^\dagger\hat{a}_c^\dagger V^{abc}), \tag{B8}$$

with the correspondence

$$V_{abc}\rightsquigarrow\int\prod_{i=1}^{3}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{-i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \tag{B9}$$

$$V^a{}_{bc}\rightsquigarrow\int\prod_{i=1}^{3}\frac{dk_i}{(2\omega_{\mathbf{k}_i}^{1/2})}\,e^{-i(-\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \tag{B10}$$

$$V^{ab}{}_c\rightsquigarrow\int\prod_{i=1}^{3}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{-i(-\mathbf{k}_1\mathbf{x}_1-\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \tag{B11}$$

$$V_{abc}\rightsquigarrow\int\prod_{i=1}^{3}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \tag{B12}$$

while the operator

$$\hat{V}=:\int dx\,\frac{\lambda}{4!}\,\hat{\phi}^4: \tag{B13}$$

reads

$$\hat{V}=\frac{\lambda}{4!}\,(V_{abcd}\hat{a}^a\hat{a}^b\hat{a}^c\hat{a}^d+4\hat{a}_a^\dagger V^a{}_{bcd}\hat{a}^b\hat{a}^c\hat{a}^d+6\hat{a}_a^\dagger\hat{a}_b^\dagger V^{ab}{}_{cd}\hat{a}^c\hat{a}^d+4\hat{a}_a^\dagger\hat{a}_b^\dagger\hat{a}_c^\dagger V^{abc}{}_d\hat{a}^d+\hat{a}_a^\dagger\hat{a}_b^\dagger\hat{a}_c^\dagger\hat{a}_d^\dagger), \tag{B14}$$

with

$$V_{abcd}\rightsquigarrow\int\prod_{i=1}^{4}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{-i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \tag{B15}$$

$$V^a{}_{bcd}\rightsquigarrow\int\prod_{i=1}^{4}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{-i(-\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \tag{B16}$$

$$V^{ab}{}_{cd}\rightsquigarrow\int\prod_{i=1}^{4}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{-i(-\mathbf{k}_1\mathbf{x}_1-\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \tag{B17}$$

$$V^{abc}{}_d\rightsquigarrow\int\prod_{i=1}^{4}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{-i(-\mathbf{k}_1\mathbf{x}_1-\mathbf{k}_2\mathbf{x}_2-\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \tag{B18}$$

$$V^{abcd}\rightsquigarrow\int\prod_{i=1}^{4}\frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}}\,e^{i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3+\mathbf{k}_4\mathbf{x}_4)}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4). \tag{B19}$$

APPENDIX C: THE COEFFICIENTS IN EQS. (4.8) AND (4.9)

Here we give the expressions for the coefficients in Eqs. (4.8) and (4.9)

$$r(\mathbf{k},\mathbf{k}';t)=\frac{3}{2}\frac{1}{\omega_{\mathbf{k}+\mathbf{k}'}\omega_{\mathbf{k}'}^2}\int_0^t d\tau e^{-i\omega_{\mathbf{k}'}\tau}\sin(\omega_{\mathbf{k}}+\omega_{\mathbf{k}+\mathbf{k}'})\tau-\frac{1}{4}\frac{1}{\omega_{\mathbf{k}'}^2\omega_{\mathbf{k}/2}}\frac{\cos\omega_{\mathbf{k}}t-1}{\omega_{\mathbf{k}}}, \tag{C1}$$

$$s(\mathbf{k},\mathbf{k}';t)=\frac{3}{2}\frac{1}{\omega_{\mathbf{k}+\mathbf{k}'}\omega_{\mathbf{k}'}^2}\int_0^t d\tau \cos\omega_{\mathbf{k}'}\tau \sin(\omega_{\mathbf{k}}+\omega_{\mathbf{k}+\mathbf{k}'})\tau-\frac{1}{2}\frac{1}{\omega_{\mathbf{k}'}^2\omega_{\mathbf{k}/2}}\frac{\cos\omega_{\mathbf{k}}t-1}{\omega_{\mathbf{k}}}, \tag{C2}$$

$$D_1(\mathbf{k};t)=3u(\mathbf{k};t)-2u'(\mathbf{k};t), \tag{C3}$$

$$D_2(\mathbf{k};t)=-3u^*(\mathbf{k};t)-2u'(\mathbf{k};t), \tag{C4}$$

$$D_3(\mathbf{k};t)=3[u(\mathbf{k};t)+u^*(\mathbf{k};t)]-4u'(\mathbf{k},t), \tag{C5}$$

$$u(\mathbf{k};t)=\frac{1}{4}\frac{1}{\omega_{\mathbf{k}}}\int\frac{dk_1}{\omega_{\mathbf{k}_1}\omega_{\mathbf{k}+\mathbf{k}_2}}\int_0^t d\tau e^{-i\omega_{\mathbf{k}}\tau}\sin(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}+\mathbf{k}_1})\tau, \tag{C6}$$

$$u'(\mathbf{k};t)=\frac{1}{2}\frac{1}{\omega_{\mathbf{k}}^3\omega_{\mathbf{k}/2}}(\cos\omega_{\mathbf{k}}t-1), \tag{C7}$$

$$K_1(bf\mathbf{k},\mathbf{k}';t)=\frac{3}{4}\frac{1}{\omega_{\mathbf{k}+\mathbf{k}'}\omega_{\mathbf{k}'}^2}\frac{\cos(\omega_{\mathbf{k}}+\omega_{\mathbf{k}+\mathbf{k}'}-\omega_{\mathbf{k}'})t-1}{\omega_{\mathbf{k}}+\omega_{\mathbf{k}+\mathbf{k}'}-\omega_{\mathbf{k}'}}-\frac{1}{2}\frac{1}{\omega_{\mathbf{k}'}^2\omega_{\mathbf{k}/2}}\frac{e^{-i\omega_{\mathbf{k}}t}-1}{\omega_{\mathbf{k}}}, \tag{C8}$$

$$K_2(\mathbf{k},\mathbf{k}';t)=\frac{3}{2}\frac{1}{\omega_{\mathbf{k}+\mathbf{k}'}\omega_{\mathbf{k}'}^2}\int_0^t e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})\tau\cos(\omega_{\mathbf{k}+\mathbf{k}'}\tau)}-\frac{1}{2}\frac{1}{\omega_{\mathbf{k}'}^2\omega_{\mathbf{k}/2}}\frac{e^{-i\omega_{\mathbf{k}}t}-1}{\omega_{\mathbf{k}}}, \tag{C9}$$

$$K_3(\mathbf{k},\mathbf{k}';t)=\frac{1}{2}\frac{1}{\omega_{\mathbf{k}+\mathbf{k}'}\omega_{\mathbf{k}'}^2}\frac{\cos(\omega_{\mathbf{k}}+\omega_{\mathbf{k}+\mathbf{k}'}+\omega_{\mathbf{k}'})t-1}{\omega_{\mathbf{k}}+\omega_{\mathbf{k}+\mathbf{k}'}+\omega_{\mathbf{k}'}}, \tag{C10}$$

$$L_1(\mathbf{k};t)=-\frac{3i}{4}\frac{1}{\omega_{\mathbf{k}}}\int\frac{d\mathbf{k}_1}{\omega_{\mathbf{k}_1}\omega_{\mathbf{k}+\mathbf{k}_2}}\int_0^t d\tau e^{-i\omega_{\mathbf{k}}\tau}\cos(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}+\mathbf{k}_1})\tau-\frac{1}{\omega_{\mathbf{k}'}^2\omega_{\mathbf{k}/2}\omega_{\mathbf{k}}}(\cos\omega_{\mathbf{k}}t-1), \tag{C11}$$

$$L_2(\mathbf{k};t)=-\frac{1}{\omega_{\mathbf{k}}^3\omega_{\mathbf{k}/2}}(\cos\omega_{\mathbf{k}}t-1)+\frac{3}{4}\frac{1}{\omega_{\mathbf{k}}}\int\frac{d\mathbf{k}_1}{\omega_{\mathbf{k}_1}\omega_{\mathbf{k}+\mathbf{k}_1}}\int_0^t d\tau e^{i\omega_{\mathbf{k}}\tau}\sin\omega_{\mathbf{k}_2+\mathbf{k}}\tau. \tag{C12}$$

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