Fine structure of Choptuik's mass-scaling relation

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(Received 28 June 1996)

We conjecture (analytically) and demonstrate (numerically) the existence of a fine structure above the power-law behavior of the mass of black holes that form in the gravitational collapse of a spherical massless scalar field. The fine structure is a periodic function of the critical separation $(p-p^*)$. We predict that the period ϖ is *universal* and that it depends on the previous universal parameters, the critical exponent β, and the echoing period Δ as $\varpi=\Delta/(2\beta)$. [S0556-2821(97)50202-5]

PACS number(s): 04.70.Bw

I. INTRODUCTION

The gravitational collapse of a spherically symmetric massless scalar field has two possible end states. Either the scalar field dissipates away, leaving a flat spacetime, or a black hole form. Numerical simulations of this model problem [1] have revealed an unexpected critical behavior when the initial conditions are close to the critical case $p=p^*$ (p is some parameter which characterizes the strength of the initial scalar field, and p^* is the threshold value). More precisely, Choptuik has found a power-law dependence of the black-hole mass on critical separation $(p-p^*)$ of the form $M_{\rm BH}\propto(p-p^*)^\beta$ for $p>p^*$, and a discrete echoing with a period Δ (a discrete self-similar behavior) for $p=p^*$.

Subsequently, similar critical behavior has been observed for other collapsing fields: axisymmetric gravitational wave packets [2], spherically symmetric radiative fluids [3], and charged complex scalar fields [4]. In all these model problems the critical exponent β turned out to be close to the value originally found by Choptuik, 0.37, suggesting a universal behavior. However, Maison [5] has shown that for fluid collapse models with an equation of state given by $p=k\rho$, the critical exponent strongly depends on the parameter k.

In this work we conjecture the existence of a small periodic correction, $\Psi[\ln(p-p^*)]$, to the power-law dependence of the black-hole mass [6]. Ψ is periodic and its period ϖ is universal and it depends on the previous universal parameters as $\varpi=\Delta/(2\beta)$. Our analytical argument predicts the existence of the fine-structure periodic term and its expected period. The argument is based upon the final stage of a supercritical evolution: from the moment when the deviation from the exact self-similar critical evolution becomes larger than some given value (and the evolution is no longer self-similar) up to the horizon formation. We then provide numerical evidence that verifies the existence of the conjectured periodic term, the universality of its period and the relation $\varpi=\Delta/(2\beta)$.

Our numerical formalism is based on the characteristic scheme of Goldwirth and Piran [7], to which we have added an expansion near the origin which is essential to achieve the extremely high accuracy needed for these computations. The evolution equations, our algorithm and numerical methods, and the discretization and error analysis are all described in a previous paper [4], and will not be repeated here.

II. THEORETICAL PREDICTIONS VS NUMERICAL RESULTS

We consider the spherical collapse of a massless scalar field. Choptuik has shown that for a critical parameter p^* there is a critical solution which has an infinite discrete self-similar behavior. The scalar field oscillates with a period Δ. The physical quantities, and in particular M/r, depend quadratically on the field's derivatives and hence have a period $\Delta/2$, i.e., there are two physical echoes for each echo in the scalar field. In the following we will be interested only in these physical echoes.

The critical solution by itself does not yield the scaling relation of the black-hole mass in which we are interested. We perturb, therefore, the critical initial conditions. This leads to a dynamical instability—a growing deviation from the critical evolution toward either a subcritical dissipation or a supercritical black-hole formation.

Let $f(u)$ be a function of u, the time coordinate of an observer at rest at the origin, that characterize the solution along the outgoing null geodesic that leaves the origin at u. The function, f, could be, for example, the maximal value of $M(u,r)/r$ along this geodesic. Following Evans and Coleman [3] and Maison [5] we describe the runaway of the perturbed solution from the critical evolution (described by f_c) as a power law

$$f(u)-f_c(u)=\lambda(u^*-u)^{-\alpha}, \tag{1}$$

where the critical solution reaches the zero-mass singularity at $u=u^*$. The prefactor λ satisfies $\lambda\propto(p-p^*)$.

We assume that the range of validity of the perturbation theory is restricted to some maximal deviation, χ, from the exact critical evolution; i.e., the evolution is approximately self-similar until $|f-f_c|=\chi$. From here on, the evolution is outside the scope of the perturbation theory—there is subcritical dissipation of the field or supercritical black-hole formation. In either case, the evolution from this stage onwards loses its self-similar character. We choose now $p>p^*$ so that the perturbed initial conditions develop into a black hole. The time $u_\chi(p-p^*)$ required in order to reach the maximal deviation is given simply by the relation

$$\lambda(u^*-u_\chi)^{-\alpha}=\chi. \tag{2}$$

Of course, a larger initial perturbation requires a shorter time to reach this value. We define now the logarithmic time, $T \equiv -\ln[(u^*-u)/u^*]$, in which the critical solution is periodic. The logarithmic time T_χ, which corresponds to the loss of self-similarity, is given by

$$T_\chi = -\alpha^{-1}\ln(p-p^*) + b_k, \tag{3}$$

where b_k depends on χ, u^*, and k. The index k denotes the family of initial conditions considered. We conjecture now that the logarithmic time until the horizon formation equals T_χ plus a periodic term $F[\ln(p-p^*)]$, or

$$T_{\rm BH} = -\alpha^{-1}\ln(p-p^*) + b_k + F[\ln(p-p^*)]. \tag{4}$$

The period, ϖ, is universal and it depends on the previous universal parameters according to

$$\varpi = \alpha\Delta/2. \tag{5}$$

Consider now two different initial conditions, which lead to n and $n+1$ echoes, respectively (until the deviation from the critical self-similar evolution reaches χ). These solutions are related to each other by an exact scale transformation with a factor $e^{(\Delta/2)}$. The final stages of these two supercritical evolutions, from the stage when the deviation from the exact critical evolution reaches χ (and the evolution ceases to be periodic in T), up to the horizon formation, are equal up to a scaling transformation. The periodic nature of the function F arises from this final stage: The period of the function F is the amount that should be added to the quantity $\ln(p-p^*)$, in order to reduce the number of echoes by one. This will reduce $T_{\rm BH}$ by $\Delta/2$. From Eq. (4) this amounts to a period $\varpi = \alpha\Delta/2$ in F.

This conjecture is verified by numerical simulations of four families of initial data (two neutral and two charged). In all those families we have found that $-T_{\rm BH}$ as a function of $\ln(p-p^*)$ was well fit by a straight line with a slope $1/\alpha \approx 0.37$. On top of this straight line there was a small modulation. The deviation from a straight line is shown in Fig. 1, which provides numerical evidence for the existence of the periodic term F in Eq. (4). We see that the function F is indeed periodic, with a universal period $\varpi = \alpha\Delta/2 \approx 4.6$.

We have to relate now the exponent α to the critical exponent β (which describes the power-law dependence of the black-hole mass), and then generalize Choptuik's scaling relation by proving that one should also add a periodic term to Choptuik's mass scaling relation. We write $T_{\rm BH}$ in the form

$$T_{\rm BH} = T_{\rm init} + n\Delta + F, \tag{6}$$

where $T_{\rm init}$ is the initial logarithmic time required for the system to settle down to a periodic behavior in T, and n is the number of echoes. We assume that $T_{\rm init}$ is independent of $(p-p^*)$. Using Eq. (4) we obtain

$$n\Delta = -\alpha^{-1}\ln(p-p^*) + d_k, \tag{7}$$

where d_k is a family-dependent constant. We define $M^{(n)}$ as the mass after n echoes (note that this is not the final black-hole mass). Since M decreases in each echo by a factor $e^{-\Delta}$ we have, using Eq. (7),

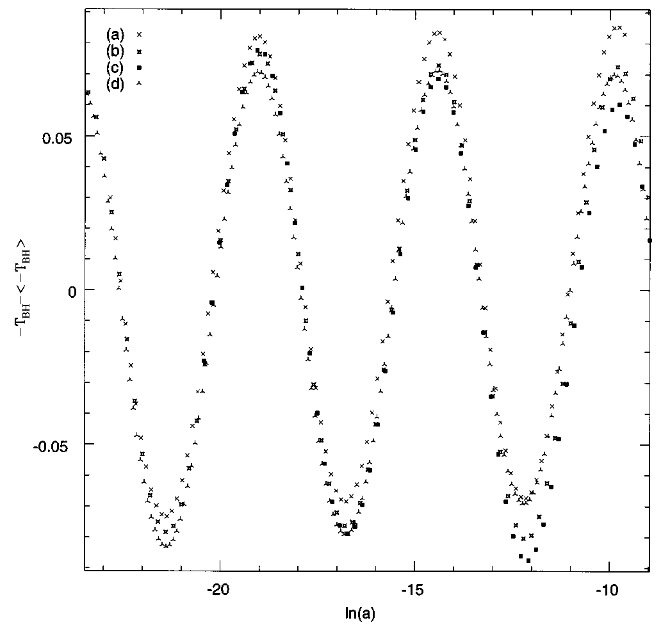

FIG. 1. Illustration of the conjectured *universal periodic fine structure* of $-T_{\rm BH}$. The quantity $[-T_{\rm BH} - \langle -T_{\rm BH} \rangle]$ is plotted as a function of ln(a), where $a \equiv (p-p^*)/p^*$, for the four families. The curves were shifted horizontally (but not vertically) in order to overlap the first oscillation of each family with the first one of family (a). $\langle T_{\rm BH} \rangle$ is the value of $T_{\rm BH}$ determined from a straight line approximation, i.e., $\langle T_{\rm BH} \rangle = {\rm const} + \beta\ln(p-p^*)$. The numerical results agree with the *predicted* relation $\varpi = \alpha\Delta/2 \approx 4.6$.

$$M^{(n)} = M^{(0)}e^{-n\Delta} = M^{(0)}e^{-d_k}(p-p^*)^\beta, \tag{8}$$

from which it follows that $\beta = 1/\alpha$.

To obtain $M_{\rm BH}$, the final black-hole mass, one should multiply $M^{(n)}$ by a periodic function $G[\ln(p-p^*)]$ which measures the change of mass, from the stage when the evolution is no longer periodic in T, until the horizon forms. The function G depends only on the field configuration at the moment when the deviation from the exact self-similar evolution reaches χ (and the evolution is no longer self-similar). Thus, G is expected to have the same value each time the system completes another echo, i.e., each time n increases by unity. Using Eq. (6) we find that the function $G[\ln(p-p^*)]$ is expected to have a period of $\varpi = \Delta/(2\beta)$. Thus, we obtain

$$\ln(M_{\rm BH}) = \beta\ln(p-p^*) + c_k + \Psi[\ln(p-p^*)], \tag{9}$$

where c_k is a family-dependent constant and $\Psi[\ln(p-p^*)]$ is a periodic function with a *universal* period, ϖ.

Figure 2 depicts this periodic fine structure for our four families of solutions mentioned earlier. In all four families we obtain the basic power law behavior with $\beta \approx 0.37$. Figure 2 displays the deviation of $\ln(M_{\rm BH})$ from this straight line as a function of $\ln(p-p^*)$. The agreement between the four families shows that the fine structure is indeed universal with the expected period. The periodic functions F and Ψ are universal in shape. However, there is a family-dependent horizontal offset (which depends on β and on the previous family-dependent constants) in these functions; see also [8].

One may worry, of course, whether this fine structure is real or if it could arise from some numerical errors. In a

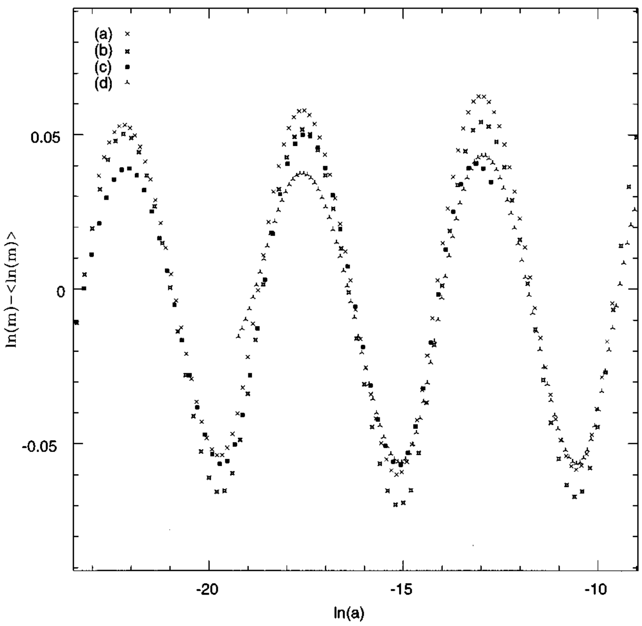

FIG. 2. Illustration of the conjectured *universal periodic fine-structure* generalization of Choptuik's mass-scaling relation. $\ln(m)-\langle\ln(m)\rangle$ is plotted as a function of $\ln(a)$ for the four families, where $m\equiv M_{\rm BH}/M_{i,c}$ is the normalized black-hole mass in units of the initial mass in the critical solution $M_{i,c}$. $\langle\ln(m)\rangle$ is the value of $\ln(m)$ determined from a straight line approximation. The curves were shifted horizontally (but not vertically) in order to overlap the first oscillation of each family with the first one of family (a). The numerical results agree with the *predicted* relation $\varpi=\Delta/(2\beta)\approx 4.6$.

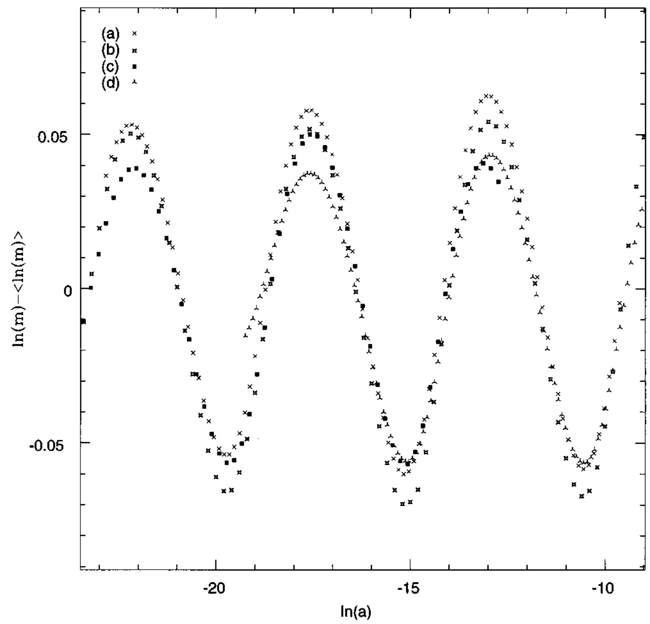

FIG. 3. $\ln(m)-\langle\ln(m)\rangle$ is plotted as a function of $\ln(a)$ for family (a), and with it, five different resolution grids with 100, 200, 400, 800, and 1600 grid points. The five curves overlap and all show the same periodic behavior.

previous paper [4] we have established the stability and convergence of our code with numerous tests. Still, because of the importance of this issue, we demonstrate here the physical character of this fine structure. Figure 3 depicts the deviations of $\ln(M_{\rm BH})$ from a straight line as a function of $\ln(p-p^*)$ for five different calculations with 100, 200, 400, 800, and 1600 grid points for the same initial data. The same features were found on all grids even though the grid sizes differ by a factor of 16. The five curves overlap and all show the same periodic behavior. Numerical convergence (the 800 and 1600 curves are nearer than the 100 and 200 curves, for example) is clearly seen.

III. SUMMARY AND CONCLUSIONS

We have studied the spherical gravitational collapse of a massless scalar-field, both for the uncharged case and for the charged configurations. Our main interest was the supercritical $(p>p^*)$ feature of Choptuik's solution, i.e., the power-law dependence of the black-hole mass on the critical separation. We have shown the existence of a fine-structure above this power-law dependence in the form of a periodic term with a universal period, ϖ. We are not aware of such a fine-structure periodic term in any other phase transitions in statistical mechanics. Our periodic term with its period strongly depends on the discrete echoing character of the critical solution. This discrete self-similarity has been seen only in the works of Choptuik [1], Abrahams and Evans [2] (collapse of axisymmetric vacuum gravitational field), and in our work concerning the gravitational collapse of a charged (complex) scalar field [4]. Abrahams and Evans [2] have found an echoing period of $\Delta\approx 0.6$. Thus, using our analytical argument, we conjecture that a careful analysis will reveal a periodic fine structure (to the power-law behavior), with a period of $\varpi\approx 0.8$, in the model problem of the collapse of axisymmetric gravitational wave packets.

We have learned, after submission, that Gundlach [8] has shown independently that such a periodicity should exist.

ACKNOWLEDGMENTS

We thank Amos Ori for helpful discussions. This research was supported by a grant from the US-Israel BSF and a grant from the Israeli Ministry of Science.

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