Quantum fields in disequilibrium: Squeezed states

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A spacetime description of squeezed states in quantum fields is presented, revealing the connection between squeezing and nonequilibrium dynamics. Squeezings in configuration space, occupation number space, and phase space are distinguished; generating transformations and criteria for their physical realization are discussed. The results have an immediate applicability to atoms and ions in traps, as well as to quantum optics in relativistic and nonequilibrium systems. Squeezing, sub-Poissonian statistics, and antibunching are all shown to be a direct consequence of spacetime inhomogeneities in the quantum field. The finite speed of communication between separate regions of the field (finite speed of light) places a lower limit on the attainable spectral width of squeezed states. The squeezing parameter for field quadratures has the appearance of a chemical potential in an inhomogeneous field, and through a renormalization may be generated by a Chern-Simons-like term. [S0556-2821(97)04902-3]

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I. INTRODUCTION

Demonstrations of the field as a truly quantum-mechanical entity have come most convincingly from quantum optics in the last two decades [1]. Light, or the photon-radiation field, exhibits several phenomena which admit a complete description only in quantum, many-particle theory. Examples include squeezed states, sub-Poissonian statistics, and antibunching [2]. The micromaser has proven to be an invaluable testing ground for these phenomena, since individual atoms may effectively be used as measurement probes of the cavity field. Squeezing, in general, concerns the reduction of fluctuations (noise) in a canonical pair $\mathbf{p},\mathbf{q}$, such that their uncertainty product is minimized *and* so that one of the pair attains a value which is lower than the value for coherent states. Since the quantum field possesses several position-momentum pairs, this property may arise in both the single and many-particle theories. Antibunching signals an anticorrelation of atoms in the sub-Poissonian regime so that squeezing is enhanced for events arriving at broadly separated intervals from the source [2]. In this paper it is shown that these phenomena can be understood as features of a nonequilibrium quantum field theory.

In a previous paper [3] (hereafter referred to as paper I), the problem of inhomogeneous quantum fields was discussed by introducing a covariant method based on the Schwinger action principle and the various Green functions of the field. There it was only hinted as to the possible applications of such a formalism and the connection with ''canonical'' field theory was left implicit. Since one of the key areas of interest for nonequilibrium field theory and inhomogeneous systems is nonlinear, quantum optics, it is important to bridge the gap between old-style canonical formalism and the more modern approach in terms of Green functions. Moreover, the connection between squeezing and nonequilibrium field theory does not seem to have been appreciated in the literature.

Single-particle modes in field theory are an idealization which does not convincingly describe interacting, finite systems away from equilibrium. What we are really interested in, in quantum optical systems, is the observable properties of the field as an entity, divorced from speculations as to the microscopic details which give rise to the observables. One of the principal aims of this paper is to eliminate single-particle momentum-space creation and annihilation operators, with definite momentum, in favor of more general Green functions for the field. This should lead to a more generally applicable theory and also a new calculational scheme which makes contact with quantum electrodynamics and all of the accumulated wisdom therein. To do so it is helpful to first establish a more direct correspondence between the canonical approach of plane wave expansions and creation-annihilation operators, and the covariant operator approach [4] in a way which is conducive to generalization [3]. Finally, the finite speed of light is also an issue which has been previously ignored and an appreciation of its consequences is forthcoming. One cannot exclude the possibility that extreme astrophysical quantum-light sources would require a fully relativistically covariant analysis [5].

It was shown in Ref. [6] that quadrature-squeezed states are related to the coherent states (a vacuum or quasivacuum in the presence of a source) by a unitary transformation in the space of the relevant canonical pair $\mathbf{p},\mathbf{q}$. The case most often considered in connection with sub-Poissonian statistics is that of number squeezing in creation-annihilation operator (occupation number) space. There one finds that an effective action or Hamiltonian for stable-squeezed states involves the combinations $a^\dagger a$, a^2, and $(a^\dagger)^2$. Converting these Fourier transformed expressions into configuration space, it is found that the latter two terms correspond to interactions of the form $\Phi(x)\Phi(-x)$, for all x, which appears to be unphysical: an interaction between t and $-t$ violates causality and has no invariant meaning (since it singles out a special origin); similarly, an interaction between two points $\mathbf{x}$ and $\mathbf{x}'$ singles out a special point on each spacelike cross section of the field and implies a notion of absolute simultaneity which is forbidden in special relativity, owing to the finite speed of light. It is, therefore clear that terms such as a^2 and $(a^\dagger)^2$ are idealizations valid over small aparati. Moreover, the identification of a special origin in spacetime is a highly specialized case, the true meaning of the selection of a special point is that one requires a notion of *inhomogeneity* or loss of translational invariance in space and/or time so that at any arbitrary position one can discuss a distribution of events around that position. This reasoning makes the formalism of paper I immediately interesting.

A study of inhomogeneous systems, avoiding free-field perturbation theory, requires a theory of nonlinear, nonlocal fields with complex dispersion relations. (The formal apparatus required to begin a discussion of such systems requires a certain initial patience on the part of the reader, but the benefits of the fully covariant approach are well worth the effort. This paper will also serve to clarify the relationship to more traditional canonical field theory.) Such a theory was discussed in paper I for a single-component bosonic field. A real space description of squeezing was previously given in Ref. [7], but these authors neglect the possibility of nonlinear effects such as hysteresis and do not explore the dynamical aspect presented in this paper.

Clearly the radiation field is a vector quantity which deserves a fully covariant treatment; in keeping with the literature, we shall consider mostly a single polarization (a real scalar field) in the present work, and a full vector field theory will be given elsewhere. A brief summary of the radiation field is given in the final section. The present results are not especially confined to the radiation field, but are confined to any bosonic field, and the straightforward generalization to fermions can be expected to lead to an improved description of relativistic system such as the free electron laser [8]. Recent experiments on trapped ions may also be modeled by a real bosonic field interacting with a harmonic potential [9].

The outline of this paper is as follows. In Sec. II, an approximate implementation of a creation-annihilation formalism for inhomogeneous field theory in slowly varying external potentials is presented. The aim of this section is mainly to serve as a bridge between the general spacetime approach of paper I and methods which are in widespread use. Section III presents a clarification of the notions of momentum and position for single-particle excitations of inhomogeneous quantum fields, and motivates the introduction of a phase-space connection. Section IV discusses the issue of squeezing of the Fourier decomposition of a quantum field in an inhomogeneous background. This is often the case which is more closely associated with experiments. The message of this analysis is that monochromatic squeezing is a physical impossibility owing to the finiteness of systems, either due to boundary conditions or ultimately due to the finite speed of communication signals imposed by special relativity. Section V shows how the foregoing results can be obtained in a spacetime formulation, avoiding the approximate notions of Sec. II and finds the width of the spectral distribution for squeezed states. The time evolution of squeezed states, viewed as a nonequilibrium quantum field, is presented for both the real scalar field (neutral atoms) and in Sec. VI for the electromagnetic field. Finally, some order of magnitude estimates of the importance of the corrections described in this paper are presented in Sec. VII.

II. OSCILLATOR STATES

As a first step towards a nonlocal, inhomogeneous theory in which a and $a^\dagger$ are absent, one must establish the meaning of a number of conventional relations in terms of inhomogeneous, nonlocal oscillator states. These are a first order perturbation of the pure momentum-space objects, and serve as a temporary bridge between the canonical creation-annihilation formalism and the fully covariant inhomogeneous case. We begin, therefore, by retracing some features of the discussion for the bosonic Klein-Gordon field in paper I in terms of creation and annihilation operators for a field satisfying a complex dispersion relation.

The real scalar field $\Phi(x)$ satisfies an equation of motion of the form

$$(-\Box + A^\mu \partial_\mu + B)\Phi(x) + \int dV_{x'} C(x,x')\Phi(x') = 0 \quad (1)$$

for some real source terms A,B,C. The sources may be thought of as a phenomenological interaction with matter, the details of which are to be specified for each model. An inhomogeneous system is characterized by a $C(x,x')$ which is not translationally invariant, i.e., $C = C(x-x', \underline{x+x'})$ the source conceals a dependence on an average position $\underline{x}$ with respect to some origin. The following notation is used for symmetric and antisymmetric objects, and all conventions follow those in paper I:

$$\overline{x} = \frac{1}{2}(x+x'),$$

$$\tilde{x} = \frac{1}{2}(x-x'). \quad (2)$$

The translational noninvariance of the source C implies that the field is a nonlocal object, depending not only on x but on neighboring positions within a radius which is governed by the behavior of $C(x,x')$. In such a system, a conventional momentum-space expansion is not possible. However, locally, about a point $\overline{x}$, one may establish momentum coordinates through the partial Fourier transform.

The scalar product of any two functions of only positive frequency is given by [10]

$$(f_\alpha, f_\beta) = \int d\sigma^\mu [f_\alpha^\dagger i(\partial_\mu f_\beta) - i(\partial_\mu f_\alpha^\dagger) f_\beta], \quad (3)$$

where

$$f_\alpha(x) = \int \frac{d^n k}{(2\pi)^n} \theta(k_0)\, \delta(-k_0^2 + \omega^2) e^{ikx} \tilde{f}_\alpha(k) \quad (4)$$

which satifies the dispersion relation, $-k_0^2 + \omega^2 = 0$. The mode decomposition of a real scalar field $\Phi(x)$ may be written

$$\Phi(x) = \int \frac{d^n k}{(2\pi)^n} \theta(k_0)\, \delta(-k_0^2 + \omega^2) \times [a(\mathbf{k},x) e^{ikx} + a^\dagger(\mathbf{k},x) e^{-ikx}] \quad (5)$$

$$= \int \frac{(dk)}{2|\omega|} [a(\mathbf{k},x) e^{ikx} + a^\dagger(\mathbf{k},x) e^{-ikx}], \quad (6)$$

where $(dk)\equiv d^{n-1}k/(2\pi)^{n-1}$ and $\exp(ikx)$ is a short hand for $\exp[i(\mathbf{k}\mathbf{x}-\omega t)]$. In the limit of plane waves, the creation and annihilation operators $a^\dagger,a$ are independent of x. One may envisage their x dependence as a measure of the inappropriateness of a plane-wave expansion of the field. The reality of the field is secured by the relation $a^\dagger(\mathbf{k},x)=a(-\mathbf{k},x)$, provided the frequencies ω are real. If ω is a complex quantity, as is the case for systems off equilibrium, then one must write

$$\Phi(x)=\int (dk)\left\{\frac{a(\mathbf{k},x)}{2\omega(\mathbf{k},x)}e^{ikx}+\frac{a^\dagger(\mathbf{k},x)}{2\omega^*(\mathbf{k},x)}e^{-ikx}\right\}, \quad (7)$$

so that the reality of the field requires

$$\left(\frac{a(\mathbf{k},x)}{2\omega(\mathbf{k},x)}\right)^*=\left(\frac{a(-\mathbf{k},x)}{2\omega(-\mathbf{k},\mathbf{x})}\right). \quad (8)$$

The two terms of positive and negative frequency in Eq. (4) may also be denoted by $\Phi(x)=\Phi^{(+)}+\Phi^{(-)}$ indicating the annihilation and creation parts in real space. Their orthogonality properties can be investigated by computing $(\Phi^{(+)},\Phi^{(+)})$ and $(\Phi^{(+)},\Phi^{(-)})$:

$$(\Phi^{(-)},\Phi^{(+)})=-i\int d^{n-1}\mathbf{x}(dk)(dp)\times\left(\frac{a(\mathbf{p},x)}{2\omega(\mathbf{p},x)}e^{ipx}\overleftrightarrow{\partial_t}\frac{a(\mathbf{k},x)}{2\omega(\mathbf{k},x)}\right)e^{ikx}. \quad (9)$$

In the usual case where A and ω are independent of x, it is possible to integrate $\exp[i(\mathbf{p}+\mathbf{k})\mathbf{x}]$ to obtain a δ function $\delta(\mathbf{p}+\mathbf{k})$. This secures the immediate vanishing of Eq. (9), which indicates the mutual orthogonality of the positive and negative frequency parts of the field. However, we are interested in nontrivial x dependence, and it is clear that the integral over $\mathbf{x}$ is not a δ function if the exponential is modulated by an $\mathbf{x}$-dependent coefficient. In the case above, this makes no difference since the integral vanishes by the symmetry of the factors. However, this issue raises its head at several stages in the discussion and it is useful to deal with it now. There are three possibilities for inhomogeneous systems: (i) the $\mathbf{x}$ dependence of $a(\mathbf{k},x)$ is so slow that it can be ignored so that one still obtains a δ function by approximation, (ii) a and $a^\dagger$ do not depend on the point x but on a neighboring point, nonlocally detached from x; this also results in a δ function, assuming the $\mathbf{x}$ dependence is ‘‘slow enough,’’ and (iii) the x dependence is strong and cannot be ignored.

The second of these points is somewhat vaguely stated. Its precise meaning acquires a firmer footing in the Green function approach of paper I, in which one works entirely in terms of two-point functions. One writes

$$\Phi(x)=\int (dk)\left\{\frac{a(\mathbf{k},\overline{x})}{2\omega(\mathbf{k},\overline{x})}e^{ikx}+\frac{a^\dagger(\mathbf{k},\overline{x})}{2\omega^*(\mathbf{k},\overline{x})}e^{-ikx}\right\}, \quad (10)$$

which compares to Eq. (7). In the third, most general case, the $\mathbf{x}$ dependence is not ignorable and a modification of the δ-function result is inevitable. In this case one must ask what sort of $\mathbf{x}$ dependence is physically acceptable. The δ-function property arises in the even part of the small $\mathbf{k}$ expansion of the exponential

$$\int \frac{d^{n-1}\mathbf{x}}{(2\pi)^{n-1}}e^{i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{x},\mathbf{k})\sim\delta(\mathbf{k}) \quad (11)$$

and finite behavior is only secured because the odd part vanishes in the limits. However, even a small modulating function which is odd in $\mathbf{x}$ can destroy this property and lead to divergent behavior. Thus, provided the function which modulates the exponential is even in $\mathbf{x}$, the integral transforms smoothly into a δ function with no divergences. In general, one has to deal with the issue of quantum interference through a convolution: for instance,

$$\Delta(k)=\int (dx)e^{ikx}f(k,x)=\int (dx)(dp)e^{i(k+p)x}f(k)g(p)=f(k)g(-k), \quad (12)$$

which illustrates the conection between nonlocality in momentum space and the inhomogeneities in configuration space for future reference.

The expansion of the field in terms of creation and annihilation operators $a^\dagger$ and a is only possible (or even plausible) in an approximation in which the inhomogeneities are sufficiently weak to make the δ-function assumption an approximately valid one. In general, something much more (colloquially) ‘‘coherent’’ occurs and one must deal with the field as an entity as in paper I. Proceeding on the assumption that the δ-function approximation is indeed valid, we obtain

$$(\Phi^{(+)},\Phi^{(+)})=\int (dk)\frac{e^{2\mathrm{Im}(\omega)t}}{4|\omega|^2}\times\left(a^\dagger a\left[\omega+i\frac{\partial_t a}{a}-i\frac{\partial_t\omega}{\omega}\right]+\text{c.c.}\right), \quad (13)$$

where we again assume the slowness of ω as a function of time. For real frequencies (in the absence of dissipation or amplification), one may write this in the form

$$(\Phi^{(+)},\Phi^{(+)})=\int (dk)[N(\mathbf{k},\overline{\mathbf{x}})+R(\mathbf{k},\overline{\mathbf{x}})], \quad (14)$$

where

$$N(\mathbf{k},\overline{\mathbf{x}})=a^\dagger(\mathbf{k},\mathbf{x})a(\mathbf{k},\mathbf{x}'), \quad (15)$$

$$R(\mathbf{k},\overline{\mathbf{x}})=\frac{1}{2|\omega|}(a^\dagger\partial_t a-a\partial_t a^\dagger), \quad (16)$$

which represent the number of particles at the average position $\overline{x}$ and the flux of particles created or destroyed, respectively, over the interval $x-x'$. Note that, as the δ-function approximation becomes poor, the δ function broadens to a finite width and one has $a^\dagger(k)a(k')$, i.e., particles created with one-momentum and returned to the vacuum with another, indicating dissipative or antidissipative scattering from the sources.

The same effect is seen for complex frequencies in Eq. (13) where the imaginary part $\mathrm{Im}(\omega(\mathbf{k},x))$ leads to absorption or amplification of the total particle number with momentum $\mathbf{k}$ at spacetime location x. It is interesting to observe the analogue of the covariant derivative from paper I, in the above expressions:

$$i\omega \to i\omega + \frac{\partial_t a}{a} - \frac{\partial_t \omega}{\omega},$$

$$i\omega^* \to i\omega^* - \frac{\partial_t a^\dagger}{a^\dagger} + \frac{\partial_t \omega^*}{\omega^*}, \tag{17}$$

where a^{-1} represents the inverse of a. In paper I, the $a,a^\dagger$ terms were represented by the gradient of a number operator (actually, its ensemble average, the Wigner function). This occurs due to the nonlocality of the inhomogeneous field over a spacing of the order $\tilde{x} \sim x - x'$, where one notes that

$$\frac{\tilde{\partial}_\mu (a^\dagger(\overline{x - \tilde{x}}) a(\overline{x + \tilde{x}}))}{a^\dagger(\overline{x - \tilde{x}}) a(\overline{x + \tilde{x}})} \equiv \frac{\partial_\mu N(\mathbf{k},\overline{x})}{N(\mathbf{k},\overline{x})} = \frac{\partial_\mu a}{a} - \frac{\partial_\mu a^\dagger}{a^\dagger}. \tag{18}$$

That the ensemble average of the nonlocal number operator in Eq. (15) is related to the Wigner function $f(\overline{x},k)$, is clear from

$$\{\Phi(x),\Phi(x')\} = \int \frac{(dk)}{2|\omega|} e^{ik(x-x')}[1 + 2f(k,\overline{x})]. \tag{19}$$

It now seems clear that the nonlocal generalization of the commutation relations for $a,a^\dagger$ must be given by

$$[a(\mathbf{k},x),a^\dagger(\mathbf{k},x')] = (2\pi)^{n-1} 2\omega(\mathbf{k},\overline{x})\delta(\mathbf{k}-\mathbf{k}') \tag{20}$$

for real frequencies and by

$$\omega^*(\overline{x}) a(\mathbf{k},x) a^\dagger(\mathbf{k},x') - \omega(\overline{x}) a^\dagger(\mathbf{k},x') a(\mathbf{k},x) = (2\pi)^{n-1} 2|\omega(\mathbf{k},\overline{x})|^2 \delta(\mathbf{k}-\mathbf{k}') \tag{21}$$

for complex frequencies. To show these one need the inversion formulas for the creation-annihilation operators. For real frequencies and homogeneous fields, one has the relations

$$\frac{1}{(2\pi)^{n-1}} \frac{1}{2\omega} a^\dagger a = -\int d\sigma^\mu \Phi^{(-)} i\overleftrightarrow{\partial_\mu} \Phi(x),$$

$$\frac{1}{(2\pi)^{n-1}} \frac{1}{2\omega} a a^\dagger = \int d\sigma^\mu \Phi^{(+)} i\overleftrightarrow{\partial_\mu} \Phi(x). \tag{22}$$

In inhomogeneous fields, complex case one simply replaces the partial derivative by its covariant analogue, $\partial_\mu \to D_\mu = \partial_\mu + \overline{a_\mu}$, where

$$\overline{a_\mu} = \frac{\partial_\mu \omega}{\omega} - \frac{\partial_\mu a}{a} + (\partial_\mu k_\nu) x^\nu, \tag{23}$$

where all operators are evaluated at ($\mathbf{k}$,x), so that

$$\int \frac{(dk)}{2|\omega|^2} \omega(\overline{x}) a(\mathbf{k},x')^\dagger a(\mathbf{k}',x) = -\int d\sigma^\mu \Phi^{(-)} i\overleftrightarrow{D_\mu} \Phi(x),$$

$$\int \frac{(dk)}{2|\omega|^2} \omega^*(\overline{x}) a(\mathbf{k}',x) a(\mathbf{k},x')^\dagger = \int d\sigma^\mu \Phi^{(+)} i\overleftrightarrow{D_\mu} \Phi(x). \tag{24}$$

One should now check that $N(\mathbf{k},\overline{x})$ has the property of a number operator. This is confirmed through the relations

$$[N(\mathbf{k},\overline{x}),a^\dagger(\mathbf{k},x)] = a^\dagger(\mathbf{k},x), \tag{25}$$

$$[N(\mathbf{k},\overline{x}),a(\mathbf{k},x')] = -a(\mathbf{k},x'), \tag{26}$$

from which one derives

$$N(\mathbf{k},\overline{x}) a^\dagger(\mathbf{k},x)|n(\mathbf{k},x)\rangle = [N(\mathbf{k},\overline{x}) + 1]|n(\mathbf{k},x)\rangle, \tag{27}$$

$$N(\mathbf{k},\overline{x}) a(\mathbf{k},x')|n(\mathbf{k},x')\rangle = [N(\mathbf{k},\overline{x}) - 1]|n(\mathbf{k},x')\rangle. \tag{28}$$

These relations hold also for complex frequencies, since the commutation relations have the same structure in the real and complex cases. Finally, when the ensemble average over occupation number states $|n(\mathbf{k},x)\rangle$ is introduced, for a given density matrix, one has

$$\mathrm{Tr}\langle \rho N(\mathbf{k},\overline{x})\rangle = f(|\omega(\mathbf{k},\overline{x})|,\overline{x}), \tag{29}$$

since the occupation number states satisfy

$$\begin{aligned}\langle n|n'\rangle &= \langle 0|a^\dagger(\mathbf{k},\mathbf{x}) a(\mathbf{k}',\mathbf{x}')|0\rangle \\ &= \langle 0|[a^\dagger(\mathbf{k},\mathbf{x}),a(\mathbf{k},\mathbf{x}')]|0\rangle \\ &= (2\pi)^{n-1} 2\omega(\mathbf{k},\overline{x})\delta(\mathbf{k}-\mathbf{k}'),\end{aligned} \tag{30}$$

and thus

$$\langle n'(\mathbf{k}',x')|N(\mathbf{k},\overline{x})|n(\mathbf{k},x)\rangle \tag{31}$$

or, inserting the trace over the statistical ensemble,

$$\begin{aligned}&\mathrm{Tr}\langle n'(\mathbf{k}',x')|\rho N(\mathbf{k},\overline{x})|n(\mathbf{k},x)\rangle \\ &= \mathrm{Tr}(\rho n(\mathbf{k},\overline{x}))\langle n|n'\rangle = \mathrm{Tr}(\rho n(\mathbf{k},\overline{x}))\langle n|n'\rangle \\ &= f(|\omega(\mathbf{k},\overline{x})|,\overline{x})\langle n|n'\rangle.\end{aligned} \tag{32}$$

To summarize this section, one finds that conventional notions of particles with definite momentum can be generalized to inhomogeneous systems by adopting an adiabatic approximation,. In this approximation, we make contact with the Wigner function and number operators in their more familiar forms. To exceed this approximation, however, we must deal with the field variables directly.

III. POSITION AND MOMENTUM

Having rewritten the nearly local limit of an inhomogeneous, nonlocal field theory in terms of familiar canonical quantities, the next step is to determine the meaning of position and momentum for excitations of the field in the presence of inhomogeneities. There are three distinct interpretations of position and momentum in a field theory and each pair satisfies canonical commutation relations.

Squeezed states of the field are characterized by the fact that they minimize the uncertainty product

$$\Delta q \Delta p \geqslant \frac{1}{2}. \tag{33}$$

In order to discuss such minimal packets we need to establish the significance of each of the canonical pairs: (i) $[\Phi(\mathbf{x}),\Pi(\mathbf{x}')] = i\delta(\mathbf{x},\mathbf{x}')$, which refers to the field as a dynamical entity and the minimality of zero-point fluctuations, (ii) $[Q(k),P(k)] = i$, a canonical transformation of the commutator for creation and annihilation operators in Eq. (20), which refers to minimality in occupation number space, and (iii) $[\mathbf{x},\mathbf{p}] = i$, which refers to the localizability of individual one-particle observations of the field. This final case gives rise to the relativistic version of the usual Heisenberg uncertainty product for quantum mechanics. For the inhomogeneous field there is a fourth measure of position and momentum which relates to the inhomogeneity scale and is closely linked to the covariant connections for position and momentum. This will emerge from the discussion of covariance. We begin, therefore, with the third of these quantities.

Consider the normalized plane-wave one-particle eigenfunctions in the homogeneous system, $\phi_1(x) = 1/\sqrt{\mathcal{N}}\exp(ikx)$, normalized by the inner product:

$$(\phi_1(\mathbf{x},t),\phi_1(\mathbf{x}',t)) = 1,$$

$$-i\int \frac{d^{n-1}\mathbf{x}}{\mathcal{N}}[e^{-ikx}\overleftrightarrow{\partial_t}e^{ikx}] = 1,$$

$$\frac{2\mathrm{Re}\omega}{\mathcal{N}}e^{-2\mathrm{Im}\omega}\sigma = 1, \tag{34}$$

where σ is the spatial volume of the system. Thus, $\mathcal{N} = 2\mathrm{Re}e^{-2\mathrm{Im}\omega}\sigma$ corresponds to a single particle in the total volume, $\mathcal{N} = 2\mathrm{Re}\omega e^{-2\mathrm{Im}\omega}$ corresponds to one particle per unit volume, and so on. In what follows, we shall simply define the one-particle normalization to be

$$\mathcal{N}_1 = 2\mathrm{Re}(\omega)\sigma, \tag{35}$$

which corresponds to one particle in the total volume, in the absence of dissipative or antidissipative processes.

In nonrelativistic quantum mechanics, one is used to the notion that a differential realization of the momentum operator is given by $\mathbf{p} = -i\partial$ and, similarly for the position $\mathbf{x} = -i\partial/\partial\mathbf{k}$. However, these differential operators do not commute with the normalization factors and, therefore, do not constitute Hermitian operators. They can be made Hermitian by introducing connections Γ_x and Γ_p, such that

$$\mathbf{x} = -i\frac{\partial}{\partial\mathbf{k}} + \Gamma_x,$$

$$\mathbf{p} = -i\partial + \Gamma_p, \tag{36}$$

so that

$$(\phi_1,\mathbf{x}\phi_1) = (\mathbf{x}\phi_1,\phi_1). \tag{37}$$

Symmetry under integration by parts implies that

$$\Gamma_x = -\frac{i}{2}\frac{1}{\omega}\frac{\partial\omega}{\partial\mathbf{k}} = -\frac{i}{2}\frac{\mathbf{v}_g}{\omega}, \tag{38}$$

where $\mathbf{v}_g$ is the group velocity of the total wave packet.

The connection for the momentum operator on these one-particle waves is, by analogy

$$\Gamma_p = -\frac{i}{2}\frac{\partial\omega}{\omega}. \tag{39}$$

The connection for the position operator is well known from relativistic quantum field theory [10,11]; the connection for the momentum is new in the inhomogeneous system. Slight modifications to the connections must be made for the full quantum field. Here, one must deal with general superpositions of wave functions which also (by necessity of relativistic invariance) involve timelike variations. Quantum wave packets must now satisfy the dispersion relation for the full field, which is both spacetime dependent and highly nonlinear.

The invariant inner product for the field, at a given point, is

$$\begin{aligned}(\Phi(x),\Phi(x)) &= -i\int d\sigma^{\mu}[\Phi(x),\partial_{\mu}\Phi(x)]\\ &= -i\int d\sigma^{\mu}[\Phi(x),\Pi_{\mu}(x)]\\ &= 1\end{aligned} \tag{40}$$

and

$$(\Phi(x),\Phi(x)) = (\Phi^{(+)}(x),\Phi^{(+)}(x)) + (\Phi^{(-)}(x),\Phi^{(-)}(x)). \tag{41}$$

The expression for the momentum operator's covariant connection is, therefore, the previously introduced vector field $\overline{a_{\mu}}$ [see Eq. (23) and the final section of paper I]. To determine the modified connection for the position operator, one considers $(\Phi^{(+)}(x),\Phi^{(+)}(x))$ in Eq. (40). A straightforward repetition of the argument for one-particle wave functions leads to the expression

$$\begin{aligned}\Gamma_x^{\mu} &= -\frac{i}{2}\left[2\frac{\partial}{\partial\mathbf{k}_{\mu}}\mathrm{Im}(\omega) + \frac{1}{N}\frac{\partial N}{\partial\mathbf{k}_{\mu}} - \mathrm{Re}\frac{v_g^{\mu}}{\omega}\right]\\ &= -\frac{i}{2}\left[2\frac{\partial}{\partial\mathbf{k}_{\mu}}\mathrm{Im}(\omega) + T^{\mu} - \mathrm{Re}\frac{v_g^{\mu}}{\omega}\right].\end{aligned} \tag{42}$$

This may be compared to Eq. (96) of paper I, where it arises as a length scale of ''inhomogeneity fluctuations'' Δx.

Since one is dealing with modified position and momentum operators, it is pertinent to ask whether the commutation relations are preserved by the introduction of covariant connections. Writing

$$[x^{\mu},D_{\nu}] = i\delta^{\mu}_{\nu} + \Delta^{\mu}_{\nu}, \tag{43}$$

one has

$$\Delta^\mu_\nu = \partial^\mu \Gamma_{x\nu} - \frac{\partial}{\partial k^\nu}\Gamma^\mu_k, \tag{44}$$

where $\Gamma^\mu_k = \overline{a^\mu}$. This can be further rewritten in the form

$$\Delta^\mu_\nu = \left[\partial^\mu, \frac{\partial}{\partial k^\nu}\right](f, \omega, \ldots). \tag{45}$$

In other words, it is the commutator of the position and momentum derviatives acting on the complex frequency, the Wigner function, and possible other x-dependent quantities (which arise in connection with dissipation and the density matrix). Thus, provided phase space contains no singularities, Δ^μ_ν must vanish. This has the physical implication that a minimal uncertainty packet of the field can never violate the well-known minimum Heisenberg uncertainty value.

In obtaining dispersion relations in paper I, we were led to conclude that a natural length scale for the ''granularity'' of the field, and the extent of position or momentum fluctuations were given by

$$L_\mu = P_\mu^{-1} = \overline{\partial_\mu} H(k, \overline{x}), \tag{46}$$

$$\overline{\Delta x} = \frac{1}{H(k,x)}\frac{\partial H}{\partial k} = \Gamma_x, \tag{47}$$

$$\overline{\Delta p} = \frac{1}{H(k,x)}\frac{\partial H}{\partial x} = \Gamma_p, \tag{48}$$

where

$$\langle \Phi(x)\Phi(x')\rangle = \int \frac{d^n k}{(2\pi)^n} e^{ik(x-x')} H(k, \overline{x}). \tag{49}$$

The length scale $L \to 0$ in the homogeneous limit, indicating an infinite resolution of the field or no inhomogeneity. This also implies either granularity nor weak localization since the corresponding momentum scale becomes infinite, leading to a continuous spectrum of frequencies. The other two scales represent an approximate measure of the degree of localization about the average position $\overline{x}$ and the spread of momentum uncertainty of a wave packet. $\overline{\Delta x}$ vanishes when the k dependence of $H(k,x)$ vanishes. This corresponds to white-noise fluctuations of the field in the momentum (all frequencies equally likely) as one would expect from the usual uncertainty relation. In other words, as the system is completely delocalized in the momentum (as a result of statistical and vacuum fluctuations), the uncertainty in the corresponding position is minimized. k independence implies a purely local limit. $\overline{\Delta p}$ vanishes only when the system is completely homogeneous, or the $\overline{x}$ dependence vanishes. This implies that all positions of a plane wave are equally likely, since there is no potential to single out a special point. This is also in accord with familiar ideas about uncertainty.

One should be wary not to confuse these approximate measures of uncertainty with the more familiar $\Delta x = x - \langle x\rangle$ and $\Delta p = p - \langle p\rangle$. The scale in Eqs. (47) and (48) refers directly to the connections Γ_x and Γ_p and represents statistical and kinematical modifications to the minimum width of particle excitations in phase space, at given values of k and $\overline{x}$. The function $H(k, \overline{x})$ summarizes the deviation of the dynamics from that for one-particle excitations, and these two scales must be understood as additional uncertainties, on top of those which appear in the conventional Heisenberg relation. Moreover, they do not satisfy a Heisenberg-type inequality themselves since the k dependence arises as a result of statistical and vacuum fluctuations, whereas the $\overline{x}$ dependence arises as a result of dynamics (transport) and external boundary conditions. The two dependences are related by a Boltzmann-Vlasov-type equation, which we shall not discuss here [12].

$\overline{\Delta x}$ encompasses, amongst other things, thermal broadening of the noise in position localization:

$$\begin{aligned}\overline{\Delta x}^\mu &= \frac{\partial}{\partial k_\mu}\left(\frac{1+f}{2\omega}\right)\frac{2\omega}{1+f}\\ &= \left((1+f)^{-1}\frac{\partial f}{\partial w} - \frac{1}{\omega}\right)\frac{\partial \omega}{\partial k_\mu}\\ &= -\left(\frac{\beta}{e^{\beta\omega}-1} + \frac{1}{\omega}\right)v_g^\mu,\end{aligned} \tag{50}$$

where $1/f = \exp(\beta\omega) - 1$ and $1 + f = e^{\beta\omega} f$ and v_g^μ is the group velocity of wave packets.

$\overline{\Delta p}$ accounts for a contribution from the $\overline{x}$ dependence, a broadening of the momentum distribution due to the inhomogeneous distribution of the field. It is tempting to refer to this as ''inhomogeneous broadening.'' This name is usually reserved for Doppler broadening of frequency in gaseous matter, so we should be cautious in adopting such a name. In fact, the two quantities are related. Since this term represents a localized correction to the momentum of the field, induced by a microscopic disequilibrium (not the averaged field), it has precisely the property of a generalized Doppler width. Indeed, $\overline{\Delta p}$ is related to the rate of change of frequency due to localized disequilibrium $\partial_t \omega/\omega$. Thus, this fluke of nomenclature is, for once, a lucky one.

IV. FOURIER MODES

The second definition of canonical variables arises in occupation number space, or the Fock space of the particle creation-annihilation operators $a^\dagger, a$. Often in experiments it is this Fourier decomposition of the field which can be measured. Here, one may define position and momentum variables $\hat{Q}$ and $\hat{P}$ in such a way that the free-particle Hamiltonian takes on the appearance of an array of harmonic oscillators in occupation number space. In discussions of squeezed states of the radiation field, it is normally this pair of operators to which one refers, not to the observable position and momentum. $\hat{Q}$ and $\hat{P}$ are related to $a^\dagger, a$ by [6]

$$a = \frac{1}{\sqrt{2}}\left(s\hat{Q} + \frac{i}{s}\hat{P}\right),$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(s\hat{Q} - \frac{i}{s}\hat{P}\right), \tag{51}$$

where $s=\sqrt{\omega}$ for an unperturbed oscillator array (free field). In general, one may regard s as a squeezing parameter, reducing $\hat{Q}$ at the expense of $\hat{P}$ or vice versa. This may be seen more clearly by inverting the relations

$$\hat{Q}=\frac{1}{s\sqrt{2}}(a+a^{\dagger}),$$

$$\hat{P}=\frac{s}{i\sqrt{2}}(a-a^{\dagger}), \tag{52}$$

so that the Hamiltonian

$$H(k)=\hat{P}^2+s^4\hat{Q}^2=s^2a^{\dagger}a, \tag{53}$$

which is manifestly a harmonic oscillator for the kth mode, with natural frequency s^2. It can be noted that the transformation preserves the form of the commutation relation (20):

$$[\hat{Q},\hat{P}]=2i\omega\delta(\mathbf{k}-\mathbf{k}')(2\pi)^{n-1}. \tag{54}$$

Note that the commutator is independent of s and thus s is a kinematical symmetry of the system which preserves the minimality of the uncertainty product

$$\Delta Q\Delta P\geqslant\frac{1}{2} \tag{55}$$

since, by the Schwarz inequality, any two operators satisfying $[X,Y]=iA$ also satisfy $\Delta X\Delta Y\geqslant\frac{1}{2}A$. For $s>1$ the position $\Delta Q=1/\sqrt{2s}$ is squeezed, and for $s<1$, the momentum $\Delta P=i\sqrt{s/2}$ is squeezed. The structure of these relations is unaffected by a functional dependence on the average position $\overline{x}$ and thus no immediate problems arise in connection with inhomogeneous systems. As shown by Stoler [6], the unitary operator

$$U_z=\exp\left(\frac{1}{2}[za^2-z^*(a^{\dagger})^2]\right) \tag{56}$$

may be thought of as the exponentiated generator of this symmetry, so that if $|\Phi,k\rangle$ is a minimal field configuration (i.e., it saturates the uncertainty inequality in the kth mode), then so is $U_z|\Phi,k\rangle$. Although this tells us nothing about minimality in real space, it is related to squeezing of the number distribution, or the issue of sub-Poissonian statistics and is, therefore, of great interest in quantum optics.

The suitability of this operator as a symmetry generator is not so straightforward for inhomogeneous systems and must be reinvestigated. To do this one notes that minimal uncertainty packets are eigenfunctions of the dimensionless operator [13,14]

$$\hat{M}=\frac{1}{2\gamma}(\hat{Q}+i\gamma\hat{P}), \tag{57}$$

which is clearly equal to the annihilation operator $a(\mathbf{k},x)$ when $s^2=1/\gamma$. The eigenfunctions of this operator satisfy

$$\hat{M}|M\rangle=\lambda_M|M\rangle, \tag{58}$$

where $\lambda_M=(1/2\gamma)\langle Q\rangle+i\gamma\langle P\rangle$. When $s^2=1/\gamma$ and $M=a$, these are clearly the coherent states. The symmetry of the commutator now suggests that all other minimal states might be obtainable from the coherent states with the aid of a similarity transformation. Let

$$\hat{M}=J_+a(\mathbf{k},x)+J_-a^{\dagger}(\mathbf{k},x), \tag{59}$$

so that

$$J_+=\frac{1}{2}\left(\frac{1+s^2\gamma}{s\sqrt{\gamma}}\right),$$

$$J_-=\frac{1}{2}\left(\frac{1-s^2\gamma}{s\sqrt{\gamma}}\right). \tag{60}$$

The argument for homogeneous systems now proceeds by proving the relations [6]

$$U_zaU_z^{-1}=a\cosh|z|+a^{\dagger}e^{-i\arg z}\sinh|z|,$$

$$U_za^{\dagger}U_z^{-1}=a^{\dagger}\cosh|z|+ae^{i\arg z}\sinh|z|, \tag{61}$$

with the help of the formula

$$e^ABA^{-A}=B+[A,B]+\frac{1}{2!}\{A,[A,B]\}\cdots \tag{62}$$

and by making the identification $J_+=\cosh|z|$, $J_-=\sinh|z|$ and $\arg z=0$. In evaluating Eq. (61) one uses the commutator for $a,a^{\dagger}$. These formulas remain correct provided the creation and annihilation operators are all evaluated at the same spacetime point $\overline{x}$. An approximation in which the derivative is slowly varying can only be accomplished by assuming that the operators are effectively independent of $\overline{x}$. However, this is no hindrance and it simply reflects the essential locality of each ''domain'' or ''cell'' of the field, which is labeled by $\overline{x}$.

Using the unitary operator U_z for real z we are, therefore, able to generate all of the squeezed states

$$|M\rangle=U_z|a\rangle \tag{63}$$

characterized by three real parameters: the real and imaginary parts of a and z. If $a=n\exp(i\theta)$, then n is the occupation number of the coherent state and θ is its phase. For $|z|\rightarrow 0$ one obtains the coherent states.

The dynamical evolution of a minimal state, in momemtum space, is determined by

$$|M,t\rangle=\exp(-iHt)U_z|a\rangle. \tag{64}$$

One can ask what is the general form of a Hamiltonian which preserves the minimality of a set states in the decomposition of the quantum field? An evaluation of the transformed free Hamiltonian leads to

$$UHU^{-1}=\omega(a^{\dagger}a+\tfrac{1}{2})+2\omega\cosh|z|\sinh|z|[a^2+(a^{\dagger})^2]. \tag{65}$$

Terms quadratic in the creation and annihilation operators are, therefore, important here. Many authors refer to a Hamiltonian with such an operator as a squeezed Hamiltonian, indicating that this is sufficient to obtain quadrature squeezing [1,15]. This, however, is not, in the strictest sense, well defined, since it violates locality of interactions (while one may certainly write down such an interaction in Fourier space, it is impossible to obtain such an operator through any physical process). In fact, that such a term can exist at all, requires a notion of inhomogeneity such as is described in this paper and its predecessor. To show this, and to obtain a more satisfactory formulation, it is necessary to abandon the particle approach and consider the quantum field as an entity in real space.

In the local limit of the creation and annihilation operators (the limit in which they are functions of **k** only), one may use the inversion relation

$$a(\mathbf{k})=-i\int\frac{(d\mathbf{x})}{2\omega(\mathbf{k})}[e^{ikx}\overleftrightarrow{\partial_0}\Phi(x)] \tag{66}$$

to show that a term of the form a^2 in the momentum-space action or Hamiltonian has the following form as a position-space term in the action:

$$\begin{aligned}a^2=\int(d\mathbf{x})(d\mathbf{x}')dt\{&[\partial_0\Phi(\mathbf{x},t)]g(\mathbf{x},\mathbf{x}')[\partial_0\Phi(\mathbf{x}',t)]\\&+iJ(\mathbf{x}',t)[\partial_0\Phi(\mathbf{x},t)]\Phi(\mathbf{x}',t)+\Phi(\mathbf{x},t)\\&\times\Phi(-\mathbf{x},t)\delta(\mathbf{x}-\mathbf{x}')\}.\end{aligned} \tag{67}$$

Thus, the unitary operator involves the form

$$a^2-(a^\dagger)^2=2i\int(d\mathbf{x})(d\mathbf{x}')dtJ(\mathbf{x}',t)[\partial_0\Phi(\mathbf{x},t)], \tag{68}$$

which may be identified with the generally covariant form $\phi\gamma^\mu\partial_\mu\phi$ from paper I, where it represented a phase current in the internal space of the field. These expressions involve integrals of the form

$$\begin{aligned}&\int(dk)\frac{\exp[i\mathbf{k}(\mathbf{x}+\mathbf{x}')]}{\omega^2}=g(\mathbf{x}+\mathbf{x}'),\\&\int(dk)\frac{\exp[i\mathbf{k}(\mathbf{x}+\mathbf{x}')]}{\omega}=iJ(\mathbf{x}+\mathbf{x}'),\\&\int(dk)\exp[i\mathbf{k}(\mathbf{x}+\mathbf{x}')]=\delta(\mathbf{x}+\mathbf{x}').\end{aligned} \tag{69}$$

Notice that Eq. (67) is highly nonlocal: in particular, the latter term depends on diametrically opposed point in space. Such a term cannot be supported for arbitrary spatial separations in a relativistic theory, owing to the finite speed of communication imposed by special relativity. It represents an instantaneous interaction over infinite distance. Moreover, this expression singles out the arbitrary origin as a special point and is, therefore, a special case, rather than a covariant expression. It seems clear that the squeezing operator equation (56) can only be an approximation, valid over small separations. A correct starting point must involve a generally covariant, nonlocal expression which contains an inhomogeneous potential. Without such a potential, no special points in space can be singled out and the action or Hamiltonian can never contain terms other than $a^\dagger a$. We begin again, therefore, with an action of the form

$$\begin{aligned}S_{\rm sq}=\int dV_x dV_{x'}\{&(\overset{g}{D}{}^\mu\Phi)g_{\mu\nu}(x,x')[\overset{g}{D}{}^\nu\Phi(x')]\\&+\Phi(x)\gamma^\mu(x,x')[\overset{\gamma}{D}_\mu\Phi(x')]-\Phi(x')\gamma^\mu(x,x')\\&\times[\overset{\gamma}{D}{}'_\mu\Phi(x)]+\Phi(x)A(x,x')\Phi(x')\},\end{aligned} \tag{70}$$

where the notation of paper I has been used: $\overset{\gamma}{D}_\mu\equiv\partial_\mu-\Gamma^\gamma_\mu$ is a derivative which commutes with the function $\gamma(x,x')$, etc.

One can now examine the form of this action in momentum space and compare it to the idealized Hamiltonian in Eq. (65). Substituting Eq. (7) for $\Phi(x)$ and separating variables into $\tilde{x}$ and $\overline{x}$, one obtains

$$\begin{aligned}S_{\rm sq}=2\,{\rm Re}\int dV_{\overline{x}}(dk)(dp)&\frac{a(k,\overline{x})a(p,\overline{x})}{\omega(k,\overline{x})\omega(p,\overline{x})\omega\left(\frac{p-k}{2}\right)}\\&\times\exp[i(p+k)\overline{x}]\Big\{(-k^\mu p^\nu-ik^\mu\Gamma^\nu_g-ip^\nu\Gamma^\mu_g+\Gamma^\mu_g\Gamma^\nu_g)\\&\times g_{\mu\nu}\left(\frac{p-k}{2},\overline{x}\right)+2\gamma^\mu\left(\frac{p-k}{2},\overline{x}\right)(ik_\mu-\Gamma^\gamma_\mu)\\&+A\left(\frac{p-k}{2},\overline{x}\right)\Big\}.\end{aligned} \tag{71}$$

Clearly, in the limit that $A,\gamma_\mu,g_{\mu\nu}$ become independent of $\overline{x}$, one can integrate over $\overline{x}$, obtaining $\delta(\mathbf{k}+\mathbf{p})$ and subsequently p to obtain the usual local result

$$S_{\rm loc}=\int\frac{(dk)}{\omega(\mathbf{k})}a(\mathbf{k})a^\dagger(\mathbf{k})[g_{\mu\nu}(k)k^\mu k^\nu+2A(k)]. \tag{72}$$

Note that this purely local operator contains no squeezing terms, thus an action or Hamiltonian which admits squeezed solutions must necessarily be nonlocal.

To produce a term in Eq. (71) which reproduces the standard a^2 terms as closely as possible, we require that the functions $A((p-k)/2,\overline{x})$ etc. be concentrated strongly about $p=k$ to counteract the tendency of the exponential to lead to a δ function (imposing $p=-k$). The functions or distributions must also lead to a well-behaved reverse Fourier transform in order to lead to physically acceptable solutions.

This tendency of the interactions to turn an annihilation operator into a creation operator (and vice versa) reflects the ability of the medium to store and rechannel energy by stimulated scattering. Instead of particles being created and then destroyed, two quanta are created and two quanta are destroyed as independent, but balanced, processes. Clearly, this requires a very special physical environment; it also requires a special function $A((p-k)/2,\overline{x})$. Clearly, the $\overline{x}$ dependence plays a crucial role in frustrating the δ function, by producing noise in $\overline{x}$. In the absence of such noise, no function would survive the δ function resulting from the exponential. Perhaps, the simplest case is presented by a periodic function of $\overline{x}$, such as would be generated by a periodic pulsing or a periodic background potential. This provides a connection between the formalism and the phenomenon of antibunching of photons in sub-Poissonian light (see below). In either case, the importance of a spacetime inhomogeneity is emphasized. Another possibility is a rapidly changing function of $\overline{x}$ such as one could find in a system far from mechanical or thermal equilibrium.

While this property of the sources is important for the description of squeezing, a more fundamental limitation must be satisfied in order to preserve causality. The finite speed of communication between spatially separated parts of the system is imposed only if the nonlocal sources are the Green functions of some mediating interaction. For example, they will satisfy an equation of the form

$$[-\Box+\mathcal{M}^2(\overline{x})]A(x,x')=\delta(x-x), \tag{73}$$

for some *positive* mass-energy squared, $\mathcal{M}^2\geqslant 0$. The precise equation will be determined by the procedure described in paper I, together with a knowledge of the microscopic interactions of the field. We examine the effect of this assumption more closely in Sec. VII.

Before presenting an improved formulation of the foregoing results, it is of special interest to consider the width of field-intensity fluctuations, $\Delta N=N-\langle N\rangle$, since this is a more readily observed quantity than P and Q. The properties of ΔN may be inferred from the true momentum of the field. Following Schwinger [16], we note that

$$\langle p_\mu\rangle=\langle(-i\partial_\mu-\Gamma_{p\mu})(\Phi^{(+)},\Phi^{(+)})\rangle=\left\langle\int(dk)\widetilde{N}(\mathbf{k},\overline{x})\right\rangle, \tag{74}$$

where $\widetilde{N}=N(\mathbf{k},\overline{x})+R(\mathbf{k},\overline{x})$ [see Eq. (14)] and the angular brackets represent the expectation value of a field operator (see paper I). The connected correlations of the momentum operator are described by

$$\begin{aligned}\langle(p_\mu-\langle p_\mu\rangle)&(p_\nu-\langle p_\nu\rangle)\rangle\\&=\langle p_\mu p_\nu\rangle-\langle p_\mu\rangle\langle p_\nu\rangle\\&=(-i\partial_\mu-\Gamma_{p\mu})(-i\partial_\nu-\Gamma_{p\nu})\langle(\Phi^{(+)},\Phi^{(+)})\rangle\\&=\int(dk)(k_\mu k_\nu+i\partial_\mu\Gamma_{p\nu})\widetilde{N}(\mathbf{k},\overline{x}).\end{aligned} \tag{75}$$

This can also be interpreted as [16]

$$\begin{aligned}\langle\widetilde{N}(\mathbf{k},\overline{x})\widetilde{N}(\mathbf{k}',\overline{x})\rangle&-\langle\widetilde{N}(\mathbf{k},\overline{x})\rangle\langle\widetilde{N}(\mathbf{k}',\overline{x})\rangle\\&=\delta_{\mathbf{k}\mathbf{k}'}[\langle\widetilde{N}(\mathbf{k},\overline{x})\rangle+i\partial_\mu\Gamma_{p\nu}(k_\mu k'_\nu)^{-1}]\end{aligned} \tag{76}$$

on extracting factors of the momentum. The integral over momenta leads to a modification of the standard result for the Poisson distribution of random particles

$$\langle\widetilde{N}^2\rangle-\langle\widetilde{N}\rangle^2=\langle\widetilde{N}\rangle+\int(dk)\partial_\mu\Gamma_{p\nu}(k_\mu k'_\nu)^{-1}. \tag{77}$$

As the $\overline{x}$ dependence disappears, $\widetilde{N}\rightarrow N$ and the last term involving the connection Γ_p vanishes, leaving the well-known property of the Poisson distribution.

In the literature, it is common to specify the statistics of the field by quoting the Mandel parameter Q or Fano number $f=1+Q$ [17]. The latter is obtained from the above by rearranging:

$$f=\frac{\langle\widetilde{N}^2\rangle-\langle\widetilde{N}\rangle^2}{\langle\widetilde{N}\rangle}-\Delta f, \tag{78}$$

where $\Delta f\langle\widetilde{N}\rangle$ is defined to be equal to the last term in Eq. (77). This number is equal to unity for a Poisson distribution. A number less than one indicates sub-Poisson statistics; a number greater than one indicates super-Poisson statistics. Here, one sees that the result of the inhomogeneous connection can lead to a ''squeezing'' of the field statistics.

It is worth noting from the discussion in this section that the inhomogeneity of the field is required to explain both anomalous statistics and notion of squeezed states. The fact that a broad (periodic) $\overline{x}$ distribution of the sources is required to satisfy the criteria for these explains the phenomenon of antibunching, or the tendency of source photons to be distributed evenly in spacetime (the opposite of a δ function).

V. SPACETIME FORMULATION

We are now in a position to reformulate inhomogeneous field theory and present all the preceding results in a compact form, based on the quantum action principle [4]. This eliminates not only the need for the localized creation or annihilation operators, but also the introduction of a special squeezing transformation U. The latter may now be understood as a subclass of the usual finite unitary transformations induced by the dynamics of the field.

In moving to a spacetime approach we open the door to additional definitions of squeezing which describe the time evolution of fields. Before examining these, we shall consider the analogue of the momentum-space squeezing in Sec. IV. We begin with a phenomenological action of the form

$$\begin{aligned}S=\int dV_x dV_{x'}\{&(D^\mu\phi(x))^T g_{\mu\nu}(x,x')(D^\nu\phi(x'))\\&+\phi(x)A(x,x')\phi(x')+J(x)\phi(x)\},\end{aligned} \tag{79}$$

where the superscript T on the derivative indicates that the sign of the connection term is reversed, and the sources $g_{\mu\nu}(x,x')$ and $A(x,x')$ are taken to be symmetrical in their indices, and may also be split up into a local part and a nonlocal part if desired. D_μ is a derivative which commutes with the source $g_{\mu\nu}(x,x')$.

According to the quantum action principle, the variation of any quantum transformation function (amplitude) is given by

$$\delta\langle\phi'|\phi\rangle=i\langle\phi'|\delta S|\phi\rangle. \tag{80}$$

From this relation one infers both the operator equations of motion $\delta S/\delta\phi=0$, for dynamical variables ϕ and the generator of infinitesimal unitary transformations G which is obtained from the total time dervitative in δS. S is an action symmetrized with respect to the kinematical derivatives of the dynamical variables. From this, one obtains the variation of any operator A on the basis $|\phi\rangle$

$$\delta A = -i[A,G]. \tag{81}$$

Applying this principle to the action in Eq. (79), by varying the action with respect to $\phi(x)$, one obtains

$$\delta S = \int dV_x dV_{x'} \{2\,\delta\phi(x)\cdot - D^\mu g_{\mu\nu}(x,x')D^\nu\phi(x') + 2\,\delta\phi(x)\cdot A(x,x')\phi(x') + \delta\phi\cdot J\} + \int dV_x d\sigma^\mu[\delta\phi(x) g_{\mu\nu}(x,x')D^\nu\phi(x')]. \tag{82}$$

From this we infer the nonlocal field equation

$$\int dV_{x'}\{-D^\mu g_{\mu\nu}(x,x')D^\nu\phi(x') + A(x,x')\phi(x')\} = 0, \tag{83}$$

which implies a nonlocal Hamiltonian of the form in Eq. (65), and generator of infinitesimal unitary transformations

$$G = \int dV_x d\sigma^\mu[\delta\phi(x) g_{\mu\nu}(x,x')D^\nu\phi(x')]. \tag{84}$$

The equal-time commutation relations are suitably modified by the external interaction with the source, and follow directly from Eq. (81):

$$\left[\phi(x), \int dV_{x'} g_{\mu\nu}(x,x')[D^\nu\phi(x')]\right] = i\,\delta(\mathbf{x},\mathbf{x})\hat{n}_\mu. \tag{85}$$

The canonical choice for the unit vector $\hat{n}_\mu$ is $\mu=0$, pointing in a pure time direction. It is now clear that the generator G leads to the unitary transformation matrix

$$U = \exp(iG) \tag{86}$$

in infinitesimal form. The finite transformation can now be compared to

$$U = \exp\left(\frac{1}{2} g[a^2 - (a^\dagger)^2]\right) = \exp\left(i\int dV_x d\sigma^\mu g_{\mu\nu}(x,x')\phi(x)D^\nu\phi(x')\right), \tag{87}$$

which, referring to Eq. (56), is seen to be of the same form as the squeezing transformation. When applied to a coherent state (displaced vacuum),

$$|c\rangle = \exp\left(i\int dV_x J\Phi\right)|0\rangle = \exp[-(1/2)|J_k|^2]\frac{J_k^n}{\sqrt{n!}}|n\rangle, \tag{88}$$

this results in a squeeze. Thus, the squeezing transformation is to be interpreted as simply one of a class of standard unitary transformations on the field, in a field theory with an inhomogeneous bilocal interaction. Note also that $\phi D_\mu \phi$ has the interpretation of an invariant probability P on the manifold of positive energy solutions for the field, so that the unitary transformation has the form of a weight

$$U \sim e^{gP}, \tag{89}$$

which acts on the mixture of states to which the transformation function in Eq. (80) refers. This will supply a relation between the statistical distribution (density matrix or Wigner function) and the unitary evolution of the field.

Although we have been seeking to eschew the notion of creation-annihilation operators, it is useful to return to momentum space. In terms of this improved, fully covariant derivation, we can now attempt to identify the true nature of the idealized squeezing parameter s in Eq. (51). Crudely speaking, it is now a nonlocal quantity, depending on both the momentum k and the inhomogeneity coordinate $\overline{x}$ in its Fourier-transformed form. One might also wonder whether the resulting nonlocal transformation is even a symmetry of the system any longer. In fact, the derivation above in terms of the action principle demonstrates this unequivocally, but we can also show the deviation from the well-known form in Eq. (61), by assuming the adiabatic approximation and employing the inhomogeneous oscillator states. The symmetry then rests on the idenitification of the transformed operator $Ua(k,\overline{x})U^{-1}$ with $\hat{M} = J_+ a(\mathbf{k},x) + J_- a^\dagger(\mathbf{k},x)$. The transformation is possible provided the coefficients J_+ and J_- are expressible in terms of the symmetry parameter, and the relation closes under the algebra of $a, a^\dagger$.

Returning to momentum space, the true form of a permissible squeezing transformation is nonlocal in the momentum

$$U(k) = \exp\left(\frac{1}{2}\int (dk')[a(k,\overline{x})g_{00}(k,k',\overline{x})a(k',\overline{x}) - a^\dagger(k,\overline{x})g^*_{00}(k,k',\overline{x})a^\dagger(k',\overline{x})]\right). \tag{90}$$

Since the creation-annihilation operators commute except when $k=k'$, one now finds [using the identity in Eq. (62)] that

$$U(k)a(k,\overline{x})U(k)^{-1} = a(k,\overline{x})\cosh\left(\frac{1}{2}\int (dk')g_{00}(k,k',\overline{x})(1+\delta_{kk'})\right) + e^{-i\arg g_{00}}a^\dagger(k,\overline{x}) \times \sinh\left(\frac{1}{2}\int (dk')g_{00}(k,k',\overline{x})(1+\delta_{kk'})\right), \tag{91}$$

$$U(k)a^\dagger(k,\overline{x})U(k)^{-1} = a^\dagger(k,\overline{x})\cosh\left(\frac{1}{2}\int (dk')g_{00}(k,k',\overline{x})(1+\delta_{kk'})\right) + e^{i\arg g_{00}}a(k,\overline{x})\sinh\left(\frac{1}{2}\int (dk')g_{00}(k,k',\overline{x})(1+\delta_{kk'})\right), \tag{92}$$

where the hyperbolic sines and cosines are to be regarded as abbreviations for their power-series expansions. Although this formal expression cannot be evaluated any further without more specific details, it clearly has the correct properties to act as a unitary transformation, transforming any minimal packet of the field into any other. The transformation is now highly nonlinear. Moreover, one identifies the $k,\overline{x}$-dependent parameter s by

$$\frac{1}{2}\left(\frac{1+s^2\gamma}{s\sqrt{\gamma}}\right)=\cosh\left(\frac{1}{2}\int (dk')g_{00}(k,k',\overline{x})(1+\delta_{kk'})\right), \tag{93}$$

which reduces to the well-known result in Eq. (61) when $g_{00}(k,k',\overline{x})\rightarrow|z|\delta(k-k')$ and $\arg g_{00}=0$. Rearranging this formula, one finds an inevitably exponential average-time (or space) dependence in the squeezing parameter,

$$s(k,\overline{x})=\frac{2}{\sqrt{\gamma}}\exp\left(\pm\frac{1}{2}\int (dk')g_{00}(k,k',\overline{x})(1+\delta_{kk'})\right), \tag{94}$$

such as is observed in the squeezing of trapped ions in a recent experiment [9] by Meekhof *et al.* The specific nature of this exponential variation depends on environmental factors (boundary conditions). Reference [9] does not provide sufficient details for a more detailed comparison. Note that, in this relativistic formulation, the squeezing parameter is a position, time- and freqency-dependent quantity, which is peaked sharply around $k=k'$. Only in an idealized system is it possible to obtain pure mode squeezing. The width of the distribution is determined by the width of the dipolelike driving force $g_{00}(k,k',\overline{x})$, which is, in turn, found from the dispersion relation for the interacting field (see Sec. VII). An order of magnitude estimate is $\Delta k=\omega/v_g$. The distribution will be broadest when the dispersion relation of the field has large gaps (an effective mass) in the energy spectrum. The effect of this splitting of a monochromatic field into several components is reminiscent of parametric down conversion [18]. This frequency and spacetime dependence of the squeezing will tend to lead to squeezing concentrated in specific areas of phase space, i.e., specific discrete momenta at specific times and places, typically with an oscillatory character. This could be observed as a collapse and revival of occupation numbers of a given frequency, or as standing waves in space.

In addition to the Fourier space squeezing of Sec. IV, there are three other nonzero commutators which can be used to substantiate the notion of squeezing the field. We have the usual equal-time commutators $[x,p]$ and $[\phi,\Pi]$ and there is the commutator of the positive and negative frequency parts of the field [10,3] which is summarized by

$$[\phi(x),\phi(x')]=i\widetilde{G}(x,x'), \tag{95}$$

where $\widetilde{G}(x,x')$ is the commutator function, expressible as the sum of the positive and negative frequency Wightman functions [19]. This commutator vanishes on a spacelike hypersurface, i.e., in canonical language it only exists for causally connected fields at different times and is independent of the statistical state of the system (though it does depend on the rate of change of the statistical state). It is initimately connected with the time development of the quantum field. This expression also illustrates the essential locality of the squeezing, in spite of the fact that the quadratic interaction is bilocal.

The case of $[x,p]$ squeezing is no different from the discussion in Sec. III, so there is no need to repeat it here. Squeezing of the zero-point fluctuations in the field and conjugate momentum can be presented in a real space form without introducing the Fourier components as in Sec. IV. A squeezing operator can be constructed in the usual way by constructing variables

$$A_{\pm}=\frac{1}{\sqrt{2}}\left(s\phi(x)\pm\frac{i}{s}\Pi(x)\right), \tag{96}$$

such that

$$[A_+,A_-]=\delta(\mathbf{x},\mathbf{x}') \tag{97}$$

at equal times. The quadratic operator may now be written

$$U=P\exp\left(\int dV_x\gamma^0(A_+^2-A_-^2)\right), \tag{98}$$

which may also be written in the generally covariant form

$$U=P\exp\left(\int dV_x\gamma^\mu\phi(\partial_\mu\phi)\right). \tag{99}$$

This term also appeared in paper I as an off-diagonal source term, which was directly related to the changing of the occupation number in different number states in momentum space. Notice that the term, as presented, violates parity and time-reversal invariance, but that the combined transformation of variables involves both U and U^{-1} so that parity is preserved in any physical expectation values. We thus identify this source term as an operator which will squeeze minimal states of the field. This makes an interesting connection in the case of the electromagnetic field, as seen below. Before leaving this case, it may be noted that this operator is not even bilocal: it serves as the generator of squeezings even as a purely local quantity. This can be understood as follows. Such a source term does not occur naturally in the action of a physical theory. It violates parity and time-reversal invariance and can, therefore, only arise as an expression of external boundary conditions. In paper I it was shown that the coefficient γ^μ was equivalent to the time variation of the dispersion relation (Ω^μ) and the time variation of the density matrix (F^μ), indeed, all of these played the role of a ''gauge field'' or chemical potential for the quadratures of the real scalar field. Thus, while there is no need for an inhomogeneity at the formal level, one should understand that the physical origin of such a term is precisely a result of the inhomogeneous (time-dependent) development of the system. Such a term may be viewed as a renormalization of such an inhomogeneous system.

The unequal time commutator in Eq. (95) admits another internal transformation of the field over a two-time interval. The Schwarz inequality then implies that

$$\Delta\phi(x)\Delta\phi(x')\geq\frac{1}{2}\langle\widetilde{G}(x,x')\rangle, \tag{100}$$

i.e., the fluctuation width in the field between any two times is bounded by an expression which is independent of the state $f(k,\overline{x})$ of the system, but which can depend on the rate of change of $f(k,\overline{x})$ and the spectrum $\omega(k,\overline{x})$ away from equilibrium. The width of these fluctuations is determined by the spectral content of the field as $1/\omega(k,\overline{x})$. An increased mass or gap in the spectrum typically damps the value of the right-hand side (RHS) exponentially, instead of like a power law; thus the tendency for fluctuations to increase over a short time interval is reduced by the gap. The form of the gap was derived in paper I and is given by $m^2+\overline{A}(k,\overline{x})+(F-N)^2$, where F_μ is the gradient of the Wigner function and N_μ is the rate of expansion of a cavity to which the field is confined. The form of this expression indicates that rapid (nonadiabatic) changes in the occupation numbers of particles in momentum states (nonequilibrium), or an decrease in the size of trap or cavity would tend to increase the possibility for squeezing.

It is also possible to view Eq. (100) as two separate commutators for the positive and negative frequency components of the field. The relation

$$[\phi^{(+)}(x),\phi^{(-)}(x')]=iG^{(+)}(x,x') = \int\frac{(dk)}{2\omega}\frac{(dk')}{2\omega'}[a(k),a^\dagger(k')]e^{ikx+ik'x'} \tag{101}$$

shows that this is directly related to the squeezing expressed in terms of a and $a^\dagger$ in Sec. IV. To make the squeezing explicit, we may introduce new real variables

$$a_\pm=\frac{1}{\sqrt{2}}\left(s\phi(x)\pm\frac{1}{s}\phi(x')\right) \tag{102}$$

such that

$$[a_+(x_1,x_2),a_-(x_3,x_4)]=C(x_1,x_2,x_3,x_4), \tag{103}$$

and

$$[a_+(x_1,x_2),a_-(x_1,x_2)]=-i\widetilde{G}(x,x'). \tag{104}$$

The corresponding operator U can be constructed and represents the tendency of the field quadratures to be squeezed dynamically over the time interval concerned:

$$U(x,x')=P\exp\left(i\int_x^{x'}dV_x dV_{x'}\times[a_+^2(x_1,x_2)-a_-^2(x_1,x_2)]g(x,x')\right) =P\exp\left(i\int_x^{x'}dV_x dV_{x'}\phi(x)g(x,x')\phi(x')\right), \tag{105}$$

where $g(x,x')$ is introduced to make a causal connection between the field at x and x'. It represents a ‘‘vacuum polarization’’ of the field. The squeezing transformation resulting from this operator is now

$$U(x,x')a_+(x,x')U^{-1}(x',x) =\int_x^{x'}dV_y dV_{y'}a_+(y,y')\cosh[C(x,x',y,y')g(y,y')] +a_-(y,y')\sinh[C(x,x',y,y')g(y,y')],$$

$$U(x,x')a_-(x,x')U^{-1}(x',x) =\int_x^{x'}dV_y dV_{y'}a_-(y,y')\cosh[C(x,x',y,y')g(y,y')] +a_+(y,y')\sinh[C(x,x',y,y')g(y,y')]. \tag{106}$$

If the function $g(y,y')$ is sharply peaked at $y=y'$, this transformation has no squeezing effect. This reflects the need for an inhomogeneity in the development of the system. This transformation does not provide a sharp relationship between the squeezed and nonsqueezed fields, rather there is a dependence on the entire history of the field’s development. This is equivalent to the finite width in momentum space in Eq. (91).

This transformation is simply a nonequilibrium perturbation of the field. It is caused by a bilocal interaction and represents a correlation between the field at different space-time points. Such correlations are related to the notion of off-diagonal long range order (ODLRO) or Bose-Einstein condensation, and it is interesting to speculate on how perturbations might lead to squeezed states of motion in atomic and ionic condensates. (Note: such long range phase correlations, or global symmetry breaking, give the appearance of nonlocal correlations in the field. This is to be understood as a collective effect which in no way violates the assumption of finite speed of communication.)

In summary: by describing every manifestation of squeezing in terms of distinct noncommuting pairs, one gains an insight into the necessity of time-dependent or inhomogeneous interactions from several different perspectives. Although formally distinct, these different squeezings are all related to one another and may be thought of as different manifestations of the same phenomenon. The intimate relationship with the nonequilibrium development of the field plays a central role in the squeezing of the modes.

VI. THE ELECTROMAGNETIC FIELD

The real scalar field has been used as the center point of the discussion thus far. Scalar fields are the relevant variables for atomic and some ionic systems and are more closely related to the single-polarization models commonly discussed in the literature. The most important field, however, from an experimental viewpoint, is the electromagnetic (em) field. This case has been discussed in a real space formulation in Ref. [7], where the authors emphasize the importance of time-dependent interactions in optical media, but do not properly discuss the effect of this inhomogeneity on the field equations. In fact, it is a special property of Maxwell’s equations in 3+1 dimensions which makes them invariant under inhomogeneity (conformal) transformations, so the conclusions of Ref. [7] can be trusted in spite of the apparent omission in their analysis. As a result, it would be superfluous to reproduce their discussion, even though it takes a different approach from the present paper. Instead, to round off this discussion, the foregoing results are only summarized for the specific case of the em field. The results of Ref. [7] are not relativistically covariant, so we extend them and write down generally covariant forms which encompass the time-dependent behavior found in nonlinear media, such as hysteresis. The electric and magnetic fields are really the components of a rank-two tensor; in 3+1 dimensions they happen to have the characteristics of a vector and a pseudovector, respectively, while in 2+1 dimensions, the magnetic field is a pseudoscalar. In covariant language, the interconvertibility of **E** and **B** in relativistic systems due to relative motion are automatically accounted for. Such an interchangeability is particularly important in relativistic systems such as the free electron laser.

To express Maxwell's equations in *covariant* form, we introduce the fields **D** and **H** and a tensor $D_{\mu\nu}$, given by

$$D_{\mu\nu}(x,x')=\begin{pmatrix} 0 & -D_1 & -D_2 & -D_3 \\ D_1 & 0 & H_3 & -H_2 \\ D_2 & -H_3 & 0 & H_1 \\ D_3 & H_2 & -H_1 & 0 \end{pmatrix} \tag{107}$$

in 3+1 dimensions. We see that this tensor has the same structure as

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{108}$$

but with **D** replacing **E** and **H** replacing **B**. This tensor is a nonlocal quantity since the polarization of the medium is a nonlocal relation

$$\mathbf{P}(t)=\int dt\chi(t-t')\mathbf{E}. \tag{109}$$

In terms of this tensor, we can write the action

$$S=\int dV_x dV_{x'}\left\{\frac{1}{4}F^{\mu\nu}(x)D_{\mu\nu}(x,x')-J^\mu A_\mu \delta(x,x')\right\}. \tag{110}$$

The variation of the action is given by

$$\delta S=\int dV_x dV_{x'}\{-\delta A^\nu\partial^\mu D_{\mu\nu}-J^\mu\delta A_\mu\} + \int dV_{x'}d\sigma^\mu\{\delta A_\nu D_{\mu\nu}\}=0. \tag{111}$$

The continuity condition implies that the canonical momentum is $(\mu=0)$

$$\Pi_\mu=D_{0\mu}, \tag{112}$$

and that the condition for continuity across a surface dividing two regions of space is $\mu=i$ divides into two cases:

$$\Delta D_{i0}=\Delta\mathbf{D}=0,$$

$$\Delta D_{ij}=\Delta\mathbf{H}=0. \tag{113}$$

These are the well-known continuity conditions for the field at a dielectric boundary and serve as a check on perhaps unfamiliar formalism. From the surface term in the variation of the action, we obtain the covariant form of the commutation relations for the field:

$$\left[A_\lambda(x),\int dV_{x''}D_{\mu\nu}(x',x'')\right]=ig_{\lambda\nu}\delta_\mu(x,x'), \tag{114}$$

where $\delta_\mu(x,x')$ is the δ function on a spacelike hypersurface pointing in the μ direction and $g_{\mu\nu}$ here is the (local) spacetime metric. The canonical value for μ is zero. This commutator can also be expressed in gauge-invariant form,

$$\left[F_{\rho\lambda}(x),\int dV_{x''}D_{\mu\nu}(x',x'')\right]=i\partial_\rho g_{\lambda\nu}\delta_\mu(x,x') - i\partial_\lambda g_{\rho\nu}\delta_\mu(x,x'), \tag{115}$$

so that the canonical limit, in 3+1 dimensions, gives

$$[\mathbf{B},\mathbf{D}]=i\nabla\times\delta(\mathbf{x},\mathbf{x}'). \tag{116}$$

It is more useful though to use the dual of the field strength $F_{\mu\nu}$ to express this, since this gives a more accurate and compact impression of which components are conjugate to one another. It also makes transparent an algebraic step below. Thus, in 3+1 dimensions one has

$$\left[F^*_{\alpha\beta},\int dV_{x''}D_{\mu\nu}(x',x'')\right]=\frac{1}{2}ig_{\lambda\nu}\epsilon_{\alpha\beta}{}^{\lambda\rho}\partial_\rho\delta_\mu(x,x'), \tag{117}$$

where the dual field strength in 3+1 dimensions is defined by

$$F^*_{\alpha\beta}=\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}. \tag{118}$$

In 2+1 dimensions, one has

$$\left[F^*_\alpha,\int dV_{x''}D_{\mu\nu}(x',x'')\right]=\frac{1}{2}ig_{\lambda\nu}\epsilon^\lambda_{\alpha\beta}\partial^\beta\delta_\mu(x,x'), \tag{119}$$

where the dual is given by

$$F^*_\mu=\frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}. \tag{120}$$

We are now interested in constructing minimal uncertainty states by forming combinations of these conjugate variables. An important point to note here, which is sometimes presented in a misleading manner in the literature, is that the coupling of sources to the electromagnetic field has the form

$$S_{\rm int}=\int dV_x J^\mu A_\mu+\int dV_x dV_{x'}J^{\mu\nu}D_{\mu\nu}(x,x'), \tag{121}$$

i.e., currents couple to the vector potential and not to the electric field. The term $J^{\mu\nu}$ behaves like an external ''dipole force'' and is related to the atomic polarization or susceptibility. This means that, rather than forming the combinations $\mathbf{B}\pm i\mathbf{D}$ as in Ref. [7] (which is a pseudovector in $3+1$ dimensions, and cannot be generalized), a cleaner and more general form can be constructed using the scalar

$$a_{\pm}=sj^{\mu}A_{\mu}(x)\pm\frac{i}{s}\int dV_{x'}j^{\mu\nu}D_{\nu\mu}(x,x'), \qquad (122)$$

where j^{μ} is a conserved spacelike current and $j^{\mu\nu}$ is antisymmetric. The construction of the transformation term now involves the following forms, up to a gauge transformation:

$$a_{+}^{2}-a_{-}^{2}\sim j^{\mu}A_{\mu}\cdot j^{\alpha\beta}\int dV_{x''}D_{\beta\alpha}(x,x'). \qquad (123)$$

Now, from the relative orientation of the sources combined with gauge invariance, one may write, in $2+1$ dimensions,

$$j^{\mu}j^{\nu\lambda}\sim\theta(x)\epsilon^{\mu\nu\lambda}, \qquad (124)$$

and in $3+1$ dimensions one has

$$j^{\mu}j^{\nu\lambda}\sim\theta_{\rho}(x)\epsilon^{\mu\nu\lambda\rho}. \qquad (125)$$

This means that the Chern-Simons term,

$$S_{\mathrm{CS}}=\frac{1}{4}\int dV_{x}dV_{x'}\,\theta(x)\epsilon^{\mu\nu\lambda}A_{\mu}D_{\nu\lambda}(x,x'), \qquad (126)$$

can be regarded as a source term or driving force which tends to squeeze minimal packets of the field in $(2+1)$-dimensional B, $\mathbf{D}$ space. Thus, squeezing of the Fourier modes arises directly from the nonlocal susceptibility $\chi(x,x')$ and permeability $\mu(x,x')$ contained in $D_{\mu\nu}(x,x')$, by analogy with Eqs. (79)–(92). That the Chern-Simons term tends to implement a symmetry transformation (squeezing) might be expected since it is known to have no independent dynamics of its own. This is interesting since the Chern-Simons term is cited in connection with a number of physical systems, most notably the quantum Hall effect. The above result prompts immediate speculation as to whether squeezed variables can be identified in these systems. The converse speculation that photonic solitons might obey fractional statistics has also been proposed recently [20]. The Chern-Simons term is the quantity analogous to $\gamma^{\mu}\phi(\partial_{\mu}\phi)$ introduced in paper I with γ_{μ} as a ''gauge field'' or a chemical potential for the field. This also lends credence to the view of the Chern-Simons coefficient as a nonequilibrium parameter in Refs. [21,22].

VII. DISCUSSION OF THE INHOMOGENEOUS THEORY

The preceding sections present a detailed justification of the way in which squeezing interactions result from the nonequilibrium evolution of the quantum field. It is a ''microscopic'' rather than a phenomenological description, involving Green functions and observable expectation values. In this section we shall use the results to determine the magnitude of the corrections which can be expected to the single-mode, idealized models presented in the literature.

There are two independent issues to be addressed: (i) the corrections arising from the essentially inhomogeneous nature of the interaction, and (ii) corrections arising from a proper observance of the laws of relativity and locality of interactions. The first of these may be dealt with by comparing the orders of magnitude of the frequency of radiation with the frequency (or rate of change) of the time- or space-dependent interaction causing the inhomogeneous development. The latter might be the rate of change of a spatially heterogeneous medium, or it might be the characteristic transition or hysteresis times of atoms with nonlinear susceptibilities; the nature of the spacetime dependence has been kept general in this paper. For simplicity we, shall assume that all spacetime variations are harmonic (sinusoidal) in time and talk about the frequency of the modulating inhomogeneity Ω.

The effect of the inhomogeneous corrections is twofold: first, there is a shift in the frequency (timelike) or wave number (spacelike), and second, there is an effective mass or gap in the classical dispersion relation. Since the frequency and wave number corrections add or subtract directly from the actual frequency of radiation, they give rise to a shift of each mode. The sign of the shift depends upon the gradient of the change in (i) the effective frequency found from the dispersion relation, and (ii) the rate of change of the Wigner function or density matrix for the field. Thus, if a physical system has a time-dependent interaction with a characteristic frequency which is, say, one percent of the frequency of radiation, then the shift can be expected to be maximally of this same order, multiplied by a frequency-dependent exponential damping factor. A similar argument holds for spatial gradients and the wave number, though the effect for spatial inhomogeneity is smaller by a factor of c. Some typical examples could include the modulation of atomic energies by lightwaves (with typically 10 000 times longer wavelength) leading to shifts of the order of 10^{6}–10^{7} Hz at optical frequencies, which is of the order of photon recoil energies [23] and should be observable along with the Lamb shift [24]. For nonlinear susceptibilities one would expect shifts on the order of MHz and above at optical frequencies.

The effective mass may be estimated by using $\hbar\Omega\sim mc^{2}$, where Ω is the rate of the time-dependent interaction. For optical frequencies and microwaves, the mass is extremely small (around one thousandth of the electron rest mass), but for energetic x-ray time scales, it can clearly approach the electron rest mass. This may play a significant role in free electron lasers.

The second of the distinct issues addressed here concerns the finiteness of the speed of light in mediating explicit interactions. In this paper we assume that the interaction term in the action or Hamiltonian must be of a local form. Only interactions which are in concord with special relativity are permitted. The fact that such (bi)local interactions lead to nonlocal correlations is well known and should not be confused with the essential locality of the interaction terms. We shall discuss this point further below. As derived in Sec. IV, the extent to which squeezing can be realized in a two-photon (two-field) interaction depends on the spectral content of an interaction term of the form

$$S_{\rm int}=\int dV_x dV_{x'}\,\phi(x)A(x,x')\,\phi(x'). \tag{127}$$

This depends on the nature of the interaction kernel $A(x,x')$ in Fourier space, which, in turn, depends on the details of the microscopic system being modeled. To answer the question of the intrinsic width of participating modes required for squeezing, we assume the following Lorentzian form for this kernel:

$$A(x,x')=\int \frac{d^n k}{(2\pi)^n}\,e^{ik(x-x')}\,\frac{1}{k^2+\mathcal{M}^2(\overline{x})}. \tag{128}$$

This form incorporates the assumption of limited c. The effective mass $\mathcal{M}(\overline{x})$ is possibly the inhomogeneous gap in the dispersion relation for the waves. The boundary conditions on this propagation kernel do not play a role in our analysis, but one would clearly expect to restrict the sum to include only causal times in the action [3]. Given the above form, we can now transform to Fourier space. If one assumes that the effective mass is independent of $\overline{x}$, implying a homogeneous theory, then the action takes the form

$$S_{\rm int}=\int \frac{d^n k}{(2\pi)^n}\,\frac{\phi(k)\,\phi(-k)}{k^2+\mathcal{M}^2}. \tag{129}$$

The idealized form for the squeezing operator, used in the literature, $a^2-(a^\dagger)^2$ gives not $\phi(k)\,\phi(-k)$ in the numerator, but $\phi(k)\,\phi(k)$, which is completely nonlocal and cannot be generated by the given $A(x,x')$ for any $\mathcal{M}$. However, if the mass depends on the average position, one may reduce the interaction term to the form

$$S_{\rm int}=\int dV_{\overline{x}}\,\frac{d^n k_1}{(2\pi)^n}\,\frac{d^n k_2}{(2\pi)^n}\,e^{i(k_1+k_2)\overline{x}}\,\frac{\phi(k_1)\,\phi(k_2)}{\frac{1}{4}(k_1-k_2)^2+\mathcal{M}^2(\overline{x})}. \tag{130}$$

The meaning of this term is most easily seen if we take the inhomogeneity to be purely in time and express in even and odd variables

$$S_{\rm int}\sim\int dx^0\int \frac{d^{n-1}k\,d\overline{\omega}\,d\widetilde{\omega}}{(2\pi)^{n+1}}\,e^{-2i\widetilde{\omega}\overline{x}^0}\,\frac{\phi(\overline{\omega}+\widetilde{k})\,\phi(\overline{\omega}-\widetilde{k})}{(\widetilde{\mathbf{k}}^2-\widetilde{\omega}^2)+\mathcal{M}^2(\overline{x^0})}, \tag{131}$$

where $\widetilde{k}=\frac{1}{2}(k_1-k_2)$, and $\overline{k}=\frac{1}{2}(k_1+k_2)$, and so on. It is now seen that squeezing depends on $\overline{\omega}$ being large compared to $\widetilde{\omega}$. Thus, outgoing photons (fields) with identical frequency profiles are squeezed best. The denominator ensures that the interaction term always includes a finite width, however. This is to do with the Lorentzian form and not to do with the inhomogeneity. Another feature to observe in the above expression is that the denominator vanishes for on-shell (classical) photons and thus peaks sharply at this value. The interaction looks most like the ideal squeezing interaction when the variation of the Lorentzian is slow, however. Thus, it is the flattest parts of the Lorentzian which are squeezed most. This occurs (i) at resonance maxima and minima, and (ii) for highly nonclassical (off-shell) states. If one takes, as an illustrative case, a harmonic inhomogeneity, $\mathcal{M}(\overline{t})=\mathcal{M}\sin(\overline{t})$, then it is possible to evaluate the above integral analytically. The specific expression is not particularly interesting; the behavior of the result is no longer a sharp δ-function interaction at a single frequency, but a peak of finite width with oscillating side bands (see Fig. 1). It is also interesting to note the asymmetry in the shape of the peaks which arises from the finite limits on the integral over t. This should be observable for strongly time-dependent interactions in a finite cavity.

We may thus conclude from the above that a single-mode, idealized model is a good approximation for strongly time-dependent interactions: squeezing is maximal when the interacting photons have similar frequency profiles and is maximal at a strong resonance, such as in a high-Q cavity. The instrinsic line width is of the order $\mathcal{M}c^2/\hbar$ for $\mathcal{M}\neq 0$. Finally, the term ''nonlocal'' is often used in connection with squeezing.

It be clear from the analysis in this paper that ''nonlocality'' in a relativistically covariant field theory is a derived concept: squeezing is accomplished only through propagated interactions which obey the theory of relativity. [The term ''nonlocal'' is sometimes used to refer to function with more than one variable, such as $A(x,x')$, but here we mean faster than light communication.] Since the wave packets have an intrinsic width, there is a probable time delay in measurement of the order $1/\Delta\omega$ between two photons or fields. The inhomogeneity of the theory does not play a special role here. Nonlocal correlations may indeed be observable (phase correlations which move with the phase velocity, for instance), but no direct interaction (communication) is either necessary or possible between points lying outside the light cone.

VIII. SUMMARY

Squeezed states are conventionally explained as two-particle emission and absorption processes in momentum space, using a formalism of single-mode creation and destruction operators [18]. In this work, the notion of squeezing is addressed from several related perspectives. A spacetime approach is employed to elucidate the physical reason for squeezing, in terms of nonequilibrium dynamics of the quantum field. Momentum-space results are presented for their relationship with experimental situations. Results are given for the real scalar field (representing atoms and certain ionic systems) and the electromagnetic field. The formally distinct commutators of interest are (i) $[x,p]$, the true position and momentum of one-particle excitations; these are modified in the presence of spacetime-dependent interactions such as those which generate squeezings; (ii) $[a,a^\dagger]$ and $[Q,P]$, the single-mode quadratures of the field, related to antibunching, intrinsically polychromatic with minimum width limited by the weight $1/\omega$; (iii) $[\phi,\Pi]$, the canonical field variables which express the same as (ii) in real space, and make contact with nonequilibrium, inhomogeneous field theory as described in paper I; and finally (iv) $[\phi(x),\phi(x')]$, the unequal time commutator, which contains the dynamical evolution of squeezed packets. All of these are expressions of the same underlying dynamics, and merely reflect different aspects of the evolution through alternative variables. The transformation operators U which squeeze minimal field configurations

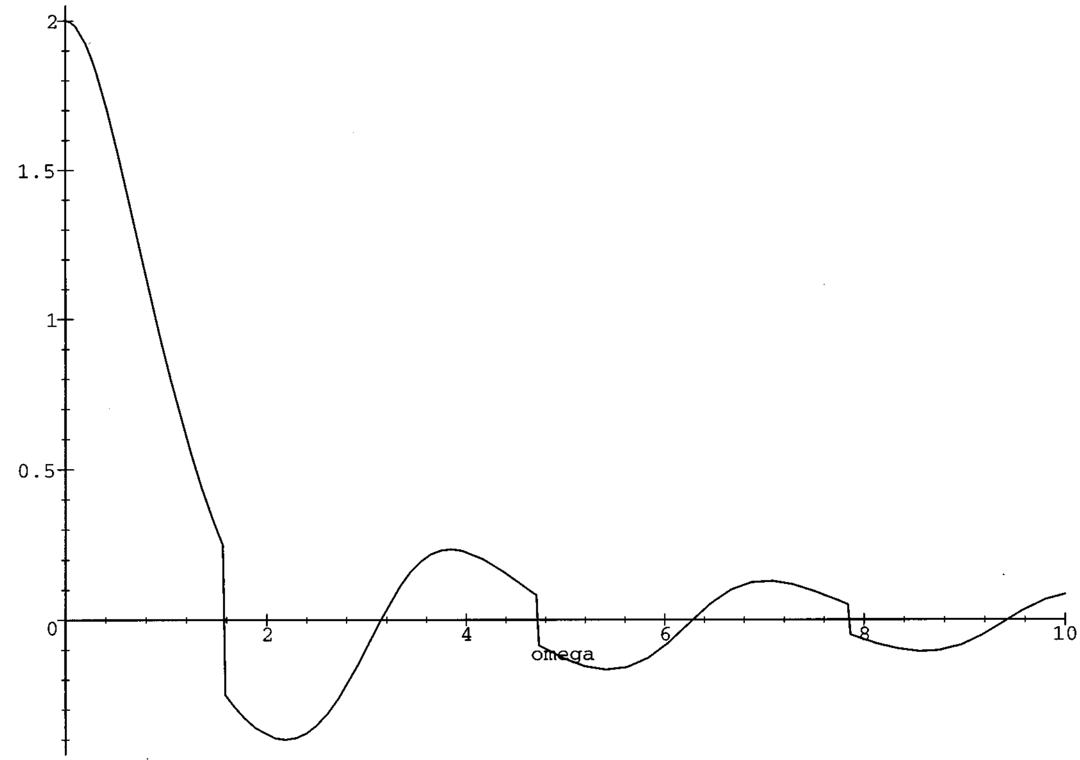

FIG. 1. A plot of the integral $\int_{-L}^{L}\cos(2\omega t)dt/[\sin^2(t)+1]$, over a range of ω. This represents a harmonic inhomogeneity of half the frequency of the field components, for illustrative purposes. The asymmetry of the curve is a result of the finite correlation length combined with the time dependence of the effective mass, $\mathcal{M}(\overline{t})=\sin(\overline{t})$, in scaled units. This particular integral may be solved analytically, but in general a numerical investigation is required. The shape of this graph is to be compared to a δ function for a monochomatic theory.

are best viewed in real space, and can be identified with off-diagonal source terms, analogous to the Chern-Simons terms in $(2+1)$-dimensional field theory through a change of variables. The basic physical reason which underlies squeezing is an inhomogeneous (time-dependent) interaction of the field, expressed by nonlocal interactions (hysteresis and electron recoil). Environment and decoherence are intrinsic issues in squeezed systems. Two-particle creation processes (down conversion, etc.) are the first term in a ''nonlocal source'' expansion of the evolving density matrix [12,3]. In spite of the apparent nonlocal appearance of squeezing, it is shown here that it arises from purely local interactions.

New in this paper is the covariant, unified picture of squeezing; the relation to nonequilibrium field theory, and the attention to finite speed of light and finite system size (shorter coherence length), leading to an intrinsic line width and sidebands. The latter might well be an important issue in solid state lasers, where spatial inhomogeneities are more pronounced and coherence over many optical wavelengths is perturbed by inhomogeneous structure in the medium [25]. It could also be important in extremely relativistic systems such as astrophysical sources and the free electron laser where the degree of coherence is less acute [8]. The connection with off-diagonal long range order suggests that squeezed velocity distributions might well characterize a Bose-Einstein condensate formed in a trap, if the condensate were formed rapidly enough.

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[19] Note that there is a typographical error in paper I in the definitions of $\widetilde{G}(x,x')$ and $\widetilde{G}(x,x')$. The definitions should read $-i[\phi(x),\phi(x')]=\widetilde{G}(x,x')$ and $-i\{\phi(x),\phi(x')\}=\widetilde{G}(x,x')$.

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