PHYSICAL REVIEW D VOLUME 55, NUMBER 12 15 JUNE 1997

BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than four printed pages and must be accompanied by an abstract.

Translation of multipoles for a 1/*r* potential

Christian D'Urso and E. G. Adelberger
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 3 February 1997)

We outline an analytic method for calculating inner and outer multipoles of a $1/r$ potential about an arbitrary point in terms of known multipoles about a given point. [S0556-2821(97)01712-8]

PACS number(s): 04.40.−b, 04.80.Cc

I. INTRODUCTION

Numerical integrations are often used to calculate the inner and outer spherical multipoles q_{lm} and Q_{lm} of distributions of mass, charge, etc. Problems arise when asymmetrical shapes are involved and high accuracy is required. This problem is frequently encountered in experimental gravitation where the multipole formalism is used to calculate torques and forces [1,2] on test bodies in the fields of attractors with realistic geometries. Many cases of practical interest deal with distributions that can be accurately approximated as a linear superposition of basic geometric shapes such as cylindrical annuli, rectangular prisms, trapezoidal slabs, etc. To calculate the inner and outer multipoles of these distributions it is necessary to know the multipoles about the point of interest for each of the superposed shapes. We will show that if the inner and outer multipoles of these simple shapes are known about any one point then analytic solutions can be obtained for the multipoles of a superposed distribution about an arbitrary point. This allows one to calculate, without numerical integration, inner and outer multipoles of complex systems and, perhaps more importantly, to study the effect of perturbations of the system arising from misalignments, etc. Rotations of the spherical multipoles are easily carried out using standard techniques of angular momentum algebra. This paper addresses the problem of translations.

II. FORMALISM

Consider an arbitrary distribution of mass, charge, etc., that can be separated into two disjoint bodies. One body, the "object," is considered to be in the field produced by the other body, the "source." Integrating over this distribution, the object has inner multipoles

$$q_{lm}=\int \rho_o(\vec{r})r^l Y^*_{lm}(\hat{r})d^3r \tag{1}$$

and the source has outer multipoles

$$Q_{lm}=\int \rho_s(\vec{r})r^{-(l+1)}Y_{lm}(\hat{r})d^3r, \tag{2}$$

where Y^*_{lm} and Y_{lm} are spherical harmonics and the density functions $\rho_o(\vec{r})$ and $\rho_s(\vec{r})$ correspond to the object and source bodies, respectively. Assuming that both q_{lm} and Q_{lm} are known about a coordinate origin, O, we now wish to know the corresponding inner and outer multipoles about an arbitrary point, $\mathcal{P}(r',\theta',\phi')$:

$$\tilde{q}_{LM}=\int \rho_o(\vec{r}'')r''^L Y^*_{LM}(\hat{r}'')d^3r'' \tag{3}$$

and

$$\tilde{Q}_{LM}=\int \rho_s(\vec{r}'')r''^{-(L+1)}Y_{LM}(\hat{r}'')d^3r''. \tag{4}$$

To express the moments about $\mathcal{P}$ in terms of the moments about O it is necessary to expand the solid harmonics in the integrands of Eqs. (3) and (4) using the relations [3]

$$r''^L Y_{LM}(\hat{r}'')=\sum_{l',l=0}^{L}\sqrt{\frac{4\pi(2L+1)!}{(2l'+1)!(2l+1)!}}\times r'^{l'}r^l\{Y_{l'}(\hat{r}')\otimes Y_l(\hat{r})\}_{LM}\delta_{L,l+l'} \tag{5}$$

and

$$r''^{-(L+1)}Y_{LM}(\hat{r}'')=\sum_{l',l=0}^{\infty}\sqrt{\frac{4\pi(2l)!}{(2L)!(2l'+1)!}}\times\frac{r'^{l'}}{r^{l+1}}\{Y_{l'}(\hat{r}')\otimes Y_l(\hat{r})\}_{LM}\delta_{L,l-l'}, \tag{6}$$

where the tensor product is given by

$$\{Y_{l'}(\hat{r}')\otimes Y_l(\hat{r})\}_{LM}=\sum_{m',m} C(l',m',l,m,L,M)\times Y_{l'm'}(\hat{r}')Y_{lm}(\hat{r}), \tag{7}$$

and C is a Clebsch-Gordan coefficient. These equations differ from those given in Ref. [3]. We found by doing explicit calculations, such as the ones that follow, that the equations in Ref. [3] give incorrect results by a factor of $(-1)^l$. The inner multipole moments about $\mathcal{P}$ now have the form

$$\tilde{q}_{LM}=\sum_{l',m',l,m}\sqrt{\frac{4\pi(2L+1)!}{(2l'+1)!(2l+1)!}}r'^{l'}Y^*_{l'm'}(\hat{r}')\times C(l',m',l,m,L,M)\delta_{L,l+l'}\int\rho_o(\vec{r})r^lY^*_{lm}(\hat{r})d^3r, \tag{8}$$

while the outer multipoles are

$$\tilde{Q}_{LM}=\sum_{l',m',l,m}\sqrt{\frac{4\pi(2l)!}{(2L)!(2l'+1)!}}r'^{l'}\times Y_{l'm'}(\hat{r}')C(l',m',l,m,L,M)\times\delta_{L,l-l'}\int\rho_s(\vec{r})r^{-(l+1)}Y_{lm}(\hat{r})d^3r. \tag{9}$$

It is easy to see that the integrals in Eqs. (8) and (9) are simply the multipole moments about O, given in Eqs. (1) and (2). Therefore, we have

$$\tilde{q}_{LM}=\sum_{l',m',l,m}\sqrt{\frac{4\pi(2L+1)!}{(2l'+1)!(2l+1)!}}r'^{l'}\times Y^*_{l'm'}(\hat{r}')C(l',m',l,m,L,M)\delta_{L,l+l'}q_{lm} \tag{10}$$

and

$$\tilde{Q}_{LM}=\sum_{l',m',l,m}\sqrt{\frac{4\pi(2l)!}{(2L)!(2l'+1)!}}r'^{l'}\times Y_{l'm'}(\hat{r}')C(l',m',l,m,L,M)\delta_{L,l-l'}Q_{lm}. \tag{11}$$

These expressions provide analytic solutions for the multipole moments about an arbitrary point in terms of known moments about a given point. Equation (10) is equivalent to one given in Ref. [2].

Equations (10) and (11) show that the inner moments $\tilde{q}_{LM}$ depend on the finite set of moments q_{lm} with $l<L$, whereas, the outer moments $\tilde{Q}_{LM}$ depend on the infinite set of moments Q_{lm} with $l>L$. However, the infinite sum in Eq. (11) does not produce problems in practical applications. Because the Q_{lm} scale like $1/R$, where R is a characteristic distance from O to the object, one sees that the $\tilde{Q}_{LM}$ moments induced by a displacement r' are smaller by a factor of $(r'/R)^{l'}$ compared to the characteristic size of the Q_{LM} moments of the same order. So, in general, the series converges with only a small number of terms.

III. SELECTION RULES

In this section we mention selection rules that are useful in understanding the effects on the multipole moments due to perturbations in the position of the objects. For small displacements the leading order terms in Eqs. (10) and (11) have $l'=1$. As a result, a small translation of an object with moments q_{lm} and Q_{lm} induces new moments $\tilde{q}_{l+1,m'}$ and $\tilde{Q}_{l-1,m'}$ where $m'=m$ if the displacement is along $\hat{z}$ and $m'=m\pm1$ if the displacement is in the x-y plane. In second order in the displacement one induces new moments $\tilde{q}_{l+2,m''}$ and $\tilde{Q}_{l-2,m''}$ where $m''=m$ for displacements along $\hat{z}$ and $m''=m,m\pm2$ for displacements in the x-y plane.

IV. APPLICATIONS

As an illustration of the effectiveness of Eqs. (10) and (11) consider two unit charges, of opposite sign, initially located at $\pm a$ on the x axis, respectively. The inner moments of this charge distribution calculated about the origin using Eq. (1) have odd l and m and the leading terms are

$$q_{11}=-a\sqrt{\frac{3}{2\pi}}$$

and

$$q_{33}=-a^3\sqrt{\frac{35}{16\pi}}.$$

If we now displace these point charges by an amount r' along the positive x axis, moments with even l and m are generated. As an example, consider the q_{44} moment, which can be calculated analytically using Eq. (3) with

$$\rho_o(\vec{r})=\frac{\delta(\theta-\pi/2)[\delta(r-(a+r'))\delta(\phi)-\delta(r-(a-r'))\delta(\phi-\pi)]}{r^2}, \tag{12}$$

so that

$$\tilde{q}_{44}=(a+r')^4Y^*_{44}(\pi/2,0)+(a-r')^4Y^*_{44}(\pi/2,\pi)$$
$$=3\sqrt{\frac{35}{8\pi}}(a^3r'+ar'^3). \tag{13}$$

For comparison we calculate the $\tilde{q}_{44}$ moment using Eq. (10). From the discussion above, $\tilde{q}_{44}$ depends only on the q_{11} and q_{33} moments. The q_{33} moment is the first-order contribution to the sum and the q_{11} moment is the third-order contribution, the second-order term is absent because of the symmetries of the q_{lm}'s calculated about the origin. Therefore, Eq. (10) gives

$$\tilde{q}_{44}=\sqrt{\frac{4\pi 9!}{3!7!}}r'Y^*_{11}(\pi/2,0)C(1,1,3,3,4,4)q_{33}$$
$$+\sqrt{\frac{4\pi 9!}{7!3!}}r'^3Y^*_{33}(\pi/2,0)C(3,3,1,1,4,4)q_{11}$$
$$=3\sqrt{\frac{35}{8\pi}}(a^3r'+ar'^3). \tag{14}$$

In this case the result to third order is exact because it includes all the nonvanishing terms in Eq. (10).

The outer multipole moments are also easily calculated for the system of displaced charges. As an illustration, we calculate the $\tilde{Q}_{22}$ moment using Eq. (4) and the density function given by Eq. (12) to obtain the displaced outer multipole moment

$$\tilde{Q}_{22}=\frac{1}{4}\sqrt{\frac{15}{2\pi}}[(a+r')^{-3}-(a-r')^{-3}]. \tag{15}$$

To calculate $\tilde{Q}_{22}$ using Eq. (11) we need all moments Q_{lm} with $l\geqslant 2$. To first order, we require the Q_{33} and Q_{31} moments,

$$Q_{33}=-\frac{1}{a^4}\sqrt{\frac{35}{16\pi}},\quad Q_{31}=\frac{1}{a^4}\sqrt{\frac{21}{16\pi}}.$$

The second-order terms vanish because $Q_{4m}=0$. To third order we need the Q_{55}, Q_{53}, Q_{51}, and Q_{5-1} moments:

$$Q_{55}=-\frac{1}{4a^6}\sqrt{\frac{693}{16\pi}},\quad Q_{53}=\frac{1}{4a^6}\sqrt{\frac{385}{16\pi}},$$
$$Q_{51}=-\frac{15}{8a^6}\sqrt{\frac{11}{30\pi}},\quad Q_{5-1}=\frac{15}{8a^6}\sqrt{\frac{11}{30\pi}}.$$

The $\tilde{Q}_{22}$ moment calculated from Eq. (11) is

$$\tilde{Q}_{22}=\sqrt{\frac{4\pi 6!}{4!3!}}r'[Y_{1-1}(\pi/2,0)C(1,-1,3,3,2,2)Q_{33}$$
$$+Y_{11}(\pi/2,0)C(1,1,3,1,2,2)Q_{31}]$$
$$+\sqrt{\frac{4\pi 10!}{4!7!}}r'^3[Y_{3-3}(\pi/2,0)C(3,-3,5,5,2,2)Q_{55}$$
$$+Y_{3-1}(\pi/2,0)C(3,-1,5,3,2,2)Q_{53}$$
$$+Y_{31}(\pi/2,0)C(3,1,5,1,2,2)Q_{51}$$
$$+Y_{33}(\pi/2,0)C(3,3,5,-1,2,2)Q_{5-1}]. \tag{16}$$

For a ten percent displacement, $a=10$, $r'=1$, the values of $\tilde{Q}_{22}$ and $\tilde{Q}_{22}$ from Eqs. (15) and (16) are

$$\tilde{Q}_{22}=-2.39655\times 10^{-4}$$

and

$$\tilde{Q}_{22}=-2.3949\times 10^{-4},$$

respectively, which differ by 0.07%.

V. CONCLUSION

We have shown that if the inner and outer multipoles about a given point are known for any distribution of mass, charge, etc., then it is always possible to construct solutions for $\tilde{q}_{LM}$ and $\tilde{Q}_{LM}$ about a nearby point that are linear combinations of the inner and outer multipoles about the original point. This eliminates the need for error-inducing numerical integrations when dealing with perturbations of distributions whose unperturbed multipole moments are known and, for example, to center precisely objects in the field of a source using purely gravitational means (see Ref. [4]). With a catalog of solutions for multipoles of simple geometric distributions, it is possible to construct arbitrarily complicated distributions and obtain analytic solutions for their multipole moments about arbitrary points.

ACKNOWLEDGMENTS

We thank B. R. Heckel for very helpful comments on this paper. This work was supported by the National Science Foundation (Grants Nos. PHY-9104541 and PHY-9602494).

[1] E. G. Adelberger, C. W. Stubbs, B. R. Heckel, Y. Su, H. E. Swanson, G. Smith, J. H. Gundlach, and W. F. Rogers, Phys. Rev. D **42**, 3267 (1990).

[2] Y. Su, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, M. Harris, G. L. Smith, and H. E. Swanson, Phys. Rev. D **50**, 3614 (1994).

[3] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988), p. 167.

[4] J. H. Gundlach, G. L. Smith, E. G. Adelberger, B. R. Heckel, and H. E. Swanson, Phys. Rev. Lett. **78**, 2523 (1997).