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Focusing of timelike world sheets in a theory of strings

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An analysis of the generalized Raychaudhuri equations for string world sheets is shown to lead to the notion of *focusing* of timelike world sheets in the classical Nambu–Goto theory of strings. The conditions under which such effects can occur are obtained. Explicit solutions as well as the Cauchy initial value problem are discussed. The results closely resemble their counterparts in the theory of point particles which were obtained in the context of the analysis of spacetime singularities in general relativity many years ago.
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I. INTRODUCTION

The theory of extended objects such as strings and higher branes embedded in an ambient background spacetime has been extensively studied in the recent past. The string or membrane viewpoint has found useful applications in seemingly diverse fields ranging from a theory of fundamental strings [1] in the context of quantum gravity and unification to two-dimensional objects (hypersurfaces) embedded in an Euclidean background, examples of which are abundant in the active area of biological (amphiphilic) membranes [2]. The consequences of the generalization of those equations which describe various features of a point particle theory to the case of strings and membranes are therefore worth investigating.

To get into the relevant context we must first ask what these equations are. For any theory, the starting point is almost always the action. The action for the relativistic point particle is the integral of the arc length, ds. The first variation results in the equation of motion which in a general background is the *geodesic equation*. Solutions to this equation are the geodesic curves of the corresponding background geometry. The second variation of the action is related to the *Jacobi or geodesic deviation* equation which governs the separation of one geodesic from another in a generic curved background. In general relativity (GR), where spacetime curvature is related to matter, geodesic deviation provides a measure of the gravitational force. An alternative set of equations which contain further information about the nature of a one parameter family of geodesics is composed of the *Raychaudhuri equations* [3]. These deal with the issue of the focusing or defocusing of geodesic congruences and play a major role in the proofs of the singularity theorems of GR.

Each of the above-mentioned equations have generalizations for the case of strings as well as higher branes. The geodesic equation is replaced by the string or membrane equations of motion and constraints which emerge out of the first variation of the area functional (Nambu-Goto action). The Jacobi equation has also been extended recently by evaluating the second variation [4,5] (see also [6] for an earlier reference in the mathematics literature). Finally, generalized Raychaudhuri equations also exist today due to the efforts of Capovilla and Guven [7]. However, not much attention has been devoted towards understanding the general features of the solutions of the Jacobi and Raychaudhuri equations in string or membrane theories in a way similar to their treatment in the context of GR. Our main aim in this paper would therefore be to analyze the Raychaudhuri equation for strings and derive the world sheet analogue of geodesic focusing.

II. THE RAYCHAUDHURI EQUATIONS

In introducing the Raychaudhuri equations and their generalizations we shall prefer writing down the equations first and then explaining the relevance and geometrical meaning of the various quantities which appear.

For the case of families of timelike geodesic curves the Raychaudhuri equation for the quantity known as the expansion θ is given as

$$\frac{d\theta}{d\lambda}+\frac{1}{3}\theta^2+2\sigma^2-2\omega^2=-R_{\mu\nu}\xi^\mu\xi^\nu. \tag{1}$$

The expansion θ measures the rate of change of the cross sectional area of a family of geodesics. $\sigma^{\mu\nu}$ and $\omega^{\mu\nu}$ are known as the shear and rotation of the congruence. Thus, if the expansion is negative (positive) somewhere, we can conclude that the congruence or family is converging (diverging). Moreover, if the expansion goes to $-\infty$, we have focusing of geodesics—a generalization of which is the main topic here.

An alternative way to look at Eq. (1) is to convert it into a second order, linear, ordinary differential equation. This is done by a simple change of variables $\theta=(3/F)dF/d\lambda$ which yields the equation

$$\frac{d^2F}{d\lambda^2}+\frac{1}{3}H(\lambda)F=0, \tag{2}$$

where $H(\lambda)=R_{\mu\nu}\xi^\mu\xi^\nu+2\sigma^2-2\omega^2$. The focusing theorem which originates from an analysis of either version of the

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Raychaudhuri equation states that if $\omega^2=0$ and matter satisfies an energy condition [usually $R_{\mu\nu}\xi^\mu\xi^\nu \geqslant 0$ or, using Einstein's field equation, $(T_{\mu\nu}-\frac{1}{2}Tg_{\mu\nu})\xi^\mu\xi^\nu \geqslant 0$], then converging ($\theta$ negative) families of timelike or null geodesics must necessarily focus within a finite value of the affine parameter λ. Note that the existence of zeros in the class of solutions of Eq. (2) implies the divergence of the expansion. Detailed analysis of the focusing theorems can be found in [8–10]. Physically, focusing is a natural consequence of the *attractive* nature of gravitating matter and acts as a pointer to the existence of spacetime singularities.

A generalization of the above equation is achieved by considering families of surfaces as opposed to families of curves. These surfaces are timelike (i.e., they have a Lorentzian induced metric on the world sheet) and extremal with respect to variations of the Nambu-Goto action. The original derivation for the most general case of D-dimensional timelike, extremal, Nambu-Goto surfaces embedded in an N-dimensional Lorentzian background is due to Capovilla and Guven [7]. The form of the equation for string world sheets given below [11] is obtained by using certain properties of two-dimensional surfaces (the choice of isothermal coordinates) and simplifications achieved by implementing the Gauss-Codazzi integrability conditions. We have

$$-\frac{\partial^2 F}{\partial \tau^2}+\frac{\partial^2 F}{\partial \sigma^2}+\frac{1}{N-2}\Omega^2(-{}^2R+R_{\mu\nu}E^\mu_a E^{\nu a})F=0, \quad (3)$$

where σ,τ are the world sheet coordinates, ${}^2R$ is the world sheet Ricci curvature, $R_{\mu\nu}$ is the spacetime Ricci tensor, and E^μ_a are the tangent vectors to the world sheet in the frame basis $[g(E_a,E_b)=\eta_{ab}]$. $\Omega^2(\sigma,\tau)$ is the conformal factor in the metric induced on the world sheet from the background geometry. We shall denote the coefficient of F in the third term collectively as $\alpha(\sigma,\tau)=[1/(N-2)]\Omega^2(-{}^2R+R_{\mu\nu}E^\mu_a E^{\nu a})$.

The above generalized equation is a second order, linear, hyperbolic partial differential equation. It is the parallel of Eq. (2). We now have two quantities θ_τ and θ_σ which represent the generalized expansions along the τ and σ directions of the world sheet and are obtained by taking the partial derivative of $\ln F$ with respect to the τ and σ variables, respectively. Our objective now is to obtain and analyze the solutions of this equation.

III. SOLUTIONS IN LIGHT-CONE COORDINATES

In order to arrive at and extract information about the solutions of Eq. (3) it is useful to make a few assumptions about the quantity $\alpha(\sigma,\tau)$. We can think of two possibilities straightaway. The first of these is to assume that α is separable in the σ,τ variables. On the other hand, one may prefer going over to light-cone coordinates and assume separability in that system. The conclusions related to the former case have already been discussed in a previous paper by this author [11]. We therefore concentrate on the latter.

In light-cone coordinates defined by

$$\sigma_+=\tfrac{1}{2}(\sigma-\tau), \quad \sigma_-=\tfrac{1}{2}(\sigma+\tau), \quad (4)$$

the generalized Raychaudhuri equation takes the form

$$\frac{\partial^2 F}{\partial\sigma_+\partial\sigma_-}+\alpha(\sigma_+,\sigma_-)F=0, \quad (5)$$

where F and α are functions of the σ_+,σ_- variables. A class of solutions of this equation can be easily obtained by inspection. We first note that the usual general solution of the wave equation in $1+1$ dimensions which involves the superposition of functions of σ_+ and σ_- does not work here because of the presence of the second term in the equation.

Assuming $\alpha(\sigma_+,\sigma_-)=\alpha_+(\sigma_+)\alpha_-(\sigma_-)$ we may choose

$$F(\sigma_+,\sigma_-)=\exp\left(a\int\alpha_+(\sigma_+)d\sigma_+ + b\int\alpha_-(\sigma_-)d\sigma_-\right), \quad (6)$$

where a, b are two constants which must satisfy the condition $ab+1=0$ if Eq. (6) has to be a solution of Eq. (5).

It is easily seen that the following four possibilities exist for choices of a and b:

$$(1)\, a=1,\ b=-1, \quad (2)\, a=-1,\ b=1, \quad (7)$$

$$(3)\, a=i,\ b=i, \quad (4)\, a=-i,\ b=-i. \quad (8)$$

Note in the above that there are both oscillatory as well as exponential solutions. For the former, we need to look into the real and imaginary parts (the cosine and sine solutions, respectively) which are

$$F(\sigma_+,\sigma_-)=\cos\left(\int\alpha_+(\sigma_+)d\sigma_+ + \int\alpha_-(\sigma_-)d\sigma_-\right), \quad (9)$$

$$F(\sigma_+,\sigma_-)=\sin\left(\int\alpha_+(\sigma_+)d\sigma_+ + \int\alpha_-(\sigma_-)d\sigma_-\right). \quad (10)$$

Introduce the quantities θ_+ and θ_- (expansions along the light-cone directions σ_+ and σ_-) which are related to θ_σ and θ_τ as

$$\theta_+=\theta_\sigma-\theta_\tau, \quad \theta_-=\theta_\sigma+\theta_\tau. \quad (11)$$

For the exponential solutions we therefore have

$$\theta_+=\frac{1}{F}\frac{\partial F}{\partial\sigma_+}=\pm\alpha_+(\sigma_+), \quad (12)$$

$$\theta_-=\frac{1}{F}\frac{\partial F}{\partial\sigma_-}=\mp\alpha_-(\sigma_-), \quad (13)$$

$$\theta_+\theta_-=\theta_\sigma^2-\theta_\tau^2=-\alpha(\sigma_+,\sigma_-). \quad (14)$$

The upper and lower signs refer to the choices (1) and (2) in Eq. (7), respectively.

For positive α [i.e., (a) $\alpha_\pm>0$ or (b) $\alpha_\pm<0$] one can have the following alternatives [we take the lower sign in the previous expressions, i.e., (2) in Eq. (7)]: θ_+ negative and θ_- positive [for (a)] and θ_+ positive and θ_- negative [for (b)]. On the other hand, for negative α [i.e., (c) $\alpha_+>0$, $\alpha_-<0$ or (d) $\alpha_+<0$, $\alpha_->0$] the following possibilities exist: $\theta_\pm$ negative [for (c)] and $\theta_\pm$ positive [for (d)]. Additionally, for this class of solutions, a divergence in α_+ or α_- is necessary to have divergent expansions. This implies a divergence in world sheet curvature or the spacetime Ricci tensor when evaluated on the world sheet.

Let us now turn to the oscillatory solutions. The expressions for θ_+ and θ_- for them can be obtained in a similar fashion. We choose to work with the cosine solution for which we have

$$\theta_+ = -\alpha_+(\sigma_+)\tan\left(\int \alpha_+(\sigma_+)d\sigma_+ + \int \alpha_-(\sigma_-)d\sigma_-\right), \tag{15}$$

$$\theta_- = -\alpha_-(\sigma_+)\tan\left(\int \alpha_-(\sigma_+)d\sigma_+ + \int \alpha_-(\sigma_-)d\sigma_-\right), \tag{16}$$

$$\theta_+\theta_- = \theta_\tau^2 - \theta_\sigma^2 = \alpha(\sigma_+,\sigma_-)\tan^2\left(\int \alpha_-(\sigma_+)d\sigma_+ + \int \alpha_-(\sigma_-)d\sigma_-\right). \tag{17}$$

First let us assume $\alpha>0$ which implies the constraints (a) and (b) mentioned before on $\alpha_\pm$. Consequently we have $\theta_\pm>0$ or $\theta_\pm<0$ depending on the sign of the tangent function. On the contrary, if $\alpha<0$, i.e., cases (c) and (d), we find that for both cases θ_+ and θ_- can only have opposite signs. However, in contrast to the oscillatory solutions $\theta_\pm$ can diverge for finite values of $\sigma_\pm$ even though α may be completely regular there.

If $\alpha=0$, one has to analyze the solutions of the ordinary wave equation which are given by the function $f(\sigma_+)$ or $g(\sigma_-)$ or their linear superposition. By specializing to exponential or oscillatory cases it is easy to arrive at focusing effects at least for the latter. Note, however, that solutions to the $\alpha\neq 0$ case may not go over smoothly to those for $\alpha=0$. The simplest example of this type of behavior can be noted for the ordinary differential equation for the simple harmonic oscillator which has solutions of the form $\cos kx$, $\sin kx$, k being the frequency. Putting $k=0$ in the solutions yields trivial results whereas we know that the differential equation for $k=0$ has a solution of the form $ax+b$ where a,b are two arbitrary constants.

We now construct an explicit example of an embedding which is such that the quantity α is separable in light-cone coordinates.

The background metric is assumed to be conformally flat—the line element is taken as

$$ds^2 = f(x_0,x_1)[-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2]. \tag{18}$$

An embedding which satisfies the Nambu-Goto equations and constraints could be

$$x_0 = C_1\tau + C_2\sigma,\quad x_1 = C_2\tau + C_1\sigma,\quad x_2 = \text{const},\quad x_3 = \text{const}, \tag{19}$$

where C_1, C_2 are constants with $C_1^2 > C_2^2$.

One therefore needs to write down the expression for the quantity α which turns out to be

$$\alpha = -2\frac{1}{\sqrt{f}}\partial_+\partial_-\sqrt{f}. \tag{20}$$

Defining the induced metric on the world sheet as

$$ds_I^2 = e^{2\rho}[-d\tau^2 + d\sigma^2], \tag{21}$$

with $e^{2\rho} = f(C_1^2 - C_2^2)$, we can convert the expression above into the form

$$\alpha = -2e^{-\rho}\partial_+\partial_- e^{\rho} = -2(\partial_+\partial_-\rho + \partial_+\rho\partial_-\rho). \tag{22}$$

Choosing a generic form of $\rho = A(\sigma_+) + B(\sigma_-)$ we can easily see that it is possible to get an α which is separable in light-cone coordinates. Note that in this entire discussion we have never really chosen an explicit form for the function $f(x_0,x_1)$. This is not necessary as is apparent from the calculation. The separability of ρ which ultimately results in the separability of α, however, yields a world sheet metric which is flat ($^2R = -2e^{-2\rho}\partial_+\partial_-\rho$ turns out be zero).

Also, if the background geometry had been chosen such that the conformal factor was associated as a factor with the x_0, x_1 part of the metric [more precisely, $ds^2 = f(x_0,x_1)(-dx_0^2 + dx_1^2) + dx_2^2 + dx_3^2$], then the same embedding would have resulted in an α identically equal to zero.

IV. FOCUSING THEOREM

We now move on to the more important question of analyzing the generalized Raychaudhuri equation in string theory from the viewpoint of a Cauchy initial value problem. Note that the discussion presented in the previous sections has been largely aimed at obtaining specific solutions with the assumption of separability in light-cone variables.

Fortunately, we have several oscillation theorems due to Pagan and Stocks [12,13] which are essentially tailored to our requirements. We mention below one such theorem which we shall use subsequently.

Theorem (Pagan and Stocks 1975). Let $F(\sigma_+,\sigma_-)$ satisfy the partial differential equation

$$F_{+-} + \alpha(\sigma_+,\sigma_-)F = 0, \tag{23}$$

with the initial conditions

$$F(\sigma_+,\sigma_+) = r(\sigma_+),\qquad \frac{\partial F}{\partial\sigma_-}\Big|_{\sigma_+=\sigma_-} = t(\sigma_+), \tag{24}$$

in the domain $\sigma_- - \sigma_+ \geqslant 0$ (i.e. $\tau\geqslant 0$). Let the following conditions also hold:

$$\text{(i)}\quad \alpha(\sigma_+,\sigma_-) \geqslant k^2 > 0, \tag{25}$$

$$\text{(ii)}\quad \alpha_+(\sigma_+,\sigma_-)\geqslant 0,\qquad \text{(iii)}\quad \alpha_-(\sigma_+,\sigma_-)\geqslant 0, \tag{26}$$

$$\text{(iv)}\quad \left|F^2(\sigma_+,\sigma_+) - \frac{F_+^2(\sigma_+,\sigma_+)}{\alpha(\sigma_+,\sigma_+)}\right|,$$

$$\left|F^2(\sigma_-,\sigma_-) - \frac{F_-^2(\sigma_-,\sigma_-)}{\alpha(\sigma_-,\sigma_-)}\right| \text{are bounded as } \sigma_\pm\to\infty; \tag{27}$$

then, F changes sign (i.e., develops a zero nodal line) somewhere in the domain

$$D \equiv \{\sigma_+, \sigma_- | \Sigma_- \leqslant \sigma_- < \infty, \Sigma_+ \leqslant \sigma_+ < \infty, \Sigma_- - \Sigma_+ \geqslant 0\}. \tag{28}$$

The possible existence of the nodal line (a curve along which F is zero) is the basic result of the above-stated theorem. In the language of GR the nodal line is a generalization of the focal point—we might call it the *focal curve* along which families of timelike world sheets intersect. We can see straightaway that there are several conditions which have to be obeyed in order to ensure the existence of a nodal line. We now briefly discuss the implications of each of them.

Condition (i) is the analogue of the usual energy condition in the theory of geodesic curves although the right-hand side (RHS) of the inequality has a *positive number* instead of zero. However, since the $\alpha = 0$ case leads to a simple wave equation (whose solutions always have zeros), we can extend this condition to the analogue of the usual energy condition with k^2 being replaced by zero.

The second condition [i.e., (ii)] imposes restrictions on the derivatives of α. Translated in the language of σ, τ coordinates one can easily check that the following have to hold true:

$$\frac{\partial \alpha}{\partial \sigma} \geqslant \frac{\partial \alpha}{\partial \tau}, \quad \frac{\partial \alpha}{\partial \sigma} \geqslant -\frac{\partial \alpha}{\partial \tau}. \tag{29}$$

Therefore, if α is only a function of σ, one requires $d\alpha/d\sigma \geqslant 0$, whereas if α is only a function of τ, then one actually ends up in a contradiction, the only resolution of which is to assume α as a constant or zero.

Finally, the third condition, which is on the function F, implies, for instance, for the oscillatory solutions derived earlier, the boundedness of the quantity α as $\sigma_\pm$ approaches $\pm\infty$. This can be seen by substituting the solution in the expression for the condition.

Based on the above theorem we can now frame our focusing theorem for timelike world sheets.

If θ_+ or θ_- is negative somewhere, then they tend to $-\infty$ within a finite value of the worldsheet parameters σ_+ or σ_- provided all conditions on the α are obeyed. The negativity of θ_+ or θ_- is dependent on the negativity of the functions r and t which appear in the initial conditions.

It is perhaps easier to visualize the notion of focusing for the case of a family of closed string world sheets. Assume a family of cylindrical world sheets which meet along some curve $\sigma_+ = f(\sigma_-)$. This curve is the nodal line mentioned before. It may happen that this curve (nodal line) degenerates to a point. For example, if the equation of the curve turns out to be $\sigma_+^2 + \sigma_-^2 = 0$, then the only real solution is $\sigma_+ = \sigma_- = 0$. For such cases we have a family of cones emerging out of that point—the common vertex of the cones being the focal point of the congruence of world sheets. This basically means that the world sheet geometries have a conical singularity in the sense of unbounded curvature at that specific point. In GR, the focal point of a congruence indicates a singularity in the congruence of geodesics. We may find that the spacetime singularity (in the sense of unbounded curvature) coincides with the focal point of a geodesic congruence in the spacetime—for example, this happens in the universe models which exhibit curvature singularities . However, this is not always true. At this stage, it is not completely clear whether a notion of incompleteness of string world sheets can be derived and related to a singularity in the background spacetime. Moreover, we do not know precisely if the conical world sheet singularity which may arise if the focal curve degenerates to a point has any relation to the background spacetime singularities. Further analysis is essential if one wishes to arrive at a better understanding.

It must be mentioned that this is *a* focusing theorem—i.e., with the assumptions on the various quantities one can conclude that a focal curve can exist. The results of the previous section are different from two angles—first, we do *not* frame an initial value problem there, and second the solutions are obtained *ad hoc*, largely by inspection. It is possible that under different assumptions on the variables as well as other initial conditions one may also be able to prove the existence of a nodal curve. The author, unfortunately, is unaware of such results either in the mathematics or in the physics literature.

V. OFFSHOOTS

Before we conclude let us point out certain applications of the formalism of the generalised Jacobi and Raychaudhuri equations in a completely different context—that of biological membranes. Here, we consider two-dimensional hypersurfaces embedded in a Euclidean (flat) background space. The first variation yields the surface configurations which can be minimal (zero mean curvature) or Willmore (constant mean curvature) depending on the choice of the action functional. The corresponding Jacobi equations contain information about the normal deformations of these two-dimensional surfaces and are thereby linked to the question of stability. Much of the basic notions along these directions have been pursued in a mathematical context in a number of papers [14]. However, an application-oriented analysis with an emphasis on specific, physically relevant cases has not yet been performed. On the other front, solutions of the Raychaudhuri equations, which are in a certain sense nonperturbative, would indicate the formation of cusps and kinks on the membrane [focusing along a degenerate curve (point)]. A major difference with the analysis presented in this paper and that required to understand membranes in a Euclidean background is the appearance of elliptic equations as opposed to hyperbolic ones. Therefore, to analyze the Jacobi equations or when focusing one has to utilize the oscillation theorems for elliptic equations. Fortunately, once again such theorems do exist [15]. A detailed presentation of these ideas and their consequences in the context of biological membranes will be reported elsewhere [16].

VI. CONCLUSIONS

In conclusion we summarize and raise a few questions of related interest.

We have obtained a *focusing theorem* for string world sheets. This is illustrated through exact solutions as well as an analysis of the Cauchy initial value problem. The conditions for focusing are outlined—these constrain world sheet as well as spacetime properties. An analysis of the Jacobi equation as well as a more detailed presentation of the ideas here is in progress and will be reported in the near future [17].

It is a somewhat pleasing fact that most of the results for point particle theories have their generalization for the case of strings. However, GR as a theory of gravity has a unique feature—the equations of motion for test particles (i.e. the geodesic equation) can be derived from the Einstein field equations for the field $g_{\mu\nu}$ [18]. We may therefore ask—given the string equation of motion—can one find the corresponding ''Einstein equation'' which would lead to it under the suitable assumptions which may define a *test* string?

Finally, of course, one has to address the question of background spacetime singularities—does a string description as opposed to the point particle resolve the issue at the classical level? As a first step towards this (following the path of GR) we now have a focusing theorem. It would perhaps be worthwhile to attempt a derivation of the analogues of the Hawking-Penrose theorems for the case of strings and thereby demonstrate the existence or nonexistence of spacetime singularities in the classical theory of strings. If the answer remains the same as in GR (i.e., spacetime singularities exist under quite general conditions), then one can proceed towards examining how quantum string theory can help us solve the problem.

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