Minimal length uncertainty relation and ultraviolet regularization

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(Received 9 December 1996)

Studies in string theory and quantum gravity suggest the existence of a finite lower limit Δx_0 to the possible resolution of distances, at the latest on the scale of the Planck length of 10^{-35} m. Within the framework of the Euclidean path integral we explicitly show ultraviolet regularization in field theory through this short distance structure. Both rotation and translation invariance can be preserved. An example is studied in detail. [S0556-2821(97)06510-7]

PACS number(s): 11.25.Sq, 04.60.Gw, 11.10.Gh, 11.30.Cp

I. INTRODUCTION

As has long been known, the combination of relativistic and quantum effects implies that the conventional notion of distance breaks down the latest at the Planck scale, which is about 10^{-35} m. The basic argument is that the resolution of small distances requires test particles of short wavelength and thus of high energy. At sufficiently small scale, i.e., close to the Planck scale, the gravitational effect of the test particle's energy significantly disturbs the space-time structure which was tried. Studies on gedanken experiments therefore suggest the existence of a finite limit Δx_0 to the possible resolution of distances. String theory, as a theory of quantum gravity, should allow a deeper understanding of what could happen at such extreme scales. Indeed, several studies in string theory yielded a certain type of correction to the uncertainty relation

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}[1+\beta(\Delta p)^2+\cdots], \quad \beta>0, \tag{1}$$

which, as is easily verified, implies a finite minimal uncertainty $\Delta x_0=\hbar\sqrt{\beta}$. Therefore, $\Delta x_0>0$ can be viewed as a fuzziness of space, or also as a consequence of the nonpointlikeness of the fundamental particles. It seems that, in string theory, intuitively, the input of more energy does eventually no longer allow to improve the spatial resolution, as this energy starts to enlarge the probed string. References are, e.g., [1–7]; see also [8]. For recent reviews, see, e.g., [9,10].

Using the usual definition of uncertainties ($|\psi\rangle$ normalized)

$$(\Delta x)_{|\psi\rangle}=\langle\psi|(\mathbf{x}-\langle\psi|\mathbf{x}|\psi\rangle)^2|\psi\rangle^{1/2}, \tag{2}$$

the uncertainty relation Eq. (1) implies a small correction term to the commutation relation in the associative Heisenberg algebra:

$$[\mathbf{x},\mathbf{p}]=i\hbar(1+\beta\mathbf{p}^2+\cdots). \tag{3}$$

For studies on the technical and conceptual implications of these and more general types of correction terms, see [11–16]. We remark that those studies arose from work (e.g., [17]) in the seemingly unrelated field of quantum groups, in which this type of commutation and uncertainty relations had appeared independently (first in [18]). A standard reference on quantum groups is [19].

For the general case of n dimensions it appears that no consensus has been reached in the literature on which generalization of Eq. (3), i.e., which particular correction terms to the uncertainty relations could arise as a gravity effect in the ultraviolet, or as a string effect. Let us therefore here consider small correction terms of a general form ($\mathbf{x}_i^\dagger=\mathbf{x}_i, \mathbf{p}_i^\dagger=\mathbf{p}_i$)

$$[\mathbf{x}_i,\mathbf{p}_j]=i\hbar(\delta_{ij}+\beta_{ijkl}\mathbf{p}_k\mathbf{p}_l+\cdots) \tag{4}$$

with the coefficients β_{ijkl} (and also possible terms of higher power in the $\mathbf{p}_i$) chosen such that the corresponding uncertainty relations imply a finite minimal uncertainty $\Delta x_0>0$. We will for simplicity normally assume $[\mathbf{p}_i,\mathbf{p}_j]=0$, but we allow $[\mathbf{x}_i,\mathbf{x}_j]\neq 0$. Let us keep in mind that it is the correction terms to the $\mathbf{x},\mathbf{p}$ commutation relations which induce $\Delta x_0>0$. A noncommutativity of the $\mathbf{x}_i$ will not be necessary for the appearance of a finite minimal uncertainty Δx_0.

In short, the key mechanism which leads to ultraviolet regularization in the presence of a minimal uncertainty Δx_0 is the following.

In the case of the ordinary commutation and uncertainty relations underlying, the states of maximal localization are position eigenstates $|x\rangle$, for which the uncertainty in position vanishes. Crucially, these maximal localization states are nonnormalizable. Therefore, their scalar product is not a function but the Dirac δ distribution $\langle x|x'\rangle=\delta(x-x')$. As is well known (for a recent reference, see [20]), in the formulation of local interaction in field theory it is the ill definedness of the product of these and related distributions which give rise to ultraviolet divergencies.

A finite minimal uncertainty Δx_0 will yield normalizable maximal localization states, and thereby regularize the ultra-

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violet. More precisely, as we will see, there exist generalized commutation relations of the type of Eqs. (4) such that there exists a minimal uncertainty $\Delta x_0 > 0$, with the vectors of maximal localization $|x^{\mathrm{ml}}\rangle$ obeying

$$\langle x^{\mathrm{ml}}|x^{\mathrm{ml}}\rangle = 1, \quad \Delta x_{|x^{\mathrm{ml}}\rangle} = \Delta x_0, \quad \langle x^{\mathrm{ml}}|\mathbf{x}|x^{\mathrm{ml}}\rangle = x \quad \text{with} \quad x \in \mathbb{R}. \tag{5}$$

It follows that due to their normalizability, the scalar product

$$\tilde{\delta}(x,y) := \langle x^{\mathrm{ml}}|y^{\mathrm{ml}}\rangle \tag{6}$$

is a function rather than a distribution.

A simple example is the one-dimensional case of Eq. (3) with $\beta > 0$ and no higher order corrections. For this case the scalar product of the maximal localization states has been calculated in [13]:

$$\tilde{\delta}(x,y) = \frac{1}{\pi}\left[\frac{x-y}{2\hbar\sqrt{\beta}} - \left(\frac{x-y}{2\hbar\sqrt{\beta}}\right)^3\right]^{-1} \sin\left(\frac{x-y}{2\hbar\sqrt{\beta}}\pi\right). \tag{7}$$

Note that the poles of the first factor are cancelled by zeros of the sine function, so that $\tilde{\delta}$ is a regular function. For a graph see Fig. 3 in [13]. The analogous result for the case with also a finite minimal uncertainty in momentum has been worked out in [15].

We consider it to be an attractive feature of this short distance structure that it will not require the breaking of translation and rotation invariance, while also being compatible with possible (e.g., quantum group) generalizations of these symmetries. Also, this regularization will not require to cut momentum space.

A general approach for the formulation of quantum field theory with generalized **x,p**-commutation relations underlying has been developed in [12], with a general result on infrared regularization in [16], and preliminary results on ultraviolet regularization in [14]. Our aim here is to show the general mechanism, both abstractly and explicitly, by which a minimal uncertainty in position regularizes the ultraviolet, i.e., we show how a Δx_0 could indeed provide a natural ultraviolet cutoff in quantum field theory. While we will focus here on commutation relations which induce a finite $\Delta x_0 > 0$, the general framework does allow for generic commutation relations. Let us therefore also mention some of those studies which suggest such more general commutation relations.

For example, the approach by Doplicher *et al.* [21] suggests the existence of specific corrections to the $\mathbf{x},\mathbf{x}$ commutation and uncertainty relations. One of the arguments there is that the improvement of a position measurement in one direction ultimately requires a delocalization in orthogonal directions, in order to reduce the gravitationally disturbing energy density of the probing particle. A possible noncommutativity of the position operators was probably first discussed in [22], developing a line of thought which has been followed since, mainly by Russian schools; see, e.g., [23]. In the context of noncommuting position operators, see also [24]. Other studies, e.g., [25], suggest a length dependence of the minimal uncertainty in length measurements. Correction terms specifically to the $\mathbf{p},\mathbf{p}$ commutation relations have been discussed, e.g., in [1,26]. The approach of ''generalized quantum dynamics'' by Adler [27] allows for generic commutation relations and a possible generalization of the underlying Hilbert space to a quaternionic space. In this approach the ordinary canonical commutation relations have been derived as a first order approximation in a statistical averaging process [28].

Further, it should also be of interest to apply the noncommutative geometric concepts developed in [29], in particular to study the modifications to the differential and integral calculus over such generalized quantum phase spaces.

We note that, technically, the appearance of correction terms to the canonical commutation relations can generally also be viewed as a nontrivial and nonunique change of generators from the $\mathbf{x_0},\mathbf{p_0}$ which obey $[\mathbf{x_0},\mathbf{p_0}] = i\hbar$ to new sets of generators. Examples for such algebra homomorphisms ρ for the case of Eq. (3) are ρ: $\mathbf{x_0} \rightarrow \mathbf{x} = \mathbf{x_0} + \beta\mathbf{p_0}\mathbf{x_0}\mathbf{p_0}$, $\mathbf{p_0} \rightarrow \mathbf{p_0}$, or also ρ: $\mathbf{x_0} \rightarrow \mathbf{x} = \mathbf{x_0}$, $\mathbf{p_0} \rightarrow \mathbf{p} = \beta^{-1/2}\tan(\mathbf{p_0}\beta^{1/2})$.

The reason why a slight change in the commutation relations is able to introduce a drastically new short distance structure is not only that expectation values of a function of operators generally do not equal the function of the expectation values. Technically, the reason is of course that algebra homomorphisms ρ which change the commutation relations of the generators are necessarily noncanonical transformations, i.e., unlike symmetries, the ρ cannot be implemented as unitary (nor as antilinear antiunitary) transformations. Unitaries U generally preserve any chosen commutation relations, say $h(\mathbf{x},\mathbf{p}) = 0$, since $h(\mathbf{x},\mathbf{p}) = 0 \Rightarrow h(\mathbf{x}',\mathbf{p}') = h(U\mathbf{x}U^{\dagger}, U\mathbf{p}U^{\dagger}) = Uh(\mathbf{x},\mathbf{p})U^{\dagger} = 0$. Thus any change in the commutation relations introduces new features into the theory, such as the appearance of a $\Delta x_0 > 0$, which we will here focus on.

II. GENERAL FRAMEWORK

A. Partition function

Let us consider the example of Euclidean charged scalar ϕ^4 theory, in its formulation on position space:

$$Z[J] := N \int D\phi \ \exp\biggl(-\int d^4x \ \Bigl[\phi^*(-\partial_i\partial^i + m^2c^2)\phi + \frac{\lambda}{4!}(\phi\phi)^*\phi\phi - \phi^*J - J^*\phi \Bigr] \biggr), \tag{8}$$

with N a normalization factor. Fourier transformation allows us to express the action functional in momentum space, which is of course to choose the plane waves as a Hilbert basis in the space of fields which is formally being summed over. Equivalently, the action functional can be expressed in any arbitrary other Hilbert basis, such as, e.g., a Hilbert basis of Hermite functions. In fact, it is not necessary to specify any choice of basis. Fields can be identified as vectors in the representation space F of the associative Heisenberg algebra $\mathcal{A}$ with the canonical commutation relations:

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\,\delta_{ij}, \quad i,j = 1, \ldots, 4. \tag{9}$$

Since the functional analytic structure is analogous to the situation in quantum mechanics, we formally extend the Dirac notation for states to fields, i.e., $\phi(x) = \langle x|\phi\rangle$ and $\phi(p) = \langle p|\phi\rangle$. We recall that, via $\langle x|p\rangle = (2\pi\hbar)^{-2}\exp(ixp/\hbar)$, the $\hbar$ which appears in the Fourier factor $e^{ixp/\hbar}$ of the transformation from position to momentum space stems from the $\hbar$ of Eq. (9). Of course, the simple quantum mechanical interpretation of fields $|\phi\rangle$ and in particular of the position and momentum operators of Eq. (9) does not simply extend, due to the relativistically necessary existence of antiparticles; see [30]. However, this formulation clarifies the functional analytic structure of the action functional [12,16]:

$$Z[J] = N\int D\phi \exp\left(-\frac{l^2}{\hbar^2}\langle\phi|\mathbf{p}^2 + m^2c^2|\phi\rangle - \frac{\lambda l^4}{4!}\langle\phi*\phi|\phi*\phi\rangle + \langle\phi|J\rangle + \langle J|\phi\rangle\right). \tag{10}$$

The pointwise multiplication * of fields is crucial for the description of local interaction. It maps two fields onto one field, i.e., *: $F\otimes F \to F$, and it normally reads

$$* = \int d^4x \ \ |x\rangle\otimes\langle x|\otimes\langle x| \tag{11}$$

so that, in our notation,

$$(\phi_1 * \phi_2)(y) = \langle y|\phi_1 * \phi_2\rangle = \int d^4x \ \ \langle y|x\rangle\langle x|\phi_1\rangle\langle x|\phi_2\rangle = \phi_1(y)\phi_2(y). \tag{12}$$

In the case of generalized commutation relations we read Eq. (11) with the $|x\rangle$ denoting the vectors of maximal localization, i.e., we are integrating over the position expectation values of the maximal localization vectors:

$$* = \int d^4x \ \ |x^{\mathrm{ml}}\rangle\otimes\langle x^{\mathrm{ml}}|\otimes\langle x^{\mathrm{ml}}|. \tag{13}$$

In Eq. (10), in order to make the units more transparent, we introduced an arbitrary unit length l, so that the fields $|\phi\rangle$ become unitless. l could trivially also be reabsorbed in the definition of the fields. As is easily seen, in the case of the ordinary commutation relations the vectors $|x\rangle$ have units *length*$^{-2}$, so that $|\phi_1 * \phi_2\rangle$ has units *length*$^{-2}$, implying that the coupling constant λ (of the unregularized theory) is unitless. As is to be expected in a regularized situation, this changes in the cases of generalized commutation relations with normalizable maximal localization vectors. Due to $\langle x^{\mathrm{ml}}|x^{\mathrm{ml}}\rangle = 1$ the $|x^{\mathrm{ml}}\rangle$ do not carry units, so that the coupling λ is no longer unitless.

We recall that in the case of the ordinary commutation and uncertainty relations the position eigenvectors are the maximal localization vectors, implying that the application of the definition Eq. (11) for * in the partition function describes the maximally local interaction. The apparent ''nonlocality'' introduced in Eq. (13) is only of the size of the now underlying finite minimal position uncertainty. Within the framework, physical processes, including measurement processes, obey the uncertainty relations. We therefore conclude that the so-defined interactions are observationally strictly local since the apparent nonlocality could not be observed—due to the fuzziness Δx_0 introduced through the generalized uncertainty relations.

In our formulation of quantum field theories with underlying generalized **x,p** commutation relations, we will stick to the abstract form of the action functional and the partition function, as, e.g., given in Eq. (10), i.e., we will not introduce any changes ''by hand'' into the form of the action functional. The switching on of corrections to the underlying uncertainty relations will automatically manifest itself in the explicit form of the resulting Feynman rules. The correction terms to the commutation relations induce modifications to the action of the operator $(\mathbf{p}^2 + m^2)$, and to the properties of the maximally localized fields $|x^{\mathrm{ml}}\rangle$, which will both crucially enter into the Feynman rules.

We remark that, as a new feature, some generalized commutation relations will have nontrivial unitarily nonequivalent representations, as the well-known theorem by von Neumann no longer applies. It has been suggested that such cases could correspond to manifolds with horizons or nontrivial topology [26].

B. Feynman rules

For explicitness, let us specify some arbitrary Hilbert basis $\{|n\rangle\}_n$ in the space F of fields on which the generalized commutation relations are represented. While this basis can be continuous, discrete, or generally a mixture of both, we here use the convenient notation for n discrete. We recall that the discreteness or continuousness of the choice of basis is unrelated to the issue of regularization. F is separable even in the case of the ordinary commutation relations, i.e., discrete Hilbert bases (such as the Fock basis) also exist in the case of the ordinary commutation relations. We remark that, in the case of the ordinary commutation relations, when choosing the position space representation the situation is slightly subtle since the propagator and the vertex are then distributions. The situation will become simpler for $\Delta x_0 > 0$, as the distributions will turn into regular functions.

Fields, operators, and * are expanded in the $\{|n\rangle\}$ basis as

$$\phi_n = \langle n|\phi\rangle \quad \text{and} \quad (\mathbf{p}^2 + m^2c^2)_{nm} = \langle n|\mathbf{p}^2 + m^2c^2|m\rangle \tag{14}$$

and

$$* = \sum_{n_i} L_{n_1,n_2,n_3}|n_1\rangle\otimes\langle n_2|\otimes\langle n_3|. \tag{15}$$

Thus

$$|\phi*\phi'\rangle = \sum_{n,m,r} L_{nmr}\langle m|\phi\rangle\langle r|\phi'\rangle|n\rangle, \tag{16}$$

i.e.,

$$(\phi*\phi')_n = L_{nrs}\phi_r\phi'_s\,. \tag{17}$$

In this Hilbert basis the partition function Eq. (10) thus reads, summing over repeated indices,

$$Z[J]=N\int_F D\phi\ \exp\left(-\frac{l^2}{\hbar^2}\ \phi^*_{n_1}(\mathbf{p}^2+m^2c^2)_{n_1n_2}\phi_{n_2}\right.$$
$$\left.-\frac{\lambda l^4}{4!}L^*_{n_1n_2n_3}L_{n_1n_4n_5}\phi^*_{n_2}\phi^*_{n_3}\phi_{n_4}\phi_{n_5}+\phi^*_nJ_n+J^*_n\phi_n\right). \tag{18}$$

Pulling the interaction term in front of the path integral, completing the squares, and carrying out the Gaussian integrals yields

$$Z[J]=N'\exp\left(-\frac{\lambda l^4}{4!}L^*_{n_1n_2n_3}L_{n_1n_4n_5}\frac{\partial}{\partial J_{n_2}}\frac{\partial}{\partial J_{n_3}}\frac{\partial}{\partial J^*_{n_4}}\frac{\partial}{\partial J^*_{n_5}}\right.$$
$$\left.\times e^{-(\hbar^2/l^2)\,J^*_n(\mathbf{p}^2+m^2c^2)^{-1}_{nm}J_m}\right). \tag{19}$$

We can therefore read off the Feynman rules for the propagator and the vertex

$$G_{nm}=\left(\frac{\hbar^2/l^2}{\mathbf{p}^2+m^2c^2}\right)_{nm},\quad \Gamma_{rstu}=-\frac{\lambda l^4}{4!}L^*_{nrs}L_{ntu}. \tag{20}$$

Note that the earlier arbitrarily introduced constant l drops out of the Feynman rules since each vertex attaches to four propagators.

Explicitly, Eq. (13) yields the structure constants

$$L_{n_1,n_2,n_3}=\int d^4x\ \ \langle n_1|x^{\rm ml}\rangle\langle x^{\rm ml}|n_2\rangle\langle x^{\rm ml}|n_3\rangle. \tag{21}$$

For the case of the ordinary commutation relations, we recover with $|x^{\rm ml}\rangle=|x\rangle$, and, e.g., choosing the position representation $|n\rangle=|x\rangle$,

$$L^{(\Delta x_0=0)}_{x,x',x''}=\delta^4(x-x')\,\delta^4(x-x''). \tag{22}$$

In the general case with $\Delta x_0>0$, as we said, the coupling constant picks up units. We can however still define a unitless λ by splitting off suitable factors of l. Let us also choose $l=\Delta x_0$. Any other choice for l would amount to a redefinition of the coupling constant λ.

As abstract operators, i.e., without specifying a Hilbert basis in the space of fields, the free propagator and the lowest order vertex then read, using the definition Eq. (6),

$$G=\frac{\hbar^2}{(\Delta x_0)^2(\mathbf{p}^2+m^2c^2)}, \tag{23}$$

$$\Gamma=-\frac{\lambda}{4!}\int\frac{d^4xd^4y}{(\Delta x_0)^8}\tilde{\delta}^4(y^{\rm ml},x^{\rm ml})|y^{\rm ml}\rangle\otimes|y^{\rm ml}\rangle\otimes\langle x^{\rm ml}|\otimes\langle x^{\rm ml}|. \tag{24}$$

We can now use the Feynman rules Eqs. (23) and (24) to explicitly check for UV regularization in the cases of Heisenberg algebras $\mathcal{A}$ generated by operators $\mathbf{x},\mathbf{p}$ which obey generalized commutation implying $\Delta x_0>0$.

C. Regularization

Let us first consider the tadpole graph (see Fig. 1). Using Eqs. (19)–(21), or directly Eqs. (23) and (24), yields its expression as an operator:

$$\frac{2\lambda\hbar^2}{(4!)^2(\Delta x_0)^2}\int\frac{d^4xd^4y}{(\Delta x_0)^8}\tilde{\delta}^4(x,y)$$
$$\times\left\langle x^{\rm ml}\left|\frac{1}{\mathbf{p}^2+m^2c^2}\right|y^{\rm ml}\right\rangle|y^{\rm ml}\rangle\otimes\langle x^{\rm ml}|. \tag{25}$$

As is well known, ordinarily this graph is quadratically divergent for large momenta. On position space the divergence, or rather the ill definedness of this graph, arises not through the large scale integrals, but instead at short distances, i.e., as $x\rightarrow y$.

For our cases of generalized commutation relations this graph is however well defined: Due to the normalizability of the maximal localization vectors, their scalar product $\tilde{\delta}^4$ is a function bounded by 1, rather than a distribution. In the second factor, which consists of matrix elements of the propagator, the operator $(\mathbf{p}^2+m^2c^2)^{-1}$ is bounded. Therefore, again due to the normalizability of the $|x^{\rm ml}\rangle$ also these matrix elements are bounded functions of x and y. Thus the short-distance divergence is indeed removed in the case of the generalized commutation relations.

In the case $m=0$ the operator $1/\mathbf{p}^2$ is unbounded, which, as is well known, can lead to infrared divergencies at large distances. A relevant question in this context is of course whether in cases of generalized commutation and uncertainty relations with a finite minimal uncertainty in momentum this infrared problem could be avoided. Indeed, as has been shown in [16], the existence of a finite $\Delta p_0>0$ implies that the operator $1/\mathbf{p}^2$ is as well behaved as if it contained a mass term, i.e., it is a bounded self-adjoint operator. Since we are here primarily interested in the ultraviolet behavior, let us in the following assume the infrared to be regularized either through $m>0$, or, e.g., through $\Delta p_0>0$ (examples of generalized commutation relations which imply both, finite minimal uncertainties in position Δx_0 and in momentum Δp_0 are known, see [11]).

The tadpole graph could of course have been avoided by normal ordering the interaction Lagrangian. Let us therefore consider the further example of the normally logarithmically divergent ''fish'' graph (see Fig. 2).

It requires two vertices and two propagators:

$$\frac{2\lambda^2\hbar^4}{(4!)^2(\Delta x_0)^4}\int\frac{d^4x_1\ \ d^4x_2\ \ d^4x_3\ \ d^4x_4}{(\Delta x_0)^{16}}$$
$$\times\left\langle x_2^{\rm ml}\left|\frac{1}{\mathbf{p}^2+m^2c^2}\right|x_3^{\rm ml}\right\rangle^2\tilde{\delta}^4(x_1,x_2)\,\tilde{\delta}^4(x_3,x_4)$$
$$\times|x_1^{\rm ml}\rangle\otimes|x_1^{\rm ml}\rangle\otimes\langle x_4^{\rm ml}|\otimes\langle x_4^{\rm ml}|. \tag{26}$$

Ordinarily, in position space, the propagator $\langle x_2|(\mathbf{p}^2+m^2c^2)^{-1}\ \ |x_3\rangle$ is divergent for $x_2\rightarrow x_3$. Neverthe-

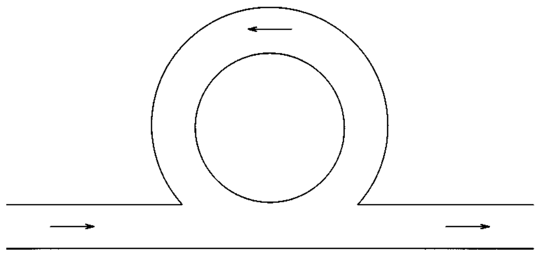

FIG. 1. The tadpole graph. The notation is meant to indicate $\Delta x_0 > 0$, i.e., the fuzziness of space-time, or the particles' nonpointlikeness.

less, it is well defined as a distribution. However, its square $\langle x_2 | (\mathbf{p}^2 + m^2c^2)^{-1} \ | x_3 \rangle^2$ is not.[1]

In contrast, since in the case of the generalized commutation relations the matrix elements of the propagator $\langle x_2^{\mathrm{ml}} | (\mathbf{p}^2 + m^2c^2)^{-1} | x_3^{\mathrm{ml}} \rangle$ are bounded, also for $x_2 \rightarrow x_3$, arbitrary high powers $\langle x_2^{\mathrm{ml}} | (\mathbf{p}^2 + m^2c^2)^{-1} | x_3^{\mathrm{ml}} \rangle^r$, $r \in \mathbb{N}$, are also well defined functions of x_2 and x_3. Again, the short distance structure is found to be regularized.

In fact, it is obvious that the short distance structure of all graphs is regularized, since in arbitrary graphs at most finite powers of matrix elements of the propagator, and powers of $\tilde{\delta}$ can appear, which both are now bounded regular functions.

We should note, however, that although we have seen that the ultraviolet divergencies are absent, we cannot generally exclude that some sets of generalized commutation relations could introduce new types of divergencies. This will have to be investigated case by case.

D. External symmetry

The one-dimensional uncertainty relation Eq. (1) has no unique n-dimensional generalization. Therefore, any particular choice for the corrections to the commutation relations in n dimensions will require motivation from string theory or quantum gravity. There is also the possibility of generalized external and internal symmetry groups (e.g., quantum groups) at the Planck scale, see, e.g., [11,31–33]. We will here not attempt to develop such arguments any further. Let us here instead consider the constraints which can be posed by requiring conventional translation and rotation invariance of the commutation relations.

We start with a general ansatz for $\mathbf{x},\mathbf{p}$ commutation relations in n dimensions:

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar \Theta_{ij}(\mathbf{p}), \tag{27}$$

where we require that only the ultraviolet is affected, i.e., $\Theta_{ij}(p)$ shall be allowed to significantly differ from δ_{ij} only for large momenta.

As we said, we assume $[\mathbf{p}_i, \mathbf{p}_j] = 0$. (We remark that it has been argued that if the final theory of quantum theory on curved space does contain momentum operators, these should be generators of a generalized definition of translation on curved space, in which case $[\mathbf{p}_i, \mathbf{p}_j] = 0$ would express the absence of curvature on position space [26].)

[1]We remark that the ansatz of differential renormalization, see, e.g. [20], starts here by replacing the ill defined square of the propagator (nonuniquely) by the derivative of a well defined distribution, thereby introducing a length scale.

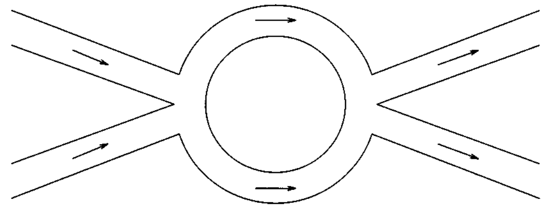

FIG. 2. The fish graph.

The remaining commutation relations among the $\mathbf{x}_i$ are then determined through the Jacobi identities, yielding [26]

$$[\mathbf{x}_i, \mathbf{x}_j] = i\hbar \{ \mathbf{x}_a, \Theta^{-1}_{ar} \Theta_{s[i} \Theta_{j]r,s} \}. \tag{28}$$

For simplicity we adopted the geometric notation, with $\{,\}$ and $[,]$ standing for (anti) commutators and with $Q_{,s} = \partial / \partial p_s Q$.

We observe that the $\mathbf{x},\mathbf{p}$ commutation relations Eqs. (27) are translation invariant in the sense that they are preserved under the transformations

$$\mathbf{x}_i \rightarrow \mathbf{x}_i + d_i, \quad \mathbf{p}_i \rightarrow \mathbf{p}_i, \quad d_i \in \mathbb{R}, \ i = 1, \ldots, n. \tag{29}$$

On the other hand, for generic Θ, the commutation relations Eqs. (28) are not invariant under translations, i.e., the generators obtained through the transformations Eqs. (29) do not obey Eq. (28). We can, however, enforce translation invariance by requiring Θ to yield $[\mathbf{x}_i, \mathbf{x}_j] = 0$. We read off from Eq. (28) that a sufficient and necessary condition for this to hold is (summing over i)

$$\Theta_{ia} \partial_{p_i} \Theta_{bc} = \Theta_{ib} \partial_{p_i} \Theta_{ac} \tag{30}$$

which may be viewed as expressing the absence of curvature on momentum space, by the same arguments as above. Of course, central correction terms may still be added on the right-hand side (RHS) of the $\mathbf{x},\mathbf{x}$ commutation relations, without spoiling translation invariance, e.g., terms of the form suggested in [21].

The requirement of rotation invariance further imposes

$$\Theta_{ij}(p) = f(p^2)\delta_{ij} + g(p^2) p_i p_j \tag{31}$$

so that Eq. (30) takes the form [34]

$$g = \frac{2ff'}{f - 2p^2 f'}, \tag{32}$$

where the prime denotes d/dp^2. Under these conditions translations and rotations do respect the commutation relations, i.e., they are quantum canonical transformations, and can indeed be implemented as unitary transformations. The translations are given by

$$U(d) := e^{d \cdot T} \tag{33}$$

with $[T_i, \mathbf{x}_j] = \delta_{ij}$, and where we denoted the scalar product $\Sigma_{i=1}^n d_i T_i$ by $d \cdot T$. Since in the ''naive'' definition of translations in Eqs. (29) there is no explicitly built-in ''knowledge'' of the new short distance structure, the anti-Hermitian generators T_i are not given by the $-\mathbf{p}_i/i\hbar$ directly. Instead, they are

$$T_i=\frac{\mathbf{p}_i}{-i\hbar f(\mathbf{p}^2)} \tag{34}$$

as is not difficult to verify. As a consequence of the new short distance structure the translators T_i will be found to be bounded operators, technically as we will see [Eq. (42)], because f eventually goes linearly with p for large p.

Analogously, rotations

$$U(\Theta)=e^{\Theta_{ij}M_{ij}} \tag{35}$$

are generated by the operators

$$M_{ij}=\frac{1}{-i\hbar f(\mathbf{p}^2)}\ (\mathbf{p}_i\mathbf{x}_j-\mathbf{p}_j\mathbf{x}_i) \tag{36}$$

which obey

$$[\mathbf{p}_i,\mathbf{M}_{jk}]=\delta_{ik}\mathbf{p}_j-\delta_{ij}\mathbf{p}_k, \tag{37}$$

$$[\mathbf{x}_i,\mathbf{M}_{jk}]=\delta_{ik}\mathbf{x}_j-\delta_{ij}\mathbf{x}_k, \tag{38}$$

$$[\mathbf{M}_{ij},\mathbf{M}_{kl}]=\delta_{ik}\mathbf{M}_{jl}-\delta_{il}\mathbf{M}_{jk}+\delta_{jl}\mathbf{M}_{ik}-\delta_{jk}\mathbf{M}_{il} \tag{39}$$

as usual.

III. EXPLICIT EXAMPLE

In the following we will illustrate the formalism with an explicit example of generalized commutation relations.

A. Choice of commutation relations

If we require our generalized commutation relations to obey translation and rotation invariance, there still appears to be considerable freedom in choosing the functions f and, through Eq. (32), the function g. Many choices may not lead to generalized commutation relations that imply a minimal uncertainty $\Delta x_0>0$. In particular, Eq. (32) indicates that g can develop singularities. A detailed investigation into the various possibilities is in progress [35]. Here, in order to obtain a well behaved example of generalized commutation relations we simply force there not to appear a singularity by imposing, as the simplest choice ($\beta>0$),

$$g=\beta. \tag{40}$$

Thus, Eq. (32) then reads

$$f'=\frac{\beta f}{2(f+\beta p^2)} \tag{41}$$

which is solved by

$$f=\frac{\beta p^2}{\sqrt{1+2\beta p^2}-1}. \tag{42}$$

The Taylor expansion around the origin is well behaved:

$$f=1+\frac{\beta}{2}p^2+O((\beta p^2)^2), \tag{43}$$

so that, if we choose β, e.g., at around the Planck scale $\beta^{-1/2}\approx p_{\mathrm{Pl}}$, then f significantly deviates from the identity only for large momenta of that scale.

We therefore obtain the commutation relations

$$[\mathbf{x}_i,\mathbf{p}_j]=i\hbar\ \left(\frac{\beta\mathbf{p}^2}{(1+2\beta\mathbf{p}^2)^{1/2}-1}\ \delta_{ij}+\beta\mathbf{p}_i\mathbf{p}_j\right), \tag{44}$$

$$[\mathbf{x}_i,\mathbf{x}_j]=0, \tag{45}$$

$$[\mathbf{p}_i,\mathbf{p}_j]=0. \tag{46}$$

We remark that, assuming translation and rotation invariance, the correction terms to the commutation relations are in fact unique to first order in β: Eq. (32) yields $f=1+\beta/2p^2+O(\beta^2)$ and $g=\beta+O(\beta^2)$, so that

$$[\mathbf{x}_i,\mathbf{p}_j]=i\hbar[(1+\beta/2\mathbf{p}^2)\delta_{ij}+\beta\mathbf{p}_i\mathbf{p}_j+O(\beta^2)] \tag{47}$$

and $[\mathbf{x}_i,\mathbf{x}_j]=0+O(\beta^2)$, $[\mathbf{p}_i,\mathbf{p}_j]=0$, which of course coincides with what we obtain from Eqs. (44) to first order in β.

We remark that concerning the possible choices of commutation relations it should generally be interesting to investigate the interplay of the technical constraints with the input and physical intuition from string theory and quantum gravity. In particular, as follows from the relation between the translators and the momenta, Eq. (34), the rule for the addition of extremely large momenta is modified through $(p\hat{+}k)_i=p_i f^{-1}(p^2)+k_i f^{-1}(k^2)$. There should exist an interpretation in terms of the effects of gravity at the Planck scale, similar to the well-known effect of momentum nonconservation through gravity on large scales ($T^{\mu\nu}{}_{;\nu}=0$ rather than $T^{\mu\nu}{}_{,\nu}=0$). This may, e.g., be related to the old idea of possible curvature in momentum space, in which a generalized parallelogram rule for the addition of momenta has been discussed; see [22], and more recently [23]. It has of course long been suggested that, more drastically, both rotation and translation invariance may be generalized or broken at the Planck scale. Any physical intuition for this could and should then also provide guidance for the generalization of Eq. (13) to account for the then position (and possibly orientation) dependence of the short distance structure. This will at first require a case by case study.

B. Hilbert space representations

The commutation relations Eqs. (44), (45), and (46) still find a Hilbert space representation in the spectral representation of the momenta $\mathbf{p}_i$ (since momentum space is still commutative and there is no finite minimal uncertainty in momentum, $\Delta p_0=0$):

$$\mathbf{x}_i\cdot\psi(p)=i\hbar\left[\left(f'+p^2g'+\frac{n+1}{2}g\right)p_i+f\partial_{p_i}+gp_ip_j\partial_{p_j}\right]\psi(p), \tag{48}$$

$$\mathbf{p}_i \cdot \psi(p) = p_i \psi(p), \tag{49}$$

$$\langle \psi_1 | \psi_2 \rangle = \int d^n p \, \psi_1^*(p) \psi_2(p), \tag{50}$$

where $\psi(p) = \langle p | \psi \rangle$ and $\langle p | p' \rangle = \delta(p - p')$. $\mathbf{x}_i$ and $\mathbf{p}_i$ are symmetric operators on the dense domain $D := S_\infty$. This representation holds for any choice of f and g, as can be checked directly. The case of commutation relations with general Θ is covered in [26].

A further representation of the commutation relations Eqs. (44)–(46), which will prove convenient for practical calculations, is obtained by using that the translators T_i are anti-Hermitian and have a spectral representation on the Hilbert basis $\{|\rho\rangle | \rho \in I_n\}$ of vectors obeying $T_i \cdot |\rho\rangle = \rho_i / i\hbar |\rho\rangle$ with

$$I_n = \{\rho \in \mathbb{R}^n | \rho^2 < 2/\beta\}, \tag{51}$$

i.e., the T_i are bounded operators. The unitary transformation which maps from momentum space to the spectral representation of the T_i has the matrix elements

$$\langle \rho | p \rangle = (1 - \beta\rho^2/2)^{-(n+1)/2} (1 + \beta\rho^2/2)^{1/2} \times \delta^n\left(p_i - \frac{\rho_i}{1 - \beta\rho^2/2} \right). \tag{52}$$

The operator representations and the scalar product then read in ρ space

$$\mathbf{x}_i \cdot \psi(\rho) = i\hbar \partial_{\rho_i} \psi(\rho), \tag{53}$$

$$\mathbf{p}_i \cdot \psi(\rho) = \frac{\rho_i}{1 - \beta\rho^2/2} \psi(\rho), \tag{54}$$

$$\langle \psi_1 | \psi_2 \rangle = \int_{I_n} d^n \rho \, \psi_1^*(\rho) \psi_2(\rho), \tag{55}$$

where $\psi(\rho) = \langle \rho | \psi \rangle$ and $\langle \rho | \rho' \rangle = \delta^n(\rho_i - \rho_i')$. Note that, as is easy to see in this representation, the momentum operators $\mathbf{p}_i$ are still unbounded.

We note also that the momentum operators $\mathbf{p}_i$ no longer coincide with the generators of translations $-i\hbar T_i$, but that they differ from them for large momenta, i.e., for small distances. Related to our discussion at the end of Sec. I, Eqs. (53) and (54) then suggest to take the point of view that the introduction of a Planck scale minimal uncertainty in positions amounts to sticking to the usual position operators while giving up the usual momentum operators $\mathbf{p}_{0i}$ for new momentum operators $\mathbf{p}_i$, thereby leading to ultraviolet regularization. (We are here neglecting functional analytic details such as the changing domain and defect indices of the position operators.)

On the other hand, using the eigenbasis of the $\mathbf{p}_i$, Eqs. (48) and (49) show that, alternatively, the introduction of a finite minimal uncertainty in positions can be viewed as keeping the conventional momentum operators $\mathbf{p}_i = \mathbf{p}_{0i}$ and instead replacing the conventional position operators $\mathbf{x}_{0i}$ by new position operators $\mathbf{x}_i = \mathbf{x}_i(\mathbf{x_0}, \mathbf{p_0})$ which are given via Eq. (48) (with $p_i = \mathbf{p}_{0i}$ and $\partial_{pi} = \mathbf{x}_{0i} / i\hbar$).

Generally, within the Hilbert space representation of the generalized commutation relations there exist an arbitrary number of Hilbert bases in which the $\mathbf{x}_i$ and $\mathbf{p}_i$ are represented in terms of multiplication and differentiation operators:

$$\mathbf{x}_i \cdot \psi(v) = f_i(v, d/dv) \psi(v), \tag{56}$$

$$\mathbf{p}_i \cdot \psi(v) = g_i(v, d/dv) \psi(v). \tag{57}$$

Since the d/dv_i and v_i obey the Leibniz rule $[d/dv_i, v_j] = \delta_{ij}$, each one of these Hilbert bases offers an alternative viewpoint according to which the introduction of a minimal uncertainty in positions is the replacement of operators $\mathbf{x}_{0i}, \mathbf{p}_{0i}$ which obey the conventional commutation relations by new operators $\mathbf{x}_i$ and $\mathbf{p}_i$

$$\mathbf{x}_i = f_i(\mathbf{x_0}, -\mathbf{p_0}/i\hbar), \quad \mathbf{p}_i := g_i(\mathbf{x_0}, -\mathbf{p_0}/i\hbar) \tag{58}$$

(identifying $\mathbf{x}_{0i} = v_i, \mathbf{p}_{0i} = -i\hbar d/dv_i$). The action, including the definitions of maximally localized fields [Eq. (5)] and maximally local interaction [Eq. (13)], is dependent on the generalized uncertainty relations and the thereby generalized commutation relations, but it is of course independent of the choice of Hilbert basis in which it is calculated. Therefore, while some choices of Hilbert bases can provide conceptually interesting ''points of view,'' none of these is canonical.

We still have to prove that the generalized commutation relations Eqs. (44)–(46) do in fact imply a finite minimal uncertainty $\Delta x_0 > 0$, rather than, e.g., a discretization of position space. Before we do this in the next section, let us note an important representation theoretic consequence of the existence of a minimal uncertainty $\Delta x_0 > 0$.

A general argument shows that commutation relations which imply a finite minimal uncertainty in position cannot find a Hilbert space representation on a spectral representation of the position operators: The uncertainty relations hold in all * representations of the commutation relations. On the other hand, as is easily seen, e.g., in the example of Eq. (2), an eigenvector to an observable necessarily has vanishing uncertainty in this observable. Thus, if the uncertainty relations imply a finite uncertainty in positions, they exclude the existence of any position eigenvectors in any physical domain, i.e., on any domain on which the commutation relations are represented. In particular, in cases where $\Delta x_0 > 0$ and $\Delta p_0 > 0$ both position and momentum representations are ruled out and one has to resort to other Hilbert bases, as, e.g., in [12,14].

To be precise, let us assume that the commutation relations are represented on some dense domain $D \subset H$ in a Hilbert space H. Ordinarily, there would exist sequences $\{|\psi_n\rangle \in D\}$ with position uncertainties decreasing to zero (e.g., Gaussian approximations to the position eigenvectors). In the presence of a finite $\Delta x_0 > 0$, however, there exists a minimal uncertainty ''gap,'' i.e., there are no vectors $|\psi\rangle \in D$ which would have an uncertainty in positions in the interval $[0, \Delta x_0[$, so that now

$$\nexists \quad \{|\psi_n\rangle \in D\}: \lim_{n \to \infty} (\Delta x_0)_{|\psi_n\rangle} = 0. \tag{59}$$

Technically, the position operators are merely symmetric on representations D of the commutation relations. Their deficiency indices are nonvanishing and equal, implying the existence of a family of self-adjoint extensions in H, though, crucially of course, not in D. This functional analytic structure was first found in [11].

As is easily seen, there do exist formal position eigenvectors in H:

$$\psi_\xi(\rho)=\left[\left(\frac{\beta}{2\pi}\right)^{n/2}\frac{n\Gamma(n/2)}{2}\right]^{1/2}e^{-i\xi\cdot\rho/\hbar}. \tag{60}$$

Concerning the normalization, recall that the surface of the $(n-1)$-dimensional unit sphere reads $S_n=\int d\Omega_n=2\pi^{n/2}/\Gamma(n/2)$. The scalar product can be calculated to be

$$\begin{aligned}\langle\psi_\xi|\psi_\eta\rangle&=\left[\left(\frac{\beta}{2\pi}\right)^{n/2}\frac{n\Gamma(n/2)}{2}\right]\int_{I_n}d^n\rho e^{-i(\eta-\xi)\cdot\rho/\hbar}\\&=\left(\frac{\sqrt{2}\hbar\sqrt{\beta}}{|\xi-\eta|}\right)^{n/2}\Gamma\left(\frac{n}{2}+1\right)J_{n/2}\left(\frac{\sqrt{2}|\xi-\eta|}{\hbar\sqrt{\beta}}\right),\end{aligned} \tag{61}$$

where $J_{n/2}$ is the Bessel function of the first kind of order $n/2$. The zeros of the scalar product determine the self-adjoint extensions of the on D densely defined $\mathbf{x}_i$ (for any chosen ξ, all η's such that $|\xi-\eta|$ is a zero of $J_{n/2}$ correspond to the eigenvectors of one self-adjoint extension). However, as is readily verified, none of these vectors is in the domain of the $\mathbf{p}_i$. Thus, as is to be expected when $\Delta x_0>0$, none of the family of self-adjoint extensions of the $\mathbf{x}_i$ is in the domain of the representation of the commutation relations. In the one-dimensional case $n=1$ we recover the results obtained in [13], in particular the scalar product of the ''formal position eigenvectors'' (technically of eigenvectors of the adjoints $\mathbf{x}_i^*$, which are not self-adjoint, nor symmetric):

$$\langle\psi_\xi|\psi_\eta\rangle=\frac{\hbar\sqrt{\beta}\,\sin\left(\frac{\sqrt{2}|\xi-\eta|}{\hbar\sqrt{\beta}}\right)}{\sqrt{2}|\xi-\eta|}. \tag{62}$$

There is, however, a natural generalization of the position space representation. To this end we define a Hilbert space representation of the commutation relations on ''quasiposition space,'' see [13,15]:

$$\psi(x):=\langle x^{\mathrm{ml}}|\psi\rangle. \tag{63}$$

These quasiposition functions $\psi(x)$ are obtained by projecting the fields $|\psi\rangle$ onto the fields of maximal localization $|x^{\mathrm{ml}}\rangle$ and they do of course turn into the ordinary position space representation for $\Delta x_0\to 0$.

C. Maximally localized fields

Let us now prove that $\Delta x_0>0$ by explicitly calculating the maximally localized fields.

As is well known the $\Delta x_i\Delta p_i$ uncertainty relations are derived from the positivity of the norm:

$$||(\mathbf{x}_i-\langle\mathbf{x}_i\rangle)+ik(\mathbf{p}_i-\langle\mathbf{p}_i\rangle)|\psi\rangle||\geq 0. \tag{64}$$

Thus the vectors on the boundary of the region allowed by the uncertainty relations obey the squeezed state equation:

$$(\mathbf{x}_i-\langle\mathbf{x}_i\rangle)+ik(\mathbf{p}_i-\langle\mathbf{p}_i\rangle)|\psi\rangle=0. \tag{65}$$

Due to the symmetry of the underlying generalized commutation relations we do not lose generality by calculating the maximally localized field $|x^{\mathrm{ml}}\rangle$, around the origin, i.e., with $\langle 0^{\mathrm{ml}}|\mathbf{x}_i|0^{\mathrm{ml}}\rangle=0$ and $\langle 0^{\mathrm{ml}}|\mathbf{p}_i|0^{\mathrm{ml}}\rangle=0$. In ρ space Eq. (65) reads

$$\left(i\hbar\partial_{\rho_i}+ik\frac{\rho_i}{1-\beta\rho^2/2}\right)\psi_k(\rho)=0. \tag{66}$$

Due to rotational symmetry, $|0^{\mathrm{ml}}\rangle$ can only depend on p^2, so that Eq. (66) becomes

$$\partial_{\rho^2}\psi_k(\rho^2)=-\frac{k}{2\hbar(1-\beta\rho^2/2)}\psi_k(\rho^2) \tag{67}$$

whose normalized solutions read $(k>0)$

$$\psi_k(\rho^2)=\left[\left(\frac{\beta}{2\pi}\right)^{n/2}\frac{\Gamma\left(\frac{2k}{\hbar\beta}+\frac{n}{2}+1\right)}{\Gamma\left(\frac{2k}{\hbar\beta}+1\right)}\right]^{1/2}(1-\beta\rho^2/2)^{k/\hbar\beta}. \tag{68}$$

We can now calculate the squared uncertainty in position as a function of k:

$$(\Delta x)^2_{|\psi_k\rangle}=\frac{\hbar k}{4}\frac{4k+n\hbar\beta}{2k-\hbar\beta}. \tag{69}$$

The minimum is reached for

$$k_0=\frac{\hbar\beta}{2}\left(1+\sqrt{1+\frac{n}{2}}\right). \tag{70}$$

We therefore find the finite minimal uncertainty Δx_0:

$$(\Delta x_0)^2=(\Delta x)^2_{|\psi_{k_0}\rangle}=\frac{\hbar^2\beta}{4}\sqrt{1+n/2}(\sqrt{1+n/2}+1)^2. \tag{71}$$

The field $|0^{\mathrm{ml}}\rangle=|\psi_{k_0}\rangle$ of maximal localization around the origin therefore reads, in the ρ representation,

$$\begin{aligned}\langle\rho|0^{\mathrm{ml}}\rangle=\psi_{k_0}(\rho^2)&=N^{1/2}(n)(\beta/2\pi)^{n/4}\\&\times(1-\beta\rho^2/2)^{1/2+\sqrt{1/4+n/8}},\end{aligned} \tag{72}$$

where we defined

$$N(n):=\frac{\Gamma(2+n/2+\sqrt{1+n/2})}{\Gamma(2+\sqrt{1+n/2})}. \tag{73}$$

The ρ-space representation of the fields of maximal localization around arbitrary position expectation values ξ now follow by translation:

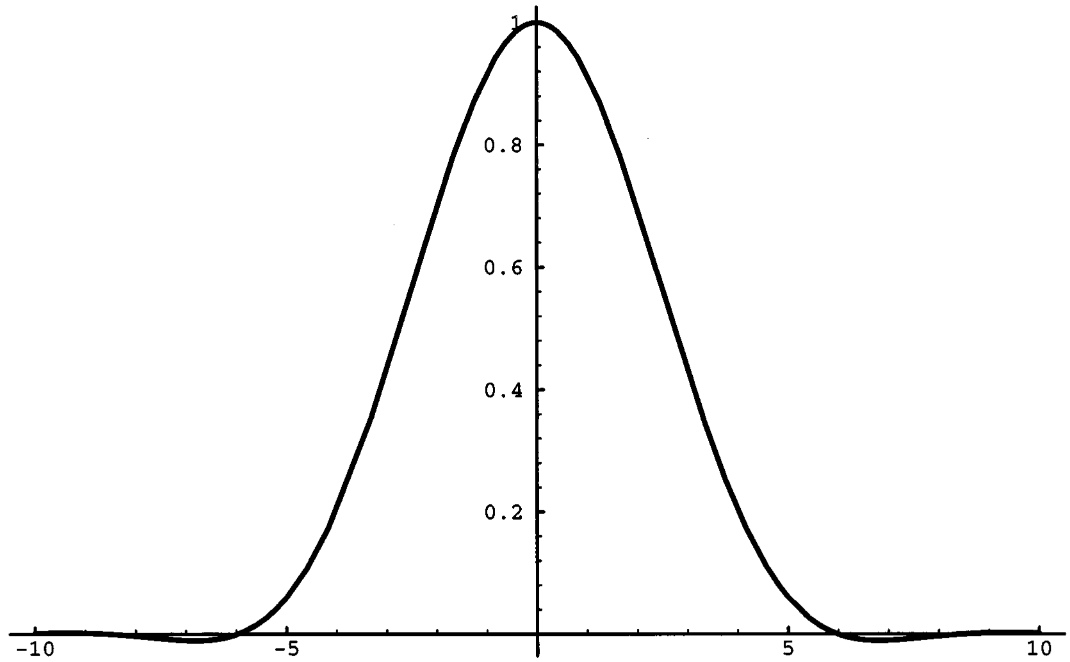

FIG. 3. Plot of the scalar product of maximally localized fields $\tilde{\delta}(x-y)$ vs $|x-y|/\hbar\sqrt{\beta}$. $\tilde{\delta}$ generalizes the Dirac δ distribution for $\Delta x_0 > 0$.

$$\langle \rho | x^{\mathrm{ml}} \rangle = \langle \rho | e^{x \cdot T} | 0^{\mathrm{ml}} \rangle = \langle \rho | 0^{\mathrm{ml}} \rangle e^{-ix \cdot \rho/\hbar}. \tag{74}$$

Using Eq. (52), we eventually obtain the maximally localized fields in momentum space,

$$\langle p | x^{\mathrm{ml}} \rangle = (\beta/2\pi)^{n/4} \; N^{1/2}(n) \left(\frac{\beta p^2}{1 + 2\beta p^2 - \sqrt{1+2\beta p^2}} \right)^{1/2} \times \left(\frac{\sqrt{1+2\beta p^2} - 1}{\beta p^2} \right)^{1+n/2+\sqrt{1/4+n/8}} \times \exp\left(-i \frac{x \cdot p}{\hbar} \frac{\sqrt{1+2\beta p^2} - 1}{\beta p^2} \right). \tag{75}$$

This expression is of course also the quasiposition representation of the plane wave with momentum p. We observe that in quasiposition space the fields can now not have arbitrarily fine ripples. Indeed, from the argument in the exponent in Eq. (75) we read off that for increasing momentum the wavelength in quasiposition space only tends towards a finite minimal wavelength

$$\lambda_0 = \pi\hbar\sqrt{2\beta} \tag{76}$$

which is reached as the momentum p_i tends to infinity. The situation is perhaps comparable to the speed of light as a fundamental limit, in which case also the kinematics lets the energy diverge as the fundamental limit is approached. In the Appendix we give the unitary transformation from the ρ representation to the quasiposition representation and back, and we prove the completeness of the set of maximally localized fields.

D. Feynman rules

As we saw in Sec. II B, the Feynman rules are composed of two basic functions, related to the vertex and to the propagator, respectively,

$$\tilde{\delta}(x^{\mathrm{ml}}, y^{\mathrm{ml}}) := \langle x^{\mathrm{ml}} | y^{\mathrm{ml}} \rangle$$

and

$$G(x,y) := \frac{\hbar^2}{(\Delta x_0)^2} \langle x^{\mathrm{ml}} | (\mathbf{p}^2 + m^2 c^2)^{-1} | y^{\mathrm{ml}} \rangle. \tag{77}$$

The calculation of $\tilde{\delta}$, i.e., of the scalar product of maximally localized fields is straightforward, in particular in the ρ representation. The result is of course independent of the choice of Hilbert basis. Choosing spherical coordinates:

$$\begin{aligned} \tilde{\delta}(x-y) &= \langle x^{\mathrm{ml}} | y^{\mathrm{ml}} \rangle \\ &= (\beta/2\pi)^{n/4} N^{1/2}(n) \int_{I_n} d^n \rho (1 - \beta\rho^2/2)^{1+\sqrt{1+n/2}} \\ &\quad \times e^{i(x-y)\cdot\rho/\hbar} \\ &= \left(\frac{\hbar\sqrt{2\beta}}{|x-y|} \right)^{1+n/2+\sqrt{1+n/2}} \Gamma\left(2 + \frac{n}{2} + \sqrt{1+n/2} \right) \\ &\quad \times J_{1+n/2+\sqrt{1+n/2}}\left(\frac{\sqrt{2}|x-y|}{\hbar\sqrt{\beta}} \right). \end{aligned} \tag{78}$$

We note that $\tilde{\delta}(x-y)$ is also the quasiposition wave function of the around y maximally localized field $|y^{\mathrm{ml}}\rangle$. Because of the finite norm of the maximally localized fields, $\tilde{\delta}(x-y)$ had of course to come out as a regular function. Its graph is plotted in Fig. 3.

Recall that ordinarily, i.e., when $\Delta x_0=0$, the propagator in position space $G(x-y)$ can only be defined as a distribution, and that it is the ill definedness of its square (as well as of higher powers) which gives rise to ultraviolet divergencies; see, e.g., [20]. We know from Sec. II C that $G(x-y)$ must now be a well-defined function without singularities. Explicitly, let us consider the free propagator matrix elements:

$$G(x-y)=\frac{\hbar^2}{(\Delta x_0)^2}\langle x^{\mathrm{ml}}|(p^2+m^2c^2)^{-1}|y^{\mathrm{ml}}\rangle$$
$$=\frac{N(n)\hbar^2}{(\Delta x_0)^2}\left(\frac{\beta}{2\pi}\right)^{n/2}\int_{I_n} d^n\rho\,\frac{(1-\beta\rho^2/2)^{3+\sqrt{1+n/2}}}{\rho^2+m^2c^2(1-\beta\rho^2/2)^2}$$
$$\times e^{i(x-y)\cdot\rho/\hbar}. \tag{79}$$

The massive propagator cannot be simply expressed in terms of elementary or special functions. However, for arbitrary nonvanishing mass, $G(x-y)$ can be uniformly bounded by

$$|G(x-y)|\leqslant\frac{\hbar^2}{m^2c^2(\Delta x_0)^2} \tag{80}$$

which shows that the propagator is well behaved for all distances, in particular also for $|x-y|\rightarrow 0$.

Since the small distance behavior is independent of the mass, let us also consider the simpler massless propagator. Using spherical coordinates and introducing the dimensionless variables

$$t=\rho\sqrt{\beta/2},\qquad s=\cos\theta, \tag{81}$$

we obtain

$$G(x-y)=\frac{N(n)\hbar^2\beta}{2(\Delta x_0)^2}\left(\frac{1}{\pi}\right)^{n/2}S_{n-1}\int_0^1 dt$$
$$\times\int_0^\pi d\theta(\sin\theta)^{n-2}t^{n-3}(1-t^2)^{3+\sqrt{1+n/2}}e^{idt\,\cos\theta}$$
$$=\frac{N(n)\hbar^2\beta}{2(\Delta x_0)^2}\left(\frac{1}{\pi}\right)^{n/2}S_{n-1}\int_0^1 dt\int_{-1}^1 ds2$$
$$\times(1-s^2)^{(n-3)/2}t^{n-3}(1-t^2)^{3+\sqrt{1+n/2}}e^{idts}, \tag{82}$$

where $d:=\sqrt{2}|x-y|/(\hbar\sqrt{\beta})$. Performing the integration over s and then over t, and after some simplification, one finally obtains, for $n>2$,

$$G(x-y)=2^n\ \frac{(3+\sqrt{1+n/2})(2+\sqrt{1+n/2})}{(n-2)\sqrt{1+n/2}(1+\sqrt{1+n/2})^2(2+n/2+\sqrt{1+n/2})}{}_1F_2(-1+n/2;\ 3+n/2+\sqrt{1+n/2},\ n/2;$$
$$-|x-y|^2/(2\hbar^2\beta)), \tag{83}$$

where we have used the explicit expressions for $(\Delta x_0)^2$ and $N(n)$ given in Eqs. (71) and (73).

In the particular case of four Euclidean dimensions the last expression can be cast in a much simpler form. For $n=4$, Eqs. (71) and (73) for $(\Delta x_0)^2,N(n)$ and the definition of $\tilde{\delta}(x-y)$ yield

$$G(x-y)=\frac{36+20\sqrt{3}}{6+4\sqrt{3}}\,\frac{\hbar^2\beta}{|x-y|^2}(1-\tilde{\delta}(x-y)). \tag{84}$$

Therefore, the propagator can be expressed as the product of the usual zero mass propagator and a smooth *cutoff* function, which has the following behavior for large and short distances:

$$\hbar^2\beta(1-\tilde{\delta}(x-y))\sim\frac{1}{8+2\sqrt{3}}|x-y|^2\left\{1+O\left[\left(\frac{\sqrt{2}|x-y|}{\hbar\sqrt{\beta}}\right)^2\right]\right\}\qquad\text{for}\ \ |x-y|\ll\hbar\sqrt{\beta},$$
$$\hbar^2\beta(1-\tilde{\delta}(x-y))\sim\hbar^2\beta\left\{1+O\left[\left(\frac{\sqrt{2}|x-y|}{\hbar\sqrt{\beta}}\right)^{-7/2-\sqrt{3}}\right]\right\}\qquad\text{for}\ \ |x-y|\gg\hbar\sqrt{\beta}. \tag{85}$$

In particular $G(x-y)$ is a well behaved function in the short distance regime and tends to a finite limit for $|x-y|=0$, while for distances larger than $\hbar\sqrt{\beta}$ it rapidly approaches the well known $|x-y|^{-2}$ behavior of the free massless propagator of the case of the ordinary commutation relations.

IV. SUMMARY AND OUTLOOK

Studies in string theory and quantum gravity provide theoretical evidence for various types of correction terms to the canonical commutation relations. Measurable effects could in principle appear anywhere between the presently resolvable scale of 10^{-18} m and the Planck scale of 10^{-35} m, although expected close to the Planck scale.

Our aim here was to show that the existence of even an at present unmeasurably small Δx_0, for example at about the Planck length, could have a drastic effect in field theory, namely by rendering the theory ultraviolet finite. We note that this new short distance behavior would truly be a quantum structure, in the sense that it has no classical analog. Further, the presence of a $\Delta x_0 > 0$ is compatible with both, generalized symmetries as well as with conventional rotation and translation symmetry.

The existence of this short distance structure would raise a number of conceptual issues, in particular, it would lead to a generalized notion of local interaction. Strictly speaking, the maximally local interaction term in the case of generalized commutation relations is neither local nor nonlocal in the conventional sense. Instead, it is ''observationally'' local in the sense that it is local as far as distances can be resolved, given the generalized uncertainty relations.

Similarly, questions such as whether the unitarity of time evolution is broken or conserved, or whether local gauge invariance is broken or conserved also seem not to be applicable in the usual sense. Instead, in the case of the generalized uncertainty relations the notions of ''time evolution'' or ''local'' gauging may need to be redefined, analogously to how local interaction is generalized into ''observationally'' local interaction. This is under investigation.

There exist a number of immediate technical issues which need to be addressed, for example the significance of Eq. (32) and its pole structure, and Wick rotation. It should of course also be worth exploring the possible usefulness of the approach as a mere regularization method.

APPENDIX

We prove the completeness of the set of maximally localized fields and we give the unitary transformation which connects the ρ representation with the quasiposition representation for the example of generalized commutation relations considered in Sec. III.

In order to see that the set of maximally localized fields $\{|x^{\mathrm{ml}}\rangle\}$ is complete, we use that a set of vectors $|\psi_\lambda\rangle$ in a Hilbert space is complete if from $\langle\phi|\psi_\lambda\rangle = 0,\ \forall|\psi_\lambda\rangle$ follows that $\langle\phi| = 0$. Consider now $\langle\phi|x^{\mathrm{ml}}\rangle$:

$$\langle\phi|x^{\mathrm{ml}}\rangle = N^{1/2}(n)(\beta/2\pi)^{n/4}\int_{I_n} d^n\rho \times (1-\beta\rho^2/2)^{1/2+\sqrt{1/4+n/8}}e^{ix\cdot\rho/\hbar}\phi(\rho). \quad (\mathrm{A1})$$

Using the mean value theorem

$$\begin{aligned}\langle\phi|x^{\mathrm{ml}}\rangle &= N^{1/2}(n)(\beta/2\pi)^{n/4}(1-\beta\overline{\rho}^2/2)^{1/2+\sqrt{1/4+n/8}} \\ &\quad\times\int_{I_n} d^n\rho e^{ix\cdot\rho/\hbar}\phi(\rho) \\ &= N^{1/2}(n)\frac{2}{n\Gamma(n/2)}(1-\beta\overline{\rho}^2/2)^{1/2+\sqrt{1/4+n/8}} \\ &\quad\times\langle\phi|\psi_x\rangle, \end{aligned} \quad (\mathrm{A2})$$

where the $|\psi_x\rangle$ are eigenvectors to the (non-Hermitian) operators $\mathbf{x}_i^*$ [see Eq. (60)] and $\overline{\rho}$ is some value in the open interval $]0,\sqrt{2/\beta}[$. The set of vectors $|\psi_x\rangle$ is the collection of all eigenbases to the self-adjoint extensions of the $\mathbf{x}_i$, and is therefore overcomplete (for the details of the functional analysis see [11,13,15,18]). Further, the factor in front of the scalar product in Eq. (A2) never vanishes nor diverges for any $\overline{\rho}$. Thus, if the right-hand side of Eq. (A2) vanishes for all x this implies $|\phi\rangle = 0$, which had to be shown.

The completeness of the set of maximally localized fields means that we obtain the full information on any $|\psi\rangle$ when collecting its projections $\langle x^{\mathrm{ml}}|\psi\rangle$ on the $|x^{\mathrm{ml}}\rangle$, i.e., the quasiposition representation truly represents the fields. Indeed, the mapping, e.g., from ρ space to quasiposition space is invertible.

Using the explicit expressions Eqs. (72) and (74) for the maximally localized fields in the ρ representation we obtain for arbitrary $|\psi\rangle$ the quasiposition wave function $\psi(x)$ expressed in terms of its ρ representation $\psi(\rho)$ as

$$\psi(x) = \langle x^{\mathrm{ml}}|\psi\rangle = N^{1/2}(n)(\beta/2\pi)^{n/4} \times\int_{I_n} d^n\rho(1-\beta\rho^2/2)^{1/2+\sqrt{1/4+n/8}}e^{ix\cdot\rho/\hbar}\psi(\rho) \quad (\mathrm{A3})$$

with the inverse:

$$\psi(\rho) = N^{-1/2}(n)(\beta/2\pi)^{-n/4}(2\pi\hbar)^{-n} \times(1-\beta\rho^2/2)^{-1/2-\sqrt{1/4+n/8}}\int d^nx\ \ e^{-ix\cdot\rho/\hbar}\psi(x). \quad (\mathrm{A4})$$

Let us denote the mapping from ρ space to quasiposition space [Eq. (A3)] by U. The identity $U^{-1}U = 1$ is easily verified by inserting Eq. (A3) into Eq. (A4).

In the quasiposition representation the scalar product and the action of the position and momentum operators then read

$$\langle\psi|\phi\rangle = N^{-1}(n)(\beta/2\pi)^{-n/2}(2\pi\hbar)^{-2n} \times\int_{I_n} d^n\rho\ \ (1-\beta\rho^2/2)^{-1-\sqrt{1+n/2}} \times\int\int d^nx\ d^ny\ e^{i(x-y)\cdot\rho/\hbar}\psi^*(x)\phi(y), \quad (\mathrm{A5})$$

$$\mathbf{p}^i\cdot\psi(x) = -i\hbar\sum_{r=0}^{\infty}(\hbar^2\beta\Delta/2)^r\frac{\partial}{\partial x_i}\psi(x), \quad (\mathrm{A6})$$

$$\mathbf{x}^i\cdot\psi(x) = \left(x^i + i\frac{\hbar\beta}{2}\left(1+\sqrt{1+\frac{n}{2}}\right)\mathbf{p}^i\right)\psi(x), \quad (\mathrm{A7})$$

where $\Delta = \Sigma_i\partial^2/\partial x_i^2$. Note that the action of $\mathbf{p}_i$ given in Eq. (A6) [and also used in Eq.(A7)] is well defined on quasiposition wave functions $\psi(x)$, since they Fourier decompose into wavelengths not smaller than the finite minimal wavelength λ_0 [Eq. (76)]. In this context see also [13] where the concept of quasiposition representation was first introduced.

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