Partition function for (2+1)-dimensional Einstein gravity

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Taking (2+1)-dimensional pure Einstein gravity for arbitrary genus $g \geqslant 1$ as a model, we investigate the relation between the partition function formally defined on the entire phase space and the one written in terms of the reduced phase space. The case of $g=1$ (torus) is analyzed in detail and it provides us with good lessons for quantum cosmology. We formulate the gauge-fixing conditions in a form suitable for our purpose. Then the gauge-fixing procedure is applied to the partition function Z for (2+1)-dimensional gravity, formally defined on the entire phase space. We show that basically it reduces to a partition function defined for the reduced system, whose dynamical variables are (τ^A, p_A). (Here the τ^A's are the Teichmüller parameters and the p_A's are their conjugate momenta.) As for the case of $g=1$, we find out that Z is also related with another reduced form, whose dynamical variables are not only (τ^A, p_A), but also (V, σ). [Here σ is a conjugate momentum to the two-volume (area) V of a spatial section.] A nontrivial factor appears in the measure in terms of this type of reduced form. This factor is understood as a Faddeev-Popov determinant associated with the time-reparametrization invariance inherent in this type of formulation. In this manner, the relation between two reduced formulations becomes transparent in the context of quantum theory. As another result for the case of $g=1$, one factor originating from the zero modes of a differential operator P_1 can appear in the path-integral measure in the reduced representation of Z. It depends on how to define the path-integral domain for the shift vector N_a in Z: If it is defined to include ker P_1, the nontrivial factor does not appear. On the other hand, if the integral domain is defined to exclude ker P_1, the factor appears in the measure. This factor can depend on the dynamical variables, typically as a function of V, and can influence the semiclassical dynamics of the (2+1)-dimensional spacetime. These results shall be significant from the viewpoint of quantum gravity and quantum cosmology. [S0556-2821(97)06402-3]

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I. INTRODUCTION

Because of both its simplicity and nontrivial nature, (2+1)-dimensional Einstein gravity serves as a good test case for pursuing quantum gravity in the framework of general relativity. In particular, because of the low dimensionality, the global degrees of freedom of a space can be analyzed quite explicitly in this case [1–4].

Recently, back-reaction effects from quantum matter on the global degrees of freedom of a semiclassical universe were analyzed explicitly [5]. In this analysis, the (2+1)-dimensional homogeneous spacetime with topology $\mathcal{M} \simeq T^2 \times \mathbf{R}$ was chosen as a model. This problem was investigated from a general interest on the global properties of a semiclassical universe, whose analysis has not yet been pursued sufficiently [5–7].

In this analysis, it was also investigated whether the path integral measure could give a correction to the semiclassical dynamics of the global degrees of freedom [5]. By virtue of several techniques developed in string theory, one can give a meaning to a partition function, formally defined as

$$Z = \mathcal{N} \int [dh_{ab}][d\pi^{ab}][dN][dN_a] \ \exp(iS). \qquad (1)$$

Here h_{ab} and π^{ab} are a spatial metric and its conjugate momentum, respectively; N and N_a are the lapse and shift functions, respectively; S is the canonical action for Einstein gravity. It is expected that Z reduces to the form

$$Z = \mathcal{N} \int [dV \, d\sigma][d\tau^A \ dp_A][dN'] \times \mu(V, \sigma, \tau^A, p_A) \exp(iS_{\text{reduced}}).$$

Here V, σ, τ^A, and p_A ($A=1,2$) are, respectively, the two-volume (area) of a torus, its conjugate momentum, the Teichmüller parameters, and their conjugate momenta; N' is the spatially constant part of N; S_{reduced} is the reduced action written in terms of V, σ, τ^A, and p_A. The factor $\mu(V, \sigma, \tau^A, p_A)$ is a possible nontrivial measure, which can cause a modification of the semiclassical evolution determined by S_{reduced}. The result of Ref. [5] was that $\mu(V, \sigma, \tau^A, p_A) = 1$: The partition function defined as in Eq. (1) is equivalent, after a suitable gauge fixing, to the one defined directly from the reduced system, S_{reduced}. Though this result looks natural at first sight, it is far from trivial. One needs to extract a finite dimensional reduced phase space from an infinite dimensional original phase space. Therefore, it is meaningful to show that such a natural reduction is really achieved by a suitable gauge fixing.

The main interest in Ref. [5] was the explicit analysis of the semiclassical dynamics of the tractable model, $\mathcal{M} \simeq T^2 \times \mathbf{R}$. Therefore, the analysis of the reduction of the partition function was inevitably restricted to the special model in question. Namely it was the case of $g=1$, where g is a genus of a Riemann surface. Furthermore, the model was set to be spatially homogeneous from the outset. It is then desirable for completeness to generalize the analysis in Ref. [5] to the general case of any $g \geqslant 1$.

More significantly, there is one issue remaining to be clarified in the case of $g=1$: The relation between the reduced system of the type of Ref. [2] and the one of the type of Ref. [3] in the context of quantum theory. For brevity, let us call the former formulation as the τ-form, while the latter one as the (τ,V)-form. The τ-form takes (τ^A,p_A) as fundamental canonical pairs and the action is given by [2]

$$S[\tau^A,p_A]=\int d\sigma\{p_A \; d\tau^A/d\sigma-V(\sigma,\tau^A,p_A)\}. \qquad (2)$$

On the other hand the (τ,V)-form uses (V,σ) as well as (τ^A,p_A) and the action is given in the form [3]

$$S[(\tau^A,p_A),(V,\sigma)]=\int dt\{p_A \; \dot{\tau}^A+\sigma\dot{V}-NH(\tau^A,p_A,V,\sigma)\}. \qquad (3)$$

(The explicit expression for H shall be presented later [Eq. (23)].) The key procedure in deriving the (τ,V)-form (in the classical sense) is to choose $N=$ spatially constant [3]. Since the compatibility of this choice with York's time-slicing is shown by means of the equations of motion [3], one should investigate the effect of this choice in quantum theory. Furthermore, the condition $N=$ spatially constant is not in the standard form of the canonical gauge, so that the analysis of its role in the quantum level requires special cares. Since the model analyzed in Ref. [5] was chosen to be spatially homogeneous, this issue did not make its appearance. We shall make these issues clarified.

Regarding the (τ,V)-form, there is another issue which is not very clear. In this formulation, (V,σ) joins to (τ^A,p_A) as one of the canonical pairs. Therefore [5], $\int[d\sigma]$ should appear in the final form of Z as well as $\int[dV]$. Since the adopted gauge-fixing condition is $\pi/\sqrt{h}-\sigma=0$ (York's gauge [8]), σ plays the role of a label parametrizing a family of allowed gauge-fixing conditions, so that it is not dynamical in the beginning. Therefore, the appearance of $\int[d\sigma]$ is not apparent, and worthy of tracing from a viewpoint of a general procedure of gauge fixing. We shall investigate these points.

Independently from the analysis of Ref. [5], Carlip also investigated the relation between two partition functions, one being defined on the entire phase space, and the other one on the reduced phase space in the sense of the τ-form [9]. With regard to this problem, his viewpoint was more general than Ref. [5]. He showed that, for the case of $g\geqslant 2$, the partition function formally defined as in Eq. (1) is equivalent to the one for the reduced system in the τ-form. On the other hand, the exceptional case of $g=1$ was not analyzed so much. Indeed, we shall see later that the case of $g=1$ can yield a different result compared with the case of $g\geqslant 2$. In this respect, his analysis and the analysis in Ref. [5] do supplement each other. Furthermore, his way of analysis is quite different from the one developed in Ref. [5]. In particular, it looks difficult to trace the appearance of $\int[d\sigma]$ if his analysis is applied to the case of $g=1$ in the (τ,V)-form. It may be useful, therefore, to investigate all the cases of $g\geqslant 1$ from a different angle, namely by a developed version of the method of Ref. [5].

In view of these situations of previous work, we shall present here the full analysis for all the cases $g\geqslant 1$. In particular, a more detailed investigation for the case of $g=1$ shall be performed.

In Sec. II, we shall investigate for $g\geqslant 1$ the reduction of the partition function of Eq. (1), to the one for the reduced system in the τ-form. In Sec. III, we shall investigate how the (τ,V)-form emerges when $g=1$ in the course of the reduction of the partition function, Eq. (1). We shall find out that a nontrivial measure appears in the formula defining a partition function, if the (τ,V)-form is adopted. We shall see that this factor is understood as the Faddeev-Popov determinant associated with the reparametrization invariance inherent in the (τ,V)-form. Furthermore we shall see that another factor can appear in the measure for the case of $g=1$, originating from the existence of the zero modes of a certain differential operator P_1. It depends on how to define the path-integral domain for the shift vector N_a in Z: If it is defined to include ker P_1, the nontrivial factor does not appear, while it appears if the integral domain is defined to exclude ker P_1. We shall discuss that this factor can influence the semiclassical dynamics of the (2+1)-dimensional spacetime with $g=1$. These observations urge us to clarify how to choose the integral domain for N_a in quantum gravity. Section IV is devoted to several discussions. In the Appendixes, we shall derive useful formulas which shall become indispensable for our analysis.

II. THE PARTITION FUNCTION FOR (2+1)-GRAVITY

Let us consider a (2+1)-dimensional spacetime, $\mathcal{M}\simeq\Sigma\times\mathbf{R}$, where Σ stands for a compact, closed, orientable two-surface with genus g. The partition function for (2+1)-dimensional pure Einstein gravity is formally given by

$$Z=\mathcal{N}\int [dh_{ab}][d\pi^{ab}][dN][dN_a] \times\exp\left(i\int dt\int_\Sigma d^2x \;\; (\pi^{ab}\dot{h}_{ab}-N\mathcal{H}-N_a\mathcal{H}^a)\right), \qquad (4)$$

where[1]

$$\mathcal{H}=(\pi^{ab}\pi_{ab}-\pi^2)\sqrt{h^{-1}}-({}^{(2)}R-2\lambda)\sqrt{h}, \qquad (5)$$

$$\mathcal{H}^a=-2D_b\pi^{ab}. \qquad (6)$$

Here, λ is the cosmological constant which is set to be zero if it is not being considered.

Taking $\mathcal{H}=\mathcal{H}(\sqrt{h})$, a canonical pair $(\sqrt{h},\pi/\sqrt{h})$ can be chosen to be gauge-fixed. One natural way to fix the gauge is to impose a one-parameter family of gauge-fixing conditions:

[1]We have chosen units such that $c=\hbar=1$ and such that the Einstein-Hilbert action becomes just $\int R\sqrt{-g}$ up to a boundary term. The spatial indices $a,b,\ldots$ are raised and lowered by h_{ab}. The operator D_a is the covariant derivative with respect to h_{ab}, and ${}^{(2)}R$ stands for a scalar curvature of the two-surface Σ. Unless otherwise stated, the symbols π and h stand for $h_{ab}\pi^{ab}$ and det h_{ab}, respectively, throughout this paper.

$$\chi_1 := \frac{\pi}{\sqrt{h}} - \sigma = 0 \quad (\exists \sigma \in \mathbf{R}), \tag{7}$$

where σ is a spatially constant parameter (York's gauge [8]). Let us make clear the meaning of the gauge Eq. (7).

We adopt the following notations: $(P_1^\dagger w)^a := -2D_b w^{ab}$ for a symmetric traceless tensor w^{ab}; $\tilde{\pi}^{ab} := \pi^{ab} - \frac{1}{2}\pi h^{ab}$ is the traceless part of π^{ab} and in particular $\tilde{\pi}'^{ab}$ stands for $\tilde{\pi}'^{ab} \notin \ker P_1^\dagger$.

Now, let $Q := \pi/\sqrt{h}$ and $Q' := \int_\Sigma d^2x \sqrt{h} Q / \int_\Sigma d^2x \sqrt{h}$, which is the spatially constant component of Q. Therefore, $\mathcal{P}'(\cdot) := \int_\Sigma d^2x \sqrt{h}\,(\cdot)/\int_\Sigma d^2x \sqrt{h}$ forms a linear map which projects Q to Q'. On the other hand, $\mathcal{P} = 1 - \mathcal{P}'$ projects Q to its spatially varying component. Note that $(\mathcal{P}Q, \mathcal{P}'Q) = 0$ with respect to the natural inner product (Appendix A). Then, Eq. (7) can be recast as

$$\chi_1 := \mathcal{P}\left(\frac{\pi}{\sqrt{h}}\right) = 0. \tag{8}$$

Next, we note that $\mathcal{H}^a = -2D_b \tilde{\pi}^{ab} =: (P_1^\dagger \tilde{\pi})^{ab}$ under the condition of Eq. (8). Taking $\mathcal{H}^a = \mathcal{H}^a(\tilde{\pi}'^{ab})$, a pair $(h_{ab}/\sqrt{h}, \tilde{\pi}'^{ab}\sqrt{h})$ shall be gauge-fixed. Thus we choose, as a gauge-fixing condition,

$$\chi_2 := \frac{h_{ab}}{\sqrt{h}} - \hat{h}_{ab}(\tau^A) = 0 \quad (\exists \tau^A \in \mathcal{M}_g), \tag{9}$$

where $\hat{h}_{ab}$ is an m-parameter family of reference metrics ($m = 2$, $6g-6$ for $g=1$, $g \geqslant 2$, respectively) such that $\det \hat{h}_{ab} = 1$; τ^A $(A = 1, \ldots, m)$ denote the Teichmüller parameters parametrizing the moduli space $\mathcal{M}_g$ of Σ [10].

At this stage, we recall [10] that a general variation of h_{ab} can be decomposed as $\delta h_{ab} = \delta_W h_{ab} + \delta_D h_{ab} + \delta_M h_{ab}$, where $\delta_W h_{ab}$ is the trace part of δh_{ab} (Weyl deformation), $\delta_D h_{ab} = (P_1 v)_{ab} := D_a v_b + D_b v_a - D_c v^c h_{ab}$ for $\exists v^a$ (the traceless part of a diffeomorphism), and $\delta_M h_{ab} = \mathcal{T}_{Aab}\delta\tau^A := (\partial h_{ab}/\partial \tau^A - \frac{1}{2} h^{cd}(\partial h_{cd}/\partial\tau^A) h_{ab})\delta\tau^A$ (the traceless part of a moduli deformation).[2] It is easy to show that [10] the adjoint of P_1 with respect to the natural inner product (Appendix A) becomes $(P_1^\dagger w)^a := -2D_b w^{ab}$, acting on a symmetric traceless tensor w^{ab}. [Therefore the notation "P_1" is compatible with the notation "$P_1^\dagger$" introduced just after Eq. (7).]

Now, the meaning of the gauge Eq. (9) is as follows. The variation of $h_{ab}/\sqrt{h}$ in the neighborhood of $\hat{h}_{ab}(\tau^A)$ is expressed as[3]

$$\delta\{h_{ab}/\sqrt{h}\} = \delta_D\{h_{ab}/\sqrt{h}\} + \delta_M\{h_{ab}/\sqrt{h}\}.$$

Let $\mathrm{Riem}_1(\Sigma)$ denote the space of unimodular Riemannian metrics on Σ. We introduce projections of the tangent space of $\mathrm{Riem}_1(\Sigma)$ at $\hat{h}_{ab}(\tau^A)$, $T_{\hat{h}_{ab}(\tau^A)}(\mathrm{Riem}_1(\Sigma))$:

$$\mathcal{P}_D(\delta\{h_{ab}/\sqrt{h}\}) = \delta_D\{h_{ab}/\sqrt{h}\},$$

$$\mathcal{P}_M(\delta\{h_{ab}/\sqrt{h}\}) = \delta_M\{h_{ab}/\sqrt{h}\},$$

$$\mathcal{P}_D + \mathcal{P}_M = 1.$$

Then, the gauge Eq. (9) is recast as

$$\chi_2 = \mathcal{P}_D(\delta\{h_{ab}/\sqrt{h}\}) = 0. \tag{10}$$

On $\mathrm{Riem}_1(\Sigma)$ we can introduce a system of coordinates in the neighborhood of each $\hat{h}_{ab}(\tau^A)$. Then $[dh_{ab}]$ in Eq. (4) is expressed as $[d\sqrt{h}][d\delta\{h_{ab}/\sqrt{h}\}]$. (It is easy to show that the Jacobian factor associated with this change of variables is unity.)

Finally let us discuss about the integral domain for N_a in Eq. (4) for the case of $g=1$.[4] Let us note that, under the gauge Eq. (8), we get

$$\int_\Sigma d^2x\, N_a \mathcal{H}^a = 2\int_\Sigma d^2x\, (P_1 N)_{ab}\tilde{\pi}^{ab}.$$

Thus, when $N_a \in \ker P_1$, N_a does not work as a Lagrange multiplier enforcing the momentum constraint Eq. (6). Then there are two possible options for the path-ingegral domain of N_a:

(a) All of the vector fields on Σ, including $\ker P_1$.

(b) All of the vector fields on Σ, except for $\ker P_1$.

If we choose the option (a), the integral over N_a in Eq. (4) yields the factors $\det^{1/2}(\varphi_\alpha, \varphi_\beta)\,\delta(P_1^\dagger\tilde{\pi})$. Here $\{\varphi_\alpha\}_{\alpha=1,2}$ is a basis of $\ker P_1$ for the case of $g=1$.[5] The factor $\det^{1/2}(\varphi_\alpha, \varphi_\beta)$ appears here since it is proportional to the volume of $\ker P_1$ with respect to the natural inner product.

If we choose the option (b), the integral over N_a yields just a factor $\delta(P_1^\dagger\tilde{\pi})$.

Integrating over the Lagrange multipliers N and N_a, (4) reduces to

$$Z = \mathcal{N}\int [dh_{ab}][d\pi^{ab}]\mathcal{B} \times \delta(\mathcal{H})\delta(\mathcal{H}^a)\exp\left(i\int dt\int_\Sigma d^2x\, \pi^{ab}\dot{h}_{ab}\right), \tag{11}$$

where

[2]Needless to say, these quantities are defined for h_{ab}, a spatial metric induced on Σ. Therefore, under the condition (9), they are calculated using $\sqrt{h}\hat{h}_{ab}(\tau^A)$, and not just $\hat{h}_{ab}(\tau^A)$.

[3]The symbol $\delta\{\cdot\}$ shall be used to represent a variation whenever there is a possibility of being confused with the delta function $\delta(\cdot)$.

[4]The author thanks S. Carlip for valuable remarks on this point.

[5]Let us recall that $\dim\ker P_1 = 6$, 2, and 0 for $g=0$, $g=1$, and $g \geqslant 2$, respectively. On the other hand, $\dim\ker P_1^\dagger = 0$, 2, and $6g-6$ for $g=0$, $g=1$, and $g\geqslant 2$, respectively. There is a relation $\dim\ker P_1 - \dim\ker P_1^\dagger = 6 - 6g$ (Riemann-Roch theorem) [10]. [Throughout this paper, $\dim W$ indicates the real dimension of a space W, regarded as a vector space over $\mathbf{R}$.]

$$\mathcal{B}=\begin{cases}\det^{1/2}(\varphi_\alpha,\varphi_\beta) & \text{when } g=1 \text{ with the option (a)}\\ 1 & \text{otherwise.}\end{cases}$$

According to the Faddeev-Popov procedure [11], we insert into the right-hand side of Eq. (11) the factors

$$|\det\{\mathcal{H},\chi_1\}||\det\{\mathcal{H}^a,\chi_2\}|\,\delta(\chi_1)\,\delta(\chi_2).$$

Note that, because $\{\int v_a\mathcal{H}^a,\chi_1\}=-v_a\mathcal{H}^a-v^cD_c\chi_1=0$ mod $\mathcal{H}^a=0$ and $\chi_1=0$, the Faddeev-Popov determinant separates into two factors as above.[6] The determinants turn to simpler expressions if we note the canonical structure of our system:

$$\int_\Sigma d^2x\ \pi^{ab}\delta h_{ab}$$

$$=\int_\Sigma d^2x\left(\tilde{\pi}^{ab}+\frac{1}{2}\pi h^{ab}\right)(\delta_W h_{ab}+\delta_D h_{ab}+\delta_M h_{ab})$$

$$=\int_\Sigma d^2x\left(\frac{\pi}{\sqrt{h}}\delta\sqrt{h}+(P_1^\dagger\tilde{\pi}')^a v_a+\tilde{\pi}^{ab}\delta_M h_{ab}\right).$$

Thus,

$$\det\{\mathcal{H},\chi_1\}=\frac{\partial\mathcal{H}}{\partial\sqrt{h}},$$

$$\det\{\mathcal{H}^a,\chi_2\}=\left(\det\frac{\partial\mathcal{H}^a}{\partial\tilde{\pi}'^{ab}}\right)\cdot\frac{\partial\chi_2}{\partial(\delta_D h_{ab})}=\det{}'P_1^\dagger\sqrt{h}.$$

Thus we get

$$Z=\mathcal{N}\int[d\sqrt{h}\ \ d(\delta\{h_{ab}/\sqrt{h}\})d(\pi/\sqrt{h})\ \ d\tilde{\pi}^{ab}]\mathcal{B}$$

$$\times\frac{\partial\mathcal{H}}{\partial\sqrt{h}}\det{}'P_1^\dagger\sqrt{h}\,\delta(\mathcal{H})\,\delta(P_1^\dagger\tilde{\pi})\,\delta(\mathcal{P}(\pi/\sqrt{h}))$$

$$\times\delta(\mathcal{P}_D(\delta\{h_{ab}/\sqrt{h}\}))$$

$$\times\exp\left[i\int dt\int_\Sigma d^2x\left(\tilde{\pi}^{ab}+\frac{1}{2}\pi h^{ab}\right)\dot{h}_{ab}\right].\tag{12}$$

We can simplify the above expression. First, the path integral with respect to $\pi/\sqrt{h}$ in Eq. (12) is of the form

$$I_1=\int d(\pi/\sqrt{h})\,\delta\,(\mathcal{P}(\pi/\sqrt{h}))F(\pi/\sqrt{h}),$$

so that Eq. (B1) in Appendix B can be applied. Note that ker $\mathcal{P}=$ a space of spatially constant functions, which forms a one-dimensional vector space over **R**. Now dim ker $\mathcal{P}=1$, so that dp_A and $d(p_A\vec{\Psi}^A)$ are equivalent, following the notation in Appendix B. Furthermore $\mathcal{P}$ is a projection. Thus no extra Jacobian factor appears in this case. Thus we get

$$I_1=\int[d\sigma]F(\mathcal{P}(\pi/\sqrt{h})=0,\sigma),$$

where σ denotes a real parameter parametrizing ker $\mathcal{P}$.

Second, the path integral with respect to $\delta\{h_{ab}/\sqrt{h}\}$ is of the form

$$I_2=\int d\delta\{h_{ab}/\sqrt{h}\}\delta\,(\mathcal{P}_D(\delta\{h_{ab}/\sqrt{h}\}))G(\delta\{h_{ab}/\sqrt{h}\}).$$

Note that ker $\mathcal{P}_D=\delta_M\{h_{ab}/\sqrt{h}\}=\sqrt{h^{-1}}\delta_M h_{ab}$ and $\det'\mathcal{P}_D=1$. Let $\{\xi^A\}$ $(A=1,\ldots,\dim\ker P_1^\dagger)$ be a basis of ker $\mathcal{P}_D$. Then the factor $\det^{1/2}(\xi^A,\xi^B)$ [see Eq. (B1)] is given as

$$\det{}^{1/2}(\xi^A,\xi^B)=\det(\mathcal{T}_A,\Psi^B)\det{}^{-1/2}(\Psi^A,\Psi^B)\sqrt{h^{-1}},\tag{13}$$

where $\{\Psi^A\}$ $(A=1,\ldots,\dim\ker P_1^\dagger)$ is a basis of ker $P_1^\dagger$. This expression results in as follows. Carrying out a standard manipulation [10,9,5],[7]

$$\delta h_{ab}=\frac{\delta\sqrt{h}}{\sqrt{h}}h_{ab}+(P_1v)_{ab}+\mathcal{T}_{Aab}\delta\tau^A$$

$$=\frac{\delta\sqrt{h}}{\sqrt{h}}h_{ab}+(P_1\tilde{v})_{ab}$$

$$+(\mathcal{T}_A,\Psi^B)(\Psi^\cdot,\Psi^\cdot)^{-1}{}_{BC}\Psi^C\delta\tau^A.\tag{14}$$

For the present purpose, the first and second terms are set to be zero. (See footnote 2.) According to Appendix A, then, it is easy to get Eq. (13). Then with the help of Eq. (B1), we get

$$I_2=\int d\tau^A\ \det(\mathcal{T}_A,\Psi^B)\ \det{}^{-1/2}(\Psi^A,\Psi^B)\sqrt{h^{-1}}$$

$$\times G(\delta_D\{h_{ab}/\sqrt{h}\}=0,\ \tau^A).$$

Here we understand that the integral domain for $\int d\tau^A$ is on the moduli space $\mathcal{M}_g$, and not the Teichmüller space, which is the universal covering space of $\mathcal{M}_g$ [10]. This is clear because τ^A appears in the integrand G only through $\hat{h}_{ab}$ [Eq. (9)].

We note that the kinetic term in Eq. (12) becomes

[6]For notational neatness, the symbol of absolute value associated with the Faddeev-Popov determinants shall be omitted for most of the cases.

[7]Because $P_1^\dagger$ is a Fredholm operator on a space of symmetric traceless tensors $\mathcal{W}$, $\mathcal{W}$ can be decomposed as $\mathcal{W}=\mathrm{Im}\,P_1\oplus\ker P_1^\dagger$ [12]. Therefore $\mathcal{T}_{Aab}\delta\tau^A\in\mathcal{W}$ is uniquely decomposed in the form of $P_1u_0+(\mathcal{T}_A,\Psi^B)(\Psi^\cdot,\Psi^\cdot)^{-1}{}_{BC}\Psi^C\delta\tau^A$. Then, $(P_1\tilde{v})_{ab}:=(P_1(v+u_0))_{ab}$.

$$\int_\Sigma d^2x\left(\tilde{\pi}^{ab}+\frac{1}{2}\pi h^{ab}\right)\dot{h}_{ab}\bigg|_{\chi_1=\chi_2=0}$$

$$=\int_\Sigma d^2x\left(\tilde{\pi}^{ab}+\frac{1}{2}\sigma\sqrt{h}h^{ab}\right)\dot{h}_{ab}\bigg|_{\chi_2=0}$$

$$=(\tilde{\pi}'^{ab}+p_A\Psi^{Aab}\sqrt{h},\mathcal{T}_{Bcd})\dot{\tau}^B+\sigma\dot{V}.$$

Here $V:=\int_\Sigma d^2x\sqrt{h}$, which is interpreted as a two-volume (area) of Σ. (See Appendix A for the inner product of densitized quantities.)

Finally, the path integral with respect to $\tilde{\pi}^{ab}$ in Eq. (4) is of the form

$$I_3=\int d\tilde{\pi}^{ab}\;\;\delta(P_1^\dagger\tilde{\pi})H(\tilde{\pi}^{ab}).$$

Using Eq. (B1), this is recast as

$$I_3=\int dp_A\;\;\det{}^{1/2}(\Psi^A,\Psi^B)(\det{}'P_1^\dagger)^{-1}H(\tilde{\pi}'^{ab}=0,\,p_A).$$

Combining the above results for I_1, I_2, and I_3, the expression in Eq. (12) is recast as

$$Z=\mathcal{N}\int[d\sqrt{h}\,d\sigma\,d\tau^Adp_A]\;\frac{\partial\mathcal{H}}{\partial\sqrt{h}}\;\delta(\mathcal{H})\frac{\det(\mathcal{T}_A,\Psi^B)}{\det^{1/2}(\varphi_\alpha,\varphi_\beta)}\mathcal{B}$$
$$\times\exp\left(i\int_\Sigma dt\{p_A(\Psi^A,\mathcal{T}_B)\dot{\tau}^B+\sigma\dot{V}\}\right).\qquad(15)$$

The reason why the factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ appears in Eq. (15) for $g=1$ shall be discussed below. [For the case of $g\geqslant 2$, the factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ should be set to unity.] Without loss of generality, we can choose a basis of $\ker P_1^\dagger$, $\{\Psi^A\}$, as to satisfy $(\mathcal{T}_A,\Psi^B)=\delta_A{}^B$.

Under our gauge choice, the equation $\mathcal{H}=0$ considered as being an equation for $\sqrt{h}$, has a unique solution, $\sqrt{h}=\sqrt{h}(\cdot;\sigma,\tau^A,p_A)$, for fixed σ, τ^A, and p_A [2]. We therefore obtain

$$Z=\mathcal{N}\int[d\sigma\;d\tau^Adp_A]\det{}^{-1/2}(\varphi_\alpha,\varphi_\beta)\mathcal{B}$$
$$\times\exp\left(i\int dt\{p_A\dot{\tau}^A+\sigma\dot{V}(\sigma,\tau^A,p_A)\}\right),\qquad(16)$$

where $V(\sigma,\tau^A,p_A):=\int_\Sigma d^2x\;\;\sqrt{h}(x;\sigma,\tau^A,p_A)$, which is regarded as a function of σ, τ^A, and p_A.

It is clear that there is still the invariance under the reparametrization $t\to f(t)$ remaining in Eq. (16). From the geometrical viewpoint, this corresponds to the freedom in the way of labeling the time-slices defined by Eq. (8). (This point is also clear in the analysis of Ref. [2]. The treatment of this point seems somewhat obscure in the analysis of Ref. [9].) The present system illustrates that the time-reparametrization invariance still remains even after choosing the time-slices [Eq. (7) or Eq. (8)].

Equation (16) is equivalent to

$$Z=\mathcal{N}\int[d\sigma\;dp_\sigma\;d\tau^Adp_A][dN']\det{}^{-1/2}(\varphi_\alpha,\varphi_\beta)\mathcal{A}$$
$$\times\exp\left(i\int\;\;dt\;\;\{p_A\dot{\tau}^A+p_\sigma\dot{\sigma}-N'(p_\sigma+V(\sigma,\tau^A,p_A))\}\right),$$
$$(17)$$

where the integration by parts is understood. This system has a similar structure to a system of a relativistic particle and a system of a nonrelativistic particle in a parametrized form [13]. We shall discuss this point in detail in the final section. One can gauge-fix the reparametrization symmetry by choosing $\sigma=t$, i.e., by imposing a condition $\chi=\sigma-t=0$. The Faddeev-Popov procedure [11] in this case reduces to simply inserting $\delta(\sigma-t)$ into Eq. (17). Thus we obtain

$$Z=\mathcal{N}\int[d\tau^Adp_A]\mathcal{A}$$
$$\times\exp\left(i\int d\sigma\{p_A\;d\tau^A/d\sigma-V(\sigma,\tau^A,p_A)\}\right).\qquad(18)$$

Here

$$\mathcal{A}=\begin{cases}\det^{-1/2}(\varphi_\alpha,\varphi_\beta) & \text{when}\;\; g=1\;\;\text{with the option (b)}\\ 1\;\;\text{otherwise.}\end{cases}$$

Looking at the exponent in Eq. (18), we see that $V(\sigma,\tau^A,p_A)$ plays the role of a time-dependent Hamiltonian in the present gauge [2]. We see that the partition function formally defined by Eq. (4) is equivalent to the partition function defined by taking the reduced system as a starting point, as can be read off in Eq. (18). However, there is one point to be noted. For the case of $g=1$ with the option (b), for which $\dim\ker P_1=2$, the factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ appears. This factor can cause a nontrivial effect. We shall come back to this point in the next section. Typically, this factor can be a function of $V(\sigma,\tau^A,p_A)$ [see below, Eq. (22)]. On the contrary, for the case of $g\geqslant 2$, and the case of $g=1$ with the option (a), this factor does not appear.[8]

Let us discuss the possible factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ in Eq. (18).

In the case of $g=1$, the space $\ker P_1$, which is equivalent to a space of conformal Killing vectors, is nontrivial. Now a special class of Weyl deformations represented as $\delta_W h_{ab}=D\cdot v_0\;h_{ab}$, where $v_0\in\ker P_1$, is translated into a diffeomorphism: $D\cdot v_0\;h_{ab}=(P_1v_0)_{ab}+D\cdot v_0\;h_{ab}=\mathcal{L}_{v_0}h_{ab}$. (Here $\mathcal{L}_{v_0}$ denotes the Lie derivative with respect to v_0.) Thus, $\delta_W h_{ab}=D\cdot v_0\;h_{ab}$, $v_0\in\ker P_1$ is generated by $\mathcal{H}^a$ along the gauge orbit. Therefore it should be removed from the integral domain for $\int[d\sqrt{h}]$ in Eq. (12). One easily sees that the volume of $\ker P_1$, which should be factorized out from the whole volume of the Weyl transformations, is proportional to $\det^{1/2}(\varphi_\alpha,\varphi_\beta)$. Therefore the factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ appears in Eq. (18).

[8] The author thanks S. Carlip for helpful comments on this point.

There is another way of explaining the factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ [5]. Let us concentrate on the diffeomorphism invariance in Eq. (4) characterized by $\mathcal{H}^a=0$. The Faddeev-Popov determinant associated with this invariance can be related to the Jacobian for the change $h_{ab}\to(\sqrt{h},v^a,\tau^A)$. By the same kind of argument as in Eq. (13), one finds the Faddeev-Popov determinant to be

$$\Delta_{\rm FP}=\det(\mathcal{T}_A,\Psi^B)\ \ \det{}^{-1/2}(\Psi^A,\Psi^B)(\det{}' P_1^\dagger P_1)^{1/2}.$$

One way of carrying out the Faddeev-Popov procedure is to insert $1=\int d\Lambda \det(\partial\chi/\partial\Lambda)\,\delta(\chi)$ into the path-integral formula in question, where χ is a gauge-fixing function and Λ is a gauge parameter. Then the path integral in Eq. (4) reduces to the form

$$I=\int [dh_{ab}][d\sqrt{h}\ dv^a\ d\tau^A][d*]\delta(h_{ab}-\sqrt{h}\hat{h}_{ab})\ \ f(h_{ab})$$
$$=\int [d\sqrt{h}\ dv^a\ d\tau^A][d*]\ \ f(\sqrt{h}\hat{h}_{ab}),$$

where $[d*]$ stands for all of the remaining integral measures including $\Delta_{\rm FP}$.

Now, we need to factorize out V_{Diff_0}, the whole volume of diffeomorphisms homotopic to 1. This volume is related to $\int[dv^a]$ as $V_{\mathrm{Diff}_0}=(\int[dv^a])\cdot V_{\ker P_1}$, where $V_{\ker P_1}\propto\det^{1/2}(\varphi_\alpha,\varphi_\beta)$ [10]. Here we note that $\ker P_1$ is not included in the integral domain of $\int[dv^a]$: the diffeomorphism associated with $\forall v_0\in\ker P_1$ is translated into a Weyl transformation, as $\mathcal{L}_{v_0}h_{ab}=(P_1v_0)_{ab}+D\cdot v_0\ \ h_{ab}=D\cdot v_0\ \ h_{ab}$ (it is noteworthy that this argument is reciprocal to the previous one), so that it is already counted in $\int[d\sqrt{h}]$. In this manner we get

$$I=V_{\mathrm{Diff}_0}\int \frac{[d\sqrt{h}\ d\tau^A][d*]}{\det^{1/2}(\varphi_\alpha,\varphi_\beta)}\ \ f(\sqrt{h}\hat{h}_{ab}).$$

In effect, the volume of $\ker P_1$ has been removed from the whole volume of the Weyl transformations, which is the same result as the one in the previous argument. [Again, for the case of $g\geqslant 2$, the factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ should be set to unity.] Furthermore, by factorizing the entire volume of diffeomorphisms, $V_{\rm Diff}$, and not just V_{Diff_0}, the integral domain for $\int[d\tau^A]$ is reduced to the moduli space, $\mathcal{M}_g$ [10,5]. The intermediate step of factorizing V_{Diff_0} is necessary since the v^a's are labels parametrizing the tangent space of $\mathrm{Riem}(\Sigma)$, the space of all Riemannian metrics on Σ.

III. ANALYSIS OF THE $g=1$ CASE

We now investigate how the reduced canonical system in the (τ,V)-form [3] comes out in the partition function when $g=1$.

To begin with, let us recover $\int[d\sqrt{h}]$ and $\int[d\sigma]$ in Eq. (18), yielding

$$Z=\mathcal{N}\int [d\sqrt{h}][d\sigma][d\tau^A dp_A]\mathcal{A}\,\frac{\partial\mathcal{H}}{\partial\sqrt{h}}\,\delta(\mathcal{H})\,\delta(\sigma-t)$$
$$\times\exp\left(i\int dt\left\{p_A\ \frac{d\tau^A}{dt}+\sigma\frac{d}{dt}V(\sigma,\tau^A,p_A)\right\}\right). \tag{19}$$

Equation (19) is of the form

$$I=\int [d\sqrt{h}][d*]\frac{\partial\mathcal{H}}{\partial\sqrt{h}}\delta(\mathcal{H})\ f(\sqrt{h}), \tag{20}$$

where $[d*]$ stands for all of the remaining integral measures.

Now it is shown that for $g=1$ the simultaneous differential equations, Eq. (5), Eq. (6), Eq. (7) [or Eq. (8)], and Eq. (9) [or Eq. (10)], have a unique solution for $\sqrt{h}$, which is spatially constant, $\sqrt{h}_0:=F(\tau^A,p_A,\sigma)$ [2]. Thus the integral region for $\int[d\sqrt{h}]$ in Eq. (20) can be restricted to $\mathcal{D}=\{\sqrt{h}\,|\,\sqrt{h}=\text{spatially constant}\}$. Let us note that $\sqrt{h}$ is the only quantity that in principle can depend on spatial coordinates in Eq. (19). Accordingly, only the spatially constant components of the arguments of the integrand contribute to the path integral.

Thus

$$I=\int [d*]\ f(\sqrt{h}_0)$$
$$=\int_{\mathcal{D}}[d\sqrt{h}][d*]\left\{\int_\Sigma d^2x\sqrt{h}\ \frac{\partial\mathcal{H}}{\partial\sqrt{h}}\Big/\int_\Sigma d^2x\ \ \sqrt{h}\right\}$$
$$\times\delta\left(\int_\Sigma d^2x\ \mathcal{H}\Big/\int_\Sigma d^2x\right)\ f(\sqrt{h})$$
$$=\int_{\mathcal{D}}\left([d\sqrt{h}]\int_\Sigma d^2x\right)[d*]\frac{\partial H}{\partial V}\delta(H)\ \ \tilde{f}(V)$$
$$=\int [dV][d*]\ \frac{\partial H}{\partial V}\delta(H)\ \ \tilde{f}(V)$$
$$=\int [dV][d*][dN']\ \frac{\partial H}{\partial V}\,\tilde{f}(V)\exp\left(-i\int dt\ N'(t)H(t)\right),$$

where $H:=\int_\Sigma d^2x\ \ \mathcal{H}$, $V:=\int_\Sigma d^2x\ \ \sqrt{h}$, and $\tilde{f}(V):=f(\sqrt{h})$. The prime symbol in $N'(t)$ is to emphasize that it is spatially constant.

Thus we see that Eq. (19) is equivalent to

$$Z=\mathcal{N}\int [dV\ d\sigma][d\tau^A\ dp_A][dN']\mathcal{A}\,\frac{\partial H}{\partial V}\ \ \delta(\sigma-t)$$
$$\times\exp\left(i\int dt\ (p_A\ \dot{\tau}^A+\sigma\dot{V}-N'H)\right)\ , \tag{21}$$

where V and N' are spatially constant, and H is the reduced Hamiltonian in the (τ,V)-form. [See below, Eq. (23).]

We choose, as a gauge condition [see Eq. (9)] [5],

$$h_{ab}=V\hat{h}_{ab},\quad \hat{h}_{ab}=\frac{1}{\tau^2}\begin{pmatrix}1 & \tau^1\\ \tau^1 & |\tau|^2\end{pmatrix},$$

where $\tau := \tau^1 + i\tau^2$ and $\tau^2 > 0$.[9] Here we have already replaced $\sqrt{h}$ with $V = \int_\Sigma \sqrt{h}$, noting that $\sqrt{h}$ is spatially constant for the case of $g=1$. Then, it is straightforward to get

$$\mathcal{T}_{1ab} = \frac{V}{\tau^2}\begin{pmatrix} 0 & 1 \\ 1 & 2\tau^1 \end{pmatrix}, \quad \mathcal{T}_{2ab} = \frac{V}{(\tau^2)^2}\begin{pmatrix} -1 & -\tau^1 \\ -\tau^1 & (\tau^2)^2 - (\tau^1)^2 \end{pmatrix}.$$

[See the paragraph next to the one including Eq. (9) for the definition of $\{\mathcal{T}_A\}$.]

As a basis of $\ker P_1^\dagger$, $\{\Psi^A\}_{A=1,2}$, the fact that $P_1^\dagger(\mathcal{T}_A)_a := -2D_b \mathcal{T}_{Aa}{}^b = -2\partial_b \mathcal{T}_{Aa}{}^b = 0$ simplifies the situation. We can choose as $\{\Psi^A\}_{A=1,2}$

$$\Psi^1_{ab} = \frac{1}{2}\begin{pmatrix} 0 & \tau^2 \\ \tau^2 & 2\tau^1\tau^2 \end{pmatrix}, \quad \Psi^2_{ab} = \frac{1}{2}\begin{pmatrix} -1 & -\tau^1 \\ -\tau^1 & (\tau^2)^2 - (\tau^1)^2 \end{pmatrix},$$

which satisfy $(\Psi^A, \mathcal{T}_B) = \delta^A{}_B$.

Now, let us consider the case of the option (b) (Sec. II). In this case the factor $\mathcal{A}$ becomes $\mathcal{A} = \det^{-1/2}(\varphi_\alpha, \varphi_\beta)$. As a basis of $\ker P_1$, $\{\varphi_\alpha\}_{\alpha=1,2}$, we can take spatially constant vectors because $D_a = \partial_a$ for the metric in question, and because constant vectors are compatible with the condition for the allowed vector fields on T^2. (Note the fact that the Euler characteristic of T^2 vanishes, along with the Poincaré-Hopf theorem [14].) Therefore, let us take

$$\varphi_1{}^a = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_2{}^a = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where λ_1 and λ_2 are spatially constant factors. Then, we get

$$(\varphi_\alpha, \varphi_\beta) = \begin{pmatrix} \lambda_1{}^2 V^2/\tau^2 & \lambda_1\lambda_2 V^2 \tau^1/\tau^2 \\ \lambda_1\lambda_2 V^2\tau^1/\tau^2 & \lambda_2{}^2 V^2 |\tau|^2/\tau^2 \end{pmatrix}.$$

Thus, we obtain

$$\det{}^{1/2}(\varphi_\alpha, \varphi_\beta) = |\lambda_1\lambda_2| V^2. \tag{22}$$

On account of a requirement that Z should be modular invariant, $|\lambda_1\lambda_2|$ can be a function of only V and σ at most. There seems no further principle for fixing $|\lambda_1\lambda_2|$. Only when we choose as $|\lambda_1\lambda_2| = V^{-2}$, the factor $\det^{-1/2}(\varphi_\alpha, \varphi_\beta)$ in Eq. (16) or Eq. (18) has no influence. No such subtlety occurs in the string theory, since σ does not appear and since V is not important on account of the conformal invariance (except for, of course, the conformal anomaly).

It is easy to see that, in our representation, the reduced action in the (τ, V)-form becomes

$$S = \int_{t_1}^{t_2} dt (p_A \dot{\tau}^A + \sigma \dot{V} - N'(t) H),$$

$$H = \frac{(\tau^2)^2}{2V}(p_1^2 + p_2^2) - \frac{1}{2}\sigma^2 V - \Lambda V. \tag{23}$$

Here $\lambda = -\Lambda$ $(\Lambda > 0)$ corresponds to the negative cosmological constant, which is set to zero when it is not considered. [The introduction of λ (<0) may be preferable to sidestep a subtlety of the existence of a special solution $p_1 = p_2 = \sigma = 0$ for $\lambda = 0$. This special solution forms a conical singularity in the reduced phase space, which has been already discussed in Ref. [2] and in Ref. [9].] Therefore, we get

$$-\frac{\partial H}{\partial V} = \frac{(\tau^2)^2}{2V^2}(p_1^2 + p_2^2) + \frac{1}{2}\sigma^2 + \Lambda \ . \tag{24}$$

As discussed in Sec. I, the choice of $N =$ spatially constant, which is consistent with the equations of motion, is essential in the (τ, V)-form. This procedure can be however influential quantum mechanically, so that its quantum theoretical effects should be investigated. In particular we need to understand the origin of the factor $\partial H/\partial V$ in Eq. (21).

Let us start from the action in Eq. (23). It possesses a time-reparametrization invariance:

$$\delta\tau^A = \epsilon(t)\{\tau^A, H\}, \quad \delta p_A = \epsilon(t)\{p_A, H\},$$

$$\delta V = \epsilon(t)\{V, H\}, \quad \delta\sigma = \epsilon(t)\{\sigma, H\},$$

$$\delta N' = \dot{\epsilon}(t) \quad \text{with} \quad \epsilon(t_1) = \epsilon(t_2) = 0. \tag{25}$$

In order to quantize this system, one needs to fix a time variable. One possible gauge-fixing condition is $\chi := \sigma - t = 0$. Then according to the Faddeev-Popov procedure, the factor $\{\chi, H\}$ $\delta(\chi) = -(\partial H/\partial V)$ $\delta(\sigma - t)$ is inserted into the formal expression for Z. The result is equivalent to Eq. (19) up to the factor $\mathcal{A}$.

Now we understand the origin of the nontrivial factor $\partial H/\partial V$ in Eq. (19). In order to shift from the (τ, V)-form to the τ-form, it is necessary to demote the virtual dynamical variables V and σ to the Hamiltonian and the time parameter, respectively. Then, the factor $\partial H/\partial V$ appears as the Faddeev-Popov determinant associated with a particular time gauge $\sigma = t$.

In this manner, we found that the (τ, V)-form is equivalent to the τ-form even in the quantum theory, provided that the time-reparametrization symmetry remnant in the (τ, V)-form is gauge-fixed by a particular condition $\chi := \sigma - t = 0$. In particular the key procedure of imposing $N =$ spatially constant [3] turned out to be independent of the equations of motion themselves and valid in the quantum theory. (Of course the fact that it does not contradict with the equations of motion is important.)

Finally it is appropriate to mention the relation of the present result with the previous one obtained in Ref. [5]. In Ref. [5] also, the case of $g=1$ was analyzed although the model was restricted to be spatially homogeneous. The result there was that the factor $\partial H/\partial V$ did not appear in the measure although the (τ, V)-form was adopted. This result is reasonable because in Ref. [5] only the spatial diffeomorphism symmetry associated with $\mathcal{H}^a$ was gauge-fixed explic-

[9]Throughout this section, τ^2 always indicates the second component of (τ^1, τ^2), and never the square of $\tau := \tau^1 + i\tau^2$.

itly. As for the symmetry associated with $\mathcal{H}$, the Dirac-Wheeler-DeWitt procedure was applied instead of the explicit gauge-fixing. [Alternatively, one can regard that the symmetry associated with $\mathcal{H}$ was gauge-fixed by a noncanonical gauge $\dot{N}=0$ [15].] Therefore it is reasonable that $\partial H/\partial V$ did not appear in the analysis of Ref. [5]. Thus the result of Ref. [5] is compatible with the present result.

IV. DISCUSSIONS

We have investigated how a partition function for (2+1)-dimensional pure Einstein gravity, formally defined in Eq. (4), yields a partition function defined on a reduced phase space by gauge fixing. We have shown that Eq. (4) reduces to Eq. (18), which is interpreted as a partition function for a reduced system in the τ-form. For the case of $g\geqslant 2$, this result is compatible with Carlip's analysis [9].

For the case of $g=1$, with the option (b), a factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ can arise as a consequence of the fact that dim ker $P_1\neq 0$. This factor can be influential except when the choice $\det^{1/2}(\varphi_\alpha,\varphi_\beta)=1$ is justified. The requirement of the modular invariance is not enough to fix this factor.

Furthermore Eq. (4) has turned out to reduce to Eq. (21), which is interpreted as a partition function for a reduced system in the (τ,V)-form with a nontrivial measure factor $\partial H/\partial V$ as well as the possible factor $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$. The former factor was interpreted as the Faddeev-Popov determinant associated with the time gauge $\sigma=t$, which was necessary to convert from the (τ,V)-form to the τ-form. The choice of $N=$ spatially constant was the essential element to derive the (τ,V)-form in the classical theory. In particular the equations of motion were used to show its compatibility with York's gauge [3]. Therefore the relation of the (τ,V)-form with the τ-form in the quantum level was required to be clarified. Moreover, since the condition $N=$ spatially constant is not in the form of the canonical gauge, the analysis of its role in the quantum level was intriguing. Our analysis based on the path-integral formalism turned out to be powerful for studying these issues. Our result shows that the (τ,V)-form is equivalent to the τ-form even in the quantum theory, as far as the time-reparametrization symmetry in the (τ,V)-form is gauge-fixed by $\chi:=\sigma-t=0$. The postulation of $N=$spatially constant in deriving the (τ,V)-form turned out to be independent of the equations of motion and harmless even in the quantum theory.

These results are quite suggestive to quantum gravity and quantum cosmology.

First, the measure factor similar to $\det^{-1/2}(\varphi_\alpha,\varphi_\beta)$ is likely to appear whenever a class of spatial geometries in question allows conformal Killing vectors ($\ker P_1\neq\emptyset$). This factor can be influential on the semiclassical behavior of the Universe.

The issue of the two options (a) and (b) regarding the path-integral domain of the shift vector (Sec. II) is interesting from a general viewpoint of gravitational systems. If one imposes that there should be no extra factor in the path-integral measure for the reduced system, then the option (a) is preferred. There may be other arguments which prefer one of the two options.

As another issue, the variety of representations of the same system in the classical level and the variety of the gauge-fixing conditions result in different quantum theories in general, and the relation between them should be more clarified. The model analyzed here shall be a good test case for the study of this issue.

To summarize what we have learned and to recognize what needs to be clarified more, it is helpful to place our system beside a simpler system with a similar structure. The system of a relativistic particle [13] is an appropriate model for illustrating the relation between the τ-form and the (τ,V)-form.

Let $x^\alpha:=(x^0,\ \vec{x})$ and $p_\alpha:=(p_0,\ \vec{p})$ be the world point and the four momentum, respectively, of a relativistic particle. Taking x^0 as the time parameter, the action for the (positive energy) relativistic particle with rest mass m is given by

$$S=\int dx^0\left(\vec{p}\cdot\frac{d\vec{x}}{dx^0}-\sqrt{\vec{p}^2+m^2}\right). \tag{26}$$

Equation (26) corresponds to the τ-form [Eq. (2)]. Now one can promote x^0 to a dynamical variable:

$$S=\int dt\ \{p_\alpha\dot{x}^\alpha-N(p_0+\sqrt{\vec{p}^2+m^2})\}. \tag{27}$$

Here t is an arbitrary parameter such that $x^0(t)$ becomes a monotonic function of t; N is the Lagrange multiplier enforcing a constraint $p_0+\sqrt{\vec{p}^2+m^2}=0$. The action Eq. (27) corresponds to the action appearing in Eq. (17).

It is possible to take $p^2+m^2=0$ with $p_0<0$ as a constraint instead of $p_0+\sqrt{\vec{p}^2+m^2}=0$. Then an alternative action for the same system is given by

$$S=\int_{t_1}^{t_2}dt\ \{p_\alpha\dot{x}^\alpha-N\ H\},$$

$$H=p^2+m^2. \tag{28}$$

Equation (28) corresponds to the (τ,V)-form [Eq. (3) or Eq. (23)].

The system defined by Eq. (28) possesses the time reparametrization invariance similar to Eq. (25):

$$\delta x^\alpha=\epsilon(t)\{x^\alpha,H\},\quad \delta p_\alpha=\epsilon(t)\{p_\alpha,H\},$$

$$\delta N=\dot{\epsilon}(t)\quad\text{with}\ \ \epsilon(t_1)=\epsilon(t_2)=0. \tag{29}$$

Thus the gauge-fixing is needed in order to quantize this system. Here let us concentrate on two kinds of the gauge-fixing condition:

(1) $\chi_{\rm I}:=x^0-t=0$ (canonical gauge),

(2) $\chi_{\rm II}:=\dot{N}=0$ (noncanonical gauge).

Choosing the gauge condition (i), one inserts the factors $\{\chi_{\rm I},H\}\,\delta(\chi_{\rm I})=-2p_0\ \delta(x^0-t)$ into the path-integral formula according to the Faddeev-Popov procedure [11]. More rigorously, the factors $\theta(-p_0)\ \{\chi_{\rm I},H\}\,\delta(\chi_{\rm I})$, or alternatively, $\theta(N)\{\chi_{\rm I},H\}\,\delta(\chi_{\rm I})$ should be inserted in order to obtain the equivalent quantum theory to the one obtained by Eq. (26) [13]. The factor $\theta(-p_0)$ selects the positive energy solution

$-p_0=\sqrt{\vec{p}^2+m^2}$ among the two solutions of $H=p^2+m^2=0$ with respect to p_0. This gauge (i) corresponds to the gauge $\chi=\sigma-t=0$ in the previous section. We observe that a pair $(x^0,\ p_0)$ corresponds to the pair $(\sigma,\ -V)$ which is obtained from an original pair $(V,\ \sigma)$ by a simple canonical transformation. [The relation $-p_0=\sqrt{\vec{p}^2+m^2}$ corresponds to the relation $V=V(\sigma,\tau^A,p_A)$.] Thus the additional restriction factor $\theta(-p_0)$ should correspond to $\theta(V)$, which is identically unity because of the positivity of V. It is quite suggestive that one solution among the two solutions of $H=0$ [Eq. (23)] with respect to V is automatically selected because V is the two-volume of Σ.

As for the other gauge (ii) $\chi_{\rm II}:=\dot{N}=0$, it is quite different in nature compared with (i) $\chi_{\rm I}:=x^0-t=0$. Apparently the path-integral measure becomes different. This point becomes clear if the transition amplitude $(\ x_2^\alpha\ \mid\ x_1^\alpha\ )$ for the system Eq. (28) is calculated by imposing (i) and by imposing (ii). By the canonical gauge (i), one obtains

$$(x_2^\alpha \mid x_1^\alpha)_{\rm I}=\int d^4p\ \exp\{ip_\alpha(x_2^\alpha-x_1^\alpha)\}\ \ |-2p^0|\delta(p^2+m^2),$$

if the simplest skeletonization scheme is adopted as in Ref. [13]. [Here we set aside the question about the equivalence with the system described by Eq. (26) so that the factor $\theta(-p^0)$ is not inserted.] The gauge (ii) can be handled [15] by the Batalin-Fradkin-Vilkovisky method [16] instead of the Faddeev-Popov method, and the result is

$$(x_2^\alpha \mid x_1^\alpha)_{\rm II}=\int d^4p\ \exp\{ip_\alpha(x_2^\alpha-x_1^\alpha)\}\delta(p^2+m^2).$$

Both $(\ x_2^\alpha\ \mid\ x_1^\alpha\ )_{\rm I}$ and $(\ x_2^\alpha\ \mid\ x_1^\alpha\ )_{\rm II}$ satisfy the Wheeler-DeWitt equation but they are clearly different. One finds that if another gauge (i′) $\chi_{\rm I'}:=-(x^0/2p_0)-t=0$ is adopted instead of (i), the resultant $(\ x_2^\alpha\ \mid\ x_1^\alpha\ )_{\rm I'}$ is equivalent to $(\ x_2^\alpha\ \mid\ x_1^\alpha\ )_{\rm II}$. One sees that $-(x^0/2p_0)\propto x^0\sqrt{1-(v/c)^2}$ under the condition $H=0$, which is interpreted as the proper-time.

Even in the present simple model, it is already clear that only solving the Wheeler-DeWitt equation is not enough to reveal the quantum nature of the spacetime. Then it is intriguing what the relation there is between the gauge conditions and the boundary conditions for the Wheeler-DeWitt equation. Apparently more investigations are needed regarding the gauge-fixing conditions, especially the relation between the canonical gauges and the noncanonical gauges. The system of (2+1)-dimensional Einstein gravity shall serve as a good test candidate to investigate these points in the context of quantum cosmology.

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APPENDIX A: THE JACOBIAN ASSOCIATED WITH CHANGE OF INTEGRAL VARIABLES

We often need to change integral variables in path integrals. Let X^A and $X^{A'}$ $(A,\ A'=1,\ldots,n)$ denote the original and the new variables, respectively, in terms of which the line element is given as $ds^2=G_{AB}dX^AdX^B(=:(dX,dX))=G_{A'B'}dX^{A'}dX^{B'}$. Then, a natural invariant measure becomes $[dX]=d^nX\sqrt{\det G}=d^nX'\sqrt{\det G'}$. In other words, we define a measure in an invariant manner to satisfy $1=\int[dX]\exp\{-(\delta X,\delta X)\}$. Now, a convenient way to find out the Jacobian J associated with the change of variables, $X^A\rightarrow X^{A'}$, is [17] as follows.

(i) Represent δX^A in terms of $\delta X^{A'}$,

$$\delta X^A=(\partial X^A/\partial X^{A'})\,\delta X^{A'}\ .$$

(ii) Represent $(\delta X,\delta X)$ in terms of $\delta X^{A'}$,

$$(\delta X,\delta X)=G_{AB}\frac{\partial X^A}{\partial X^{A'}}\,\frac{\partial X^B}{\partial X^{B'}}\,\delta X^{A'}\,\delta X^{B'}\ .$$

(iii) The Jacobian is given by setting

$$1=J\int d^n\delta X'\exp\{-(\delta X,\delta X)\}, \tag{A1}$$

since this should be equivalent to $1=J(\sqrt{\det G'})^{-1}$ up to some unimportant numerical factor.

Here it may be appropriate to mention the natural line element in our case. The ‘‘kinetic term’’ K in the Hamiltonian constraint defines the geometrical structure of the configuration space. It is in the form [see Eq. (5)]

$$K=\int_\Sigma d^2x\sqrt{h}(h_{ac}h_{bd}-h_{ab}h_{cd})\frac{\pi^{ab}}{\sqrt{h}}\,\frac{\pi^{cd}}{\sqrt{h}},$$

where Σ stands for a two-surface. Therefore, the inner product between second-rank tensor fields which is compatible with the geometrical structure of the configuration space is given by

$$(w^{\cdot\cdot},\ w'^{\cdot\cdot}):=\int_\Sigma d^2x\sqrt{h}(h_{ac}h_{bd}-h_{ab}h_{cd})w^{ab}w'^{cd}\ .$$

Furthermore, the second term in the parenthesis is not important in the following sense. We observe that

$$\begin{aligned}(w^{\cdot\cdot},\ w'^{\cdot\cdot})_k:&=\int_\Sigma d^2x\sqrt{h}(h_{ac}h_{bd}+k\ \ h_{ab}h_{cd})w^{ab}w'^{cd}\\ &=\int_\Sigma d^2x\sqrt{h}\ \ \{\tilde{w}^{ab}\tilde{w}'_{ab}+(1/2+k)ww'\},\end{aligned}$$

where $\tilde{w}^{ab}$ and w stand for the traceless part of w^{ab} and $h_{ab}w^{ab}$, respectively. Therefore, as far as the path integral is concerned, the effect of the value of k is absorbed into the normalization factor $\mathcal{N}$ [10]. (We exclude the singular case

$k=-1/2.$)[10] Thus we simply set $k=0$. In this manner we are given the natural inner product between second-rank tensor fields, which is diffeomorphism invariant in accordance with the principle of relativity. Afterwards we can extend the inner product to other types of fields also. For instance

$$(f,\ g):=\int_\Sigma d^2x\sqrt{h}\ fg,$$

$$(u^{\cdot},\ u'^{\cdot}):=\int_\Sigma d^2x\sqrt{h}\ h_{ab}u^a u^b,$$

$$(w^{\cdot\cdot},\ w'^{\cdot\cdot}):=\int_\Sigma d^2x\sqrt{h}\ h_{ac}h_{bd}w^{ab}w'^{cd}\quad .$$

For the case of densitized fields, an appropriate power of $\sqrt{h}$ should be multiplied to the integrand in order to make the inner product diffeomorphism invariant.

APPENDIX B: A FORMULA FOR THE DELTA FUNCTION

Here we derive a formula which is essential in our discussions [5].

[10] The Euclidean path integral with $k<-1/2$ causes a trouble of divergence, which requires a special care. We shall not discuss this issue here and understand that the Lorentzian path integral is adopted for such a case.

Let A be a linear (Fredholm) operator possibly with zero modes. Let $\{\vec{\Psi}^A\}_{A=1,\dots,\dim\ker A}$ be a basis of ker A. Suppose we evaluate an integral $I=\int d^nX\,\delta(A\vec{X})f(\vec{X})$. Any vector $\vec{X}\in W$ in the domain of A can be uniquely decomposed as $\vec{X}=\vec{X}'+p_A\vec{\Psi}^A$, where $p_A=(X,\Psi^B)(\Psi^{\cdot},\Psi^{\cdot})^{-1}{}_{BA}$, with $(\Psi^{\cdot},\Psi^{\cdot})^{-1}$ being the inverse matrix of (Ψ^A,Ψ^B). Now, let us change the integral variables from $\vec{X}$ to $(\vec{X}',p_A)$. The Jacobian J for this change is given as follows. Noting that $(\delta X,\delta X)=(\delta X',\delta X')+(\Psi^A,\Psi^B)p_Ap_B$, we obtain $J=\det^{1/2}(\Psi^A,\Psi^B)$ [see Eq. (A1)]. Then I can be expressed as

$$I=\int dp_A d\vec{X}'\ \det{}^{1/2}(\Psi^A,\Psi^B)\,\delta(A\vec{X}')\ \ f(\vec{X}',p_A).$$

We thus obtain a formula

$$\int d^nX\,\delta(A\vec{X})f(\vec{X})$$
$$=\int dp_A\ \det{}^{1/2}(\Psi^A,\Psi^B)(\det{}'A)^{-1}f(\vec{X}'=\vec{0},\ p_A),\quad \text{(B1)}$$

where $\det' A$ denotes the determinant of A on $W/\ker A$ (i.e., the zero modes of A are removed).

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