Smooth bosonization as a quantum canonical transformation

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We consider a $(1+1)$-dimensional field theory which contains both a complex fermion field and a real scalar field. We then construct a unitary operator that, by a similarity transformation, gives a continuum of equivalent theories which smoothly interpolate between the massive Thirring model and the sine-Gordon model. This provides an implementation of smooth bosonization proposed by Damgaard, Nielsen, and Sollacher as well as an example of a quantum canonical transformation for a quantum field theory. [S0556-2821(97)05612-9]

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I. INTRODUCTION

Duality, or the quantum equivalence of field theories, allows one to relate quantities in one theory, such as the particle spectrum and Green's functions, to those of another theory. This concept is most useful when duality maps a theory with a strong coupling, in which perturbation theory is invalid, to one with a weak coupling, in which a perturbative calculation may be performed. Unfortunately, many such duality transformations are hypothetical since an explicit operator mapping is absent. Duality is usually demonstrated either by symmetry arguments, as in non-Abelian bosonization [1], by a path integral calculation, as in *S* duality [2] and Abelian bosonization [3], or by a linear canonical transformation, as in *T* duality [4].

The equivalence of two different quantum mechanical theories may be rigorously established by means of a similarity transformation of all quantum operators. These transformations were established by Dirac as the quantum version of canonical transformations in classical mechanics; the former preserving the quantum commutators and the latter the Poisson brackets. More recently, Anderson further investigated these mappings, which he named ''quantum canonical transformations,'' and emphasized that they need not be unitary as originally presumed but rather isometric [5]. Quantum canonical transformations map operators $\hat{O}_i$ in one theory to $\hat{O}_i'$ in another quantum theory by $\hat{O}_i'=\hat{U}\hat{O}_i\hat{U}^{-1}$ with $\hat{U}$ an operator composed of products of the canonical position and momentum operators. Such a transformation obviously preserves commutators of operators and both theories have the same energy spectrum with eigenstates mapped as $|E\rangle'=\hat{U}|E\rangle$. Recently, in an interesting extension of quantum canonical transformations to conformal field theories, Evans and Giannakis derived *T*-duality transformation rules for string fields on tori [6].

In this work, we investigate quantum canonical transformations within a class of $(1+1)$-dimensional quantum field theories, namely the Abelian bosonization of free bosons and fermions as well as the equivalence of the massive Thirring model and the sine-Gordon model. Abelian bosonization is an ideal test of this technique since the operator correspondences between the fermionic and bosonic theories are known [7,8]. We will consider a theory which contains both fermions and bosons. A quantum canonical transformation of a theory with only fermions, the massive Thirring model, yields a continuum of theories with interacting fermions and bosons as well as the corresponding theory containing only bosons, the sine-Gordon model.

II. QUANTUM CANONICAL TRANSFORMATIONS FOR A FIELD THEORY

We first review quantum canonical transformations in quantum mechanics and next examine how to extend this definition to a quantum field theory. We begin by considering quantum mechanics in the Schrödinger picture, with only the states evolving with time. A quantum canonical transformation is a similarity transformation of the quantum operators, which may be defined by its action on the canonical position and momentum operators q_i and p_i in the original theory, to obtain q_i' and p_i' in the new theory

$$q_i'\equiv U(q,p)q_iU^{-1}(q,p),\quad p_i'\equiv U(q,p)p_iU^{-1}(q,p). \tag{1}$$

If the states also transform as $|\psi\rangle'\equiv U(q,p)|\psi\rangle$ then matrix elements are preserved and $U(q,p)$ is an isometry. This transformation is specified by an operator $U(q,p)$, which we assume to be invertible. A quantum canonical transformation also preserves the canonical commutation relations

$$[q_i',p_j']=[q_i,p_j]=\imath\delta_{ij}, \tag{2}$$

with all other commutators vanishing. These transformations may be used to calculate some physical properties of a quantum mechanical system, for instance, the energy eigenstates and eigenvalues, by mapping a given model to another model whose solutions are known, such as the harmonic oscillator [5].

Quantum field theory is usually formulated in the interaction picture where the Hamiltonian density operator is

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divided into the sum of a free part $\mathcal{H}_0$ and an interacting part $\mathcal{H}_{\rm int}$. The free field $\phi(x)$ satisfies the operator equation $(\partial/\partial_t)\phi(\vec{x},t)=\imath[\int d\vec{y}\mathcal{H}_0(\vec{y},t),\phi(\vec{x},t)]$. We will assume that exact physical quantities, i.e., calculated to all orders in perturbation theory, are independent of the choice of this partition of the Hamiltonian. The Green's functions for the interacting fields $\phi_I(\vec{x},t)$ may be expressed in terms of the free fields using the well-known equation

$$G(x_1,\dots,x_n)\equiv\langle 0|T[\phi_I(\vec{x}_1,t_1)\cdots\phi_I(\vec{x}_n,t_n)]|0\rangle=\frac{\langle 0|T^*\Big[\phi(\vec{x}_1,t_1)\cdots\phi(\vec{x}_n,t_n)\exp\Big(-\imath\int d\vec{x}'dt'\mathcal{H}_{\rm int}[\phi(\vec{x}',t')]\Big)\Big]|0\rangle}{\langle 0|T^*\exp(-\imath\int d\vec{x}'dt'\mathcal{H}_{\rm int}[\phi(\vec{x}',t')])|0\rangle}. \tag{3}$$

T^* operator products are necessary in this equation for the theory that we will consider, the Thirring model, because of the Schwinger term in the current algebra. However, following the arguments of Ref. [9], which notes that Eq. (3) yields Lorentz covariant expressions if the T^* product is replaced by the naive time-ordered product and a covariant renormalization scheme is employed, we consider usual time-ordered products for this equation. Thus the quantities that we would like to transform are vacuum expectation values of time-ordered products of the free fields $\phi(\vec{x},t)$. Rather than using the algebra of the canonical operators at equal time in the Heisenberg picture, or equivalently in the Schrödinger picture, to construct the operator U which implements the transformation, we may use the commutator of free fields at different times $[\phi(\vec{x}_1,t_1),\phi(\vec{x}_2,t_2)]\equiv\Delta(\vec{x}_2-\vec{x}_1,t_2-t_1)$. $\Delta(\vec{x},t)$ is a c-number function. Since it appears difficult to construct U for a general theory we will consider here the simpler case of a massless free field ϕ in 1+1 spacetime dimensions for which the powerful methods of conformal field theory may be applied.

We consider the field theory defined on a circle, i.e., periodic as $x^1\to x^1+\sqrt{2}L$ to avoid the infrared divergences appearing in massless field theories. This periodicity also allows a bijective map of the spacetime coordinates to the complex plane with the origin excluded. Light cone coordinates are defined as $x^\pm\equiv(1/\sqrt{2})(x^0\pm x^1)$. After a Wick rotation from Minkowski space to Euclidean space $x_E^0=-\imath x^0$ one defines $z\equiv\exp[(2\pi\imath/L)x_E^+]$ and likewise defines $\overline{z}$ in terms of x_E^-. Hereafter we consider only time-ordered products of operators, corresponding to radial-ordered products in the conformal field theory, since only these have well-defined vacuum expectation values in the Euclidean theory for any value of spacetime coordinates of the operators [10]. We then define a general form for U in the conformal field theory as $U\equiv R\exp\imath[\oint_C(dz/2\pi\imath)\Omega(z)+\oint_C(d\overline{z}/2\pi\imath)\overline{\Omega(z)}]\equiv R\exp\imath\mathcal{A}$. The integration contour C is chosen so that all operator products are radial ordered. If the operators Ω and $\overline{\Omega}$ are Hermitian then U is unitary. The quantum canonical transformation of an operator $\mathcal{O}(z,\overline{z})$ is defined as

$$\begin{aligned}\mathcal{O}'(z,\overline{z})&=R\{U^{-1}\mathcal{O}(z,\overline{z})U\}\\&=R\Big\{\mathcal{O}(z,\overline{z})+\imath[\mathcal{A},\mathcal{O}(z,\overline{z})]\\&\quad+\frac{\imath^2}{2!}[\mathcal{A},[\mathcal{A},\mathcal{O}(z,\overline{z})]]+\cdots\Big\}.\end{aligned}\tag{4}$$

The second equality follows from the Baker-Campbell-Hausdorff relation. Since the two contour integrals in the commutator are equivalent to a single contour integral about z this transformation may be easily calculated, given the Wick contractions for the relevant fields. The resulting quantum canonical transformation for the conformal field theory looks superficially similar to the case for quantum mechanics in that the free fields $\phi(z,\overline{z})$, Hamiltonian density $\mathcal{H}[\phi(z,\overline{z})]$ and vacuum state transform as

$$\begin{aligned}\phi'(z,\overline{z})&\equiv R\{U\phi(z,\overline{z})U^{-1}\},\\\mathcal{H}'[\phi(z,\overline{z})]&\equiv R\{U\mathcal{H}[\phi(z,\overline{z})]U^{-1}\},\\|0\rangle'&\equiv U|0\rangle.\end{aligned}\tag{5}$$

Transformation by U also leaves invariant the radial-ordered commutator of two operators:

$$\begin{aligned}R\{U[\mathcal{O}_1(z),\mathcal{O}_2(w)]U^{-1}\}&=R\{U\mathcal{O}_1(z)U^{-1}U\mathcal{O}_2(w)U^{-1}\\&\quad-U\mathcal{O}_2(w)U^{-1}U\mathcal{O}_1(z)U^{-1}\}\\&\equiv R\{[\mathcal{O}_1'(z),\mathcal{O}_2'(w)]\}.\end{aligned}\tag{6}$$

Once this transformation is performed the result may then be analytically continued back to Minkowski space. We will give an explicit example of such a transformation for Abelian bosonization in the following section.

We define two quantum field theories to be equivalent if they are related by a quantum canonical transformation and a finite wave function renormalization. Thus if the renormalized field Φ' is defined as $\Phi'\equiv Z^{-1/2}\phi'$ then Green's functions for the free fields in the two theories, in Minkowski space, are related by

$$\langle 0|'T[\Phi'(\vec{x}_1,t_1)\cdots\Phi'(\vec{x}_n,t_n)]|0\rangle' = Z^{-n/2}\langle 0|T[\phi(\vec{x}_1,t_1)\cdots\phi(\vec{x}_n,t_n)]|0\rangle. \quad (7)$$

Furthermore the Green's functions for the interacting fields in the new theory may, in principle, be calculated using Eq. (7) and Eq. (3) with the interaction Hamiltonian $H'_{\rm int}$.

III. FREE FERMION AND BOSON FIELDS

We begin with a $(1+1)$-dimensional field theory containing both a complex fermion field and a real boson field periodic as $x_1 \to x_1 + \sqrt{2}L$. The Hilbert space of this theory is a tensor product of the respective Fock spaces, with the vacuum state

$$|0\rangle = |0\rangle_{\rm boson} \otimes |\,0\,\rangle_{\rm fermion}. \quad (8)$$

In terms of light cone coordinates, the mode expansion for the single real scalar field ϕ is

$$\phi(x^+,x^-) = q + \overline{q} + \frac{1}{2L}(px^+ + \overline{p}x^-) + \phi_0(x^+,x^-) \quad (9)$$

with

$$\phi_0(x^+,x^-) \equiv \frac{\imath}{2\sqrt{\pi}} \sum_{n\neq 0} \frac{1}{n}\left[\alpha_n \exp\left(-\frac{2\pi\imath}{L}nx^+\right) + \overline{\alpha}_n \exp\left(-\frac{2\pi\imath}{L}nx^-\right)\right] \equiv \phi_0(x^+) + \overline{\phi_0}(x^-). \quad (10)$$

We have split the field ϕ into its zero modes, the axial charges q and $\overline{q}$ and their conjugate momenta p and $\overline{p}$, and the remainder ϕ_0. Hermiticity of $\phi_0(x)$ implies $\alpha_n^\dagger = \alpha_{-n}$.

The nonvanishing commutation relations $[\alpha_m,\alpha_n] = m\delta_{m,-n}$, $[\overline{\alpha_m},\overline{\alpha_n}] = m\delta_{m,-n}$, and $[q,p] = [\overline{q},\overline{p}] = \imath$ give the usual equal-time commutation relation

$$[\phi(x^0,x^1),\partial_0\phi(y^0,y^1)]|_{x^0=y^0} = \imath\,\delta(x^1-y^1). \quad (11)$$

A complex fermion field may be expressed in terms of two real fields as $\psi(x) \equiv \psi^1(x) + \imath\psi^2(x)$. The mode expansion for the real fermion fields $\psi^i(x)$, $i=1,2$ is

$$\psi^i(x^+,x^-) = \frac{1}{2^{3/4}}\left(\frac{1}{L}\right)^{1/2} \sum_n \left[\begin{pmatrix}1\\0\end{pmatrix} b_n^i \exp\left(-\frac{2\pi\imath}{L}nx^+\right) + \begin{pmatrix}0\\1\end{pmatrix} \overline{b_n^i} \exp\left(-\frac{2\pi\imath}{L}nx^-\right)\right]. \quad (12)$$

Again, since $\psi^i(x)$ are Hermitian $(b_n^i)^\dagger = b_{-n}^i$ and the sum extends over integer values for Ramond (R) fields and half-integer values for Neveu-Schwarz (NS) fields. The nonvanishing anticommutators for the fermion creation and annihilation operators are $\{b_m^i,b_n^j\} = \delta^{ij}\delta_{m,-n}$ and likewise for $\overline{b_m^i}$. The equal-time anticommutator for the real fermion fields is

$$\{\psi_\alpha^i(x^0,x^1),\psi_\beta^j(y^0,y^1)\}|_{x^0=y^0} = \frac{1}{2}\delta^{ij}\delta_{\alpha\beta}\delta(x^1-y^1) \quad (13)$$

which gives for the complex field

$$\{\psi_\alpha(x^0,x^1),\psi_\beta^\dagger(y^0,y^1)\}|_{x^0=y^0} = \delta_{\alpha\beta}\delta(x^1-y^1). \quad (14)$$

The light cone fermion current is defined as[1]

$$J_f(x^+) \equiv \imath\sqrt{2}L\!:\!\psi_L^i(x^+)T_{ij}\psi_L^j(x^+)\!:, \qquad T \equiv \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad (15)$$

with $\psi_{L,R} \equiv \psi_{1,2}$. This current generates $U_L(1) \sim \underline{SO_L}(2)$ symmetry transformations $\psi_L \to e^{\imath\alpha}\psi_L$ and likewise $\overline{J}_f(x^-)$ generates $SO_R(2)$ transformations of the fermion field. Since J_f and $\overline{J}_f$ satisfy the wave equation $\Box J_f = \Box\overline{J}_f = 0$ they may each be divided into positive and negative frequency modes and a normal ordering defined for modes of each current separately.

We also define a boson current

$$J_b(x^+) \equiv \frac{L}{\sqrt{\pi}}\partial_+\phi(x^+) \quad (16)$$

that generates a shift in the boson field by a constant ϵ, $\phi(x^+) \to \phi(x^+) + \epsilon$.

Next, we will use the correspondence between a field theory of massless real bosons or fermions in Minkowski spacetime and conformal field theory in order to simplify calculations. After a Wick rotation, $x_E^0 = -\imath x^0$ and defining $z \equiv \exp[(2\pi\imath/L)x_E^+]$ we have

$$\psi_L^i(x^+) \to \psi^i(z) = \frac{1}{2^{3/4}}\left(\frac{1}{L}\right)^{1/2} \sum_n b_n^i z^{-n} \quad (17)$$

and similarly for $\psi_R^i(x^-)$. The Wick contraction for these fermion fields is defined as

$$|z| > |w|, \qquad \psi^i(z)\psi^j(w) = :\!\psi^i(z)\psi^j(w)\!: + \delta^{ij}\Delta(z,w). \quad (18)$$

For R fields,

$$\Delta(z,w) = \left(\frac{1}{2^{5/2}L}\right)\frac{z+w}{z-w} \quad (19)$$

and for NS fields,

$$\Delta(z,w) = \left(\frac{1}{2^{3/2}L}\right)\frac{\sqrt{zw}}{z-w}. \quad (20)$$

[1]Repeated indices imply summation over them throughout this paper. The following convention for the γ matrices is used: $\gamma^0 \equiv \sigma_x$, $\gamma^1 \equiv -\imath\sigma_y$, $\gamma^5 \equiv \sigma_z$.

Likewise, after the change of variables $(x^+,x^-)\to(z,\bar{z})$

$$\phi_0(x^+)\to\phi_0(z)=\frac{\imath}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{1}{n}\alpha_n z^{-n}. \tag{21}$$

The Wick contraction for $\phi_0(z)$ is

$$|z|>|w|,$$

$$\phi_0(z)\phi_0(w)=:\phi_0(z)\phi_0(w):-\frac{1}{4\pi}\ln\left(1-\frac{w}{z}\right). \tag{22}$$

The operator $U(\beta)$ which implements the quantum canonical transformation in the conformal field theory is defined as the radial-ordered exponential

$$U(\beta)\equiv R\exp\left[\imath 2\sqrt{\pi}\beta\left(\oint_C\frac{dz}{2\pi\imath z}\,\phi_0(z)J_f(z)\right.\right.$$
$$\left.\left.-\oint_C\frac{d\bar{z}}{2\pi\imath\bar{z}}\bar{\phi}_0(\bar{z})\bar{J}_f(\bar{z})\right)\right]. \tag{23}$$

The contour C is chosen to be such that the complex coordinates of all other operators lie within it; i.e., the limit $x^0_{\rm min}\to\infty$ is taken. $U(\beta)$ is unitary and its inverse results from $U(\beta)$ by taking $\imath\to-\imath$ and integrating over a contour C in which the coordinates of the other operators lie outside of it, i.e., in the limit $x^0_{\rm max}\to-\infty$.

One of the main properties of $U(\beta)$ is that it transforms the fermion current J_f and the boson current J_b into one another. This may be seen by evaluating Eq. (4) for $U(\beta)$. Using the definition of the currents Eqs. (15) and (16), and the Wick contractions Eqs. (18) and (22), we find the following transformations of the currents:

$$U(\beta)\begin{pmatrix}J_f(z)\\J_b(z)\end{pmatrix}U^{-1}(\beta)=\begin{pmatrix}\cos\beta & \sin\beta\\-\sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}J_f(z)\\J_b(z)\end{pmatrix},$$
$$U(\beta)\begin{pmatrix}\bar{J}_f(\bar{z})\\\bar{J}_b(\bar{z})\end{pmatrix}U^{-1}(\beta)=\begin{pmatrix}\cos\beta & -\sin\beta\\\sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\bar{J}_f(z)\\\bar{J}_b(z)\end{pmatrix}. \tag{24}$$

As expected, this transformation preserves the commutators of the boson and fermion currents

$$[J_f(z),J_f(w)]=[J_b(z),J_b(w)]=\frac{zw}{(z-w)^2},$$
$$[J_f(z),J_b(w)]=0 \tag{25}$$

and likewise for the antiholomorphic currents.

Starting from a theory with only a free, massless complex fermion, we examine how the Hamiltonian density transforms. In general the Hamiltonian density in terms of the light cone components of the energy-momentum tensor $\theta_{\mu\nu}$ is

$$\mathcal{H}=\frac{1}{2}[\theta_{++}(x^+)+\theta_{--}(x^-)]. \tag{26}$$

The Hamiltonian density for the fermion field is

$$\mathcal{H}^f_0=\frac{\imath}{2}[\psi^\dagger\partial_0\psi-(\partial_0\psi^\dagger)\psi] \tag{27}$$

which implies that $\theta_{\pm\pm}=\imath\sqrt{2}:\psi^i_\pm\partial_\pm\psi^i_\pm:$ in terms of the real fermion fields. The fermion Hamiltonian density may then be shown to be equal to the Sugawara form in terms of the currents of Eq. (15) [11,12]:

$$\mathcal{H}^f_0=\frac{\pi}{2L^2}[\ddagger J_f^2(x^+)\ddagger+\ddagger\bar{J}_f^2(x^-)\ddagger]. \tag{28}$$

The normal ordering $\ddagger$ is with respect to modes of $J(x^+)$ and $\bar{J}(x^-)$, i.e., with $J(x^+)\equiv\Sigma_n J_n\exp[-(2\pi\imath/L)x^+]$ the J_n with positive n are placed to the right. Choosing $\beta=\pi/2$ the transformed fermion Hamiltonian density becomes

$$U\left(\frac{\pi}{2}\right)\mathcal{H}^f_0U^{-1}\left(\frac{\pi}{2}\right)=\frac{\pi}{2L^2}[\ddagger J_b^2(x^+)\ddagger+\ddagger\bar{J}_b^2(x_-)\ddagger]$$
$$=\frac{1}{2}\{:[\partial_+\phi(x^+)]^2:+:[\partial_-\bar{\phi}(x^-)]^2:\}$$
$$=\frac{1}{2}:\partial_\mu\phi(x^0,x^1)\partial^\mu\phi(x^0,x^1):$$
$$\equiv\mathcal{H}^b_0. \tag{29}$$

The second line follows from the definition of the boson current, Eq. (16), and the fact that the usual normal ordering with respect to modes of ϕ is identical to the normal ordering with respect to modes of J_b. Thus the quantum canonical transformation $U(\pi/2)$ transforms the Hamiltonian for a free complex fermion field into one for a free real boson field. The fermion field decouples, i.e., has no dynamics, and the theory is equivalent to that for a free boson field.

IV. COMPLETE BOSONIZATION OF THE MASSIVE THIRRING MODEL

We next examine how the massive Thirring model transforms under $U(\pi/2)$, or complete bosonization. The Hamiltonian density for the massive Thirring model is

$$\mathcal{H}_{\rm Thirring}=\mathcal{H}^f_0+m:\bar{\psi}\psi:+\frac{1}{2}g:\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi:$$
$$=\mathcal{H}^f_0+\imath m(\psi^i_LT_{ij}\psi^j_R+\psi^i_RT_{ij}\psi^j_L)-2g(:\psi^i_LT_{ij}\psi^j_L:)$$
$$\times(:\psi^i_RT_{ij}\psi^j_R:) \tag{30}$$

where the Hamiltonian density is expressed in terms of the real chiral fermion fields. As discussed in Ref. [7], the coupling g must be greater than $-\pi/2$ for the Thirring model with nonzero mass in order for the theory to be well defined. Both the mass and current-current terms have been regularized by normal ordering. We also use the interaction representation with massless fields in which these terms are treated as interactions.

First considering the massless case, the Hamiltonian density is

$$\mathcal{H}_{m=0\ \text{Thirring}}=\mathcal{H}_0^f-2g(\psi_L^i T_{ij}\psi_L^j)(\psi_R^i T_{ij}\psi_R^j)$$
$$=\frac{\pi}{2L^2}\left[\ddagger J_f^2(x^+)\ddagger+\ddagger\overline{J_f^2}(x^-)\ddagger+\frac{2g}{\pi}J_f(x^+)\overline{J_f}(x^-)\right]. \qquad (31)$$

Transforming by $U(\pi/2)$ results in

$$\mathcal{H}'_{m=0\ \text{Thirring}}=U\left(\frac{\pi}{2}\right)\mathcal{H}_{m=0\ \text{Thirring}}\,U^{-1}\left(\frac{\pi}{2}\right)$$
$$=\frac{\pi}{2L^2}\left[\ddagger J_b^2(x^+)\ddagger+\ddagger\overline{J_b^2}(x^-)\ddagger-\frac{2g}{\pi}\ddagger J_b(x^+)\overline{J_b}(x^-)\right]$$
$$=\frac{1}{2}[:(\partial_+\phi)^2:+:(\partial_-\phi)^2:]-\frac{g}{\pi}:\partial_+\phi\partial_-\phi:. \qquad (32)$$

The corresponding Lagrangian density is

$$\mathcal{L}'_{m=0\ \text{Thirring}}=\left(\frac{1}{2}+\frac{\text{g}}{2\pi}\right):\partial_\mu\phi\partial^\mu\phi:, \qquad (33)$$

which after a finite wave function renormalization becomes the Lagrangian density of a free massless boson field. In order to transform the mass term we use the methods of Ref. [13] to express products of the fermion fields in terms of the currents. We next repeat the derivation for our case: a complex fermion field on a compact space and currents defined by normal-ordered products. J_f may be divided into the positive frequency modes $J_f^{(+)}$ and negative frequency modes $J_f^{(-)}$, containing, respectively, creation and annihilation operators. The zero mode of J_f is divided evenly between $J_f^{(+)}$ and $J_f^{(-)}$. The normal-ordered product of a current and a fermion field is defined as

$$\ddagger J_f(z)\psi^i(z)\ddagger\equiv J_f^{(+)}(z)\psi^i(z)+\psi^i(z)J_f^{(-)}(z). \qquad (34)$$

Using the Sugawara correspondence $\ddagger J_f(z)J_f(z)\ddagger=2^{3/2}L{:}z[\partial_z\psi^i(z)]\psi^i(z){:}+\epsilon$, with $\epsilon=0,1/4$ for, respectively, NS or R fields, and the above definition of normal ordering gives

$$\frac{1}{2}\left[\oint\frac{dz}{2\pi\imath z}\ddagger J_f^2(z)\ddagger,\psi_L^i(w)\right]=-\imath T^{ij}\ddagger J_f(w)\psi_L^j(w)\ddagger$$
$$=w\partial_w\psi_L^i(w). \qquad (35)$$

Integrating this equation and expressing the result in terms of the original complex fermion field $\psi(z)=\psi^1(z)+\imath\psi^2(z)$,

$$\psi_L(z)=\ddagger\exp\left(-\int_{z_0}^{z}\frac{d\xi}{\xi}J_f(\xi)\right)\psi_L(z_0)\ddagger. \qquad (36)$$

This suggests the expression

$$\psi_L(z)\psi_L^\dagger(w)=F(z,w)\ddagger\exp\left(-\int_w^z\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger \qquad (37)$$

with

$$F(z,w)=\exp\left(\int_w^z\frac{d\xi}{\xi}J_f^{(+)}(\xi)\right)\psi_L(z)\psi_L^\dagger(w)\exp\left(\int_w^z\frac{d\xi}{\xi}J_f^{(-)}(\xi)\right). \qquad (38)$$

$F(z,w)$ is a function of the charges since it commutes with $J(z)$ and if one assumes, as in Ref. [13], that any operator that commutes with the currents is a function only of the charges. $F(z,w)$ also commutes with $\psi(u)$ for $u\neq z,w$ so it is actually a c-number function. Finally with the commutators

$$[J_f^{(+)}(z),\psi_L(w)]=\left(\frac{w}{w-z}-\frac{1}{2}\right)\psi_L(w),$$
$$[J_f^{(-)}(z),\psi_L^\dagger(w)]=\left(\frac{z}{z-w}-\frac{1}{2}\right)\psi_L^\dagger(w), \qquad (39)$$

and the definition of $F(z,w)$ in Eq. (38) one obtains the differential equations

$$\partial_z F(z,w)=\left(-\frac{1}{z-w}+\frac{1}{2z}\right)F(z,w),$$
$$\partial_w F(z,w)=\left(\frac{1}{z-w}+\frac{1}{2w}\right)F(z,w) \qquad (40)$$

with the solution

$$F(z,w)=f_0\frac{\sqrt{zw}}{z-w}, \qquad (41)$$

where f_0 is a constant. Thus we arrive at

$$\psi_L(z)\psi_L^\dagger(w)=f_0\frac{\sqrt{zw}}{z-w}\ddagger\exp\left(-\int_w^z\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger. \qquad (42)$$

Although this solution appears to be nonlocal it is in fact bilocal since it is independent of the integration path from w to z. We next define $\sigma_+(x^+,x^-)\equiv\psi_L^\dagger(x^+)\psi_R(x^-)$ and $\sigma_-(x^+,x^-)\equiv\psi_R^\dagger(x^-)\psi_L(x^+)$. Nonzero vacuum expectation values of products of these operators must contain an equal number of σ_+ and σ_- because of the $SO_L(2)\times SO_R(2)$ symmetry. We consider the product of operators

$$\sigma_+(z,\overline{z})\sigma_-(w,\overline{w})=f_0^2\left|\frac{\sqrt{zw}}{z-w}\right|^2 \ddagger\exp\left(\int_w^z\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger\ddagger\exp\left(-\int_{\overline{w}}^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J_f}(\overline{\xi})\right)\ddagger \tag{43}$$

with the equality following from the solution of Eq. (42). The value of f_0 may be found by comparing the vacuum expectation value of this operator

$$\langle 0|\sigma_+(z,\overline{z})\sigma_-(w,\overline{w})|0\rangle=\langle 0|\psi^i(z)\psi^i(w)|0\rangle\langle 0|\overline{\psi^i}(\overline{z})\overline{\psi^i}(\overline{w})|0\rangle=\frac{1}{2L^2}\left|\frac{\sqrt{zw}}{z-w}\right|^2 \tag{44}$$

with the operator in terms of the currents in Eq. (43). Because the vacuum expectation value of normal-ordered products of current operators vanishes, $f_0=1/(\sqrt{2}L)$.

We next find the transformation of the operator in Eq. (43) by $U(\pi/2)$ to get the corresponding operator in the bosonic theory

$$\begin{aligned}U(\pi/2)\sigma_+(z,\overline{z})\sigma_-(w,\overline{w})U^{-1}(\pi/2)&=\frac{1}{2L^2}\left|\frac{\sqrt{zw}}{z-w}\right|^2 :\exp\{2\sqrt{\pi}\imath[\phi_0(z)-\phi_0(w)]\}::\exp\{2\sqrt{\pi}\imath[\overline{\phi_0}(\overline{z})-\overline{\phi_0}(\overline{w})]\}:\\ &=\frac{1}{2L^2}\left|\sqrt{\frac{w}{z}}\right|^2 :\exp\{2\sqrt{\pi}\imath[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}::\exp\{-2\sqrt{\pi}\imath[\phi_0(w)+\overline{\phi_0}(\overline{w})]\}:.\end{aligned} \tag{45}$$

The last equality may be derived using the relation

$$\begin{aligned}&:e^{2\sqrt{\pi}\imath\phi_0(z)}::e^{-2\sqrt{\pi}\imath\phi_0(w)}:=e^{2\sqrt{\pi}\imath\phi_0^{(+)}(z)}e^{2\sqrt{\pi}\imath\phi_0^{(-)}(z)}e^{-2\sqrt{\pi}\imath\phi_0^{(+)}(w)}e^{-2\sqrt{\pi}\imath\phi_0^{(-)}(w)}=e^{4\pi[\phi_0^{(-)}(z),\phi_0^{(+)}(w)]}:e^{2\sqrt{\pi}\imath[\phi_0(z)-\phi_0(w)]}:\\ &=\left(\frac{z}{z-w}\right):e^{2\sqrt{\pi}\imath[\phi_0(z)-\phi_0(w)]}:.\end{aligned} \tag{46}$$

$\phi_0^{(+)}$ and $\phi_0^{(-)}$ are, respectively, the positive and negative frequency modes of ϕ_0.

Since Eq. (45) is valid for arbitrary $|z|>|w|$ the transformation of the operators σ_+ and σ_- are

$$\begin{aligned}U\left(\frac{\pi}{2}\right)\sigma_+(z,\overline{z})U^{-1}\left(\frac{\pi}{2}\right)&=\frac{1}{\sqrt{2}L}(\sqrt{z\overline{z}})^{-1}:\exp\{2\sqrt{\pi}\imath[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}:,\\ U\left(\frac{\pi}{2}\right)\sigma_-(z,\overline{z})U^{-1}\left(\frac{\pi}{2}\right)&=\frac{1}{\sqrt{2}L}\sqrt{z\overline{z}}:\exp\{-2\sqrt{\pi}\imath[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}:.\end{aligned} \tag{47}$$

Finally, to compare with previous results in the literature we add the zero modes to ϕ_0 and $\overline{\phi_0}$,

$$\phi(z)\equiv q+p\frac{1}{4\pi\imath}\ln z+\phi_0(z),\qquad \overline{\phi}(\overline{z})\equiv\overline{q}+\overline{p}\frac{1}{4\pi\imath}\ln\overline{z}+\overline{\phi_0}(\overline{z}), \tag{48}$$

and define normal-ordered expressions to have p to the right of q. Then the transformed operators become

$$\begin{aligned}U\left(\frac{\pi}{2}\right)\sigma_+(z,\overline{z})U^{-1}\left(\frac{\pi}{2}\right)&=\frac{1}{\sqrt{2}L}\sqrt{z\overline{z}}:\exp\{2\sqrt{\pi}\imath[\phi(z)+\overline{\phi}(\overline{z})]\}:,\\ U\left(\frac{\pi}{2}\right)\sigma_-(z,\overline{z})U^{-1}\left(\frac{\pi}{2}\right)&=\frac{1}{\sqrt{2}L}\sqrt{z\overline{z}}:\exp\{-2\sqrt{\pi}\imath[\phi(z)+\overline{\phi}(\overline{z})]\}:.\end{aligned} \tag{49}$$

This result agrees with the Frenkel-Kac fermionic vertex operator construction with fermion operators Ψ in the transformed theory [14,12]

$$\begin{aligned}&\Psi_L(z)=2^{-1/4}L^{-1/2}\sqrt{z}:\exp[-2\sqrt{\pi}\imath\phi(z)]:\epsilon_L,\qquad \Psi_L^\dagger(z)=2^{-1/4}L^{-1/2}\sqrt{z}:\exp[2\sqrt{\pi}\imath\phi(z)]:\epsilon_L,\\ &\Psi_R(\overline{z})=2^{-1/4}L^{-1/2}\sqrt{\overline{z}}:\exp[2\sqrt{\pi}\imath\overline{\phi}(\overline{z})]:\epsilon_R,\qquad \Psi_R^\dagger(\overline{z})=2^{-1/4}L^{-1/2}\sqrt{\overline{z}}:\exp[-2\sqrt{\pi}\imath\overline{\phi}(\overline{z})]:\epsilon_R,\end{aligned} \tag{50}$$

and $\epsilon_{L,R}$ are coordinate-independent fermionic operators which satisfy $\{\epsilon_L,\epsilon_R\}=0$ and $(\epsilon_L)^2=(\epsilon_R)^2=1$ [9]. The operators in Eq. (50) then have the correct anticommutation relations for complex fermion fields.

All that remains to be done to obtain the complete bosonized Hamiltonian density is a finite wave function renormalization of $\phi(x)$. The renormalized field $\Phi(x)$ is defined as $\Phi(x)=\alpha^{-1}\phi(x)$. We use a method similar to that used by Coleman in order to regularize the expression $:\exp[2\sqrt{\pi}\imath\alpha\Phi(z)]:$ whereby one uses Wick's theorem [7,9]

$$\exp\left(\imath\int J(x)\phi(x)d^2x\right)=:\exp\left(\imath\int J(x)\phi(x)d^2x\right):\exp\left(-\frac{1}{2}\int J(x)\Delta(x-y;m)J(y)d^2xd^2y\right) \tag{51}$$

with $J(x)$ set equal to a δ function and the propagator $\Delta(x;m)$, which is singular for spacelike $x^2\to 0$ replaced by the regulated propagator $\Delta^R(x;m,\Lambda)$, which is finite in this limit

$$\Delta^R(x;m,\Lambda)\equiv\Delta(x,m)-\Delta(x,m=\Lambda). \tag{52}$$

Λ is a large cutoff mass. In our case $m=0$ and, with the propagator of Eq. (22), the result for the field ϕ_0 is

$$:e^{2\sqrt{\pi}\imath\phi_0(z,\overline{z})}:_{\phi_0}=\left(\frac{L^2\Lambda^2}{4\pi^2}\right)^{-\alpha^2+1}:e^{2\sqrt{\pi}\imath\alpha\Phi_0(z,\overline{z})}:_{\Phi_0}. \tag{53}$$

The subscript on the normal-ordering symbol indicates with respect to which field the expression is normal-ordered. Adding in the zero modes one obtains

$$:e^{2\sqrt{\pi}\imath\phi(z,\overline{z})}:_{\phi}=\left(\frac{L^2\Lambda^2}{4\pi^2}\right)^{-\alpha^2+1}|z|^{(1/2)(\alpha^2-1)}:e^{2\sqrt{\pi}\imath\alpha\Phi(z,\overline{z})}:_{\Phi}. \tag{54}$$

After collecting the previous results with $\alpha=(1+g/\pi)^{-1/2}$ and an analytical continuation back to Minkowski space, the Lagrangian density $\mathcal{L}_b$ corresponding to the bosonized massive Thirring model is

$$\mathcal{L}_b=\frac{1}{2}:\partial_\mu\Phi\partial^\mu\Phi:-\sqrt{\frac{2m}{L}}\left(\frac{L^2\Lambda^2}{4\pi^2}\right)^{g/(\pi+g)}\exp\left[\frac{\sqrt{2}\pi}{L}\left(1+\frac{g}{\pi}\right)^{-1}x^0\right]:\cos 2\sqrt{\pi}\left(1+\frac{g}{\pi}\right)^{-1/2}\Phi:_{\Phi}. \tag{55}$$

The coefficient a of $\cos a\Phi$ in the potential, which is renormalization scheme independent, agrees with that derived using different methods [7,8]. The x^0 dependence of the unrenormalized potential term is necessary for it to have a definite scaling dimension, i.e., in conformal field theory conformal weight $(\frac{1}{2},\frac{1}{2})$. Wave function renormalization maintains this property however changes the scaling dimension. The operator in the potential term of Eq. (55) now has scaling dimension $(1+g/\pi)$. This term is therefore superrenormalizable only within the allowed range of the coupling, $g<\pi/2$. Finally, the finite wave function renormalization gives the correctly normalized kinetic term in the Lagrangian, however the transformed fermion operators in the new theory, $\Psi_{L,R}$, no longer satisfy fermionic anticommutation relations.

V. SMOOTH BOSONIZATION OF THE MASSIVE THIRRING MODEL

The previous solution of the massive Thirring model in terms of the currents may also be used to find a continuum of equivalent theories which contain interactions between the boson and fermion field by transforming the Thirring model Hamiltonian density by $U(\beta)$. The result for the massless Thirring model is

$$U(\beta)\mathcal{H}_{m=0\ \mathrm{Thirring}}U^{-1}(\beta)=\cos^2\beta\mathcal{H}_{m=0\ \mathrm{Thirring}}+\sin^2\beta\mathcal{H}'_{m=0\ \mathrm{Thirring}}+\frac{1}{2L^2}\sin 2\beta[\pi(J_fJ_b-\overline{J_f}\overline{J_b})+g(J_f\overline{J_b}-\overline{J_f}J_b)]. \tag{56}$$

The Hamiltonian density for the massless Thirring model $\mathcal{H}_{m=0\ \mathrm{Thirring}}$ and its bosonic equivalent $\mathcal{H}'_{m=0\ \mathrm{Thirring}}$ are given in Eqs. (31) and (32), respectively. One follows the same procedure as for the case of complete bosonization to calculate the transformed mass term, namely transform the fermion currents in the operator solution of $\sigma_+\sigma_-$, giving

$$U(\beta)\sigma_+(z,\overline{z})\sigma_-(w,\overline{w})U^{-1}(\beta)=\frac{1}{2L^2}\left|\frac{\sqrt{zw}}{z-w}\right|^2\ddagger\exp\left(\int_w^z\frac{d\xi}{\xi}(\cos\beta J_f(\xi)+\sin\beta J_b(\xi)]\right)\ddagger$$
$$\times\ddagger\exp\left(\int_{\overline{w}}^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}[-\cos\beta\overline{J}_f(\overline{\xi})+\sin\beta\overline{J}_b(\overline{\xi})]\right)\ddagger. \tag{57}$$

Substituting the definition of the boson current, Eq. (16), and reordering the boson terms gives

$$U(\beta)\sigma_+(z,\overline{z})\sigma_-(w,\overline{w})U^{-1}(\beta)=\frac{1}{2L^2}|w||z|^{1-2\sin^2\beta}\left|\frac{1}{z-w}\right|^{2(1-\sin^2\beta)}$$
$$\times:\exp\{2\sqrt{\pi}\imath\sin\beta[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}::\exp\{-2\sqrt{\pi}\imath\sin\beta[\phi_0(w)$$
$$+\overline{\phi_0}(\overline{w})]\}:\ddagger\exp\left(\cos\beta\int_w^z\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger$$
$$\times\ddagger\exp\left(-\cos\beta\int_{\overline{w}}^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J_f}(\overline{\xi})\right)\ddagger. \tag{58}$$

There is no simple operator expression for the normal-ordered exponents of the fermion currents in this equation in terms of the fermion fields $\psi_{L,R}$. Therefore the fermionic sector of the transformed theory will be formulated in terms of J_f and $\overline{J_f}$ rather than ψ_L and ψ_R. This is not such a radial reformulation of the theory since operators with nonvanishing vacuum expectation values in the original theory, the massive Thirring model, may also be expressed entirely in terms of the fermion currents, using the equations given previously. Next, in order to obtain $\sigma_\pm$ in the new theory we separate the normal-ordered exponent of the fermion current, for $|z|>|w|$ as

$$\ddagger\exp\left[\left(\cos\beta\int_w^z\frac{d\xi}{\xi}J_f(\xi)\right)\right]\ddagger=\ddagger\exp\ \cos\beta\left(\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)-\int_\infty^w\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger$$
$$=\left(\frac{z}{z-w}\right)^{-\cos^2\beta}\ddagger\exp\left(\cos\beta\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger\ddagger\exp\left(\cos\beta\int_\infty^w\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger. \tag{59}$$

An adiabatic cutoff, $e^{-\epsilon\xi}$ with ϵ small, is implied in the integration. After substituting this relation in Eq. (58) the result is a product of two factors, each with dependence on only one coordinate. Therefore the transformed $\sigma_\pm$ are

$$U(\beta)\sigma_+(z,\overline{z})U^{-1}(\beta)=\frac{1}{\sqrt{2}L}|z|^{-1}:\exp\{2\sqrt{\pi}\imath\sin\beta[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}:\ddagger\exp\left[\cos\beta\left(\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)-\int_\infty^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J_f}(\overline{\xi})\right)\right]\ddagger,$$
$$U(\beta)\sigma_-(z,\overline{z})U^{-1}(\beta)=\frac{1}{\sqrt{2}L}|z|:\exp\{-2\sqrt{\pi}\imath\sin\beta[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}:\ddagger\exp\left[\cos\beta\left(-\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)+\int_\infty^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J_f}(\overline{\xi})\right)\right]\ddagger. \tag{60}$$

This implies that the transformed fermion operators, for example, $\Psi_L(z)=U(\beta)\psi_L(z)U^{-1}(\beta)$, are

$$\Psi_L(z)=2^{-1/4}L^{-1/2}\sqrt{z}:\exp[-2\sqrt{\pi}\imath\sin\beta\phi_0(z)]:\ddagger\exp\left(-\cos\beta\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger\epsilon_L,$$
$$\Psi_L^\dagger(z)=2^{-1/4}L^{-1/2}\sqrt{z}^{-1}:\exp[2\sqrt{\pi}\imath\sin\beta\phi_0(z)]:\ddagger\exp\left(\cos\beta\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)\right)\ddagger\epsilon_L,$$
$$\Psi_R(\overline{z})=2^{-1/4}L^{-1/2}\sqrt{\overline{z}}^{-1}:\exp[2\sqrt{\pi}\imath\sin\beta\overline{\phi_0}(\overline{z})]:\ddagger\exp\left(-\cos\beta\int_\infty^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J_f}(\overline{\xi})\right)\ddagger\epsilon_R,$$
$$\Psi_R^\dagger(\overline{z})=2^{-1/4}L^{-1/2}\sqrt{\overline{z}}:\exp[-2\sqrt{\pi}\imath\sin\beta\overline{\phi_0}(\overline{z})]:\ddagger\exp\left(\cos\beta\int_\infty^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J_f}(\overline{\xi})\right)\ddagger\epsilon_R. \tag{61}$$

ϵ_L and ϵ_R have the same definition as in Eq. (50). These transformed fermion operators also satisfy the correct anticommutation relations for complex fermion fields.

Next we perform a finite renormalization of both the fermion and the boson currents. The renormalization of the boson currents is equivalent to renormalization of the boson field $\phi(z,\overline{z})$. The renormalized fields are defined as

$$J_f^R(z)=J_f(z)\cos\beta,\quad \overline{J_f^R}(\overline{z})=\overline{J_f}(\overline{z})\cos\beta,\quad \Phi(z,\overline{z})=\phi(z,\overline{z})\sin\beta. \tag{62}$$

Using the same method as in the last section to express the normal-ordered exponentials of expressions containing ϕ, J_f, and $\overline{J_f}$ in terms of the renormalized quantities, Eq. (60) becomes

$$[U(\beta)\sigma_+(z,\overline{z})U^{-1}(\beta)]_{\rm ren}=\frac{1}{\sqrt{2}L}|z|^{-1}\left(\frac{4\pi^2}{L^2\Lambda^2}\right):\exp\{2\sqrt{\pi}\imath[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}:\ddagger\exp\left[\left(\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)-\int_\infty^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J}_f(\overline{\xi})\right)\right]\ddagger, \tag{63}$$

$$[U(\beta)\sigma_-(z,\overline{z})U^{-1}(\beta)]_{\rm ren}=\frac{1}{\sqrt{2}L}|z|\left(\frac{4\pi^2}{L^2\Lambda^2}\right):\exp\{-2\sqrt{\pi}\imath[\phi_0(z)+\overline{\phi_0}(\overline{z})]\}:\ddagger\exp\left[\left(-\int_\infty^z\frac{d\xi}{\xi}J_f(\xi)+\int_\infty^{\overline{z}}\frac{d\overline{\xi}}{\overline{\xi}}\overline{J}_f(\overline{\xi})\right)\right]\ddagger.$$

Collecting the transformed operators of Eqs. (56) and (63) in terms of the renormalized fields the transformed Hamiltonian density $\mathcal{H}'_{\rm ren}$ in Minkowski space becomes

$$\begin{aligned}\mathcal{H}'_{\rm ren}=&\frac{1}{2}[:(\partial_+\Phi)^2:+:(\partial_-\Phi)^2:]-\frac{g}{\pi}:\partial_+\Phi\partial_-\Phi:+\frac{\pi}{2L^2}\left(\ddagger(J_f^R)^2\ddagger+\ddagger(\overline{J_f^R})^2\ddagger+\frac{2g}{\pi}J_f^R\overline{J_f^R}\right)\\&+\frac{\sqrt{\pi}}{L}\left[J_f^R\left(\partial_+\Phi+\frac{g}{\pi}\partial_-\Phi\right)-\overline{J_f^R}\left(\partial_-\Phi+\frac{g}{\pi}\partial_+\Phi\right)\right]+\frac{m}{\sqrt{2}L}\left(\frac{4\pi^2}{L^2\Lambda^2}\right)\left[\exp\left(-\frac{2\sqrt{2}\pi}{L}x^0\right):\exp(2\sqrt{\pi}\imath\Phi):\right.\\&\times\ddagger\exp\frac{2\pi\imath}{L}\left(\int_\infty^{x^+}d\xi^+J_f^R(\xi^+)-\int_\infty^{x^-}d\xi^-\overline{J_f^R}(\xi^-)\right)\ddagger+\exp\left(\frac{2\sqrt{2}\pi}{L}x^0\right):\exp(-2\sqrt{\pi}\imath\Phi):\\&\left.\times\ddagger\exp\frac{2\pi\imath}{L}\left(-\int_\infty^{x^+}d\xi^+J_f^R(\xi^+)+\int_\infty^{x^-}d\xi^-\overline{J_f^R}(\xi^-)\right)\ddagger\right].\end{aligned} \tag{64}$$

VI. DISCUSSION

If we consider parity transformations $\mathcal{P}$ we find that $\mathcal{P}U(\beta)\mathcal{P}^{-1}=U(-\beta)$ for a scalar ϕ. However $U(\beta)$ is invariant under a parity transformation if ϕ is a pseudoscalar, i.e., $\mathcal{P}\phi(x^0,x^1)\mathcal{P}^{-1}=-\phi(x^0,-x^1)$. Therefore since we start with the parity invariant massive Thirring model, the transformed Hamiltonian density of Eq. (64) is also parity invariant, but only if ϕ is a pseudoscalar. This is particularly interesting for the suggestion of Damgaard *et al.* of using smooth bosonization to construct exact Cheshire cat bag models in $1+1$ dimensions to describe QCD bound states [15]. In these models the interior of the bag has fermion fields, namely quarks, whereas outside the bag the relevant degrees of freedom are the scalar fields, namely mesons. Smooth bosonization allows a continuous transition from the fermionic theory to the bosonic one with interactions between them only near the bag surface. This interpretation is consistent with the method of smooth bosonization presented in this paper since the meson field, ϕ, has the correct parity.

We have also presented an explicit unitary operator implementing this bosonization transformation. It would be interesting to investigate similar operators for other duality transformations. Although the quantum canonical transformation in this paper is linear (in the currents), as are all such transformations in quantum field theory presented to date, the method of transformation by a unitary operator opens up the possibility of finding transformations not within this class. Finally, it is important to understand the exact relation between the various methods of demonstrating duality: path integral manipulations, linear canonical transformations, and operator transformations. This would allow comparison between the results derived using these complementary methods.

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