Spin factor in the path integral representation for the Dirac propagator in external fields

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We study the problem of the spin factor both in 3+1 and 2+1 dimensions, two cases which are essentially different in this respect. Doing all Grassmann integrations in the corresponding path integral representations for the Dirac propagator we get representations with a spin factor in an arbitrary external field. Thus, the propagator appears to be presented by means of a bosonic path integral only. Then we use the representations with a spin factor for calculations of the propagator in some configurations of external fields: namely, in a constant uniform electromagnetic field and in its combination with a plane wave field. [S0556-2821(97)02112-7]

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I. INTRODUCTION

Propagators of relativistic particles in external fields (electromagnetic, non-Abelian, or gravitational) contain important information about the quantum behavior of these particles. Moreover, if such propagators are known in an arbitrary external field, one can find exact one-particle Green functions in the corresponding quantum field theory, taking functional integrals over all external fields. The Dirac propagator in an external electromagnetic field is distinguished from that of a scalar particle by a complicated spinor structure. The problem of its path integral representation has attracted researchers' attention for a long time. Thus, Feynman, who first wrote his path integral for the probability amplitude in nonrelativistic quantum mechanics [1] and then wrote a path integral for the causal Green function of the Klein-Gordon equation (scalar particle propagator) [2], had also made an attempt to derive a representation for the Dirac propagator via a bosonic path integral [3]. After the introduction of the integral over Grassmann variables by Berezin it turned out that it is possible to present this propagator via both bosonic and Grassmann variables; the latter describe spinning degrees of freedom. Representations of this kind have been discussed in the literature for a long time in different contexts [4]. Nevertheless, attempts to write the Dirac propagator via only a bosonic path integral continued. Thus, Polyakov [6] assumed that the propagator of a free Dirac electron in $D=3$ Euclidean space-time can be presented by means of a bosonic path integral similar to the scalar particle case, modified by the so-called spin factor (SF). This idea was developed in [7], e.g., to write the SF for Dirac fermions, interacting with a non-Abelian gauge field in D-dimensional Euclidean space-time. In those representations the SF itself was presented via some additional bosonic path integrals and its γ-matrix structure was not defined explicitly. Surprisingly, it was shown in [9] that all Grassmann integrations in the representation of the Dirac propagator in an arbitrary external field in $3+1$ dimensions can be done, so that an expression for the SF was derived as a given functional of the bosonic trajectory. Having such a representation with the SF, one can use it to calculate the propagator in some particular cases of external fields. This method of calculation provides automatically an explicit spinor structure of the propagators which can be used for concrete calculations in the Furry picture (see, for example, [10,11]).

In the recent work of [13] the propagator of a spinning particle in an external field was presented via a path integral in arbitrary dimensions. It turns out that the problem has different solutions in even and odd dimensions. In even dimensions the representation is just a generalization of the one in four dimensions mentioned above. In odd dimensions the solution was presented for the first time and differs essentially from the even-dimensional case. However, the problem of SF was not discussed there.

In the present paper we continue the consideration of the problems related to the SF conception. Namely, we discuss derivation of the SF both in even and odd dimensions on the examples of 3+1 and 2+1 cases and then we use path integral representations with the SF to calculate the propagators in some configurations of external fields. In $3+1$ dimensions we present a simple derivation of the SF, avoiding some unnecessary steps in the original brief paper [9] which themselves needed some additional justification. In this way the meaning of the surprising possibility of complete integration over Grassmann variables becomes clear. Then we use the representation with the SF for calculations of the propagator in a constant uniform electromagnetic field and its combination with a plane wave. Because of the fact that this method of calculation provides automatically an explicit γ-matrix structure of the propagator, the representations obtained differ from those found by other methods, for example, differs from the well-known Schwinger formula in a constant uniform electromagnetic field. To compare both representations we prove in Appendix B some complicated decompositions of functions on the γ matrices. In 2+1 dimensions we present a derivation of the SF for the first time using the representation for the propagator obtained in [13]. We calculate then the propagator in these dimensions in a constant

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electromagnetic field by means of the representation with the SF. The result is new and cannot be derived from the (3+1)-dimensional case by means of a dimensional reduction.

II. SPIN FACTOR IN 3+1 DIMENSIONS

A. Integration over Grassmann variables

The propagator of a relativistic spinning particle in an external electromagnetic field $A_\mu(x)$ is the causal Green function $S^c(x,y)$ of the Dirac equation in this field:

$$[\gamma^\mu(i\partial_\mu - gA_\mu) - m]S^c(x,y) = -\delta^4(x-y), \tag{1}$$

where $x=(x^\mu)$, $[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}$, $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and $\mu,\nu = 0,1,2,3$.

In Ref. [8] the following Lagrangian path integral representation for the propagator was obtained in 3+1 dimensions:

$$S^c = S^c(x_{\mathrm{out}}, x_{\mathrm{in}}) = -\tilde{S}^c\gamma^5,$$

$$\tilde{S}^c = \exp\left\{ i\tilde{\gamma}^n \frac{\partial_\ell}{\partial\theta^n}\right\} \int_0^\infty de_0 \int d\chi_0 \int_{e_0} M(e) De \int_{\chi_0} D\chi \int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}} Dx \int D\pi_e \int D\pi_\chi$$
$$\times \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{ i\int_0^1 \left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}^\mu A_\mu + iegF_{\mu\nu}\psi^\mu\psi^\nu \right.\right.$$
$$\left.\left. + i\left(\frac{\dot{x}_\mu\psi^\mu}{e} - m\psi^5\right)\chi - i\psi_n\dot{\psi}^n + \pi_e\dot{e} + \pi_\chi\dot{\chi}\right] d\tau + \psi_n(1)\psi^n(0)\right\}\Bigg|_{\theta=0}, \tag{2}$$

where $\tilde{\gamma}^\mu = \gamma^5\gamma^\mu, \tilde{\gamma}^5 = \tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3 = \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$, $[\gamma^m,\gamma^n]_+ = 2\eta^{mn}$, $m,n = 0,1,2,3,5$, and $\eta^{mn} = \mathrm{diag}(1,-1,-1,-1,-1)$; θ^n are auxiliary Grassmann (odd) variables, anticommuting by definition with the γ matrices; $x^\mu(\tau)$, $e(\tau)$, and $\pi_e(\tau)$ are bosonic trajectories of integration; $\psi^n(\tau)$, $\chi(\tau)$, and $\pi_\chi(\tau)$ are odd trajectories of integration; the boundary conditions

$$x(0) = x_{\mathrm{in}}, \quad x(1) = x_{\mathrm{out}}, \quad e(0) = e_0, \quad \psi^n(0) + \psi^n(1) = \theta^n, \quad \chi(0) = \chi_0$$

take place; the measure $M(e)$ and $\mathcal{D}\psi$ have the form

$$M(e) = \int Dp \exp\left\{\frac{i}{2}\int_0^1 ep^2 d\tau\right\}, \quad \mathcal{D}\psi = D\psi\left[\int_{\psi(0)+\psi(1)=0} D\psi \exp\left\{\int_0^1 \psi_n\dot{\psi}^n d\tau\right\}\right]^{-1}, \tag{3}$$

and $\partial_\ell/\partial\theta^n$ stands for the left derivatives. The propagator (2) can be expressed [9] only through a bosonic path integral over the coordinates x. The Grassmann integrations over ψ can be performed even without changing ψ for velocities like in [9]. Instead, after integration over π_e, π_χ, e, and χ one can go back to Grassmann variables obeying antiperiodic boundary conditions [8]

$$\psi^n = \tfrac{1}{2}(\xi^n + \theta^n). \tag{4}$$

Then introducing odd sources $\rho_n(\tau)$ for the new variables $\xi^n(\tau)$, we get

$$\tilde{S}^c = -\frac{1}{2}\exp\left\{ i\tilde{\gamma}^n\frac{\partial_\ell}{\partial\theta^n}\right\}\int_0^\infty de_0 M(e_0)\int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}} Dx \exp\left\{ -i\left[\frac{\dot{x}_\mu\star\dot{x}^\mu}{2e_0} + \frac{e_0}{2}m^2 + g\dot{x}^\mu\star A_\mu\right] - \frac{ge_0}{4}\theta^\mu\star\mathcal{F}_{\mu\nu}\star\theta^\nu\right\}$$
$$\times\left[\frac{\dot{x}_\mu}{e_0}\star\left(\frac{\delta_\ell}{\delta\rho_\mu} + \theta^\mu\right) - m\star\left(\frac{\delta_\ell}{\delta\rho_5} + \theta^5\right)\right] R[x,\rho,\theta]\Bigg|_{\rho=0,\theta=0}, \tag{5}$$

where

$$R[x,\rho,\theta] = \int_{\xi(0)+\xi(1)=0}\mathcal{D}\xi\ \exp\left\{\frac{1}{4}\xi_n\star\dot{\xi}^n - \frac{ge_0}{4}\xi^\mu\star\mathcal{F}_{\mu\nu}\star\xi^\nu - \frac{ge_0}{2}\theta^\mu\star\mathcal{F}_{\mu\nu}\star\xi^\nu + \rho_n\star\xi^n\right\}, \tag{6}$$

$$\mathcal{D}\xi = D\xi\left[\int_{\xi(0)+\xi(1)=0} D\xi\ \exp\left\{\frac{1}{4}\xi_n\star\dot{\xi}^n\right\}\right]^{-1}. \tag{7}$$

Here condensed notations are used in which $\mathcal{F}_{\mu\nu}$ is understood as a matrix with continuous indices,

$$\mathcal{F}_{\mu\nu}(\tau,\tau') = F_{\mu\nu}(x(\tau))\delta(\tau-\tau') \ , \tag{8}$$

and integration over τ is denoted by a star, e.g.,

$$\xi_n \star \dot{\xi}^n = \int_0^1 \xi_n(\tau)\dot{\xi}^n(\tau)d\tau \ .$$

Sometimes discrete indices will be also omitted. In this case all tensors of second rank have to be understood as matrices with lines marked by the first contravariant indices of the tensors and with columns marked by the second covariant indices of the tensors.

The Grassmann Gaussian path integral in Eq. (6) can be evaluated straightforwardly [14] to be

$$R[x,\rho,\theta] = \exp\left\{-e_0 \mathrm{Tr}\int_0^g dg'\mathcal{R}(g')\star\mathcal{F}\right\}\exp\left\{\left(\rho^\mu - \frac{ge_0}{2}\Theta_\kappa\star\mathcal{F}^{\kappa\mu}\right)\star\mathcal{U}_{\mu\nu}(g)\star\left(\rho^\nu + \frac{ge_0}{2}\mathcal{F}^{\nu\lambda}\star\Theta_\lambda\right) - \rho_5\star\dot{\rho}_5\right\}, \tag{9}$$

where

$$\mathcal{U}_{\mu\nu}(g) = \eta_{\mu\nu}\delta'(\tau-\tau') - ge_0\mathcal{F}_{\mu\nu}(\tau,\tau') \ , \tag{10}$$

and $\mathcal{R}(g)$ is the inverse to $\mathcal{U}(g)$, considered as an operator acting in the space of the antiperiodic functions,

$$\frac{d}{d\tau}\mathcal{R}_{\mu\nu}(g|\tau,\tau') - ge_0 F_\mu^{\ \lambda}(x(\tau))\mathcal{R}_{\lambda\nu}(g|\tau,\tau') = \eta_{\mu\nu}\delta(\tau-\tau'),$$

$$\mathcal{R}_{\mu\nu}(g|1,\tau) = -\mathcal{R}_{\mu\nu}(g|0,\tau) \ , \quad \forall\tau\in(0,1). \tag{11}$$

Substituting Eq. (9) into Eq. (5) and performing then the functional differentiations with respect to ρ_n, we get

$$\widetilde{S}^c = -\frac{1}{2}\exp\left\{i\widetilde{\gamma}^n\frac{\partial_\ell}{\partial\theta^n}\right\}\int_0^\infty de_0 M(e_0)\int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}} Dx \exp\left\{-\frac{i}{2}\left[\frac{\dot{x}^\mu\star\dot{x}^\mu}{e_0} + e_0 m^2 + g\dot{x}^\mu A_\mu\right]\right\}\left[\frac{\dot{x}^\mu}{e_0}\star K_{\mu\nu}\theta^\nu - m\theta^5\right]$$
$$\times\left[1 - \frac{ge_0}{4}B_{\alpha\beta}\theta^\alpha\theta^\beta + \frac{g^2e_0^2}{16}B_{\alpha\beta}B^{*\alpha\beta}\theta^0\theta^1\theta^2\theta^3\right]\exp\left\{-\frac{e_0}{2}\int_0^\infty dg'\mathrm{Tr}\mathcal{R}(g')\star\mathcal{F}\right\}\Bigg|_{\theta=0}, \tag{12}$$

where the following notation is used:

$$B_{\mu\nu} = F_{\mu\lambda}\star K^\lambda_{\ \nu}, \quad B^{*\mu\nu} = \tfrac{1}{2}\epsilon^{\alpha\beta\mu\nu}B_{\alpha\beta}, \quad K_{\mu\nu} = \eta_{\mu\nu} + ge_0\mathcal{R}_{\mu\lambda}(g)\star F^\lambda_{\ \nu}, \tag{13}$$

and $\epsilon^{\mu\nu\alpha\beta}$ is Levi-Cività symbol normalized by $\epsilon^{0123}=1$.

Differentiation with respect to θ^n in Eq. (12) replaces the products of the variables θ^n by the corresponding antisymmetrized products of the matrices $i\widetilde{\gamma}^n$. Finally, passing to the propagator S^c and using the identities

$$\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^\lambda\gamma^{[\mu}\gamma^{\nu]} - 2\eta^{\lambda[\mu}\gamma^{\nu]}, \quad \gamma^{[\mu_1}\cdots\gamma^{\mu_5]} = 0, \quad \sigma^{\mu\nu} = i\gamma^{[\mu}\gamma^{\nu]}, \tag{14}$$

where antisymmetrization over the corresponding sets of indices is denoted by square brackets, one gets

$$S^c(x_{\mathrm{out}},x_{\mathrm{in}}) = \frac{i}{2}\int_0^\infty de_0\int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}} Dx\ M(e_0)\Phi[x,e_0]\ \exp\{iI[x,e_0]\} \ , \tag{15}$$

where $I[x,e_0]$ is the action of a relativistic spinless particle,

$$I[x,e_0] = -\int_0^1\left[\frac{\dot{x}^2}{2e_0} + \frac{e_0}{2}m^2 + g\dot{x}A(x)\right]d\tau \ , \tag{16}$$

and $\Phi[x,e_0]$ is the SF,

$$\Phi[x,\ e_0] = \left[m + (2e_0)^{-1}\dot{x}^\mu\star K_{\mu\lambda}(2\eta^{\lambda\kappa} - ge_0B^{\lambda\kappa})\gamma_\kappa - \frac{ig}{4}(me_0 + \dot{x}^\mu\star K_{\mu\lambda}\gamma^\lambda)B_{\kappa\nu}\sigma^{\kappa\nu} + m\frac{g^2e_0^2}{16}B^*_{\alpha\beta}B^{\alpha\beta}\gamma^5\right]$$
$$\times\exp\left\{-\frac{e_0}{2}\int_0^g dg'\mathrm{Tr}\mathcal{R}(g')\star\mathcal{F}\right\} \ . \tag{17}$$

B. Propagator in a constant uniform electromagnetic field

In the case of a constant uniform field ($F_{\mu\nu}=$ const), which we are going to discuss in this section, the functionals $\mathcal{R}$, K, and B do not depend on the trajectory x and can be calculated straightforwardly:

$$\mathcal{R}(g)=\frac{1}{2}\left(\eta\varepsilon(\tau-\tau')-\tanh\frac{ge_0F}{2}\right)\exp\{e_0gF(\tau-\tau')\}\ ,$$

$$K=\left(\eta-\tanh\frac{ge_0F}{2}\right)\exp(ge_0F\tau)\ ,\quad B=\frac{2}{ge_0}\tanh\frac{ge_0F}{2}\ . \tag{18}$$

Using them in Eq. (17) and integrating over τ whenever possible, we obtain the SF in the constant uniform field:

$$\Phi[x,\ e_0]=\left(\operatorname{detcosh}\frac{ge_0F}{2}\right)^{1/2}\left\{m\left[1-\frac{i}{2}\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}+\frac{1}{4}\left(\tanh\frac{ge_0F}{2}\right)^{*}_{\mu\nu}\left(\tanh\frac{ge_0F}{2}\right)^{\mu\nu}\gamma^5\right]+\frac{1}{e_0}\left(\int_0^1\dot{x}\exp(ge_0F\tau)d\tau\right)\right.$$
$$\left.\times\left(\eta-\tanh\frac{ge_0F}{2}\right)\left[\left(\eta-\tanh\frac{ge_0F}{2}\right)\gamma-\frac{i}{2}\gamma\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right]\right\}\ . \tag{19}$$

We can see that in the field under consideration the SF is linear in the trajectory $x^{\mu}(\tau)$. That facilitates the bosonic integration in expression (15).

In spite of the fact that the SF is a gauge-invariant object, the total propagator is not. It is clear from expression (15) where one needs to choose a particular gauge for the potentials A_μ. Namely, we are going to use the potentials

$$A_\mu=-\tfrac{1}{2}F_{\mu\nu}x^\nu \tag{20}$$

for the constant uniform field $F_{\mu\nu}=$ const. Thus, one can see that the path integral (15) is quasi Gaussian in the case under consideration. Let us make there the shift $x\to y+x_{\rm cl}$, with $x_{\rm cl}$ a solution of the classical equations of motion,

$$\frac{\delta I}{\delta x}=0\Leftrightarrow\ddot{x}_\mu-ge_0F_{\mu\nu}\dot{x}^\nu=0\ , \tag{21}$$

subjected to the boundary conditions $x_{\rm cl}(0)=x_{\rm in}$, and $x_{\rm cl}(1)=x_{\rm out}$. Then the new trajectories of integration, y, obey zero boundary conditions $y(0)=y(1)=0$. Because of the quadratic structure of the action $I[x,e_0]$ and the linearity of the SF in x, one can make the following substitutions in the path integral:

$$I[y+x_{\rm cl},\ e_0]\to I[x_{\rm cl},\ e_0]+I[y,\ e_0]+\frac{e_0}{2}m^2,$$

$$\Phi[y+x_{\rm cl},\ e_0]\to\Phi[x_{\rm cl},\ e_0]=\Psi(x_{\rm out},\ x_{\rm in},\ e_0). \tag{22}$$

Doing also a convenient replacement of variables $p\to p/\sqrt{e_0}$, $y\to y\sqrt{e_0}$, we get

$$S^c=\frac{i}{2}\int_0^\infty\frac{de_0}{e_0^2}\ \Psi(x_{\rm out},\ x_{\rm in},\ e_0)\ e^{iI[x_{\rm cl},\ e_0]}$$
$$\times\int_0^0 Dy\int Dp\exp\left\{\frac{i}{2}\int_0^1(p^2-\dot{y}^2-ge_0yF\dot{y})d\tau\right\}\ . \tag{23}$$

One can see that the path integral in Eq. (23) is, in fact, the kernel of the Klein-Gordon propagator in the proper-time representation. This path integral can be presented as

$$\int_0^0 Dy\int Dp\exp\left\{\frac{i}{2}\int_0^1(p^2-\dot{y}^2-ge_0yF\dot{y})d\tau\right\}$$
$$=\left[\frac{\text{Det}\ (\eta_{\mu\nu}\partial_\tau^2-ge_0F_{\mu\nu}\partial_\tau)}{\text{Det}\ (\eta_{\mu\nu}\partial_\tau^2)}\right]^{-1/2}$$
$$\times\int_0^0 Dy\int Dp\exp\left\{\frac{i}{2}\int_0^1(p^2-\dot{y}^2)d\tau\right\}\ .$$

Canceling the factor $\text{Det}\,(-\eta_{\mu\nu})$ in the ratio of the determinants one obtains

$$\frac{\text{Det}\ (\eta_{\mu\nu}\partial_\tau^2-ge_0F_{\mu\nu}\partial_\tau)}{\text{Det}\ (\eta_{\mu\nu}\partial_\tau^2)}=\frac{\text{Det}\ (-\delta^\mu_\nu\partial_\tau^2+ge_0F^\mu{}_\nu\partial_\tau)}{\text{Det}\ (-\delta^\mu_\nu\partial_\tau^2)}\ . \tag{24}$$

One can also make the replacement

$$-\mathbf{I}\partial_\tau^2+ge_0F\partial_\tau\to-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2\ , \tag{25}$$

where $\mathbf{I}$ stands for the unit 4×4 matrix, in the right-hand side (RHS) of Eq. (24), because the spectra of both operators coincide. Indeed,

$$-\mathbf{I}\partial_\tau^2+ge_0F\partial_\tau=\exp\left(\frac{ge_0}{2}F\tau\right)\left(-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2\right)$$
$$\times\exp\left(-\frac{ge_0}{2}F\tau\right)\ , \tag{26}$$

and zero boundary conditions are invariant under the transformation $y\to\exp(ge_0F\tau/2)y$. Then, using Eq. (25) and the value of the free path integral [8],

$$\frac{i}{2}\int_0^0 Dy\int Dp\exp\left\{\frac{i}{2}\int d\tau(p^2-\dot{y}^2)\right\}=\frac{1}{8\pi^2},$$

related, in fact, to the definition of the measure, we obtain

$$S^c=\frac{1}{8\pi^2}\int_0^\infty\frac{de_0}{e_0^2}\Psi(x_{\text{out}},\,x_{\text{in}},\,e_0)\;e^{iI[x_{\text{cl}},\,e_0]}\times\left[\frac{\text{Det}\left(-\mathbf{I}\partial_\tau^2+\dfrac{g^2e_0^2}{4}F^2\right)}{\text{Det}\,(-\mathbf{I}\partial_\tau^2)}\right]^{-1/2}. \tag{27}$$

The ratio of the determinants can be written now as

$$\frac{\text{Det}\left(-\mathbf{I}\partial_\tau^2+\dfrac{g^2e_0^2}{4}F^2\right)}{\text{Det}\,(-\mathbf{I}\partial_\tau^2)}=\exp\text{Tr}\left[\ln\left(-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2\right)-\ln(-\mathbf{I}\partial_\tau^2)\right]$$
$$=\exp\text{Tr}\left[\frac{e_0^2}{2}F^2\int_0^g d\lambda\;\lambda\left(-\mathbf{I}\partial_\tau^2+\frac{\lambda^2e_0^2}{4}F^2\right)^{-1}\right]$$
$$=\exp\text{tr}\left[\frac{e_0^2}{2}F^2\int_0^g d\lambda\;\lambda\sum_{n=1}^{\infty}\left(\pi^2n^2\mathbf{I}+\frac{\lambda^2e_0^2}{4}F^2\right)^{-1}\right]. \tag{28}$$

The trace in the infinite-dimensional space in the last line of Eq. (28) is taken and only one in the four-dimensional space remains. Using the formula

$$\sum_{n=1}^{\infty}(\pi^2n^2+\kappa^2)^{-1}=\frac{1}{2\kappa}\coth\kappa-\frac{1}{2\kappa^2},$$

which is also valid if κ is an arbitrary 4×4 matrix, and integrating in Eq. (28), we find

$$\frac{\text{Det}\left(-\mathbf{I}\partial_\tau^2+\dfrac{g^2e_0^2}{4}F^2\right)}{\text{Det}\,(-\mathbf{I}\partial_\tau^2)}=\det\left(\frac{\sinh\dfrac{ge_0F}{2}}{\dfrac{ge_0F}{2}}\right). \tag{29}$$

Thus,

$$S^c=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\dfrac{e_0gF}{2}}{gF}\right)^{-1/2}\times\Psi(x_{\text{out}},\,x_{\text{in}},\,e_0)e^{iI[x_{\text{cl}},e_0]}, \tag{30}$$

where the function $\Psi(x_{\text{out}},\,x_{\text{in}},\,e_0)$ is the SF on the classical trajectory x_{cl}. The latter can be easily found solving Eq. (21):

$$x_{\text{cl}}=[\exp(ge_0F)-\eta]^{-1}[\exp(ge_0F\tau)(x_{\text{out}}-x_{\text{in}})+\exp(ge_0F)x_{\text{in}}-x_{\text{out}}]. \tag{31}$$

Substituting Eq. (31) into Eqs. (22) and (30), we obtain

$$S^c=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\dfrac{ge_0F}{2}}{gF}\right)^{-1/2}\Psi(x_{\text{out}},\,x_{\text{in}},\,e_0)\times\exp\left\{\frac{ig}{2}x_{\text{out}}Fx_{\text{in}}-\frac{i}{2}e_0m^2-\frac{ig}{4}(x_{\text{out}}-x_{\text{in}})\times F\coth\left(\frac{ge_0F}{2}\right)(x_{\text{out}}-x_{\text{in}})\right\}. \tag{32}$$

where

$$\Psi(x_{\text{out}},x_{\text{in}},e_0)=\left[m+\frac{g}{2}(x_{\text{out}}-x_{\text{in}})F\left(\coth\frac{ge_0F}{2}-1\right)\gamma\right]\times\sqrt{\det\cosh\frac{ge_0F}{2}}\left[1-\frac{i}{2}\times\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\times\left(\tanh\frac{ge_0F}{2}\right)_{\alpha\beta}\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\gamma^5\right]. \tag{33}$$

Now we are going to compare the representation (32) with the Schwinger formula [15], which has been derived in the same case of a constant field by means of the proper-time method. The Schwinger representation has the form

$$S^c(x_{\text{out}},\,x_{\text{in}})=\frac{1}{32\pi^2}\left[\gamma^\mu\left(i\frac{\partial}{\partial x_{\text{out}}^\mu}-gA_\mu(x_{\text{out}})\right)+m\right]\times\int_0^\infty de_0\left(\det\frac{\sinh\dfrac{ge_0F}{2}}{gF}\right)^{-1/2}\times\exp\left\{\frac{i}{2}\left[gx_{\text{out}}Fx_{\text{in}}-e_0m^2-(x_{\text{out}}-x_{\text{in}})\times\frac{gF}{2}\coth\frac{ge_0F}{2}(x_{\text{out}}-x_{\text{in}})-\frac{ge_0}{2}F_{\mu\nu}\sigma^{\mu\nu}\right]\right\}. \tag{34}$$

Doing the differentiation with respect to x_{out}^μ we transform formula (34) into a form which is convenient for the comparison with our representation (30):

$$S^c=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\dfrac{ge_0F}{2}}{gF}\right)^{-1/2}\Psi_S(x_{\text{out}},x_{\text{in}},e_0)\times\exp\left\{i\frac{g}{2}x_{\text{out}}Fx_{\text{in}}-\frac{i}{2}e_0m^2-i\frac{g}{4}(x_{\text{out}}-x_{\text{in}})\times F\coth\left(\frac{ge_0F}{2}\right)(x_{\text{out}}-x_{\text{in}})\right\}, \tag{35}$$

where the function Ψ_S is given by

$$\Psi_S(x_{\text{out}},x_{\text{in}},e_0)=\left[m+\frac{g}{2}(x_{\text{out}}-x_{\text{in}})F\left(\coth\frac{ge_0F}{2}-1\right)\gamma\right]\times\exp\left(-i\frac{e_0g}{4}F_{\mu\nu}\sigma^{\mu\nu}\right). \quad (36)$$

Thus one need only compare the functions Ψ and Ψ_S. They coincide, since the following formula takes place (see Appendix B), where $\omega_{\mu\nu}$ is an arbitrary antisymmetric tensor:

$$\exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)=\sqrt{\det\cosh\frac{\omega}{2}}\left[1-\frac{i}{2}\left(\tanh\frac{\omega}{2}\right)_{\mu\nu}\sigma^{\mu\nu}+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\left(\tanh\frac{\omega}{2}\right)_{\alpha\beta}\left(\tanh\frac{\omega}{2}\right)_{\mu\nu}\gamma^5\right]. \quad (37)$$

In fact, the latter formula presents a linear decomposition of a finite Lorentz transformation in the independent γ-matrix structures.

C. Propagator in a constant uniform field and a plane wave field

The four-potential

$$A_\mu^{\text{comb}}=-\tfrac{1}{2}F_{\mu\nu}x^\nu+a_\mu(nx), \quad (38)$$

where $a_\mu(\phi)$ is a vector-valued function of a real variable ϕ and n is a normalized isotropic vector $n^\mu=(1,\,\mathbf{n})$,

$$n^2=0,\quad \mathbf{n}^2=1, \quad (39)$$

produces the field

$$F_{\mu\nu}^{\text{comb}}(nx)=F_{\mu\nu}+f_{\mu\nu}(nx), \quad (40)$$

which is a superposition of the constant field $F_{\mu\nu}$ and the plane-wave field

$$f_{\mu\nu}(nx)=n_\mu a'_\nu(nx)-n_\nu a'_\mu(nx).$$

Without loss of generality we may choose a_μ to be transversal:

$$n^\mu a_\mu(\phi)=0. \quad (41)$$

The dependence of the SF $\Phi[x,e_0]$ on the trajectory $x^\mu(\tau)$ is twofold. In addition to the direct dependence [see Eq. (17)], there is an indirect one through the external field. In the case under consideration the field depends on $x_\mu(\tau)$ only through the scalar combination $nx(\tau)$. Replacing the latter by an auxiliary scalar trajectory $\phi(\tau)$ one obtains

$$\tilde{\Phi}[x,\phi,e_0]=\left[m+(2e_0)^{-1}\dot{x}^\mu\star\tilde{K}_{\mu\lambda}(2\eta^{\lambda\kappa}-ge_0\tilde{B}^{\lambda\kappa})\gamma_\kappa-\frac{ig}{4}(me_0+\dot{x}^\mu\star\tilde{K}_{\mu\lambda}\gamma^\lambda)\tilde{B}_{\kappa\nu}\sigma^{\kappa\nu}+m\frac{g^2e_0^2}{16}\tilde{B}_{\alpha\beta}\tilde{B}^{*\alpha\beta}\gamma^5\right]\times\exp\left\{-\frac{e_0}{2}\int_0^g dg'\,\mathrm{Tr}\tilde{\mathcal{R}}(g')\star\mathcal{F}^{\text{comb}}(\phi)\right\}, \quad (42)$$

where $\mathcal{F}_{\mu\nu}^{\text{comb}}(\phi|\tau-\tau')=[F_{\mu\nu}+f_{\mu\nu}(\phi(\tau))]\delta(\tau,\tau')$ and

$$\tilde{B}_{\mu\nu}=F_{\mu\lambda}^{\text{comb}}(\phi)\star\tilde{K}^\lambda{}_\nu,$$

$$\tilde{K}_{\mu\nu}=\eta_{\mu\nu}+ge_0\tilde{\mathcal{R}}_{\mu\nu}(g)\star F^{\text{comb}}(\phi), \quad (43)$$

$$\left[\eta\frac{\partial}{\partial\tau}-ge_0F^{\text{comb}}(\phi(\tau))\right]\tilde{\mathcal{R}}(q|\tau,\tau')=\eta\delta(\tau,\tau'), \quad (44)$$

$$\tilde{\mathcal{R}}(g|1,\tau)=-\tilde{\mathcal{R}}(g|0,\tau),\quad \forall\tau\in(0,1). \quad (45)$$

Obviously,

$$\tilde{\mathcal{R}}(g)|_{\phi(\tau)=nx(\tau)}=\mathcal{R}(g),\quad \tilde{K}|_{\phi(\tau)=nx(\tau)}=K,$$

$$\tilde{B}|_{\phi(\tau)=nx(\tau)}=B, \quad (46)$$

and, therefore,

$$\tilde{\Phi}[x,nx,e_0]=\Phi[x,e_0]. \quad (47)$$

Inserting the integral of a δ function,

$$\int D\phi\, D\lambda\, e^{i\lambda\star(\phi-nx)}=1,$$

into the RHS of Eq. (15) and using Eq. (47) one transforms the path integral (15) into a quasi-Gaussian one of the simple form

$$S^c(x_{\text{out}},x_{\text{in}})=\frac{i}{2}\int_0^\infty de_0\int D\phi\, D\lambda\; e^{i\lambda\star\phi}\int_{x_{\text{in}}}^{x_{\text{out}}}Dx\; M(e_0)\times\tilde{\Phi}[x,\phi,e_0]\exp\{i\tilde{I}[x,\phi,e_0]-i\lambda\star(nx)\}. \quad (48)$$

The action functional

$$\tilde{I}[x,\phi,e_0]=-\frac{1}{2e_0}\dot{x}\star\dot{x}-\frac{e_0}{2}m^2-\frac{g}{2}x\star\overline{\mathcal{F}}\star\dot{x}-ga(\phi)\star\dot{x} \quad (49)$$

[where $\overline{\mathcal{F}}(\tau,\tau')=F\delta(\tau-\tau')$] contains only linear and bilinear terms in x (and the bilinear part does not depend on the wave potential a_μ). The SF $\tilde{\Phi}[x,\phi,e_0]$ is linear in x and (following the same way of reasoning as in the case of a constant field) one finds

$$S^c=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right)^{-1/2}\int D\phi\, D\lambda\, e^{i\lambda\star(\phi-nx_q)}\times\tilde{\Phi}[x_q,\phi,e_0]e^{i\tilde{I}[x_q,\phi,e_0]}, \tag{50}$$

where x_q is the solution to the equation

$$\ddot{x}_q-ge_0F\dot{x}_q=e_0\lambda n-e_0ga'(\phi)\dot{\phi}\ , \tag{51}$$

obeying the boundary conditions

$$x_q(0)=x_{\rm in},\quad x_q(1)=x_{\rm out}. \tag{52}$$

Introducing an appropriate Green function $\mathcal{G}=\mathcal{G}(\tau,\tau')$ for the second-order operator,

$$\left[\eta\frac{\partial^2}{\partial\tau^2}-ge_0F\frac{\partial}{\partial\tau}\right]\mathcal{G}(\tau,\tau')=\eta\delta(\tau-\tau'), \tag{53}$$

$$\mathcal{G}(0,\tau)=\mathcal{G}(1,\tau)=\mathcal{G}(\tau,0)=\mathcal{G}(\tau,1)=0,\quad \forall\tau\in(0,1), \tag{54}$$

one presents this solution in the form

$$x_q=x_{\rm cl}+e_0\mathcal{G}\star[\lambda n-ga'(\phi)\dot{\phi}]. \tag{55}$$

The value of the action functional $\tilde{I}[x,\phi,e_0]$ on the solution x_q is given by

$$\begin{aligned}\tilde{I}[x_q,\phi,e_0]=&\ \overline{I[x_{\rm cl},e_0]}-ga(\phi)\star\dot{x}_{\rm cl}\\ &-\frac{e_0}{2}[ga'(\phi)\dot{\phi}-\lambda n]\star\mathcal{G}\star[ga'(\phi)\dot{\phi}-\lambda n]\\ &+\lambda n\star(x_q-x_{\rm cl}),\end{aligned} \tag{56}$$

where

$$\overline{I[x,e_0]}=-\frac{1}{2e_0}\dot{x}\star\dot{x}-\frac{e_0}{2}m^2-\frac{g}{2}x\star\overline{\mathcal{F}\star}\dot{x} \tag{57}$$

is the action in a uniform constant field F.

The functional integral over λ in Eq. (48) is a quasi-Gaussian one of a simple form [let us recall that x_q is linear in λ; see Eq. (55)] and the integration can be done explicitly. The result is a formula for the propagator in which the only functional integration is over the scalar trajectory $\phi(\tau)$. However, the latter integration is hardly to be performed explicitly in the general case [for arbitrary $a_\mu(\phi)$]. Nevertheless, there exists a specific combination [5] for which the integration can be done and explicit formulas for the propagator to be derived. The latter are comparable with the corresponding Schwinger-type formulas [10], which are also explicit in this case.

Namely, let us choose the wave vector n to coincide with a real eigenvector of the matrix F (see Appendix A):

$$F_{\mu\nu}n^\nu=-\mathcal{E}n_\mu,\quad n^2=0,\quad \mathbf{n}^2=1. \tag{58}$$

In this case $nx_q=nx_{\rm cl}$, and, moreover, the action functional (49) is ''on-shell'' invariant with respect to longitudinal shifts

$$x_q(\tau)\to x_q(\tau)+\alpha(\tau)n,\quad \alpha(0)=\alpha(1)=0, \tag{59}$$

by virtue of Eq. (58) and the transversality (41) of the wave potential a_μ. Then $\tilde{I}[x_q,\phi,e_0]$ does not depend on λ,

$$\tilde{I}[x_q,\phi,e_0]=\tilde{I}[x_{\rm tr},\phi,e_0], \tag{60}$$

where

$$x_{\rm tr}=x_{\rm cl}-ge_0\mathcal{G}\star[a'(\phi)\dot{\phi}] \tag{61}$$

is a solution to the equation

$$\ddot{x}_{\rm tr}-ge_0F\dot{x}_{\rm tr}=-ge_0a'(\phi)\dot{\phi}\ , \tag{62}$$

obeying the boundary conditions (52). However, the SF $\tilde{\Phi}[x_q,\phi,e_0]$ does not show this invariance and, therefore, is λ dependent. Presenting x_q as a sum $x_q=x_{\rm tr}+e_0\mathcal{G}n\star\lambda$, and substituting it into the expansion of the SF in the antisymmetrized products of γ matrices,

$$\begin{aligned}\tilde{\Phi}[x,\phi,e_0]=&\Bigg[e_0^{-1}\dot{x}^\mu\star\tilde{K}_{\mu\nu}\left(\gamma^\nu+\frac{ge_0}{4}\tilde{B}_{\alpha\beta}\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right)\\ &+m\left(1+\frac{ge_0}{4}\tilde{B}_{\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}\right.\\ &\left.+\frac{g^2e_0^2}{32}\tilde{B}_{\alpha\beta}\tilde{B}_{\mu\nu}\gamma^{[\alpha}\gamma^\beta\gamma^\mu\gamma^{\nu]}\right)\Bigg]\Lambda[\phi,e_0],\end{aligned} \tag{63}$$

$$\Lambda[\phi,e_0]=\exp\left\{-\frac{e_0}{2}\int_0^g dg'\,{\rm Tr}\tilde{\mathcal{R}}(g')\star\mathcal{F}^{\rm comb}(\phi)\right\}. \tag{64}$$

we obtain

$$\tilde{\Phi}[x_q,\phi,e_0]=\tilde{\Phi}[x_{\rm tr},\phi,e_0]+\lambda\star l[\phi,e_0], \tag{65}$$

$$l[\phi,e_0]=n^\kappa\mathcal{G}^{(r)}_{\kappa\mu}\star\tilde{K}^\mu_{\ \nu}\left(\gamma^\nu+\frac{ge_0}{4}\tilde{B}_{\alpha\beta}\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right)\Lambda[\phi,e_0], \tag{66}$$

$$\mathcal{G}^{(r)}(\tau,\tau')=\frac{\partial}{\partial\tau'}\mathcal{G}(\tau,\tau'). \tag{67}$$

It turns out that $l[\phi,e_0]$ does not depend on ϕ. First, expanding $\tilde{\mathcal{R}}$ in powers of f and using Eqs. (58) and (41) one derives that $\Lambda[\phi,e_0]$ coincides with the expression

$$\overline{\Lambda(e_0)}=\exp\left\{\frac{e_0}{2}\int_0^g dg'\,\mathrm{Tr}\overline{\mathcal{R}(g)}\star\overline{\mathcal{F}}\right\}. \quad (68)$$

Second,[1]

$$n_\mu\tilde{K}^\mu{}_\nu=n_\mu\overline{K^\mu{}_\nu}=\tfrac{1}{2}(n\overline{Kn})n_\nu. \quad (69)$$

Indeed, using the definitions (46) one finds that $\tilde{K}$ satisfies the equation

$$\left[\frac{\partial}{\partial\tau}-ge_0[F+f(\phi(t))]\right]\tilde{K}(\tau)=0 \quad (70)$$

and the boundary conditions

$$\tilde{K}(0)+\tilde{K}(1)=2\eta. \quad (71)$$

Multiplying Eq. (70) by n and using the properties (58) and (41) we find

$$\left(\frac{\partial}{\partial\tau}-ge_0\mathcal{E}\right)n\tilde{K}=0,\quad n\tilde{K}(0)+n\tilde{K}(1)=2n. \quad (72)$$

At the same time $n\overline{K}$ obeys Eq. (72). Therefore, $n\tilde{K}$ and $n\overline{K}$ coincide. Then using Eq. (41) and the properties of $n,\overline{n}$ (see Appendix A) one gets Eq. (69). Third, using the same properties of the electromagnetic field one can derive

$$\tilde{B}_{\alpha\beta}=\overline{B_{\alpha\beta}}+n_\alpha b_\beta-b_\alpha n_\beta, \quad (73)$$

where b_α depends on ϕ and $\overline{B}$ is given by Eq. (18). Substituting Eq. (73) into Eq. (66) and using Eq. (58) one finds

$$l[\phi,e_0]=\frac{1}{4}(n\overline{\mathcal{G}^{(r)}n})\star(n\overline{Kn})\left[n_\nu\gamma^\nu+\frac{ge_0}{4}n_\nu(\overline{B_{\alpha\beta}}+n_\alpha b_\beta-b_\alpha n_\beta)\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right]\overline{\Lambda(e_0)},$$

and the contribution of the ϕ-dependent terms vanishes by virtue of the complete antisymmetry of $\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}$. Therefore $l[\phi,e_0]$ can be replaced by

$$\overline{l(e_0)}=n^\kappa\mathcal{G}^{(r)}_{\kappa\mu}\star\overline{K^\mu{}_\nu}\left(\gamma^\nu+\frac{ge_0}{4}\overline{B_{\alpha\beta}}\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right)\overline{\Lambda(e_0)}. \quad (74)$$

Substituting Eq. (74) into Eq. (65) and then into Eq. (50), using Eq. (60) and

$$\lambda(\tau)e^{i\lambda\star\phi}=-i\frac{\delta}{\delta\phi(\tau)}e^{i\lambda\star\phi},$$

and integrating by parts we find

$$S^c=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right)^{-1/2}\int D\phi\,D\lambda\,e^{i\lambda\star(\phi-nx_{\mathrm{cl}})}$$
$$\times\left[\tilde{\Phi}[x_{\mathrm{tr}},\phi,e_0]-\left(\frac{\delta}{\delta\phi}\tilde{I}[x_{\mathrm{tr}},\phi,e_0]\right)\star\overline{l(e_0)}\right]$$
$$\times\exp\{i\tilde{I}[x_{\mathrm{tr}},\phi,e_0]\}. \quad (75)$$

Inserting the derivative

$$\frac{\delta}{\delta\phi}\tilde{I}[x_{\mathrm{tr}},\phi,e_0]=-a'(\phi)\dot{x}_{\mathrm{tr}},$$

and using Eqs. (73) and (66) we transform Eq. (75) into the form

$$S^c=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right)^{-1/2}\int D\phi\,D\lambda\,e^{i\lambda\star(\phi-nx_{\mathrm{cl}})}$$
$$\times\tilde{\Phi}[\tilde{x},\phi,e_0]\exp\{i\tilde{I}[x_{\mathrm{tr}},\phi,e_0]\}, \quad (76)$$

where

$$\tilde{x}^\mu=x^\mu_{\mathrm{tr}}+ge_0n^\mu a'{}_\kappa(\phi)\dot{x}^\kappa_{\mathrm{tr}}.$$

One can straightforwardly check that $\tilde{x}$ satisfies the equation

$$\ddot{\tilde{x}}_\mu-ge_0[F_{\mu\nu}+n_\mu a'_\nu(\phi)]\dot{\tilde{x}}^\nu=-ge_0a'_\mu(\phi)\dot{\phi} \quad (77)$$

and the boundary conditions (52). The trajectory x_{tr} in the action $\tilde{I}$ can be replaced by $\tilde{x}$ due to the invariance of the action $\tilde{I}[x,\phi,e_0]$ under the longitudinal shifts (59). The integration over λ and ϕ is straightforward now. One needs only to take into account that $\tilde{x}|_{\phi=nx_{\mathrm{cl}}}\equiv x_{\mathrm{comb}}$ is the solution [subjected to the boundary conditions (52)] to the equation of motion

$$\ddot{x}^\mu_{\mathrm{comb}}-ge_0[F+f(nx_{\mathrm{comb}})]^\mu{}_\nu\dot{x}^\nu_{\mathrm{comb}}=0. \quad (78)$$

Indeed, Eq. (77) turns out to be equivalent to Eq. (78) when $\phi=nx_{\mathrm{cl}}$ and the relation $nx_{\mathrm{comb}}=nx_{\mathrm{cl}}$ is taken into account. Therefore,

$$\tilde{\Phi}[\tilde{x},\phi,e_0]|_{\phi=nx_{\mathrm{cl}}}=\Phi[x_{\mathrm{comb}},e_0],$$
$$\tilde{I}[\tilde{x},\phi,e_0]|_{\phi=nx_{\mathrm{cl}}}=I[x_{\mathrm{comb}},e_0]. \quad (79)$$

Finally, we get

$$S^c=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right)^{-1/2}$$
$$\times\Phi[x_{\mathrm{comb}},e_0]e^{I[x_{\mathrm{comb}},e_0]}, \quad (80)$$

where

[1]In this section we denote by $\tilde{\mathcal{R}}(g)$, $\overline{K},$ and $\overline{B}$ the quantities given by Eq. (18), i.e., corresponding to the case of a constant uniform field F.

$$\Phi[x_{\rm comb},e_0]=\left[e_0^{-1}\dot{x}^\mu_{\rm comb}\star K_{\mu\nu}\left(\gamma^\nu+\frac{ge_0}{4}B_{\alpha\beta}\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right)+m\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}+\frac{g^2e_0^2}{32}B_{\alpha\beta}B_{\mu\nu}\gamma^{[\alpha}\gamma^\beta\gamma^\mu\gamma^{\nu]}\right)\right]\overline{\Lambda(e_0)}. \quad (81)$$

The vector $\dot{x}_{\rm comb}$ satisfies Eq. (78) and can be presented as

$$\dot{x}_{\rm comb}(\tau)=T_A\exp\left\{-ge_0\int_\tau^1 F^{\rm comb}(nx_{\rm cl}(\tau))d\tau\right\}\dot{x}(1), \quad (82)$$

where T_A denotes the antichronological product. On the other hand, the tensor trajectory $K(\tau)$ satisfies Eq. (70) where one has to replace ϕ by $nx_{\rm cl}$. Therefore

$$K(\tau)=T_A\exp\left\{-ge_0\int_\tau^1 F^{\rm comb}(nx_{\rm cl})d\tau\right\}K(1), \quad (83)$$

and[2]

$$\dot{x}_{\rm comb}\star K=\dot{x}_{\rm comb}(1)K(1). \quad (84)$$

Substituting Eq. (84) into Eq. (81) and taking into account the relation $B_{[\alpha\beta}B_{\mu\nu]}=\overline{B}_{[\alpha\beta}\overline{B}_{\mu\nu]}$ we find

$$\Phi[x_{\rm comb},e_0]=\left[e_0^{-1}\dot{x}^\mu_{\rm comb}(1)K_{\mu\nu}(1)\times\left(\gamma^\nu+\frac{ge_0}{4}B_{\alpha\beta}\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right)+m\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}+\frac{g^2e_0^2}{32}\overline{B}_{\alpha\beta}\overline{B}_{\mu\nu}\gamma^{[\alpha}\gamma^\beta\gamma^\mu\gamma^{\nu]}\right)\right]\overline{\Lambda(e_0)}. \quad (85)$$

A representation for the propagator in this specific field combination [characterized by the relation (58)] was given in terms of the proper-time integral only in [12,10]. Another more complicated representation has been obtained before in [5]. In our notation the representation [12,10] can be written as

$$S^c(x_{\rm out},x_{\rm in})=\left[\gamma^\mu\left(i\frac{\partial}{\partial x^\mu_{\rm out}}-gA^{\rm comb}_\mu(nx_{\rm out})\right)+m\right]\frac{1}{32\pi^2}\times\int_0^\infty de_0\left(\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right)^{-1/2}\times e^{iI[x_{\rm comb},e_0]}\Delta[nx_{\rm cl},e_0], \quad (86)$$

where

$$\Delta[nx_{\rm cl},e_0]=T\exp\left\{-\frac{ige_0}{4}\int_0^1 d\tau[F+f(nx_{\rm cl}(\tau))]_{\mu\nu}\sigma^{\mu\nu}\right\} \quad (87)$$

$$=\exp\left\{-\frac{ige_0}{4}F_{\mu\nu}\sigma^{\mu\nu}\right\}-\frac{ige_0}{4}\int_0^1 d\tau[e^{ge_0F(1-2\tau)/2}f(nx_{\rm cl}(\tau))\times e^{-ge_0F(1-2\tau)/2}]_{\mu\nu}\sigma^{\mu\nu}, \quad (88)$$

and T denotes chronological product. Taking the derivative in Eq. (86) one can use the relation

$$\frac{\partial}{\partial x^\mu_{\rm out}}I[x_{\rm comb},e_0]=-p_\mu(1), \quad (89)$$

where

$$p_\mu(\tau)=-\frac{\delta}{\delta\dot{x}^\mu}I[x,e_0]\big|_{x=x_{\rm comb}}$$

is the on-shell momentum, in particular,

$$p(1)=e_0^{-1}\dot{x}_{\rm comb}(1)+ge_0A^{\rm comb}(nx_{\rm out}). \quad (90)$$

On the other hand,

$$\gamma^\mu\frac{\partial}{\partial x^\mu_{\rm out}}\Delta[nx_{\rm cl},e_0]=0. \quad (91)$$

Indeed, one gets from Eq. (31), with the aid of Eq. (58),

$$\gamma^\mu\frac{\partial}{\partial x^\mu_{\rm out}}[nx_{\rm cl}(\tau)]=\frac{e^{ge_0\mathcal{E}\tau}-1}{e^{ge_0\mathcal{E}}-1}n_\mu\gamma^\mu. \quad (92)$$

Then, using the representation (88) for $\Delta[nx_{\rm cl},e_0]$, Eqs. (39) and (41), and the properties of the γ matrices, one easily derives Eq. (91). Differentiating in Eq. (86), one obtains, with the aid of Eqs. (88), (89), and (91),

$$S^c(x_{\rm out},x_{\rm in})=\frac{1}{32\pi^2}\int_0^\infty de_0\left(\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right)^{-1/2}\times\Psi^{\rm comb}_S(x_{\rm out},x_{\rm in},e_0)e^{iI[x_{\rm comb},e_0]}, \quad (93)$$

where

$$\Psi^{\rm comb}_S(x_{\rm out},x_{\rm in},e_0)=[e_0^{-1}\dot{x}^\mu_{\rm comb}(1)\gamma_\mu+m]\Delta[nx_{\rm cl},e_0].$$

Using the identities (B8) and (B9) one can verify that

$$\Psi^{\rm comb}_S(x_{\rm out},x_{\rm in},e_0)=\Phi[x_{\rm comb},e_0].$$

Thus the representations (80) and (86) are equivalent.

[2]The operator $T_A\exp\{-ge_0\int_\tau^1F^{\rm comb}(nx_{\rm cl})d\tau\}$ preserves the scalar product due to the antisymmetry of the stress tensor.

III. SPIN FACTOR IN 2+1 DIMENSIONS

A. Derivation of the spin factor

In 2+1 dimensions the equation for the Dirac propagator has the form

$$\{\gamma^{\mu}[i\partial_{\mu}-gA_{\mu}(x)]-m\}S^{c}(x,y)=-\delta^{3}(x-y), \quad (94)$$

where γ matrices in 2+1 dimensions can be taken, for example, in the form $\gamma^{0}=\sigma^{3}$, $\gamma^{1}=i\sigma^{2}$, $\gamma^{2}=-i\sigma^{1}$, $[\gamma^{\mu},\gamma^{\nu}]_{+}=2\eta^{\mu\nu}$, $\eta^{\mu\nu}=\mathrm{diag}(1,-1,-1)$, and $\mu,\nu=0,1,2$. In this particular case they obey the relations

$$[\gamma^{\mu},\gamma^{\nu}]=-2i\epsilon^{\mu\nu\lambda}\gamma_{\lambda}, \quad \gamma^{\mu}=\frac{i}{2}\epsilon^{\mu\nu\lambda}\gamma_{\nu}\gamma_{\lambda}. \quad (95)$$

In Ref. [13] a path integral representation for the Dirac propagator was obtained in arbitrary odd dimensions. In particular, in the case under consideration this representation reads

$$S^{c}=\frac{1}{2}\exp\left(i\gamma^{\mu}\frac{\partial_{\ell}}{\partial\theta^{\mu}}\right)\int_{0}^{\infty}de_{0}\int d\chi_{0}\int_{e_{0}}M(e)De\int_{\chi_{0}}D\chi\int_{x_{\rm in}}^{x_{\rm out}}Dx\int D\pi\int D\nu\int_{\psi(0)+\psi(1)=\theta}\mathcal{D}\psi$$
$$\times\exp\left\{i\int_{0}^{1}\left[-\frac{\dot{x}^{2}}{2e}-\frac{e}{2}m^{2}-g\dot{x}_{\mu}A^{\mu}+iegF_{\mu\nu}\psi^{\mu}\psi^{\nu}\right.\right.$$
$$\left.\left.\left.+\chi\left(\frac{2i}{e}\epsilon_{\mu\nu\lambda}\dot{x}^{\mu}\psi^{\nu}\psi^{\lambda}-m\right)-i\psi_{\mu}\dot{\psi}^{\nu}+\pi\dot{e}+\nu\dot{\chi}\right]d\tau+\psi_{\mu}(1)\psi^{\mu}(0)\right\}\right|_{\theta=0}, \quad (96)$$

where $x(\tau)$, $p(\tau)$, $e(\tau)$, and $\pi(\tau)$ are even and $\psi(\tau)$, $\chi_{1}(\tau)$, $\chi_{2}(\tau)$, $\nu_{1}(\tau)$, and $\nu_{2}(\tau)$ are odd trajectories, obeying the boundary conditions $x(0)=x_{\rm in}$, $x(1)=x_{\rm out}$, $e(0)=e_{0}$, $\chi(0)=\chi_{0}$, and $\psi(0)+\psi(1)=\theta$, and the notation used is

$$\chi=\chi_{1}\chi_{2}, \quad \nu\dot{\chi}=\nu_{1}\dot{\chi}_{1}+\nu_{2}\dot{\chi}_{2}, \quad d\chi=d\chi_{1}\,d\chi_{2},$$
$$D\chi=D\chi_{1}\,D\chi_{2}, \quad D\nu=D\nu_{1}\,D\nu_{2}.$$

The measure $M(e)$ is defined by Eq. (3) in the corresponding dimensions,[3] and

$$\mathcal{D}\psi=D\psi\left[\int_{\psi(0)+\psi(1)=0}D\psi\exp\left\{\int_{0}^{1}\psi_{\mu}\dot{\psi}^{\mu}d\tau\right\}\right]^{-1}.$$

Integrating over the Grassmann variables in the same way as in the case of 3+1 dimensions we get

$$S^{c}(x_{\rm out},x_{\rm in})=\frac{i}{2}\int_{0}^{\infty}de_{0}M(e_{0})\int_{x_{\rm in}}^{x_{\rm out}}Dx\;\Phi[x,e_{0}]\exp\{iI[x,e_{0}]\}, \quad (97)$$

where

$$\Phi[x,e_{0}]=\left[\left(m+\frac{i}{e_{0}}\int_{0}^{1}d\tau\;\epsilon_{\mu\nu\lambda}\dot{x}^{\mu}(\tau)\mathcal{R}^{\nu\lambda}(g|\tau,\tau)\right)\right.$$
$$\times\left(1+\frac{ge_{0}}{4}B_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta}\right)$$
$$\left.+\frac{i}{2e_{0}}\int_{0}^{1}d\tau\;\epsilon_{\mu\nu\lambda}\dot{x}^{\mu}(\tau)K^{\nu}{}_{\alpha}(\tau)K^{\lambda}{}_{\beta}(\tau)\gamma^{\alpha}\gamma^{\beta}\right]$$
$$\times\exp\left\{-\frac{e_{0}}{2}\int_{0}^{g}dg'\,\mathrm{Tr}\mathcal{R}(g')\star\mathcal{F}\right\} \quad (98)$$

is the SF and $I[x,e_{0}]$, $\mathcal{R}(g)\equiv\mathcal{R}(g|\tau,\tau')$, and $K\equiv K(\tau)$, B, $\mathcal{F}$ are defined by Eqs. (16), (13), and (8), respectively.

Because of the relations (95), one can also present the SF in the form

$$\Phi[x,e_{0}]=\left\{m+\frac{i}{e_{0}}\dot{x}\star r(g)+\left[\left(-i\frac{ge_{0}}{4}m+\frac{g}{4}\dot{x}\star r(g)\right)u_{\alpha}\right.\right.$$
$$\left.\left.+\frac{1}{2e_{0}}(\dot{x}\star T)_{\alpha}\right]\gamma^{\alpha}\right\}\exp\left\{-\frac{e_{0}}{2}\int_{0}^{g}dg'\,\mathrm{Tr}\mathcal{R}(g')\star\mathcal{F}\right\}, \quad (99)$$

where

$$r_{\mu}(g)\equiv r_{\mu}(g|\tau)=\epsilon_{\mu\nu\lambda}\mathcal{R}^{\nu\lambda}(g|\tau,\tau), \quad u^{\mu}=\epsilon^{\mu\alpha\beta}B_{\alpha\beta},$$
$$T_{\mu}{}^{\rho}=\epsilon_{\mu\nu\lambda}\epsilon^{\rho\alpha\beta}K^{\nu}{}_{\alpha}K^{\lambda}{}_{\beta}.$$

B. Dirac propagator in a constant uniform field in 2+1 dimensions

In the case of a constant uniform field $F_{\mu\nu}=$ const one can calculate the propagator explicitly integrating over the bosonic trajectories. Following the same method as in Sec. II and taking into account that, in 2+1 dimensions,

$$\frac{i}{2}M(e_{0})\int_{0}^{0}Dx\;\exp\left\{-\frac{i}{2e_{0}}\dot{x}\star\dot{x}\right\}=\frac{e^{i\pi/4}}{2(2\pi e_{0})^{3/2}},$$

one gets for the propagator (97),

[3] We will refer to some formulas from the previous sections without specifying that the number of dimensions is 2+1 now.

$$S^c=\frac{e^{i\pi/4}}{2(4\pi)^{3/2}}\int_0^\infty de_0\left(\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right)^{-1/2}\times e^{iI[x_{\rm cl},e_0]}\Phi[x_{\rm cl},e_0],\qquad(100)$$

where $x_{\rm cl}$, $\mathcal{R}(g)$, K,B are given by Eqs. (31) and (18).

The antisymmetric matrices $F_{\mu\nu}$ can be classified by the value of the invariant φ (see Appendix A). In the case $\varphi^2>0$ one can find a Lorentz frame in which the magnetic field vanishes. On the other hand, $\varphi^2<0$ implies that the electric field vanishes in an appropriate Lorentz frame. The case $\varphi^2=0$, $F\neq 0$ corresponds to nonvanishing electric and magnetic fields of ''equal magnitude'' (and this property is Lorentz invariant). We will consider the case $\varphi^2\neq 0$. The case $\varphi^2=0$ can be easily treated, e.g., taking the limit $\varphi\to 0$.

The integration over τ in expression (98) for $\Phi[x_{\rm cl},e_0]$ can be done. Indeed, from Eqs. (31) and (18) one derives

$$\dot{x}_{\rm cl}(\tau)=e^{ge_0F(\tau-1)}\dot{x}_{\rm cl}(1),\quad K(\tau)=e^{ge_0F(\tau-1)}K(1),$$

$$\mathcal{R}_{[\mu\nu]}(g|\tau,\tau)=-\frac{ge_0}{4}(e^{ge_0F(\tau-1)})_\mu^{\ \alpha}(e^{ge_0F(\tau-1)})_\nu^{\ \beta}B_{\alpha\beta},$$

where the operator $\exp\{ge_0F(\tau-1)\}$ respects the scalar product and $\det\exp\{ge_0F(\tau-1)\}=1$. Finally, calculating the determinants involved by means of Eq. (A7), one gets

$$S^c=\sqrt{\frac{i}{16(2\pi)^3}}\int_0^\infty\frac{de_0}{\sqrt{e_0}}\frac{g\varphi}{\sinh\frac{ge_0\varphi}{2}}e^{iI[x_{\rm cl},e_0]}\Phi[x_{\rm cl},e_0],$$

$$\Phi[x_{\rm cl},e_0]=\left\{m\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)+\frac{i}{2e_0}\dot{x}^\mu(1)\times\epsilon_{\mu\nu\lambda}\left[K^\nu_{\ \alpha}K^\lambda_{\ \beta}\gamma^\alpha\gamma^\beta-\frac{ge_0}{2}B^{\nu\lambda}\times\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)\right]\right\}\cosh\frac{ge_0\varphi}{2}.\qquad(101)$$

On the other hand, one can obtain a representation for the propagator using the Schwinger proper-time method (we do not present the calculations here). Such a representation has the form

$$S^c(x_{\rm out},x_{\rm in})=\left[\gamma^\mu\left(i\frac{\partial}{\partial x^\mu_{\rm out}}-gA_\mu(x_{\rm out})\right)+m\right]\times\sqrt{\frac{i}{16(2\pi)^3}}\int_0^\infty\frac{de_0}{\sqrt{e_0}}\frac{g\varphi}{\sinh\frac{ge_0\varphi}{2}}\times e^{iI[x_{\rm cl},e_0]}e^{ge_0F_{\alpha\beta}\gamma^\alpha\gamma^\beta/4}.\qquad(102)$$

To compare both representations we take the derivative in Eq. (102) and use

$$\frac{\partial}{\partial x^\mu_{\rm out}}I[x_{\rm cl},e_0]=-e_0^{-1}(\dot{x}_{\rm cl})_\mu(1)-gA_\mu(x_{\rm out}).$$

Then one obtains

$$S^c(x_{\rm out},x_{\rm in})=\frac{e^{i\pi/4}}{4(2\pi)^{3/2}}\int_0^\infty\frac{de_0}{\sqrt{e_0}}\frac{g\varphi}{\sinh\frac{ge_0\varphi}{2}}\times\Psi_S(x_{\rm out},x_{\rm in},e_0)e^{iI[x_{\rm cl},e_0]},\qquad(103)$$

where

$$\Psi_S(x_{\rm out},x_{\rm in},e_0)=[e_0^{-1}\gamma_\mu\dot{x}^\mu_{\rm cl}(1)+m]e^{ge_0F_{\alpha\beta}\gamma^\alpha\gamma^\beta/4}.\qquad(104)$$

Comparing Eqs. (101) and (103) using the identities (see Appendix B)

$$\exp\left\{\frac{ge_0}{4}F_{\alpha\beta}\gamma^\alpha\gamma^\beta\right\}=\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)\cosh\frac{ge_0\varphi}{2},\qquad(105)$$

$$\gamma_\mu\exp\left\{\frac{ge_0}{4}F_{\alpha\beta}\gamma^\alpha\gamma^\beta\right\}=\frac{i}{2}\epsilon_{\mu\nu\lambda}\left[K^\nu_{\ \alpha}(1)\,K^\lambda_{\ \beta}(1)\gamma^\alpha\gamma^\beta-\frac{ge_0}{2}B^{\nu\lambda}\times\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)\right]\cosh\frac{ge_0\varphi}{2},\qquad(106)$$

one can verify that both representations coincide.

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APPENDIX A: SOME PROPERTIES OF ANTISYMMETRIC TENSORS

1. Antisymmetric tensors in 3+1 dimensions

The antisymmetric matrix $F_{\mu\nu}$ has, in the case of nonvanishing invariants, four isotropic eigenvectors, namely,

$$F_{\mu\nu}n^\nu=-\mathcal{E}n_\mu,\quad F_{\mu\nu}\overline{n^\nu}=\mathcal{E}\overline{n_\mu},\quad F_{\mu\nu}m^\nu=i\mathcal{H}m_\mu,$$
$$F_{\mu\nu}\overline{m^\nu}=-i\mathcal{H}\overline{m_\mu},\qquad(A1)$$

where $\mathcal{E}$ and $\mathcal{H}$ are real numbers. The eigenvectors are supposed to be normalized,

$$\overline{n^\mu}n_\mu=-\overline{m^\mu}m_\mu=2,\qquad(A2)$$

while all the other scalar products vanish. The matrix F can be presented in the form

$$F_{\mu\nu}=\frac{\mathcal{E}}{2}(\overline{n_\mu}n_\nu-n_\mu\overline{n_\nu})+\frac{i\mathcal{H}}{2}(\overline{m_\mu}m_\nu-m_\mu\overline{m_\nu}),\qquad(A3)$$

and, therefore, F^2 has the spectral decomposition

$$F^2=\mathcal{E}^2 P_{\mathcal{E}}-\mathcal{H}^2 P_{\mathcal{H}}, \qquad (A4)$$

where $P_{\mathcal{E}}$ and $P_{\mathcal{H}}$ are orthogonal projection operators onto some two-dimensional subspaces,

$$P_{\mathcal{E}}^2=P_{\mathcal{E}}, \quad P_{\mathcal{H}}^2=P_{\mathcal{H}}, \quad P_{\mathcal{E}}P_{\mathcal{H}}=P_{\mathcal{H}}P_{\mathcal{E}}=0, \quad P_{\mathcal{E}}+P_{\mathcal{H}}=\eta,$$
$$\mathrm{tr}P_{\mathcal{E}}=\mathrm{tr}P_{\mathcal{H}}=2. \qquad (A5)$$

2. Antisymmetric tensors in 2+1 dimensions

An antisymmetric matrix has eigenvalues $0, \varphi, -\varphi$, where[4] $\varphi^2=\mathrm{tr}\; F^2$ is the invariant of the tensor. In the case of nonvanishing φ there exist three eigenvectors of F, and F^2 is proportional to a projection operator P onto some two-dimensional subspace:

$$F^2=\varphi^2 P, \quad P^2=P, \quad \mathrm{tr}P=2, \quad PF=FP=F. \quad (A6)$$

Then, for an even function h,

$$h(F)=h(0)(1-P)+h(\varphi)P, \qquad (A7)$$

while, for an odd one,

$$h(F)=\frac{F}{\varphi}h(\varphi). \qquad (A8)$$

The case of vanishing φ (and $F\neq 0$) corresponds to a nilpotent matrix, $F^3=0$.

APPENDIX B: SOME IDENTITIES INVOLVING γ MATRICES

1. γ-matrix structure of Lorentz transformation in the spinor representation

Let us denote by $M(\omega)$ the expression in the RHS of Eq. (37). We are going to check that the matrix-valued function $M(\lambda\omega)$ of a real parameter λ satisfies the differential equation

$$\frac{d}{d\lambda}M(\lambda\omega)=-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}M(\lambda\omega) \qquad (B1)$$

and the initial condition

$$M(0)=1. \qquad (B2)$$

The latter is trivial, and so let us concentrate on the proof of Eq. (B1).

Using the identity

$$4\epsilon^{\alpha\beta\mu\nu}(ST)_{\alpha\beta}T_{\mu\nu}=\epsilon^{\alpha\beta\mu\nu}T_{\alpha\beta}T_{\mu\nu}\;\mathrm{tr}S, \qquad (B3)$$

where S is an arbitrary tensor and T is antisymmetric, one can put the derivative $(d/d\lambda)M(\lambda\omega)$ into the form

$$\frac{d}{d\lambda}M(\lambda\omega)=\left(\mathrm{detcosh}\frac{\lambda\omega}{2}\right)^{1/2}\left\{\frac{1}{4}\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\right.$$
$$-\frac{i}{8}\left[\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\tanh\frac{\lambda\omega}{2}\right.$$
$$\left.+2\omega\left(\eta-\tanh^2\frac{\lambda\omega}{2}\right)\right]_{\mu\nu}\sigma^{\mu\nu}$$
$$\left.+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\omega_{\alpha\beta}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\gamma^5\right\}. \qquad (B4)$$

On the other hand, with the aid of the identities

$$\sigma^{\alpha\beta}\sigma^{\mu\nu}=\eta^{\alpha\mu}\eta^{\beta\nu}-\eta^{\alpha\nu}\eta^{\beta\mu}-i(\eta^{\alpha\mu}\sigma^{\beta\nu}+\eta^{\beta\nu}\sigma^{\alpha\mu}$$
$$-\eta^{\alpha\nu}\sigma^{\beta\mu}-\eta^{\beta\mu}\sigma^{\alpha\nu})-\epsilon^{\alpha\beta\mu\nu}\gamma^5,$$
$$\sigma^{\kappa\rho}\gamma^5=\tfrac{1}{2}\epsilon^{\kappa\rho\tau\sigma}\sigma_{\tau\sigma} \qquad (B5)$$

and using the antisymmetry property $\omega_{\mu\nu}=-\omega_{\nu\mu}$ one finds

$$-\frac{i}{4}\omega\sigma^{\mu\nu}M(\lambda\omega)$$
$$=\left(\mathrm{detcosh}\frac{\lambda\omega}{2}\right)^{1/2}\left[\frac{1}{4}\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)-\frac{i}{64}\epsilon^{\kappa\rho\mu\nu}\right.$$
$$\times\left(\tanh\frac{\lambda\omega}{2}\right)_{\kappa\rho}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\epsilon_{\alpha\beta\lambda\tau}\omega^{\alpha\beta}\sigma^{\lambda\tau}-\frac{i}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}$$
$$\left.+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\omega_{\alpha\beta}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\gamma^5\right]. \qquad (B6)$$

Then, using the identity

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\beta_1\beta_2\beta_3\beta_4}$$
$$=-\sum_P(-1)^{[P]}\delta^{\beta_1}_{P(\alpha_1)}\delta^{\beta_2}_{P(\alpha_2)}\delta^{\beta_3}_{P(\alpha_3)}\delta^{\beta_4}_{P(\alpha_4)},$$

we get

$$-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}M(\lambda\omega)$$
$$=\left(\mathrm{detcosh}\frac{\lambda\omega}{2}\right)^{1/2}\left\{\frac{1}{4}\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\right.$$
$$-\frac{i}{8}\left[\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\tanh\frac{\lambda\omega}{2}+2\omega\left(\eta-\tanh^2\frac{\lambda\omega}{2}\right)\right]_{\mu\nu}$$
$$\left.\times\sigma^{\mu\nu}+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\omega_{\alpha\beta}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\gamma^5\right\}. \qquad (B7)$$

The RHS's of Eqs. (B4) and (B7) coincide. Therefore $M(\lambda\omega)$ obeys Eq. (B1). This completes the proof of formula (37).

2. Decompositions of some functions on γ matrices

Let us consider the T exponent (87) where F is a uniform constant field, f is a plane-wave field, and Eq. (56) takes place. We are going to prove the identities

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[4]We recall that, in accordance with our notation, $\mathrm{tr}\;F^2=(F^2)^\mu{}_\mu=F^\mu{}_\nu F^\nu{}_\mu$.

$$\Delta[nx_{\rm cl},e_0]=\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}+\frac{g^2e_0^2}{16}\overline{B}_{\alpha\beta}\overline{B}^{*\alpha\beta}\gamma^5\right)\overline{\Lambda(e_0)}, \tag{B8}$$

$$\gamma^\mu\Delta[nx_{\rm cl},e_0]=K^\mu_{\ \nu}(1)\left(\gamma^\nu+\frac{ge_0}{4}B_{\alpha\beta}\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right)\overline{\Lambda(e_0)}, \tag{B9}$$

where B and K are defined by Eq. (13) for the combination as it was described while $\overline{B}$, corresponding to the case of constant uniform field, is given by Eq. (18), and

$$\overline{\Lambda(e_0)}=\exp\left\{-\frac{e_0}{2}\int_0^g dg'\,{\rm Tr}\overline{\mathcal{R}(g')}\star\overline{\mathcal{F}}\right\}=\cosh\frac{ge_0\mathcal{E}}{2}\cos\frac{ge_0\mathcal{H}}{2}.$$

Presenting the T exponent in the form (88) and using Eq. (37) one obtains

$$\Delta[nx_{\rm cl},e_0]=\left(1+\frac{ge_0}{4}B_{\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}+\frac{g^2e_0^2}{16}\overline{B}_{\alpha\beta}\overline{B}^{*\alpha\beta}\gamma^5\right)\times\overline{\Lambda(e_0)}+Q_{\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}, \tag{B10}$$

where

$$Q=\frac{ge_0}{4}\overline{\Lambda(e_0)}(\overline{B}-B)+\frac{1}{4}C, \tag{B11}$$

$$C=ge_0\int_0^1 d\tau\, e^{ge_0F(1-2\tau)/2}f(nx_{\rm cl}(\tau))e^{-ge_0F(1-2\tau)/2}. \tag{B12}$$

In order to find a convenient representation for B we present K, which is a solution, obeying Eq. (71), to Eq. (70), for $\phi=nx_{\rm cl}$, in the form

$$K(\tau)=2V(\tau)[\eta+V(1)]^{-1}, \tag{B13}$$

where

$$V(\tau)=T\ \exp\left\{ge_0\int_0^\tau F^{\rm comb}(nx_{\rm cl}(\tau'))d\tau'\right\} \tag{B14}$$

is the solution to the equation

$$\left[\frac{\partial}{\partial\tau}-ge_0F-ge_0f(nx_{\rm cl}(\tau))\right]V(\tau)=0. \tag{B15}$$

Then, using the defining equation (13) for B (in which F must be understood as $F^{\rm comb}$) and Eqs. (B13) and (B14), one derives

$$B=\frac{2}{ge_0}\{\eta-2[\eta+V(1)]^{-1}\}. \tag{B16}$$

Correspondingly, from Eq. (18) we obtain

$$\overline{B}=\frac{2}{ge_0}\{\eta-2[\eta+V_0(1)]^{-1}\},\quad V_0(\tau)=e^{ge_0F\tau}. \tag{B17}$$

Solving Eq. (B15) we find

$$V(1)=V_0^{1/2}(1)\left(\eta+C+\frac{C^2}{2}\right)V_0^{1/2}(1), \tag{B18}$$

by virtue of the nilpotency [10] of C. Then we substitute Eq. (B18) into Eq. (B16) and after straightforward transformations obtain

$$B-\overline{B}=\frac{1}{ge_0}\left(\cosh\frac{ge_0F}{2}\right)^{-1}C\left(\cosh\frac{ge_0F}{2}\right)^{-1}. \tag{B19}$$

One can verify, using the transversality (41) of a_μ, that[5]

$$C=P_{\mathcal{E}}CP_{\mathcal{H}}+P_{\mathcal{H}}CP_{\mathcal{E}}. \tag{B20}$$

On the other hand, because of the evenness of the function,

$$\left(\cosh\frac{ge_0F}{2}\right)^{-1}=\left(\cosh\frac{ge_0\mathcal{E}}{2}\right)^{-1}P_{\mathcal{E}}+\left(\cos\frac{ge_0\mathcal{H}}{2}\right)^{-1}P_{\mathcal{H}}. \tag{B21}$$

We substitute Eqs. (B20) and (B21) into Eq. (B19) to get, by virtue of the properties (A5) of the projection operators $P_{\mathcal{E}}$ and $P_{\mathcal{H}}$,

$$B-\overline{B}=\frac{1}{ge_0}\left(\cosh\frac{ge_0\mathcal{E}}{2}\cos\frac{ge_0\mathcal{H}}{2}\right)^{-1}C. \tag{B22}$$

Finally, inserting Eq. (B22) into Eq. (B11) and using

$$\overline{\Lambda(e_0)}=\left(\cosh\frac{ge_0\mathcal{E}}{2}\cos\frac{ge_0\mathcal{H}}{2}\right)^{-1}, \tag{B23}$$

one finds that $Q=0$ and that the identity (B8) takes place.

Going to the identity (B9) we use Eqs. (B8) and (14), the identity

$$\gamma^\mu\gamma^5=-\frac{1}{3!}\epsilon^\mu_{\ \kappa\rho\sigma}\gamma^\kappa\gamma^\rho\gamma^\sigma,$$

and the antisymmetry $B_{\alpha\beta}=B_{[\alpha\beta]}$ to bring the LHS into the form

$$\gamma^\mu\Delta[nx_{\rm cl},e_0]=\left[\left(\eta^\mu_{\ \beta}+\frac{ge_0}{2}B^\mu_{\ \beta}\right)\gamma^\beta+\frac{ge_0}{4}\times\left(\eta^\mu_{\ \nu}B_{\alpha\beta}-\frac{ge_0}{4!}B_{\rho\sigma}B^{*\rho\sigma}\epsilon^\mu_{\ \nu\alpha\beta}\right)\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right]\times\overline{\Lambda(e_0)}. \tag{B24}$$

From Eqs. (B13) and (B16) one obtains

$$K(1)=\eta+\frac{ge_0}{2}B, \tag{B25}$$

[5]The projection operators $P_{\mathcal{E}}$ and $P_{\mathcal{H}}$ are defined in Appendix A.

$$K_{\mu[\nu}(1)B_{\alpha\beta]}=\eta_{\mu[\nu}B_{\alpha\beta]}+\frac{ge_0}{2}B_{\mu[\lambda}B_{\alpha\beta]}. \quad (B26)$$

Because of the antisymmetry of B,

$$B_{\mu[\lambda}B_{\alpha\beta]}=B_{[\mu\nu}B_{\alpha\beta]}=-\frac{2}{4!}B_{\kappa\rho}B^{*\kappa\rho}\epsilon_{\mu\nu\alpha\beta},$$

and, substituting in Eq. (B26), we get

$$K_{\mu[\nu}(1)B_{\alpha\beta]}=\eta_{\mu[\nu}B_{\alpha\beta]}-\frac{ge_0}{4!}B_{\kappa\rho}B^{*\kappa\rho}\epsilon_{\mu\nu\alpha\beta}. \quad (B27)$$

Finally, we use Eqs. (B25) and (B27) in Eq. (B24) to get Eq. (B9).

3. Identities involving γ matrices in 2+1 dimensions

To prove the identity (105) let us introduce

$$z^{\mu}=\epsilon^{\mu\nu\lambda}F_{\nu\lambda}, \quad z^2=-4\varphi^2,$$

and transform the LHS of Eq. (105) using Eq. (95):

$$\exp\left\{\frac{ge_0}{4}F_{\mu\nu}\gamma^{\mu}\gamma^{\nu}\right\}=\cosh\frac{ge_0\varphi}{2}\left(1-\frac{iz\gamma}{2\varphi}\tanh\frac{ge_0\varphi}{2}\right).$$

Taking into account Eqs. (13) and (A8) and the relation $iz\gamma=-\gamma F\gamma$, one gets Eq. (105).

Using Eqs. (105), (18), and (95) we find that the identity (106) is equivalent to the pair of identities

$$\gamma_{\mu}B_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta}-2i\epsilon_{\mu\nu\lambda}B^{\nu}{}_{\alpha}\gamma^{[\alpha}\gamma^{\beta]}+i\epsilon_{\mu\nu\lambda}B^{\nu\lambda}=0, \quad (B28)$$

$$\epsilon^{\rho\alpha\beta}(B^{\nu}{}_{\alpha}B^{\lambda}{}_{\beta}-B^{\nu\lambda}B_{\alpha\beta})=0. \quad (B29)$$

To prove Eq. (B28) one only needs the properties of the γ matrices and Eq. (95). Because of Eqs. (18) and (A8), identity (B29) is equivalent to

$$\epsilon^{\rho\alpha\beta}(B^{\nu}{}_{\alpha}B^{\lambda}{}_{\beta}-B^{\nu\lambda}B_{\alpha\beta})=0,$$

which can be checked expressing F in terms of z:

$$F_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\lambda}z^{\lambda}.$$

[1] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).
[2] R. P. Feynman, Phys. Rev. **80**, 440 (1950).
[3] R. P. Feynman, Phys. Rev. **84**, 108 (1951).
[4] E. S. Fradkin, *Proceedings of the P. N. Lebedev Physics Institute* (Nauka, Moscow, 1965), Vol. 29, p. 7; B. M. Barbashov, Sov. Phys. JETP **48**, 607 (1965); I. A. Batalin and E. S. Fradkin, Teor. Mat. Fiz. **5**, 190 (1970); M. Henneaux and C. Teitelboim, Ann. Phys. (N.Y.) **143**, 127 (1982); N. V. Borisov and P. P. Kulish, Teor. Mat. Fiz. **51**, 335 (1982); A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic, Chur, Switzerland, 1987); V. Ya. Fainberg and A. V. Marshakov, JETP Lett. **47**, 565 (1988); Phys. Lett. B **211**, 81 (1988); Nucl. Phys. **B306**, 659 (1988); *Proceedings of PhIAN* (Nauka, Moscow, 1991), Vol. 201, p. 139; J. Ambjorn, B. Durhuus, and T. Jonsson, Nucl. Phys. **B330**, 509 (1990); E. S. Fradkin, D. M. Gitman, and Sh. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991); E. S. Fradkin and D. M. Gitman, Phys. Rev. D **44**, 3230 (1991); D. M. Gitman and A. V. Saa, Class. Quantum Grav. **10**, 1447 (1993); T. M. Aliev, V. Ya. Fainberg, and N. K. Pak, Nucl. Phys. **B429**, 321 (1994); J. W. van Holten, *ibid.* **B457**, 375 (1995).
[5] I. A. Batalin and E. S. Fradkin, Teor. Mat. Fiz. **5**, 190 (1970).
[6] A. M. Polyakov, in *Fields, Strings and Critical Phenomena*, Proceedings of the Les Houches Summer School, Les Houches, France, 1988, edited by E. Brézin and J. Zinn-Justin, Les Houches Summer School Proceedings Vol. 49 (North-Holland, Amsterdam, 1990); *Gauge Fields and Strings* (Harwood Academic, Chur, Switzerland, 1987); Mod. Phys. Lett. A **3**, 325 (1988).
[7] G. M. Korchemsky, Phys. Lett. B **322**, 334 (1989); Int. J. Mod. Phys. A **7**, 339 (1992); J. Grundberg, T. H. Hansson, and A. Karlhede, Phys. Rev. D **41**, 2642 (1990); Nucl. Phys. **B347**, 420 (1990); A. I. Karanikas and C. N. Ktorides, Phys. Rev. D **52**, 5883 (1995).
[8] Fradkin and Gitman [4].
[9] D. M. Gitman and Sh. M. Shvartsman, Phys. Lett. B **318**, 122 (1993); **331**, 449(E) (1994).
[10] E. S. Fradkin, D. M. Gitman, and Sh. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991).
[11] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol, 1992); S. D. Odintsov, Fortschr. Phys. **38**, 371 (1990); **39**, 621 (1991).
[12] D. M. Gitman, M. D. Noskov, and Sh. M. Schvartsman, Int. J. Mod. Phys. A **6**, 4437 (1991).
[13] D. M. Gitman, Nucl. Phys. B (to be published).
[14] F. A. Berezin, *Method of Second Quantization* (Nauka, Moscow, 1965).
[15] J. Schwinger, Phys. Rev. **82**, 664 (1951).