Wavy strings: Black or bright?

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Recent developments in string theory have brought forth considerable interest in time-dependent hair on extended objects. This novel new hair is typically characterized by a wave profile along the horizon and angular momentum quantum numbers l,m in the transverse space. In this work, we present an extensive treatment of such oscillating black objects, focusing on their geometric properties. We first give a theorem of purely geometric nature, stating that such wavy hair cannot be detected by any scalar invariant built out of the curvature and/or matter fields. However, we show that the tidal forces detected by an infalling observer diverge at the ''horizon'' of a black string superposed with a vibration in any mode with $l \geqslant 1$. The same argument applied to longitudinal ($l=0$) waves detects only finite leading-order tidal forces. We also provide an example with a manifestly smooth metric, proving that at least a certain class of these longitudinal waves have regular horizons. [S0556-2821(97)06610-1]

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I. INTRODUCTION

One of the most confounding puzzles about black holes in general relativity is the apparent incompatibility between their extremely simple structure, governed by the famous no-hair uniqueness theorems [1], and their generically large thermodynamic entropy [2]. The former indicates that the complexity associated with the latter is not encoded in the classical solutions. The crux of the no-hair theorems in general relativity is that regardless of how a black hole was formed it is completely described once its mass, spin, and Abelian electric or magnetic charges are determined. All other types of long-range interactions, such as scalars with or without self-interactions, massive or non-Abelian gauge fields, etc., are excluded because they would either lead to naked singularities [1] or would be unstable and thus would disappear shortly after the formation of the black hole [3]. More specifically, the machinery of the no-hair theorems rests on the assumption of regularity of a stationary null surface, i.e., the event horizon, and the demonstration that it leads to the vanishing of all matter charges, other than Abelian, which could carry long-range interactions. Because the equivalence principle demands that the fields be coupled minimally to gravity, the vanishing of charges in stationary spherically symmetric space-times implies the vanishing of the external fields or hair. A small deviation from these results are black holes with non-Abelian hair, where the hair arises because of the nonlinearities in the gauge field equations of motion; however, this hair is unstable to perturbations, with the hairy black hole rapidly decaying into a bald one [3]. The final result is that all of the structure of the configuration which collapsed into a black hole not encompassed by mass, spin, and Abelian charges must have been radiated away in the process of formation of the hole.

Recent developments in nonminimal models of gravity, and especially in string theory, where the modifications of general relativity have a firmer foundation, have brought about some explicit counterexamples to the no-hair theorems [4]. Namely, the models considered involve new nonminimal couplings which provide extra sources in the equations of motion. However, whereas the particular solutions to these modified equations carry nontrivial long-range hair exterior to black holes, there are no new constants of integrations and so these aberrations of the no-hair theorems are deviations in letter only and not in spirit. The new charges are completely determined by the charges already present in Einstein's theory, and the nonminimal hair is secondary (as opposed to the primary hair carried directly by the black hole charges).[1] Nevertheless, the secondary hair does affect thermodynamic properties of black holes, changing the expressions for both the temperature and the entropy [4].

Another kind of black hole hair may appear if we abandon the constraint of stationarity [5–11]. Recent developments in string theory have brought forth a considerable interest in time-dependent hair on black holes and extended objects. In particular it has been suggested that such hair might give a classical accounting of the black hole entropy [8,11,15]. In any event, one might expect that this situation would be a more realistic description of a black hole, because stationary black holes could exist only as ''hermits'' in complete solitude; yet their very nature precludes this, as their own gravitational field permeates the whole Universe and communicates with all of its inhabitants. The principal obstacle to studying nonstationary problems is an enormous complexity

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[1]The secondary hair will be stable because the monopole charges do not vanish, in contrast with the non-Abelian hair mentioned above.

of gravitational equations of motion, often resulting in their intractability. Indeed, there are comparatively few exact nonstationary localized solutions known to date, and none of them corresponding in all important aspects to the physical picture of black holes. Still, some of these solutions may provide useful models to study realistic hairy black holes. Most of these solutions arise because we can embed four-dimensional black holes in higher dimensions via a procedure inverse to dimensional reduction (thus, the term ''oxidation''—see, e.g., [24]). In certain special cases, the resulting oxidized solution has a null, hypersurface-orthogonal, Killing vector [12–16]. They can be used as a natural starting point for the construction of a more general family of nonstationary solutions characterized by a set of arbitrary functions. The presence of this isometry indicates that there exists a much larger family of wavelike solutions which reduces to the black-hole-like solution for the special choice of its degrees of freedom. A similar correspondence exists, for example, between the Brinkmann wave solutions in general relativity and flat space. We can get a glimpse of this larger family by using a solution-generating technique defined by Garfinkle and Vachaspati [17] as a method to restore some of the original wave degrees of freedom. The resulting solutions still possess a null hypersurface-orthogonal Killing vector, and thus describe a gravitational disturbance propagating through the original environment at the speed of light, i.e., a gravitational wave.

A fundamental question of interest is to determine if this larger family still has a regular null surface, which can be reached by causal or null geodesics, i.e., an event horizon. In this paper, we will examine this question for a certain family of five-dimensional black strings with a single wave profile function in an (l,m) multipole mode. First, however, we will give a theorem of purely geometric nature (valid in any theory of gravity which assumes a pseudo-Riemannian geometry as the basis for the description of gravitational interactions) that such wavy hair generated by the Garfinkle-Vachaspati technique [17] cannot be detected by any scalar invariant built out of the curvature and/or matter fields. However, in our example, we will show that the tidal forces measured by an infalling observer diverge at the ''horizon'' of the black string superposed with a vibration in any mode with $l \geqslant 1$. Hence the solutions with excited dipole or higher multipole modes contain null singularities. The same argument applied to the monopole mode shows that the leading order tidal forces are finite. We will also construct a class of monopole wave profiles for which the metric is manifestly smooth. Hence at least some of the wavy strings indeed have regular horizons.

The paper is organized as follows. In the next section, we give a review of the wave-generating technique, outlining the conditions the matter distribution must satisfy for the method to work. In Sec. III we present our theorem on the elusive nature of Garfinkle-Vachaspati waves, and give its detailed proof. Section IV contains the derivation of the explicit form of the wave–black-string solution and a brief discussion of some of its properties. We show that these solutions contain null singularities for all higher multipole modes ($l \geqslant 1$) later in Sec. IV. In the last section, we construct an explicit family of longitudinal wave ($l=0$) solutions for which the metric is analytic on the horizon (and not only continuous, as shown in [9]). Finally, we present our conclusions and consider directions for future investigations.

II. GARFINKLE-VACHASPATI WAVES

In this section, we review the wave-generating technique of Garfinkle and Vachaspati [17]. In doing so, we outline the situations in which it may be applied, i.e., the restrictions which must be satisfied by the metric and matter fields. The crucial requirement is that the solution possess a null, hypersurface-orthogonal, Killing vector. The resulting presence of a null coordinate allows one to effectively ''linearize'' Einstein's equations and restore hidden wave degrees of freedom. The method was originally proposed for the Yang-Mills-Higgs system coupled to gravity and applied to straight cosmic string solutions in four dimensions [17]. It was later extended to gravity coupled to a scalar and a two-form potential in five dimensions in the context of low-energy string theory [18]. It was also used to construct similar wavy axionic strings in Refs. [6,7]. The following description will consider gravity in arbitrary dimensions coupled to a matter sector including various scalars as well as a set of different p-form potentials. Thus this discussion shows that this technique is quite generally applicable in supergravity or low-energy string theories [19].

Let us assume an action of the form

$$I=\int d^Dx\sqrt{-g}\Big(R(g)-\frac{1}{2}\sum_a h_a(\phi)(\nabla\phi_a)^2 - \frac{1}{2}\sum_p f_p(\phi)F^2_{(p+1)}\Big) \ . \tag{1}$$

Thus, as well as the metric, we have included a collection of scalar fields ϕ_a, which appear with nonderivative couplings in the coefficient functions, $h_a(\phi)$ and $f_p(\phi)$. The above action also involves a set of p-form potentials $A_{(p)}$ through their field strengths $F_{(p+1)}=dA_{(p)}$. Hence there is a (Abelian) gauge invariance associated with these fields, $\delta A_{(p)}=d\lambda_{(p-1)}$. Note that this action is written in terms of the Einstein-frame metric, by which we mean that there are no couplings to any ϕ_a appearing in the Ricci scalar term of Eq. (1). Now the gravity equations of motion may be written as

$$R^\mu{}_\nu-\tfrac{1}{2}\,\delta^\mu{}_\nu R=\tfrac{1}{2}\,T^\mu{}_\nu, \tag{2}$$

where

$$T^\mu{}_\nu=\sum_a h_a(\phi)[g^{\mu\rho}\partial_\rho\phi_a\partial_\nu\phi_a-\tfrac{1}{2}\,\delta^\mu{}_\nu\,(g^{\rho\lambda}\partial_\rho\phi_a\partial_\lambda\phi_a)] + \sum_p f_p(\phi)\{(p+1)[F_{(p+1)}]^{\mu\rho_1\cdots\rho_p}[F_{(p+1)}]_{\nu\rho_1\cdots\rho_p} - \tfrac{1}{2}\,\delta^\mu{}_\nu[F_{(p+1)}]^{\rho_1\cdots\rho_{p+1}}[F_{(p+1)}]_{\rho_1\cdots\rho_{p+1}}\}, \tag{3}$$

while the matter field equations may be written as

$$0=\partial_\mu[\sqrt{g}h_a(\phi)g^{\mu\nu}\partial_\nu\phi_a]-\tfrac{1}{2}\sqrt{g}\sum_b\frac{\partial h_p(\phi)}{\partial\phi_a}\nabla^\mu\phi_b\nabla_\mu\phi_a$$

$$-\tfrac{1}{2}\sqrt{g}\sum_p \frac{\partial f_p(\phi)}{\partial \phi_a}[F_{(p+1)}]^{\rho_1\cdots\rho_{p+1}}[F_{(p+1)}]_{\rho_1\cdots\rho_{p+1}}, \tag{4}$$

$$0=\partial_\mu\{\sqrt{g}f_p(\phi)[F_{(p+1)}]^{\mu\rho_1\cdots\rho_p}\}. \tag{5}$$

For later purposes, we have been explicit about the appearance of the spacetime indices in these equations.

Now within this theory let us assume a solution $(g,\phi_a,A_{(p)})$ for which there exists a vector field k^μ which is

$$\text{null:}\quad k^\mu k_\mu=0; \tag{6}$$

$$\text{hypersurface orthogonal:}\quad \nabla_{[\mu}k_{\nu]}=k_{[\mu}\nabla_{\nu]}S; \tag{7}$$

$$\text{and Killing:}\quad \nabla_{(\mu}k_{\nu)}=0. \tag{8}$$

Combining these equations, it is easy to show that k has a vanishing Lie derivative on S, i.e., $\mathcal{L}_k S=k^\mu\partial_\mu S=0$. Since we wish k to yield an invariance of the full solution, it is also assumed to have a vanishing Lie derivative on the matter fields; i.e., the matter fields are form invariant along the flow of k. Hence,

$$\mathcal{L}_k\phi_a=k^\mu\partial_\mu\phi_a=0, \tag{9}$$

$$\mathcal{L}_k F_{(p+1)}=(di_k+i_k d)F_{(p+1)}=di_k F_{(p+1)}=0, \tag{10}$$

where the latter uses $\mathcal{L}_v=di_v+i_v d$ on forms with i_v denoting the interior product, and the Bianchi identities $dF_{(p+1)}=0$. This form invariance of the matter fields guarantees that the stress-energy tensor is form invariant, as it must be given that Einstein's equations (2) are satisfied. Note that for the form fields, the vanishing Lie derivative is imposed on the physical field strengths rather than the gauge-variant potentials — the latter may vary by a gauge transformation along the k flow, $\mathcal{L}_k A_{(p)}=d\lambda_{(p-1)}$. In the following, we will further require that these fields satisfy an additional transversality constraint, namely,

$$i_k F_{(p+1)}=k\wedge\theta_{(p-1)}. \tag{11}$$

Here, the right-hand side is the wedge product of the one-form $k_\mu dx^\mu$ with some $(p-1)$-form $\theta_{(p-1)}$, which necessarily satisfies $i_k\theta_{(p-1)}=0$ since $(i_k)^2F_{(p+1)}=0$—see Sec. III A, for details.

The solutions satisfying the above conditions can be interpreted as gravity waves. Consider the coordinate system adapted to the flow of k—as well as the cyclic coordinate v, there is a coordinate u, given ''roughly'' by the integral of the dual one-form $k=k_\mu dx^\mu$. The vanishing Lie derivatives, Eqs. (8), (9), and (10), simply means that none of the fields depend on v. Hence the only ''time'' dependence can arise through the null coordinate u and, hence, represents perturbations moving at the speed of light in a certain direction of the space-time. The Garfinkle-Vachaspati (GV) solution-generating technique extends the solution by restoring additional wave degrees of freedom. To do this, they define a new metric by [17]

$$g'_{\mu\nu}=g_{\mu\nu}+e^S\Psi k_\mu k_\nu, \tag{12}$$

while leaving all of the matter fields unchanged.[2] The configuration (g',ϕ_a,A_p) also yields a solution provided the function Ψ satisfies appropriate constraints. These restrictions will guarantee that after the metric is modified the form of the explicit fields appearing in the equations of motion remains unchanged. Hence Eq. (12) is promoted to a map on the space of solutions. In order to maintain the wave interpretation of the new solution, first one requires that k have a vanishing Lie derivative on Ψ, i.e., $k^\mu\partial_\mu\Psi=0$. One may verify that this ensures that the hypersurface orthogonal and Killing conditions, Eqs. (7) and (8), are still satisfied with the new metric (with the same S). It is also obvious that the null condition (6) still holds with g'. To determine what other restrictions must be imposed on Ψ, one must consider the changes which the map (12) induces in the equations of motion, Eqs. (2)–(5).

We begin by demonstrating that the matter field equations are invariant under Eq. (12). First note that the determinant of the metric would only be modified by terms proportional to $k^\mu k_\mu$. However, since k is null, the determinant is invariant: $\det(g')=\det(g)$. On the other hand, the inverse to the metric $g'_{\mu\nu}$ is given by

$$g'^{\mu\nu}=g^{\mu\nu}-e^S\Psi k^\mu k^\nu. \tag{13}$$

Now, given the transversality constraint (11), we see that raising all of the indices of the form fields with the new metric yields

$$\begin{aligned}[F'_{(p+1)}]^{\mu_1\cdots\mu_{p+1}}&=[F_{(p+1)}]^{\mu_1\cdots\mu_{p+1}}+p(p+1)\\&\quad\times k^{[\mu_1}k^{\mu_2}[\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]}\\&=[F_{(p+1)}]^{\mu_1\cdots\mu_{p+1}},\end{aligned} \tag{14}$$

where in the first line terms with more powers of k have automatically vanished since $i_k\theta_{(p-1)}=0$. Given this result as well as the invariance of $\det(g)$, it is clear that the equations of motion for the form fields (5) are left unchanged by the map (12). Further, given the vanishing Lie derivative (9), it is clear that $g'^{\mu\nu}\partial_\nu\phi_a=g^{\mu\nu}\partial_\nu\phi_a$ and hence the scalar equations (4) are also unchanged. Thus we have shown that the configuration (g',ϕ_a,A_p) provides a solution of all of the matter field equations.

The same considerations as above also show that the stress-energy tensor with mixed indices $T^\mu{}_\nu$, given in Eq. (3), remains unchanged when the metric is modified as in Eq. (12). The only step which remains in examining the gravity equation is to compute the change of the mixed-index Einstein tensor. This is reduced to computing the change in the Ricci tensor only, because the Ricci scalar may be eliminated from Eq. (2) using $R=[1/(2-D)]T^\mu{}_\mu$ in D dimensions. It is straightforward to calculate that the Christofel symbols for the two metrics are related by $\Gamma'^\mu_{\nu\lambda}=\Gamma^\mu_{\nu\lambda}+\Omega^\mu_{\nu\lambda}$ where

$$\Omega^\mu_{\nu\lambda}=\tfrac{1}{2}[\nabla_\nu(e^S\Psi k^\mu k_\lambda)+\nabla_\lambda(e^S\Psi k^\mu k_\nu)-\nabla^\mu(e^S\Psi k_\nu k_\lambda)]\ . \tag{15}$$

[2]Throughout this section, $k_\mu=g_{\mu\nu}k^\nu$. However, from the definition of the metric $g'_{\mu\nu}$ and the fact that k is null, it does not matter which metric we use to lower and raise the index of k.

Further, using the properties of k, as well as $k^\mu \partial_\mu S = 0 = k^\mu \partial_\mu \Psi$, one can show that $\Omega^\mu_{\mu\nu} = 0 = k^\mu \nabla_\lambda \Omega^\lambda_{\mu\nu} = \Omega^\rho_{\mu\lambda}\Omega^\lambda_{\rho\nu}$. Thus one finds that $R'_{\lambda\nu} = R_{\lambda\nu} + \nabla_\rho \Omega^\rho_{\lambda\nu}$ and hence

$$R'^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} + e^S \Psi k^\mu k^\lambda R_{\lambda\nu} + \nabla_\rho (\Omega^\rho_{\lambda\nu} g^{\mu\lambda}). \tag{16}$$

Again, using the properties of k, one can show that $k^\lambda R_{\lambda\nu} = k_\nu \nabla^2 S/2$. Substituting this into Eq. (16) and after a few more manipulations, we finally arrive at [17]

$$R'^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \tfrac{1}{2} e^S k^\mu k_\nu \nabla^2 \Psi. \tag{17}$$

Therefore, the variation of the mixed Ricci tensor $R^\mu{}_\nu$ under Eq. (12) is proportional to $\nabla^2\Psi$, and so vanishes if we demand that Ψ solve the covariant Laplace equation (in the background defined by $g_{\mu\nu}$).

Therefore, given a solution for which Eqs. (6)–(11) are satisfied, then Eq. (12) provides a map to a new solution provided

$$k^\mu \partial_\mu \Psi = 0 \quad \text{and} \quad \nabla^2 \Psi = 0. \tag{18}$$

We note that the condition $\nabla^2\Psi = 0$ is really just the spatial Laplace equation since $k^\mu \partial_\mu \Psi = 0$. Furthermore, the latter indicates that the moduli of the new solutions only depend on the retarded time—u, above—and therefore they still represent gravitational waves.

The preceding discussion was phrased in terms of the Einstein-frame metric. However, in some instances (such as the example in Sec. IV), it is more convenient to work in terms of a conformally related metric, e.g.,

$$\tilde{g}_{\mu\nu} = e^{\alpha(\phi)} g_{\mu\nu}. \tag{19}$$

Given in terms of $\tilde{g}$, the action will contain nonminimal couplings of the scalar fields to the Ricci scalar. It is straightforward to show that most of the constraints (6)–(11) are unchanged when written in terms of the conformally transformed metric. The only change is that the hypersurface orthogonal condition (7) becomes

$$\tilde{\nabla}_{[\mu}\tilde{k}_{\nu]} = \tilde{k}_{[\mu}\nabla_{\nu]}\tilde{S} \quad \text{where} \quad \tilde{S} = S - \alpha(\phi). \tag{20}$$

Here we have denoted $\tilde{k}_\mu = \tilde{g}_{\mu\nu}k^\nu$. Thus the map (12) becomes

$$\tilde{g}'_{\mu\nu} = \tilde{g}_{\mu\nu} + e^{\tilde{S}} \Psi \tilde{k}_\mu \tilde{k}_\nu, \tag{21}$$

which is equivalent to $\tilde{g}'_{\mu\nu} = e^{\alpha(\phi)} g'_{\mu\nu}$. Finally, of course, the constraints on Ψ, which ensure that Eq. (21) provides a solution of the gravitational equations of motion, are identical to those appearing in Eq. (18). The latter may be written as follows, when expressed in terms of $\tilde{g}$:

$$k^\mu \partial_\mu \Psi = 0 \quad \text{and} \quad \partial_\mu (e^{(2-D)\alpha(\phi)/2} \sqrt{-\tilde{g}}\, \tilde{g}^{\mu\nu} \partial_\nu \Psi) = 0, \tag{22}$$

where again D denotes the dimension of the spacetime.

Before concluding this section, let us mention a few simple extensions of the original action (1) for which the preceding discussion would still be applicable. First, we could add a scalar potential $U(\phi)$ (which could then include a cosmological constant) to Eq. (1). This would modify the stress energy (3) by a term proportional to $\delta^\mu{}_\nu U(\phi)$, as well as adding a $\partial_{\phi_a} U$ term to the scalar equations (4). Both of these terms are obviously invariant under the map (12) and so the above construction is still valid.

Another nonminimal coupling, which commonly arises among the form fields, is Chern-Simons-like terms appearing in the definition of the field strengths; e.g., for some choice of m and n, $F_{(m+n+1)} = dA_{(m+n)} + \alpha A_{(m)} \wedge dA_{(n)}$, where we still assume $F_{(m+1)} = dA_{(m)}$ and $F_{(n+1)} = dA_{(n)}$. This definition leads to two modifications in the equations of motion. (Note that for $F_{(m+n+1)}$ the Bianchi identity is also modified, $dF_{(m+n+1)} = \alpha F_{(m+1)} \wedge F_{(n+1)}$, but this has no consequences for the present discussion.) First, Eq. (5) is modified for $p = m$ by the introduction of a source term which is proportional to

$$\sqrt{g}\, [F_{(m+n+1)}]^{\rho_1 \cdots \rho_m \mu_1 \cdots \mu_{n+1}} [F_{(n+1)}]_{\mu_1 \cdots \mu_{n+1}}. \tag{23}$$

With the same transversality constraint on $F_{(m+n+1)}$ as above, Eq. (11), we still conclude that its form with all indices raised remains invariant, as in Eq. (14), and hence this new term is also unchanged by Eq. (12). The second change is the appearance of a source term in Eq. (5) for $p = n$. In this case, upon applying the equation of motion (5) for $p = n + m$, this second source term takes precisely the same form as above, merely interchanging the roles of m and n. Therefore it is also invariant under the map (12). Thus the GV procedure will still be valid for solutions where these Chern-Simons-like couplings make nontrivial contributions.

The final extension which we will consider is the addition of topological interactions, e.g.,

$$\int A_{(\ell)} dA_{(m)} dA_{(n)},$$

where $\ell + m + n + 2 = D$. Such an interaction again modifies the equations of motion (5) for $p = \ell, m, n$ by the introduction of source terms of the form, e.g.,

$$\sqrt{g}\, \varepsilon^{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_{m+1} \rho_1 \cdots \rho_{n+1}} [F_{(m+1)}]_{\nu_1 \cdots \nu_{m+1}} \times [F_{(n+1)}]_{\rho_1 \cdots \rho_{n+1}},$$

where ε is the Levi-Cività tensor in D dimensions. Given the invariance of the determinant of the metric, $\varepsilon' = \varepsilon$ as forms. Hence we must compare raising all of the indices with g' and g. Given Eq. (13), we see that

$$\varepsilon'^{\mu\nu\cdots\rho} = \varepsilon^{\mu\nu\cdots\rho} - D e^S \Psi k^{[\mu} k_\lambda \varepsilon^{\lambda|\nu\cdots\rho]} = \varepsilon^{\mu\nu\cdots\rho}, \tag{24}$$

where the vanishing of the second term in the first line relies on a result proved in the following section—see Eq. (40). Therefore the new source terms are again unchanged by the map (12), and so these topological interactions provide no obstruction for this solution-generating technique.

III. THE ELUSIVE WAVE

Generically, it is the case that the original solution and that carrying a wave induced by Eq. (12) are not diffeomorphic; however, the wave turns out to be very elusive. In this section, we will present a theorem of a very general nature, showing that all the scalar curvature invariants of the two metrics related by the wave-generating technique are in fact identical. Thus, no curvature invariant of the oscillating metric can be used to detect the presence of the gravitational wave. The theorem generalizes in a straightforward way to include any scalar invariants constructed from both the metric and matter fields.

To prove this result, we need only assume the existence of a metric $g_{\mu\nu}$ which admits a null, hypersurface-orthogonal, Killing vector. We do not require that the metric solve Einstein's or any other equations. Thus our result is purely geometrical in nature and holds for any metric satisfying the symmetry condition. The precise statement of the theorem is as follows.

If *$g_{\mu\nu}$ is a pseudo-Riemannian metric admitting a null, hypersurface-orthogonal, Killing vector k^μ, and $g'_{\mu\nu}=g_{\mu\nu}+\kappa\, k_\mu k_\nu$, where κ is any scalar Lie derived by k to zero, i.e., $\mathcal{L}_k\kappa=0$, then all of the scalar curvature invariants of $g'_{\mu\nu}$ are exactly identical to the corresponding curvature invariants of $g_{\mu\nu}$.*

Hence the GV transform (12) provides one example of such metrics with $\kappa=e^S\Psi$. In the following, we will refer to $g_{\mu\nu}$ as the original metric, $g'_{\mu\nu}$ as simply the primed or shifted metric, and κ as the wave profile, in analogy to the last section.

Before presenting the details, let us first give a brief sketch of our proof. Before examining the curvature invariants, we establish two crucial results. First, contracting k^μ with any tensor constructed from the original metric, its curvature, and covariant derivatives of the curvature and/or any scalars with a vanishing Lie derivative under k produces a sum of terms in each of which k appears uncontracted. Second, any tensor (e.g., the curvature or covariant derivatives thereof) in the primed background may be written as the sum of that for the original metric plus a κ-dependent term, for which all of the contributions are at least bilinear in the Killing vector k. These two results set the stage for an examination of the scalar curvature invariants. There we find that all of the new κ-dependent terms vanish by combining the previous two results with the fact that k is null. Hence, we conclude that the original and the corresponding primed invariants are identical. As a consequence, no evidence of the wave profile κ can be detected in any of the curvature invariants of the metric. This result represents a generalization of the previous studies of curvature invariants of geometries admitting null, covariantly constant, Killing vectors [20,21]. An immediate corollary of this theorem is that if $g_{\mu\nu}$ represents an extended black object with a regular horizon to which we add a GV wave as in the last section, then the oscillations will not produce new *scalar* curvature singularities—however, in later sections, we will discuss the limitations of this statement with explicit examples.

A. Useful formulas

Let us start by listing some important formulas which arise from the existence of a null, hypersurface-orthogonal, Killing vector. As stated before [in Eq. (7)], the hypersurface-orthogonal condition amounts to $\nabla_{[\mu}k_{\nu]}=k_{[\mu}\nabla_{\nu]}S$ for some scalar S which can be determined from the metric. Combining this with the Killing condition (8) yields

$$\nabla_\mu k_\nu=\tfrac{1}{2}\,(k_\mu\nabla_\nu S-k_\nu\nabla_\mu S)\ . \tag{25}$$

Further, it is not difficult to see that the Killing condition alone leads to

$$\nabla_\nu\nabla_\mu k_\lambda=k^\rho R_{\rho\nu\mu\lambda}\,, \tag{26}$$

as can be determined by considering the commutator of two covariant derivatives acting on k. We also note that we may express

$$k^\rho R_{\rho\nu\lambda\sigma}=\nabla_\sigma\nabla_\lambda k_\nu-\nabla_\lambda\nabla_\sigma k_\nu=k_{[\lambda}\nabla_{\sigma]}\nabla_\nu S-\tfrac{1}{2}k_{[\lambda}\nabla_{\sigma]}S\nabla_\nu S, \tag{27}$$

using Eq. (25). Recall the expression for the Lie derivative of a general tensor $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$:

$$\begin{aligned}\mathcal{L}_v T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}=&\,v^\lambda\nabla_\lambda T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}-T^{\lambda\cdots\mu_p}{}_{\nu_1\cdots\nu_q}\nabla_\lambda v^{\mu_1}-\cdots-T^{\mu_1\cdots\lambda}{}_{\nu_1\cdots\nu_q}\nabla_\lambda v^{\mu_p}+T^{\mu_1\cdots\mu_p}{}_{\lambda\cdots\nu_q}\nabla_{\nu_1}v^\lambda+\cdots\\&+T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\lambda}\nabla_{\nu_q}v^\lambda.\end{aligned} \tag{28}$$

From this definition and the Killing condition (8), one may show that the Lie derivative with respect to a Killing vector $\mathcal{L}_k$ commutes with the covariant derivative. Begin by considering

$$\begin{aligned}\nabla_\mu\mathcal{L}_k T_{\nu_1\cdots\nu_q}=&\,k^\lambda\nabla_\mu\nabla_\lambda T_{\nu_1\cdots\nu_q}+\nabla_\lambda T_{\nu_1\cdots\nu_q}\nabla_\mu k^\lambda+\cdots+\nabla_\mu T_{\nu_1\cdots\lambda}\nabla_{\nu_q}k^\lambda+T_{\lambda\cdots\nu_q}\nabla_\mu\nabla_{\nu_1}k^\lambda+\cdots+T_{\nu_1\cdots\lambda}\nabla_\mu\nabla_{\nu_q}k^\lambda\\=&\,k^\lambda\nabla_\lambda\nabla_\mu T_{\nu_1\cdots\nu_q}+\nabla_\lambda T_{\nu_1\cdots\nu_q}\nabla_\mu k^\lambda+\cdots+\nabla_\mu T_{\nu_1\cdots\lambda}\nabla_{\nu_q}k^\lambda\\&+[T_{\lambda\cdots\nu_q}(R^\lambda{}_{\nu_1\rho\mu}+R_{\rho\mu\nu_1}{}^\lambda)k^\rho+\cdots+T_{\nu_1\cdots\lambda}(R^\lambda{}_{\nu_q\rho\mu}+R_{\rho\mu\nu_q}{}^\lambda)k^\rho]\\=&\,\mathcal{L}_k\nabla_\mu T_{\nu_1\cdots\nu_q},\end{aligned} \tag{29}$$

using Eq. (26) and the standard commutator

$$[\nabla_\mu,\nabla_\lambda]T_{\nu_1\cdots\nu_q}=R^\rho{}_{\nu_1\lambda\mu}T_{\rho\cdots\nu_q}+\cdots.$$

Hence, $[\mathcal{L}_k,\nabla_\mu]T_{\nu_1\cdots\nu_q}=0$. The case where $[\mathcal{L}_k,\nabla_\mu]$ acts on a tensor with some raised indices is trivially related to this one because $\mathcal{L}_k g^{\nu\lambda}=0=\nabla_\mu g^{\nu\lambda}$. Similarly, one can show $\mathcal{L}_k R_{\rho\lambda\nu\mu}=0$, as well as $\mathcal{L}_k S=k^\mu\nabla_\mu S=0$. Further, recall, by definition, $\mathcal{L}_k\kappa=k^\mu\nabla_\mu\kappa=0$. We will find all of these formulas useful below when evaluating the contractions of tensors with the null vector k.

B. Contraction identities

The first step is to show that for any tensor built out of the original curvature, any scalars with vanishing Lie derivative under k (i.e., S and κ), and any number of covariant derivatives (with respect to the original connection) acting on either of these, the contraction $k^\mu T_{\nu_1\cdots\nu_p\mu\lambda_1\cdots\lambda_q}$ is at least linear in vector k_μ; i.e., it can be expressed as a linear combination of terms which factorize as some tensor of rank lower by 2 and the vector k:

$$k^\mu T_{\nu_1\cdots\nu_p\mu\lambda_1\cdots\lambda_q}=\sum_{n=1}^{p}k_{\nu_n}\theta^{(n)}_{\nu_1\cdots\underline{\nu_n}\cdots\nu_p\lambda_1\cdots\lambda_q}+\sum_{n=1}^{q}k_{\lambda_n}\theta^{(p+n)}_{\nu_1\cdots\nu_p\lambda_1\cdots\underline{\lambda_n}\cdots\lambda_q},\tag{30}$$

where underlining an index denotes its deletion from the expression. (For convenience, we will work with only covariant, i.e., ''downstairs,'' indices.) This is true in the few simple cases encountered so far, e.g., $k^\mu\nabla_\mu S=0=k^\mu\nabla_\mu\kappa$ and $k^\rho R_{\rho\nu\lambda\sigma}$ as given in Eq. (27). Let us now show that it holds in general for what we denote as ''primary'' tensors, namely, tensors obtained by an arbitrary number of covariant derivatives acting on the curvature or the scalars S,κ. The proof relies on mathematical induction and makes essential use of Eq. (25) which allows any covariant derivative of k to be reexpressed in terms on undifferentiated k's. First, we establish the result for the simplest cases: For any scalar B Lie derived to zero by k, we have $k^\mu\nabla_\mu B=0$ and further

$$\begin{aligned}k^\mu\nabla_\nu\nabla_\mu B&=k^\mu\nabla_\mu\nabla_\nu B=\nabla_\nu(k^\mu\nabla_\mu B)-\nabla_\mu B\nabla_\nu k^\mu\\&=-\tfrac{1}{2}k_\nu\,\nabla^\mu S\nabla_\mu B,\end{aligned}\tag{31}$$

using Eq. (25). Now combining $\mathcal{L}_k R_{\alpha\beta\gamma\sigma}=0$ and Eq. (27), we find

$$\begin{aligned}k^\mu\nabla_\mu R_{\alpha\beta\gamma\sigma}&=R^\mu{}_{\beta\gamma\sigma}k_{[\mu}\nabla_{\alpha]}S+\cdots+R_{\alpha\beta\gamma}{}^\mu k_{[\mu}\nabla_{\sigma]}S\\&=k_\alpha\theta^{(1)}_{\beta\gamma\sigma}+\cdots+k_\sigma\theta^{(4)}_{\alpha\beta\gamma},\end{aligned}\tag{32}$$

where in the second line we collect the like terms. Explicitly, one finds, e.g., $\theta^{(1)}_{\beta\gamma\sigma}=-\frac{1}{2}(\nabla^\mu R_{\mu\beta\gamma\sigma}+\nabla_\beta\nabla_{[\gamma}S\,\nabla_{\sigma]}S)$ and $\theta^{(2)}_{\alpha\gamma\sigma}=-\frac{1}{2}(\nabla^\mu R_{\alpha\mu\gamma\sigma}-\nabla_\alpha\nabla_{[\gamma}S\,\nabla_{\sigma]}S)$. Similarly,

$$\begin{aligned}k^\alpha\nabla_\mu R_{\alpha\beta\gamma\sigma}&=\nabla_\mu(k^\alpha R_{\alpha\beta\gamma\sigma})-R_{\alpha\beta\gamma\sigma}\nabla_\mu k^\alpha\\&=k_\mu\overline{\theta^{(1)}}_{\beta\gamma\sigma}+\cdots+k_\sigma\overline{\theta^{(4)}}_{\mu\beta\gamma},\end{aligned}\tag{33}$$

again using Eqs. (25) and (27).

So now let us assume that Eq. (30) holds for all primary tensors of rank q or less. Now, by our definition, a primary tensor of rank $q+1$ will be obtained by covariant derivative acting on a primary tensor of rank q, i.e., $\nabla_\lambda T_{\lambda_1\cdots\lambda_q}$. Hence we consider

$$\begin{aligned}k^\mu\nabla_{\lambda_1}T_{\mu\lambda_2\cdots\lambda_q}&=\nabla_{\lambda_1}(k^\mu T_{\mu\lambda_2\cdots\lambda_q})-T_{\mu\lambda_2\cdots\lambda_q}\nabla_{\lambda_1}k^\mu\\&=\sum_{n=2}^{q}\nabla_{\lambda_1}(k_{\lambda_n}\theta^{(n)}_{\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q})+\tfrac{1}{2}T_{\mu\lambda_2\cdots\lambda_q}k^\mu\nabla_{\lambda_1}S-\tfrac{1}{2}T_{\mu\lambda_2\cdots\lambda_q}k_{\lambda_1}\nabla^\mu S\\&=\sum_{n=2}^{q}(k_{\lambda_n}\nabla_{\lambda_1}\theta^{(n)}_{\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q}+\nabla_{\lambda_1}k_{\lambda_n}\theta^{(n)}_{\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q}+\tfrac{1}{2}k_{\lambda_n}\theta^{(n)}_{\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q}\nabla_{\lambda_1}S)-\tfrac{1}{2}k_{\lambda_1}T_{\mu\lambda_2\cdots\lambda_q}\nabla^\mu S\\&=\sum_{n=2}^{q}(k_{\lambda_n}\nabla_{\lambda_1}\theta^{(n)}_{\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q}+\tfrac{1}{2}k_{\lambda_1}\nabla_{\lambda_n}S\;\theta^{(n)}_{\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q})-\tfrac{1}{2}k_{\lambda_1}T_{\mu\lambda_2\cdots\lambda_q}\nabla^\mu S\\&=\sum_{n=1}^{q}k_{\lambda_n}\overline{\theta^{(n)}}_{\lambda_1\cdots\underline{\lambda_n}\cdots\lambda_q},\end{aligned}\tag{34}$$

after collecting the like terms into the tensors $\overline{\theta^{(n)}}$. Of course, the same result follows if k^μ is contracted with any of the other indices on T above. The last possibility is $k^\mu\nabla_\mu T_{\lambda_1\cdots\lambda_q}$. Here we use $\mathcal{L}_k T_{\lambda_1\cdots\lambda_q}=0$ which holds by $[\mathcal{L}_k,\nabla_\mu]=0$ and the fact that both S and the curvature have a vanishing Lie derivative under k. Thus,

$$\begin{aligned}k^\mu\nabla_\mu T_{\lambda_1\cdots\lambda_q}&=-T_{\mu\lambda_2\cdots\lambda_q}\nabla_{\lambda_1}k^\mu-\cdots-T_{\lambda_1\cdots\lambda_{q-1}\mu}\nabla_{\lambda_q}k^\mu\\&=\tfrac{1}{2}(T_{\mu\lambda_2\cdots\lambda_q}\nabla_{\lambda_1}S+\cdots+T_{\lambda_1\cdots\lambda_{q-1}\mu}\nabla_{\lambda_q}S)k^\mu-\tfrac{1}{2}(T_{\mu\lambda_2\cdots\lambda_q}k_{\lambda_1}+\cdots+T_{\lambda_1\cdots\lambda_{q-1}\mu}k_{\lambda_q})\nabla^\mu S\\&=\sum_{n=1}^{q}k_{\lambda_n}\widetilde{\theta}^{(n)}_{\lambda_1\cdots\underline{\lambda_n}\cdots\lambda_q},\end{aligned}\tag{35}$$

where we applied Eq. (30) for rank-q primary tensors in the second line and gathered the like terms. With Eqs. (34) and (35), we have shown that Eq. (30) holds for any index μ on a primary tensor of rank $q+1$. Therefore by induction this factorization property is established for all primary tensors.

At this stage, we must consider the ''secondary'' tensors, i.e., the θ tensors produced in the contractions of k with primary tensors—although here we will leave many of the details to the reader. Considering the simplest examples in Eqs. (27) and (31)–(33), one finds that the θ tensors are not simply primary tensors, but rather involve certain products and/or contractions of primary tensors. However, in those particular examples, it is not hard to show that they share two important properties in common with the primary tensors: (i) The θ's have a vanishing Lie derivative under k, and (ii) upon contraction with k, they factorize as in Eq. (30). Having established that these conditions apply in the simplest cases, it is straightforward to formulate an inductive proof to show that they also apply for the θ tensors produced from primary tensors of higher rank. One would begin by assuming that (i) and (ii) hold for the θ's arising from rank-q primary tensors, and then examine those produced at rank $q+1$. In the case considered in Eq. (34), one has

$$\overline{\theta^{(1)}}_{\lambda_2\cdots\lambda_q}=\frac{1}{2}\sum_{n=1}^{q}\nabla_{\lambda_n}S\theta^{(n)}_{\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q}-\frac{1}{2}\nabla^{\mu}ST_{\mu\lambda_2\cdots\lambda_q},$$

$$\overline{\theta^{(n)}}_{\lambda_1\cdots\underline{\lambda_n}\cdots\lambda_q}=\nabla_{\lambda_1}\theta^{(n)}_{\lambda_2\cdots\lambda_q}\quad\text{for } n>1. \tag{36}$$

It is easy to show that $\overline{\theta^{(1)}}$ will satisfy both the Lie derivative and factorization properties because all of the components of which it is comprised (i.e., $\theta^{(n)}$'s, T, ∇S) do. For $n>1$, $\overline{\theta^{(n)}}$ will be Lie derived to zero by k because $\theta^{(n)}$'s are and $[\mathcal{L}_k,\nabla_\mu]=0$. Given the vanishing Lie derivative and that factorization (30) holds for the $\theta^{(n)}$'s, one would extend this condition to the $\overline{\theta^{(n)}}$ tensors in the same way as in Eqs. (34) and (35) for the primary tensors. One should then examine the $\widetilde{\theta}$'s arising in Eq. (35) for that particular case of rank-$(q+1)$ primary tensors. One is again able to show that both (i) and (ii) apply for these θ tensors as well, although the index gymnastics is somewhat more involved. Furthermore, the proof of these properties extends in a similar way to any ''higher-order'' tensors, that is, any new θ's produced by contracting k with θ tensors.

Now this intermediate result for the θ tensors is necessary in order to show that the factorization property (30) also holds for products of primary tensors, with arbitrary contractions of pairs of indices. Our principal tool here is again mathematical induction. We begin by considering quantities of the simple product $T^{(1)}_{\nu_1\cdots\nu_{p+1}\lambda_1\cdots\lambda_q}T^{(2)\lambda_1\cdots\lambda_q}{}_{\omega_1\cdots\omega_l}$, where both $T^{(1)}$ and $T^{(2)}$ are primary tensors. Using the properties of primary tensors, we find

$$\begin{aligned}k^{\mu}T^{(1)}_{\nu_1\cdots\mu\cdots\nu_p\lambda_1\cdots\lambda_q}T^{(2)\lambda_1\cdots\lambda_q}{}_{\omega_1\cdots\omega_l}&=\sum_{n=1}^{p}k_{\nu_n}\theta^{(n,1)}_{\nu_1\cdots\underline{\nu_n}\cdots\nu_p\lambda_1\cdots\lambda_q}T^{(2)\lambda_1\cdots\lambda_q}{}_{\omega_1\cdots\omega_l}\\&+\sum_{n=1}^{q}k_{\lambda_n}\theta^{(p+n,1)}_{\nu_1\cdots\nu_p\lambda_1\cdots\underline{\lambda_n}\cdots\lambda_q}T^{(2)\lambda_1\cdots\lambda_q}{}_{\omega_1\cdots\omega_l}.\end{aligned} \tag{37}$$

Here the first term has the required form, and so for $q=0$, i.e., no contractions, we would have the desired result. However, for $q\geqslant 1$, we need to look at the cross terms in the second sum, e.g.,

$$\begin{aligned}k_{\lambda_1}\theta^{(p+1,1)}_{\nu_1\cdots\nu_p\lambda_2\cdots\lambda_q}T^{(2)\lambda_1\cdots\lambda_q}{}_{\omega_1\cdots\omega_l}&=\theta^{(p+1,1)}_{\nu_1\cdots\nu_p\lambda_2\cdots\lambda_q}\sum_{n=2}^{q}k^{\lambda_n}\theta^{(n,2)\lambda_2\cdots\underline{\lambda_n}\cdots\lambda_q}{}_{\omega_1\cdots\omega_l}\\&+\theta^{(p+1,1)}_{\nu_1\cdots\nu_p\lambda_2\cdots\lambda_q}\sum_{n=1}^{l}k_{\omega_n}\theta^{(q+n,2)\lambda_1\cdots\lambda_q}{}_{\omega_1\cdots\underline{\omega_n}\cdots\omega_l}.\end{aligned} \tag{38}$$

Here we will have arrived at the desired form if $q=1$, but for $q\geqslant 2$, we have generated further cross terms in which k is contracted back on $\theta^{(p+1,1)}$. However, since the θ's also factorize according to Eq. (30) as described above, we may continue this procedure. Now given that q, the number of contractions, is finite, this ''ladder'' of k contractions will eventually terminate, since at each step the number of contracted index pairs is reduced by one in each of the subsequent cross terms. Therefore after a finite number of steps, we arrive at a factorization without any contractions yielding the desired form. Given this result, it is obvious that we may in a similar way extend Eq. (30) to apply for an arbitrary product of primary tensors, including arbitrary contractions.

To conclude this subsection, we consider contractions of k with the Levi-Cività tensor, ε, i.e., the volume form on the D-dimensional spacetime. To analyze this case, we resort to local coordinate patches adapted to the properties of k. First, the Killing condition (8) indicates that we can find a cyclic coordinate such that $k^{\mu}\partial_{\mu}=(\partial/\partial v)$. Next the hypersurface orthogonal condition (7) indicates that we can find a dual coordinate such that $k_{\mu}dx^{\mu}=e^{-S}du$. In this local coordinate patch, one of the free indices in $k^{\mu}\varepsilon_{\mu\alpha\cdots\beta}$ must then take the value u, and hence we can write

$$i_k\varepsilon=k\wedge\theta, \tag{39}$$

where in this case θ is some $(D-2)$-form. However, as written this result is coordinate independent and so must hold in

general.[3] Thus we have shown that the Levi-Cività tensor factorizes in the same way as the primary tensors.

In fact, the resulting θ satisfies Eq. (30) as well in a trivial way since $i_k\theta=0$. This result is again most easily derived using the local coordinates introduced above. Because of the antisymmetry of the Levi-Cività tensor, none of the indices on the right-hand side of Eq. (39) correspond to the cyclic coordinate v. Hence $i_k\theta=0$ with this choice of coordinates, but this equation must then be valid in general.

As an aside, we note that the preceding discussion is equally applicable for the $\theta_{(p-1)}$ arising in the transversality constraint (11): $i_k F_{(p+1)}=k\wedge\theta_{(p-1)}$. In this case, using the adapted coordinates, the antisymmetry of the field strength $F_{(p+1)}$ ensures that $\theta_{(p-1)}$ does not carry a v index. Hence one finds that $i_k\theta_{(p-1)}=0$.

Given the previous results, our final conclusion is that Eq. (30) applies for an arbitrary product of primary tensors and Levi-Cività tensors, including arbitrary contractions.

C. Shift in tensors

Next we consider the difference between tensors calculated for the original and primed metrics. First the shift in the connection coefficients $\Omega^{\mu}_{\nu\lambda}=\Gamma'^{\mu}_{\nu\lambda}-\Gamma^{\mu}_{\nu\lambda}$ is given by

$$\Omega^{\mu}_{\nu\lambda}=\frac{1}{2}[\nabla_\nu(\kappa\, k^\mu k_\lambda)+\nabla_\lambda(\kappa\, k^\mu k_\nu)-\nabla^\mu(\kappa\, k_\nu k_\lambda])$$

$$=\frac{1}{2}(k^\mu k_\lambda\nabla_\nu\kappa+k^\mu k_\nu\nabla_\lambda\kappa-k_\nu k_\lambda\nabla^\mu\kappa) \tag{40}$$

$$-\frac{\kappa}{2}(k^\mu k_\lambda\nabla_\nu S+k^\mu k_\nu\nabla_\lambda S-2k_\nu k_\lambda\nabla^\mu S), \tag{41}$$

using Eq. (25). The corresponding shift in the curvature is then $R'^{\mu}{}_{\nu\lambda\sigma}=R^{\mu}{}_{\nu\lambda\sigma}+\nabla_\lambda\Omega^{\mu}_{\nu\sigma}-\nabla_\sigma\Omega^{\mu}_{\nu\lambda}+\Omega^{\rho}_{\nu\sigma}\Omega^{\mu}_{\rho\lambda}-\Omega^{\rho}_{\nu\lambda}\Omega^{\mu}_{\rho\sigma}$. However, the vanishing Lie derivatives $\mathcal{L}_k\kappa=0=\mathcal{L}_k S$ lead to $\Omega^{\rho}_{\nu\sigma}\Omega^{\mu}_{\rho\lambda}-\Omega^{\rho}_{\nu\lambda}\Omega^{\mu}_{\rho\sigma}=0$. Hence the shifted curvature reduces to

$$R'^{\mu}{}_{\nu\lambda\sigma}=R^{\mu}{}_{\nu\lambda\sigma}+\nabla_\lambda\Omega^{\mu}_{\nu\sigma}-\nabla_\sigma\Omega^{\mu}_{\nu\lambda}. \tag{42}$$

Hence the curvature with all covariant indices is $R'_{\mu\nu\lambda\sigma}=R_{\mu\nu\lambda\sigma}+2g_{\mu\rho}\nabla_{[\lambda}\Omega^{\rho}_{\sigma]\nu}+\kappa\, k_\mu k^\rho R_{\rho\nu\lambda\sigma}$. Now, using Eq. (27), as well as rewriting $2g_{\mu\rho}\nabla_{[\lambda}\Omega^{\rho}_{\sigma]\nu}$ with Eq. (25), we arrive at

$$\begin{aligned}R'_{\mu\nu\lambda\sigma}=R_{\mu\nu\lambda\sigma}&+2k_{[\mu}\nabla_{\nu]}\nabla_{[\lambda}\kappa\, k_{\sigma]}-4\kappa\, k_{[\mu}\nabla_{\nu]}\nabla_{[\lambda}S\, k_{\sigma]}\\&+3k_{[\mu}\nabla_{\nu]}\kappa\, k_{[\lambda}\nabla_{\sigma]}S+3k_{[\mu}\nabla_{\nu]}S\, k_{[\lambda}\nabla_{\sigma]}\kappa\\&-5\kappa\, k_{[\mu}\nabla_{\nu]}S\, k_{[\lambda}\nabla_{\sigma]}S.\end{aligned} \tag{43}$$

Here the key observation is that the primed curvature can be decomposed as $R'_{\mu\nu\lambda\sigma}=R_{\mu\nu\lambda\sigma}+\chi_{\mu\nu\lambda\sigma}$, where the tensor $\chi_{\mu\nu\lambda\sigma}$ is bilinear in the vector k_μ.

In fact, a similar decomposition holds for any covariant derivative of the curvature. To establish this result, first note that $\nabla'_\rho R'_{\mu\nu\lambda\sigma}=\nabla_\rho R_{\mu\nu\lambda\sigma}+\chi_{\rho\mu\nu\lambda\sigma}$, where $\chi_{\rho\mu\nu\lambda\sigma}=\nabla_\rho\chi_{\mu\nu\lambda\sigma}-\Omega^{\omega}_{\rho\mu}R'_{\omega\nu\lambda\sigma}-\Omega^{\omega}_{\rho\nu}R'_{\mu\omega\lambda\sigma}-\Omega^{\omega}_{\rho\lambda}R'_{\mu\nu\omega\sigma}-\Omega^{\omega}_{\rho\sigma}R'_{\mu\nu\lambda\omega}$. It is straightforward to show that the tensor $\chi_{\rho\mu\nu\lambda\sigma}$ is bilinear in the vector k. This is clear for $\nabla\chi$ using Eqs. (25) and (43), and for the ΩR terms using Eq. (40). Now similarly the $\Omega\chi$ terms are quartic in k, but a closer examination shows that the sum of these terms vanishes. Then, by induction, we see that if $\chi_{\rho_1\cdots\rho_n\mu\nu\lambda\sigma}$ is at least bilinear in k_μ, then $\chi_{\rho\rho_1\cdots\rho_n\mu\nu\lambda\sigma}$ must also be so. This is again straightforward by combining the above formulas in the definition of χ:

$$\begin{aligned}\chi_{\rho\rho_1\cdots\rho_n\mu\nu\lambda\sigma}&\equiv\nabla'_\rho\nabla'_{\rho_1}\cdots\nabla'_{\rho_n}R'_{\mu\nu\lambda\sigma}-\nabla_\rho\nabla_{\rho_1}\cdots\nabla_{\rho_n}R_{\mu\nu\lambda\sigma}\\&=\nabla_\rho\chi_{\rho_1\cdots\rho_n\mu\nu\lambda\sigma}+\Omega^{\omega}_{\rho\rho_1}\chi_{\omega\cdots\rho_n\mu\nu\lambda\sigma}+\cdots\\&\quad+\Omega^{\omega}_{\rho\rho_1}\nabla_\omega\cdots\nabla_{\rho_n}R_{\mu\nu\lambda\sigma}+\cdots.\end{aligned}$$

Hence, all of the new κ-dependent terms appearing in the primed curvature and covariant derivatives thereof, $\chi_{\rho\rho_1\cdots\rho_n\mu\nu\lambda\sigma}$, are at least bilinear in k. A detailed inspection shows that $\chi_{\rho_1\cdots\rho_n\mu\nu\lambda\sigma}$ contains a sum of terms of order k^{2p} with $p=1,2,\ldots,1+\lfloor n/2\rfloor$, where $\lfloor n/2\rfloor$ denotes the integer part of $n/2$. The precise powers will not be important below, the key point being that they all begin at k^2.

Finally we close this subsection by noting that the Levi-Cività tensor is invariant under the shift between the original and primed metrics, i.e., $\varepsilon'_{\mu\cdots\rho}=\varepsilon_{\mu\cdots\rho}$. This result follows since, as noted in Sec. II, the determinants of the two metrics are equal because k is null.

D. Scalar curvature invariants

We are now ready to consider the scalar curvature invariants which can be built for the shifted metric. The most general invariant will consist of an arbitrary product of curvatures, covariant derivatives thereof,[4] and Levi-Cività tensors, with their indices (all assumed to be covariant) contracted by the inverse metric $g'^{\mu\nu}$. Hence a generic term is of the form

$$\mathcal{I}'=\prod_{j=1}^{N}T'^{j}_{\nu_1^j\cdots\nu^j_{q_j}}\prod_{k=1}^{K}\varepsilon'_{\lambda_1^k\cdots\lambda_D^k}\prod_{l=1}^{M}g'^{\alpha_l\beta_l}, \tag{44}$$

along with a rule for contracting the upper with the lower indices. Here $\Sigma_{j=1}^{N}q_j+DK=2M$, where D is the dimension of the spacetime. In fact, one need only consider $K=0$ or 1 since the product of two Levi-Cività tensors can be reduced to a sum of products of metric tensors.

Now, from the previous subsection, we know that all of the tensors in the first product can be decomposed as $T'_{\mu_1\cdots\mu_n}=T_{\mu_1\cdots\mu_n}+\chi_{\mu_1\cdots\mu_n}$, where $\chi_{\mu_1\cdots\mu_n}$ is at least bilinear in the vector k. Further, we have $\varepsilon'=\varepsilon$ while the inverse metric is given by $g'^{\mu\nu}=g^{\mu\nu}-\kappa k^\mu k^\nu$. Hence we see that the invariant (44) can be decomposed as

[3]Equation (39) is the essential result required to prove Eq. (24).

[4]Here one might also include covariant derivatives of the scalar S which are implicitly geometric tensors derived from the original metric. Admitting these extra tensors would not change our conclusions.

$$\mathcal{I}'=\prod_{j=1}^{N}(T_{\nu_1^j\cdots\nu_{q_j}^j}+\chi_{\nu_1^j\cdots\nu_{q_j}^j})\prod_{k=1}^{K}\varepsilon_{\lambda_1^k\cdots\lambda_D^k}\prod_{l=1}^{M}(g^{\alpha_l\beta_l}-\kappa k^{\alpha_l}k^{\beta_l})=\mathcal{I}+\mathcal{J}, \tag{45}$$

where $\mathcal{I}$ is the invariant of the same algebraic structure as $\mathcal{I}'$ but constructed for the original geometry. The difference $\mathcal{J}$ then contains all of the information about the wave. Simply multiplying out the terms in Eq. (45), we may write

$$\mathcal{J}=\sum_{i=1} k^{\mu_1}k^{\nu_1}\cdots k^{\mu_i}k^{\nu_i}\hat{T}_{\mu_1\nu_1\cdots\mu_i\nu_i}, \tag{46}$$

where the tensors $\hat{T}$ are products of primary tensors and possibly Levi-Cività tensors, including contractions by $g^{\mu\nu}$. Given the results of Sec. II C, we are assured that $\mathcal{J}$ is at least bilinear in k. Note that antisymmetry of indices, e.g., in ε, may eliminate certain contributions above. However, from Sec. III B, we know that the tensors $\hat{T}$ factorize as in Eq. (30) when contracted with k. Hence we conclude that $\mathcal{J}\propto k_\mu k^\mu=0$ and so $\mathcal{I}'=\mathcal{I}$. Thus we see that any scalar curvature invariant is identical for the original and primed metrics, which concludes our proof of the theorem.

We note that we can easily generalize our theorem to cover scalar invariants constructed using both the geometry and matter fields. As in Sec. II, we consider a matter sector including various scalars ϕ_a and p-form potentials $A_{(p)}$. We also require that these fields satisfy the same constraints as there: They are form invariant along the k flow, i.e., $\mathcal{L}_k\phi_a=0=\mathcal{L}_kF_{(p+1)}$ as in Eqs. (9) and (10). The field strengths are transverse to the flow, i.e., $i_kF_{(p+1)}=k\wedge\theta_{(p-1)}$ as in Eq. (11). Recall that we also have $i_k\theta_{(p-1)}=0$. Given these results, one can further show that $\mathcal{L}_k\theta_{(p-1)}=0$—this is easily shown by referring to the local coordinate patches introduced at the end of Sec. III B.

Let us then reconsider the contraction identities proved in Sec. III B. Given the properties imposed on the matter fields above, it is straightforward to extend the discussion to include the scalars, the field strengths, and covariant derivatives of these, as primary tensors which satisfy the factorization equation (30). Similarly one may also show that all higher-order tensors and hence arbitrary products of the primary tensors satisfy the same factorization property. Next as in Sec. III C, we consider the shift in tensors calculated for the original and primed metrics. While the scalars and field strengths themselves are not affected by the shift in the metric, using Eq. (40) as well as Eq. (25), it is clear that the shifts in covariant derivatives of these fields will always be at least bilinear in the vector k. Hence both of the crucial results established for the curvature and its covariant derivatives are easily extended to the matter fields and their covariant derivatives. Thus the invariant (44) can be extended so that the T also include these latter fields, and the same final result still holds; i.e., the invariant is independent of the wave profile κ.

In summary, then, we find the rather surprising result that all scalar invariants, involving any number of covariant derivatives of the curvature and/or matter fields, are identical for both the original and the shifted metrics. The essential requirement that the original metric has to satisfy is to have a null, hypersurface-orthogonal, Killing vector, which is supplemented with certain constraints on the matter fields as well. Thus no scalar invariant contains any information about the wave profile κ, and hence to determine how the geometry has been modified, one must consider quantities such as tidal forces or nonlocal holonomies. We emphasize that our theorem is of a purely geometric nature and holds for any theory of gravity in any number of dimensions, as long as it assumes a pseudo-Riemannian geometry as the basis of the description of gravitational phenomena.

IV. FIVE-DIMENSIONAL BLACK STRING

Our original motivation in this project was to investigate the properties of extended black objects in higher dimensions carrying time-dependent or wavy hair. The theorem of the previous section tells us that coordinate-invariant probes are inadequate to examine the properties of such undulating solutions constructed through the GV technique. Hence, to study the smoothness of the horizon in the presence of a wave in the next section, we are led to consider the existence of parallelly propagated curvature singularities. In order to do so, however, we must consider a concrete example. Thus, in this section, we present a family of undulating black strings which are low-energy solutions of heterotic string theory in five dimensions. We begin with a stationary solution with a null hypersurface-orthogonal Killing vector, which results from uplifting a four-dimensional solution first written down by Cvetič and Youm [14]. Then we apply the GV technique to generate oscillations on the string. Similar oscillations of singular strings were considered in Refs. [6,5] and of black strings in Refs. [9,15,8].

The low-energy action for heterotic string theory in five dimensions includes the terms

$$I=\int d^5x\sqrt{-G}\,e^{-2\Phi}[R(G)+4(\nabla\Phi)^2-\tfrac{1}{12}H^2-(\nabla\sigma)^2-\tfrac{1}{4}e^{2\sigma}F^2-\tfrac{1}{2}e^{-2\sigma}\hat{F}^2], \tag{47}$$

as well as the metric; we have included two scalars, the dilaton Φ and a modulus field σ; two gauge fields with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\hat{F}_{\mu\nu}=\partial_\mu\hat{A}_\nu-\partial_\nu\hat{A}_\mu$; and the Kalb-Ramond field with

$$H_{\mu\nu\lambda}=\partial_\mu B_{\nu\lambda}-\tfrac{1}{2}(A_\mu\hat{F}_{\nu\lambda}+\hat{A}_\mu F_{\nu\lambda})+(\text{cyclic permutations}). \tag{48}$$

The metric above is the so-called string-frame metric. The Einstein-frame metric would be given by

$$g_{\mu\nu}=e^{-4\Phi/3}G_{\mu\nu}. \tag{49}$$

With the latter metric then, the dilaton coupling in the Einstein term is eliminated and the action becomes

$$I=\int d^5x\sqrt{-g}[R(g)-\tfrac{4}{3}(\nabla\Phi)^2-\tfrac{1}{12}e^{-8\Phi/3}H^2-(\nabla\sigma)^2-\tfrac{1}{4}e^{2\sigma-4\Phi/3}F^2-\tfrac{1}{4}e^{-2\sigma-4\Phi/3}\hat{F}^2]. \tag{50}$$

However, we choose to present our solution in terms of the string-frame metric, which has a much simpler appearance in the present case.

We will be interested in the following black string solution: The string frame metric is

$$ds^2=\frac{f}{h}\,du^2+\frac{2}{h}\,du\;dv+k\ell\,[dr^2+r^2(d\theta^2+\sin^2\theta\;d\phi^2)], \tag{51}$$

while the remaining fields are given by

$$B=\frac{1}{h}\,du\wedge dv,$$

$$A=-P_1\cos\theta\;d\phi,\quad \hat{A}=P_2\cos\theta\;d\phi,$$

$$e^{2\sigma}=\ell/k,\quad e^{4\Phi}=k\ell/h^2, \tag{52}$$

where we defined the functions

$$f\equiv 1+\frac{Q_1}{r},\quad h\equiv 1+\frac{Q_2}{r},$$

$$k\equiv 1+\frac{P_1}{r},\quad \ell\equiv 1+\frac{P_2}{r}\ . \tag{53}$$

Hence this configuration is specified by four different parameters. The solution could be simplified by setting all of these equal; however, we wish to illustrate that our results apply in the generic case. We will assume that all of the constants are positive in order that our solution properly describe a black string with a horizon at $r=0$.

Considering the asymptotic metric, one has, for large r,

$$\begin{aligned} ds^2&\to du^2+2\,du\;dv+dr^2+r^2d\Omega\\ &=-dt^2+dy^2+dr^2+r^2d\Omega, \end{aligned} \tag{54}$$

where $y=u+v$ and $t=v$. Hence we should consider y as the spatial coordinate running parallel to the string, while t is the asymptotic time. Note that as $r\to 0$, $g_{rr}\simeq P_1P_2/r^2$, indicating the presence of a degenerate horizon. Near the horizon, the metric becomes

$$ds^2\simeq\frac{Q_1}{Q_2}du^2+\frac{2r}{Q_2}\,du\;dv+P_1P_2\left[\left(\frac{dr}{r}\right)^2+d\Omega^2\right]. \tag{55}$$

The solution has two Killing vectors which are of interest

$$k^\mu\partial_\mu=\partial_v=\partial_t+\partial_y,\quad h^\mu\partial_\mu=\partial_u=\partial_y. \tag{56}$$

(These are in addition to the standard rotational Killing vectors for θ and ϕ.) The first of these is the null generator of the horizon. In this role, k has the rather unusual feature that it is null everywhere—not just at the horizon. Further, it is not given by ∂_t in the asymptotic coordinates; rather, we have $k^\mu\partial_\mu=\partial_v=\partial_t+\partial_y$. Here the ∂_y contribution is related to the presence of linear motion along the y direction. The coefficient may be interpreted as the ''horizon velocity,'' which in the present case is 1, i.e., the speed of light.

The null Killing vector k is hypersurface orthogonal as well, satisfying

$$\nabla_{[\mu}k_{\nu]}=k_{[\mu}\nabla_{\nu]}\ln h, \tag{57}$$

where h is the same function defined in Eq. (53). Hence the metric admits the symmetry desired for the wave-generating technique. It is also straightforward to show that the matter fields satisfy the appropriate conditions (9)–(11). Hence, following the discussion of Sec. II, we apply Eq. (21) to define a new string-frame metric

$$G'_{\mu\nu}=G_{\mu\nu}+h\Psi k_\mu k_\nu, \tag{58}$$

where, following Eq. (22), Ψ is chosen to satisfy

$$k^\mu\nabla_\mu\Psi=0\quad\text{and}\quad\partial_\mu(e^{-2\Phi}\sqrt{-G}G^{\mu\nu}\partial_\nu\Psi)=0. \tag{59}$$

The first condition is simply $\partial_v\Psi=0$; i.e., Ψ is independent of v. In the present case, the second condition reduces to $\nabla_F^2\Psi=0$, i.e., Laplace's equation on a flat spatial metric in the transverse coordinates (r,θ,ϕ). Thus the general solution for Eq. (59) may be written as

$$\Psi=\sum_{l,m}\,[a_{lm}(u)\;r^l+b_{lm}(u)\;r^{-(l+1)}]Y_{lm}(\theta,\phi), \tag{60}$$

where $Y_{lm}(\theta,\phi)$ are usual spherical harmonics, and a_{lm} and b_{lm} are arbitrary functions of u.

Let us consider the various perturbations in turn. The case with r^l and $l=0$ yields an asymptotically flat metric, which in fact is diffeomorphic to the original:

$$\begin{aligned} ds'^2&=\frac{1}{h}[f+a(u)]\;du^2+\frac{2}{h}\,du\;dv+k\ell(dr^2+r^2d\Omega)\\ &=\frac{f}{h}\,du^2+\frac{2}{h}\,du\;d\tilde{v}+k\ell(dr^2+r^2d\Omega), \end{aligned} \tag{61}$$

where $d\tilde{v}=dv+\frac{1}{2}a(u)du$. Note that the constant term in f is a special constant case of these perturbations, and so could also be eliminated in the same way. With the choice r^l and $l>1$, the metric is not asymptotically flat, and so we do not consider these solutions as providing perturbations intrinsic to the black string, i.e., ''wavy'' hair. Instead they would be more accurately described as embedding the string in a space filled with (asymptotic) gravitational radiation. The same is apparently true for the r^l mode with $l=1$, but in fact this solution yields an asymptotically flat metric, as is seen as follows [6,5]: First introduce Cartesian coordinates on the transverse space, in which case the wavy metric becomes

$$ds'^2=\frac{1}{h}[f+a_i(u)x^i]du^2+\frac{2}{h}\,du\;dv+k\ell\;dx_idx^i. \tag{62}$$

However, the coordinate transformation

$$\tilde{v}=v+\dot{A}_ix^i-\frac{1}{2}\int^u\dot{A}_i^2du,$$

$$\tilde{x}^i=x^i-A^i, \tag{63}$$

with $2\ddot{A}_i \equiv a_i$, produces a metric which is manifestly asymptotically flat:

$$d\tilde{s}^2 = \frac{1}{h}[f + (hk\not{\ell} - 1)\dot{A}_i^2]du^2 + \frac{2}{h}\,du[d\tilde{v} + (hk\not{\ell} - 1)\dot{A}_i d\tilde{x}^i] + k\not{\ell}\; dx_i dx^i. \tag{64}$$

These waves represent oscillations of the string in the transverse space.

The perturbations generated with $r^{-(l+1)}$ are all localized near the horizon and leave the metric asymptotically flat. Hence we may consider these deformations as candidates for ''wavy'' hair on the black string. For $l>0$, these deformations produce $g_{uu}\to\infty$ as we approach the ''horizon'' at $r=0$. Note, however, that this divergence does not effect the volume element $\sqrt{-g}$, and so we should be careful in deciding whether or not these perturbations produce a true singularity[5] at $r=0$. In the next section, we will see that in fact with $l>0$ these solutions are singular, and hence we are only left with a wavy black string for $l=0$, in which case

$$ds'^2 = \frac{1}{h}\left(f + \frac{b(u)}{r}\right)du^2 + \frac{2}{h}\,du\;dv + k\not{\ell}(dr^2 + r^2 d\Omega). \tag{65}$$

These perturbations represent longitudinal waves carrying momentum along the string without oscillations. Note that the Q_1 term in f represents a constant contribution to $b(u)$. One may wonder if this wave is really physical or merely an artifact of an awkward choice of coordinates, given our theorem on the elusiveness of the wave profile. To see that it is indeed physical, we can compute its mass per unit length. Since the oscillating string (65) is asymptotically flat, and we can use the coordinates of Eq. (54), we can determine the mass per unit length according to $dE/dy = -(1/8\pi)\int_S \varepsilon_{ybcde} dx^b dx^c \nabla^d \xi^e$, where all the tensors are defined in the Einstein frame. The vector field $\xi = \partial_t$ here is the asymptotic generator of time translations, and the integration is carried over a sphere at spatial infinity. The result is $dE/dy = [P_1 + P_2 + Q_2 + 3Q_1 + 3b(y-t)]/6$, and since the mass per unit length depends on the wave profile, we see that the solution is really a superposition of the string and the wave and so is clearly different form the stationary string, where $b=0$.

A. Parallel propagated singularities

Having applied the GV technique to the original solution above, we have apparently generated a large family of oscillating black string solutions. However, as discussed above, we have reason to worry that some of the modes may actually introduce a curvature singularity at the null surface which was originally the black string's horizon. Normally the approach to proving the existence of a horizon would be to find coordinates in which the metric is analytic at the null surface in question. For the present undulating solutions, finding such coordinates is an enormous problem (see, e.g., [9]). While we address this question for the monopole waves, i.e., those with $l=m=0$, in the following section, the task at hand is in fact much simpler. We wish to show that for $l>0$ a given null hypersurface is not a horizon, which simply requires finding any geometric quantity which diverges when the surface is approached along some geodesic. By the properties of our original black string and the approach used to generate their wavy counterparts, the theorem of Sec. III tells us that no scalar invariant involving the curvature and/or the matter fields contains any information about the waves. Therefore, given that all scalar invariants are insensitive to the oscillations and thus to any singularity which they may introduce, we resort to alternative means of probing the wavy geometry.

Tidal forces prove to be a good tool for resolving our problem. If we approach the null surface along a geodesic, and we allow our observer to be slightly nonlocal (e.g., a string which may indeed seem the natural probe in the present context), this observer will be able to determine the differences between the gravitational forces acting at different points. These forces are determined by the Riemann curvature measured in the rest frame of our observer. One may object to the notion of the observer's rest frame, as we have just said that the observer of interest is nonlocal. We will assume that the extension of the observer is controllably small, and hence that the center-of-mass frame represents a good reference frame in which to express the results.

Still, identifying a convenient geodesic trajectory to follow in the presence of a general oscillation proves to be beyond our abilities. The reason is that one cannot find enough integrals of motion to solve the problem in terms of quadratures. We therefore restrict our attention to wave profiles which are constant in u. In this case the extra Killing vector $h^\mu \partial_\mu = \partial_u$ yields an additional constant of the motion, enabling us to find analytically suitable geodesics for any mode. These solutions should be a very good approximation for backgrounds with slowly varying wave profiles. When we compute the full curvature of an undulating string, we can see that any u dependence only adds contributions of a subleading order. Hence, if the curvature turns out to be divergent as some hypersurface $r=\text{const}$ is approached, the constant profile solutions will contain all the information about the leading order of divergences.

We present our calculations in several steps. First we will show that for each mode of oscillation there exists a geodesic stretching between the null surface and the asymptotic infinity—hence showing that both of these regions belong to the space time. Next we will construct the Lorentz transformation relating a natural stationary orthonormal frame to the rest frame of the observer moving along the geodesic. Finally we will consider the orthonormal frame curvature and boost it to the frame of the infalling observer, in order to find the tidal forces which he measures. The divergences found in this way are equivalent to parallelly propagated curvature singularities. We will isolate the leading divergences of the

[5]The divergence in g_{uu} does indicate a divergence of the norm of the Killing vector $h^\mu \partial_\mu = \partial_u$, which could be interpreted as a geometric singularity [10]. One should still demonstrate that $r=0$ is accessible to causal observers, i.e., that this region ''belongs'' to the spacetime, as is done in our following analysis.

tidal forces for all modes both at $r=0$ and asymptotically, finding that all the localized multipoles with $l\geqslant 1$ have unbounded tides on the null surface and that all the growing multipole modes with $l\geqslant 2$ have divergences at asymptotic infinity.

1. Geodesics

We begin by examining the timelike geodesics of a wavy solution which is excited by a single mode. We will demonstrate that there are always geodesics extending between $r=0$ and the asymptotic region $r\to\infty$, and the former is reached in finite affine time when starting from finite r. Hence this null surface $r=0$ must be included in the manifold described by the wavy solution.

We may write the general solution as

$$ds^2=2F_2du\,dv+F_3^2du^2+\frac{1}{F_1^2}[dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)], \tag{66}$$

where

$$F_2=\frac{1}{h},\quad F_1^2=\frac{1}{kl},\quad f_3^2=\frac{f+\Psi}{h},$$

$$\Psi=B(u)r^\beta P_l^m(\cos\theta)\cos[m\phi+\delta(u)], \tag{67}$$

and f,h,k,l are defined in Eq. (53). To produce a simple real metric, we have expressed the angular dependence in terms of an associated Legendre function of the first kind, $P_l^m(\cos\theta)$, as well as the $\cos(m\phi+\delta)$ factor, rather than using spherical harmonics as in Eq. (60). Those solutions (60) are then reproduced by setting $\beta=-(l+1)$ or l, which corresponds to what we will call the localized and the growing modes, respectively. However, for much of the following analysis, we will leave this exponent as β in order to emphasize the contributions coming from the differentiation of this factor. In principle the amplitude B and the phase δ are arbitrary functions of u, but as discussed above, to simplify the analysis of the geodesics, we will set both of these to be constants in the following. This will be enough to identify the leading divergences, and should still provide good approximation in the case of a slow u dependence. In fact we will set $\delta=0$, which can be attained with a simple shift of ϕ.

As usual, to obtain the geodesic equations, we simply consider the Lagrangian $L=(ds/d\lambda)^2$, and write down the Euler-Lagrange equations. Because the Lagrangian does not contain a potential, the effective Hamiltonian is conserved, giving $(ds/d\lambda)^2=\text{const}$. Given this integral of motion, we need not consider the Euler-Lagrange equation for the radial coordinate. In addition, the two translational Killing vectors ∂_u,∂_v produce two more first integrals, which leaves us with second-order differential equations only for the two angular coordinates. Hence the equations for timelike geodesics are

$$F_2u'=p-\omega,\quad F_2v'+F_3^2u'=p,$$

$$2F_2v'u'+F_3^2u'^2+\frac{1}{F_1^2}(r'^2+r^2\theta'^2+r^2\sin^2\theta\phi'^2)=-1,$$

$$2\left(\frac{r^2\theta'}{F_1^2}\right)'=2\frac{r^2\sin\theta\cos\theta}{F_1^2}\phi'^2+\frac{B}{h}r^\beta\frac{dP_l^m(\cos\theta)}{d\theta}\cos m\phi\;u'^2, \tag{68}$$

$$2\left(\frac{r^2\sin^2\theta}{F_1^2}\phi'\right)'=-m\frac{B}{h}r^\beta P_l^m(\cos\theta)\sin m\phi\;u'^2,$$

where we expressed the integrals of motion as ω and p. When $\beta<0$, one finds that in the asymptotic region (54) these correspond to the energy (which is assumed to be positive) and linear momentum along y, respectively. The last of these equations may be solved by setting $\phi=\phi_n=(\pi/m)n$ with $n=0,1,\ldots,2m-1$ for $m\neq 0$, while for $m=0$ we may fix ϕ to be any constant. Now, with the constant ϕ, the first term on the right-hand side (RHS) of the second-to-last equation also vanishes. Hence the latter equation is solved if we now choose $\theta=\theta_0=\text{const}$, where θ_0 corresponds to an extremum of Legendre function $P_l^m(\cos\theta)$. Then the independent set of equations defining these radial geodesics may be written, using $F_3^2=[f+(-)^nBr^\beta P_l^m(\cos\theta_0)]/h$,

$$u'=\frac{p-\omega}{F_2},\quad v'=\frac{p}{F_2}+[f+(-)^nBr^\beta P_l^m(\cos\theta_0)]\frac{\omega-p}{h\,F_2^2}, \tag{69}$$

$$r'^2=[f+(-)^nBr^\beta P_l^m(\cos\theta_0)]\frac{F_1^2}{h\,F_2^2}(\omega-p)^2$$

$$+2\frac{F_1^2}{F_2}(\omega-p)p-F_1^2. \tag{70}$$

Now we must choose θ_0 and ϕ_n, as well as the constants ω and p, in such a way that our geodesics (a) extend to infinity and (b) reach the null surface $r=0$. We need the former to ensure that our observer is physically connected to recording devices infinitely far away from the gravitational source. The second condition that the geodesic does not turn before reaching $r=0$ is necessary because we want to probe this region for singularities.

We begin by considering the localized modes with $\beta=-(l+1)<0$. From the radial equation (70), we see that as $r\to\infty$, for which $h, F_{1,2}\to 1$ and $r^\beta\to 0$, and so $r'^2\to\omega^2-p^2-1$. Thus we require $\omega\geqslant\sqrt{p^2+1}$ so that the geodesics extend to infinity. Now, in the limit $r\to 0$, we have $F_2=h^{-1}\propto r$, $F_1^2\propto r^2$, and $f\propto 1/r$. Thus the leading contribution on the RHS of Eq. (70) comes from the r^β term when $\beta<-1$, and hence we must choose θ_0 and n such that $(-)^nBP_l^m(\cos\theta_0)>0$ in order that no turning points occur before reaching $r=0$. For $l\geqslant 1$ there exist many extrema of $P_l^m(\cos\theta)$, and it is straightforward to verify that one has enough freedom to choose the angles in order that the geodesic reaches $r=0$ (even for the case $m=0$). For the special case $\beta=-1$ or $l=0$, turning points are evaded[6] as long as $Q_1+B>0$.

[6]Note then that for $l=0$ and $B<-Q_1$, one has solutions for which the radial geodesics can never reach $r=0$. As pointed out in Ref. [22] however, when the y direction is compactified in this case, there appear closed timelike curves, which can communicate with observers at infinity, implying the breakdown of chronology. In any event, we will ignore this pathological case in the following.

For the growing modes with $\beta=l\geqslant 0$, the analysis is similar to that above. Note that in this case the interpretation of ω and p would not be correct since the asymptotic structure of the metric is modified. Beginning with the limit $r\to 0$, we can ignore the r^β contribution in Eq. (70) and in fact one finds that this region is always reached without imposing any constraints. In the asymptotic region $r\to\infty$, the r^β term dominates Eq. (70) for $\beta>0$, and we must again choose the angles such that $(-)^n BP_l^m(\cos\theta_0)>0$ if the geodesics are to extend all the way to infinity without turning back. For the special case $\beta=0=l$, the constraint to reach the asymptotic region becomes $(\omega-p)(\omega+p+B)\geqslant 1$ which can always be satisfied with an appropriate choice for the integrals of the motion.

In the following, it will be convenient to absorb $(-)^n P_l^m(\cos\theta_0)$ into the amplitude. Hence we define $B_0\equiv(-)^n BP_l^m(\cos\theta_0)$. For the special case with $\beta=-1$ and $l=0$, we set $B_0\equiv Q_1+B$.

2. Lorentz transformation

Hence we have defined an interesting set of radial geodesics for which the tangent vector is given by $V^\mu=dx^\mu/d\lambda$, where

$$u'=\frac{p-\omega}{F_2},\quad v'=\frac{p}{F_2}+(f+B_0 r^\beta)\frac{\omega-p}{h\,F_2^2},$$

$$r'=\pm\left((f+B_0 r^\beta)\frac{F_1^2}{h\,F_2^2}(\omega-p)^2+2\frac{F_1^2}{F_2}(\omega-p)p-F_1^2\right)^{1/2}, \tag{71}$$

and $\theta'=0=\phi'$. Here the $-$ $(+)$ sign corresponds to inward (outward) directed geodesics. We will want to examine the tidal forces in the rest frame of an observer moving with this five-velocity. Hence, as an intermediate step, we determine the Lorentz transformation which takes us from a natural stationary frame, in which the curvature is easily calculated, to the observer's freely falling frame.

First to define our stationary orthonormal frame, we complete the squares in our general metric (66):

$$ds^2=-\frac{F_2^2}{F_3^2}dv^2+F_3^2\left(du+\frac{F_2}{F_3^2}dv\right)^2+\frac{1}{F_1^2}[dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)]. \tag{72}$$

We see that an obvious orthonormal basis of one-forms is

$$e^0=\frac{F_2}{F_3}dv,\quad e^4=F_3 du+\frac{F_2}{F_3}dv,\quad e^r=\frac{dr}{F_1},$$

$$e^\theta=\frac{r\,d\theta}{F_1},\quad e^\phi=\frac{r\sin\theta\,d\phi}{F_1}. \tag{73}$$

Note that in this basis, e^0 is distinguished as the unit-time-like one-form, at least everywhere along our radial geodesics.[7] In this frame, we have $V^a=e^a{}_\mu V^\mu$, where the $e^a{}_\mu$ are the components of the fünfbein (73),

$$V^0=\frac{F_3}{F_2}(\omega-p)+\frac{p}{F_3},\quad V^4=\frac{p}{F_3},$$

$$V^r=\pm\left(\frac{F_3^2}{F_2^2}(\omega-p)^2+2\frac{(\omega-p)p}{F_2}-1\right)^{1/2}, \tag{74}$$

along with $V^\theta=0=V^\phi$. As a check, one may easily verify that $\eta_{ab}V^aV^b=-1$.

Now we wish to find a Lorentz transformation which takes a unit-time-like vector $N^a=\delta^a_0$ into the observer's five-velocity: $V^a=L^a{}_b N^b$. Then applying this transformation to our stationary fünfbein (73) would produce a natural basis of orthonormal one-forms which the observer might use in his rest frame. Of course, we are left with some ambiguity in the choice of the SO(1,4) matrix defining this Lorentz transformation. One approach to resolving this ambiguity is defining the transformation by parallell propagating the stationary frame in from infinity along our radial geodesic. Since $V^\theta=0=V^\phi$, a much less labor-intensive approach is to simply take the uniquely defined SO(1,2) matrix mixing only the 0, 4, and r directions. The latter will differ from that defined through parallel propagation by an SO(4) transformation mixing the spatial one-forms. Such a rotation, however, will not introduce any new divergences, and hence it suffices to consider the simpler boost. One can think of this transformation as the result of parallelly propagating in not our stationary fünfbein (73) but rather a rotated version of it. The simpler SO(1,2) Lorentz transformation may be written as

$$L^a{}_b=\begin{pmatrix} V^0 & V^4 & V^r & 0 & 0\\ V^4 & 1+\dfrac{(V^4)^2}{V^0(V^0+1)} & \dfrac{V^4\,V^r}{V^0(V^0+1)} & 0 & 0\\ V^r & \dfrac{V^4\,V^r}{V^0(V^0+1)} & 1+\dfrac{(V^r)^2}{V^0(V^0+1)} & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{75}$$

At this point let us comment at the behavior of the five-velocity in the regions of interest. For $\beta\leqslant -1$ and $r\to 0$, the dominant factor is $F_3\propto r^{(\beta+1)/2}$, leading to $V^0\simeq(\omega-p)\sqrt{B_0 Q_2}\,r^{(\beta-1)/2}\simeq V^r$, while $V^4\simeq 0$. Hence the observer is accelerated to almost a null radial geodesic as he nears $r=0$, and L approaches an infinite boost in the radial direction. For $\beta\geqslant 0$, F_3 and all of the components of V^a are finite at this null surface. Similarly, for $\beta>0$ and $r\to\infty$, $F_3\propto r^{\beta/2}$ again dominates to producing the five-velocity $V^0\simeq(\omega-p)\sqrt{B_0}r^{\beta/2}\simeq V^r$ and $V^4\simeq 0$. Hence Eq. (75) yields

[7] This is guaranteed since $F_2^2=h^{-2}$ is trivially positive, while we ensured that $F_3^2>0$ in order that the geodesics reached between $r=0$ and $r\to\infty$. The sole exception, which we ignore, is the case $\beta=0=l$ for which we could have $F_3^2<0$ if $B<Q_1$.

an infinite radial boost as $r\to\infty$. Finally, we note that for $\beta\leqslant 1$ and $\beta=0$, F_3 and all of the components of V^a and L remain finite in the asymptotic region.

3. Divergent tides

Now we wish to examine the tidal forces experienced by an observer following our radial geodesics. We will focus here on the curvature components with all indices in the $0,4,r$ subspace, and study the corresponding components in the observer's rest frame. Since our Lorentz transformation (75) does not mix these indices with the θ,ϕ directions, it simplifies our presentation to only consider these components. Our omission of the components carrying angle indices does not imply that they are finite. In fact the components with two indices in the θ,ϕ subspace also typically diverge whenever we find divergences in the following analysis and in qualitatively the same way, as discussed at the end of this section. Our final results are that divergent gravitational tides appear on the null surface $r=0$ for the localized modes with $\beta<-1$ and at asymptotic infinity for the growing modes with $\beta>2$.

With our definition of the orthonormal basis (73), it is straightforward to compute the curvature in this frame. As discussed above, we focus on the curvatures in the $0,4,r$ subspace for which the nonzero components are

$$R^{4r4r}=-\frac{F_1}{F_3}F_3'\,F_1'+\frac{F_1^2}{F_2F_3}F_3'\,F_2'-\frac{F_1^2}{F_3}F_3''-\frac{F_1^2}{F_3^2}(F_3')^2-\frac{F_1^2}{4F_2^2}(F_2')^2,$$

$$R^{0r0r}=-\frac{F_1}{F_3}F_3'\,F_1'+\frac{F_1^2}{F_2F_3}F_3'\,F_2'-\frac{F_1^2}{F_3}F_3''-\frac{F_1^2}{F_3^2}(F_3')^2+\frac{F_1^2}{F_2}F_2''+\frac{F_1}{F_2}F_2'\,F_1'-\frac{3F_1^2}{4F_2^2}(F_2')^2,$$

$$R^{0404}=\frac{F_1^2}{4F_2^2}(F_2')^2,$$

$$R^{4r0r}=-\frac{F_1}{F_3}F_3'\,F_1'+\frac{F_1^2}{F_2F_3}F_3'\,F_2'-\frac{F_1^2}{F_3}F_3''-\frac{F_1^2}{F_3^2}(F_3')^2+\frac{F_1^2}{2F_2}F_2''+\frac{F_1}{2F_2}F_2'F_1'-\frac{F_1^2}{2F_2^2}(F_2')^2, \tag{76}$$

where the primes indicate partial derivatives with respect to r.

Now we wish to consider $r\to 0$. To simplify our calculations further, we only consider the leading order of these curvature components. This approach is consistent, because we will see that all of the nonzero components (76) are of the same order of magnitude at the null surface. In fact these terms are all finite there, and so any divergence in the tidal forces can only arise from the boost matrix (75) upon transforming to the observer's rest frame. Given the discussion at the end of the previous subsection then, it is clear that the only possible divergences will occur for the modes $\beta=-(l+1)<0$. For these modes, as $r\to 0$, $F_1\to r/\sqrt{P_1P_2}$, $F_1'\to 1/\sqrt{P_1P_2}$, $F_1''\to-(P_1+P_2)/(P_1P_2)^{-3/2}$, $F_2\to r/Q_2$, $F_2'\to 1/Q_2$, $F_2''\to -2/Q_2^2$, and $F_3\to\sqrt{B_0/Q_2}\,r^{-l/2}$, $F_3'\to-(l/2)\sqrt{B_0/Q_2}\,r^{-l/2-1}$, and $F_3''\to[l(l+2)/4]\sqrt{B_0/Q_2}\,r^{-l/2-2}$, and so the above curvature components (76) reduce to

$$R^{4r4r}\to-\frac{1+2\beta(\beta+1)}{4P_1P_2},\quad R^{0r0r}\to\frac{1-2\beta(\beta+1)}{4P_1P_2},$$

$$R^{0404}\to\frac{1}{4P_1P_2},\quad R^{4r0r}\to-\frac{\beta(\beta+1)}{2P_1P_2}. \tag{77}$$

Transforming to the observer's rest frame, $\hat R^{abcd}=L^a{}_kL^b{}_lL^c{}_mL^d{}_nR^{klmn}$, we find

$$\hat R^{abcd}\to\frac{1}{4P_1P_2}\{\Delta_1^{abcd}-2\beta(\beta+1)\Delta_2^{abcd}\}, \tag{78}$$

where

$$\Delta_1^{4r4r}=-1,\quad \Delta_1^{0r0r}=1,$$

$$\Delta_1^{0404}\to 1,\quad \Delta_1^{4r0r}=0,$$

$$\Delta_2^{4r4r}\to(V^0)^2\simeq(\omega-p)^2B_0Q_2\,r^{\beta-1},\quad \Delta_2^{0r0r}\to 1,$$

$$\Delta_2^{0404}\to(V^r)^2\simeq(\omega-p)^2B_0Q_2\,r^{\beta-1},$$

$$\Delta_2^{4r0r}\to V^4(V^r)^2\simeq p(\omega-p)^2\sqrt{B_0Q_2^3}\,r^{(\beta-3)/2}. \tag{79}$$

Hence we see that there are divergences but that they only appear in the terms proportional to $\beta(\beta+1)$. The β-independent contributions are essentially boost invariant. These divergences are therefore absent for $\beta=-1$ or $l=0$. This result might have been expected since in this case, for a constant wave profile, the solution is essentially the original regular black string with a modified Q_1. The leading divergences are easily seen to be

$$\hat R^{4r4r}\to-\frac{l(l+1)}{2}\frac{B_0Q_2(\omega-p)^2}{P_1P_2r^{l+2}},\quad \hat R^{0r0r}\to\frac{1-2l(l+1)}{4P_1P_2},$$

$$\hat R^{0404}\to-\frac{l(l+1)}{2}\frac{B_0Q_2(\omega-p)^2}{P_1P_2r^{l+2}},$$

$$\hat R^{4r0r}\to-\frac{l(l+1)}{2}\frac{\sqrt{B_0Q_2^3}(\omega-p)^2p}{P_1P_2r^{l/2+2}}. \tag{80}$$

So we see that for all higher multipoles with $\beta=-(l+1)<-1$ or $l>0$ there appear singular tidal forces on the null surface $r=0$. Because these divergences will not be canceled by any other terms of the metric for slowly oscillating strings, we conclude that all these space-times have a null singularity at $r=0$. Hence, the excitation of these higher modes on the black string results in the appearance of naked singularities, and so the resulting solutions are no longer black strings after all. Thus we rule out all of these higher multipoles as a variety of nonstationary hair.

A similar calculation shows that the growing wave modes with $\beta=l>1$ have diverging tidal forces in the limit $r\to\infty$. In this limit, $F_1\to 1$, $F_1', F_1''\to 0$, $F_2\to 1$, $F_2', F_2''\to 0$, and $F_3\to\sqrt{B_0}r^{\beta/2}$, $F_3'\to(\beta/2)\sqrt{B_0}r^{\beta/2-1}$, and $F_3''\to(\beta(\beta-2)/4)\sqrt{B_0}r^{\beta/2-2}$, and hence to the leading order, the curvature components (76) become

$$R^{4r4r}=R^{0r0r}=-R^{4r0r}\to-\frac{\beta(\beta-1)}{2r^2}. \tag{81}$$

The boosted curvatures, to the leading order, are given by

$$\hat{R}^{abcd}\to-\frac{\beta(\beta-1)}{2r^2}\Delta_2^{abcd}. \tag{82}$$

The terms proportional to Δ_1^{abcd} are all of the subleading order. The limiting values of Δ_2^{abcd} when $r\to\infty$ are given by

$$\Delta_2^{4r4r}\to(\omega-p)^2B_0r^\beta,\quad \Delta_2^{0r0r}\to 1,$$
$$\Delta_2^{0404}\to(\omega-p)^2B_0r^\beta,\quad \Delta_2^{4r0r}\to[\omega-p+p(\omega-p)^2]B_0r^{\beta/2}, \tag{83}$$

and so the leading divergences are easily seen to be

$$\hat{R}^{4r4r}=\hat{R}^{0r0r}\to-\frac{\beta(\beta-1)}{2}(\omega-p)^2B_0r^{\beta-2},$$
$$\hat{R}^{4r0r}\to\frac{\beta(\beta-1)}{2}(\omega-p+p(\omega-p)^2)\sqrt{B_0}r^{\beta/2-2}. \tag{84}$$

Hence we find diverging tides in the asymptotic region for all $\beta>2$. For $\beta=2$ we find finite tides at infinity; however, this implies that at infinity the energy density approaches a constant, and hence the total energy per unit length of this wave diverges. Indeed, this might have been expected as these solutions are not asymptotically flat. Instead, they represent geometries with gravitational wave energy concentrated far away from the black string.

To close this section, we will discuss the results for the remaining orthonormal components of the curvature tensor. It turns out that there are two cases to consider: R^{abcd} and $R^{a\alpha b\beta}$ where α,β take values in $0,4,r$ while the remaining frame indices are θ or ϕ. Straightforward evaluation shows that the curvature components with an odd number of angle indices vanish when evaluated on our radial geodesics which were chosen so that $\partial_\theta F_3^2=0=\partial_\phi F_3^2$. Further, like the components in Eq. (76), the nonvanishing components considered here remain finite in the stationary frame (73). So clearly the boost (75), which is trivial in the angle directions, cannot introduce any divergences in $\hat{R}^{abcd}$. Now, for the $R^{a\alpha b\beta}$, we have symbolically $\hat{R}=L^2R$ upon boosting to the observer's rest frame, where we have only indicated the nontrivial components of the boost matrix. Because the $R^{a\alpha b\beta}$ are everywhere finite, the tidal forces here diverge only as badly as the leading-order divergence in L^2. As $r\to 0$, this is $1/r^{l+2}$ for $\beta=-(l+1)$, and as $r\to\infty$, this is r^l for $\beta=l$, just as found above. Note that various cancellations above reduced the singularities from what one might expect for L^4. Also, as above, the original curvature components supply a factor of $l(l+1)$ multiplying these divergent terms so that the special cases $\beta=-1,0,1$ survive without any divergent tidal forces. These qualitative arguments are confirmed with direct evaluation. Hence one finds that there are no worse divergences than found by considering only the $0,4,r$ subspace.

Again, our final conclusion is that all of the modes with $\beta=-(l+1)<-1$ in fact produce null singularities at $r=0$, while for $\beta=l>1$ singularities appear in the asymptotic region $r\to\infty$.

V. LONGITUDINAL AND TRANSVERSE OSCILLATIONS

We now return to consider the case of the longitudinal waves (65) with $l=0,\beta=-1$ and that of the transverse waves (61) with $l=1=\beta$. The preceding analysis revealed no divergent tidal forces in either of these cases. There is still the possibility that divergent tidals might occur when the analysis is extended to consider derivatives of the curvature (much like the case discussed in [23]). In the case of the transverse waves, the worry would be that problems arise in the asymptotic region where the Lorentz boost (75) is divergent. However, recall that a coordinate transformation (63) was found for which the metric became manifestly asymptotically flat.[8] Hence one will never find any divergent tidal forces as $r\to\infty$. In fact, for the transversal waves one can argue that the divergence of the Lorentz transformation (75) is not physical, but comes from an incorrect choice of gauge in the limit $r\to\infty$. Namely, in this case one could view the asymptotically flat form of the transverse wave metric (64) with $\dot{A}_i=\text{const}$ as the correct example of the leading-order behavior of slowly oscillating transverse waves. The fact that this solution is asymptotically flat implies that the Lorentz transformation (75) must be finite in the limit $r\to\infty$.

For the longitudinal waves, the potential problem would be at $r=0$ where again divergent components appear in the transformation (75). A definitive demonstration of the regularity of these solutions would require finding a coordinate transformation for which the metric (65) becomes analytic at the null surface. For a wave profile constant in u, we recover the original solution (51) with a modified charge in which case it is straightforward to find the analogue of Eddington-Finkelstein coordinates, both past and future. For the nonstationary case, Horowitz and Marolf [9] have shown that this solution has a continuous (but not necessarily smooth) metric at $r=0$. Further, our theorem shows that all the scalar curvature invariants of this solution are identical to the original black string, while our preceding analysis found no evidence of diverging tidal forces for slowly oscillating strings. In fact, we can make an even stronger statement about the ''invisibility'' of the monopole wave: If we compute the curvature of the oscillating string in an appropriate orthonormal basis (see below), we find that it does not depend on the wave at all. Given all these results, one is tempted to conjecture that the longitudinal waves are completely regular, and so we are led to attempt a construction of analytic coordinates. It turns out that we can find such coordinates at least for a certain class of wave profiles. A rather surprising result

[8]Recall that no divergent tidals were found for $l=0=\beta$ either, but in Eq. (61) this mode was shown to be purely the result of a coordinate transformation.

in our analysis is the quantization of a certain constant present in the test function, coming from the requirement that all the derivatives of the metric be continuous as the null surface is approached. It is not clear to us at present whether this quantization is an artifact of our ansatz or whether it really represents a physical effect. We will defer a more detailed investigation of this issue to future work. At this point, we can only provide some guidelines explaining how we have arrived at the analytic ansatz and the associated quantization condition.

We begin by considering the longitudinal waves described by the metric (65) which may be written

$$ds^2=\left(1+\frac{Q_1-Q_2+b(u)}{r+Q_2}\right)du^2+\frac{2r}{r+Q_2}du\ dv +(r+P_1)(r+P_2)\left(\frac{dr^2}{r^2}+d\Omega\right). \tag{85}$$

Now the coordinate transformation $dv=d\hat{v}+\{[Q_1-Q_2+b(u)]/2Q_2\}du$ simplifies the metric somewhat, producing[9]

$$ds^2=p^2(u)\ du^2+\frac{2r}{r+Q_2}du\ d\hat{v} +(r+P_1)(r+P_2)\left(\frac{dr^2}{r^2}+d\Omega\right), \tag{86}$$

where

$$p^2(u)=[Q_1+b(u)]/Q_2\,. \tag{87}$$

Implicitly we assume that $p^2(u)>0$ — see footnote 6 — and as usual $d\Omega$ is the metric on a two-sphere. We see that the location of the null surface remains at $r=0$ and that it appears to be a metric singularity: We know, however, that if p^2 were a constant, we could easily show that this singularity is just a coordinate artifact, as we have mentioned above. Let us therefore try to follow this argument as closely as possible. We can put the $(\hat{v},u)$ part of the metric in the form conformal to the constant p^2 case, by defining a new coordinate z, $dz=p^2(u)du$ and $q(z)=1/p(u)$, in which case the metric becomes

$$ds^2=q^2(z)\left[dz^2+\frac{2r}{r+Q_2}dz\ d\hat{v}\right] +(r+P_1)(r+P_2)\left(\frac{dr^2}{r^2}+d\Omega\right)\ . \tag{88}$$

Now we can introduce new coordinates $\tilde{v}$ and $\tilde{z}$ in analogy with the tortoise coordinates we would usually define for a stationary solution. A bit of algebra leads to the choice

$$d\tilde{v}=d\hat{v}+\frac{P_1+P_2}{2\sqrt{P_1P_2}}\frac{dr}{r^2}(r+Q_2)\left(r+\frac{2P_1P_2}{P_1+P_2}\right),$$
$$d\tilde{z}=dz-\sqrt{\frac{P_2}{P_1}}\frac{dr}{r}(r+P_1). \tag{89}$$

These coordinates are oriented towards the future portion of the null surface, and they clearly show the location of the future horizon as $r\to 0$ along with $\hat{v}\to+\infty$ and $z\to-\infty$. Reversing the signs of the shifts to $\hat{v}$ and z, we can go to the past horizon. These coordinate transformations are designed to absorb the metric singularity manifest in the r^{-2} factor appearing in g_{rr}. If we rewrite the metric (87) in terms of the tortoise coordinates (89), we find

$$ds^2=q^2(z)\left\{d\tilde{z}^2+2\frac{r}{r+Q_2}d\tilde{z}\ d\tilde{v}+2\sqrt{\frac{P_2}{P_1}}\frac{r+P_1}{r+Q_2}dr\ d\tilde{v} -\frac{P_1-P_2}{\sqrt{P_1P_2}}dr\ d\tilde{z}\right\}+(r+P_1)(r+P_2) \times\left[\frac{1-q^2(z)}{r^2}dr^2+d\Omega\right]. \tag{90}$$

Here z is to be understood as an implicit function of $\tilde{z}$ and r: $z=\tilde{z}+(\sqrt{P_2/P_1})[r+P_1\ln(r/P_1)]$, and it diverges to $-\infty$ as $r\to 0$ as dictated by the logarithm. Now, if $q^2=1$, the divergent g_{rr} term in Eq. (90) would be absent, and we would obtain a smooth metric at the null surface $r=0$. Can we now select a wave profile $q^2(z)$ such that a similar cancellation still occurs in the limit $r\to 0$? The answer turns out to be in the affirmative: Consider a function $q^2(z)$ which in the limit $z\to-\infty$ converges to $1-A^2\exp[(2+\alpha)z/\sqrt{P_1P_2}]$ for some positive α. Substituting the appropriate coordinate transformation $z=z(\tilde{z},r)$ in this expression, we get that in the limit $r\to 0$ the wave profile to the leading order is $q^2=1-A^2(r/P_1)^{2+\alpha}e^{F(\tilde{z},r)}$, where for convenience we have defined the linear function $F(\tilde{z},r)\equiv(2+\alpha)(\tilde{z}+\sqrt{P_2/P_1}\ r)/\sqrt{P_1P_2}$. As long as $\alpha\geqslant 0$, this is precisely of the form needed to cancel the pole in g_{rr}. The metric, in this limit, becomes

$$ds^2=\left\{1-A^2\left(\frac{r}{P_1}\right)^{2+\alpha}e^{F(\tilde{z},r)}\right\}\left\{d\tilde{z}^2+2\frac{r}{r+Q_2}d\tilde{z}\ d\tilde{v}+2\sqrt{\frac{P_2}{P_1}}\frac{r+P_1}{r+Q_2}dr\ d\tilde{v}-\frac{P_1-P_2}{\sqrt{P_1P_2}}dr\ d\tilde{z}\right\}+(r+P_1)(r+P_2) \times\left[\frac{A^2}{P_1^2}\left(\frac{r}{P_1}\right)^{\alpha}e^{F(\tilde{z},r)}dr^2+d\Omega\right], \tag{91}$$

[9]If one now constructs an orthonormal frame analogous to that of Eq. (73) found for the metric (72), one would find that the wave profile $p^2(u)$ completely disappears from the frame components of the curvature tensor.

which is smooth at $r=0$.

At this point, however, we can push these arguments one step further. A glance at the metric (91) immediately shows that all of its derivatives with respect to the angles and the coordinates $\tilde{z}$ and $\tilde{t}$ are well defined as the horizon is approached—we ignore subleading terms in q^2 for the moment. Moreover, if α is chosen to be an non-negative integer, i.e., $\alpha=0,1,2,\ldots$, it is easily verified that all the derivatives with respect to r are also well defined in this limit. Indeed, we see that the only possibly contentious terms in the metric are the factors $r^{2+\alpha}$ and r^{α}. If α were not integral with $N>\alpha>N-1$, then taking N r-derivatives of the metric would produce factors which diverge as $r\to 0$. At this point, we cannot tell whether this divergence could cause some covariant derivative of the curvature tensor to diverge as well (resulting in a null singularity as in [23]), or if it could be removed by another even more clever change of coordinates. On the other hand, for integral α, we see that after taking some number of derivatives of the metric with respect to r, the contentious factors disappear altogether, leaving an expression which is perfectly well defined as $r\to 0$ and $z\to-\infty$.

It is a matter of a simple counting of powers to convince oneself that all these conclusions remain unchanged if the function $q^2(z)$ can be written as a uniformly convergent power series $q^2(z)=1-\Sigma_{n=2}^{\infty}A_n^2 w^n$ for $w=\exp(nz/\sqrt{P_1P_2})$ and $z\to-\infty$. The subleading contributions would be integer powers of the leading term, therefore still having well-defined r derivatives. The requirement of uniform convergence ensures that the summations and derivatives commute, resulting in the conclusion that the profile q^2 is itself analytic on the horizon. This allows us to extend our arguments in order to analytically continue the solution through the past horizon $z\to\infty$, as well as the future horizon at $z\to-\infty$. For example, consider a wave profile of the form $q^2(z)=1-A^2/\cosh(kz/\sqrt{P_1P_2})$ for some positive integer[10] $k\geqslant 2$. Expanding this profile in the limit $z\to-\infty$ (future horizon), we find $q^2(z)=1-A^2\Sigma_{m=0}^{\infty}(-1)^m\exp[(2m+1)kz/\sqrt{P_1P_2}]$, precisely of the form guaranteeing the existence of the limit $r\to 0$. If we approach the past horizon instead, with $z\to\infty$, we find $q^2(z)=1-A^2\Sigma_{m=0}^{\infty}(-1)^m\exp[-(2m+1)kz/\sqrt{P_1P_2}]$, again rendering the limit well defined. Hence we see that with this example we really have constructed a wavy black string with regular future and past horizons. Of course, this example is easily generalized e.g., by taking linear combinations of inverse cosh's with different integers k or by taking a wave profile of the form $A^2/(w^k+w^{-k'})$ with independent k and k'. Hence, as we have claimed above, we see that at least for a certain class of wave profiles, the longitudinal wave solutions are analytic at $r=0$, which therefore can be identified as a regular event horizon. Hence we can think of these waves as time-dependent hair on the black string. Clearly, our argument is only an existence proof. It would be interesting to prove that the general family of longitudinal waves is regular or to determine the precise conditions which the wave profile must satisfy in order to ensure regularity. We note here that for all of our examples above the profiles have essentially compact support in the z coordinate, and therefore in the horizon limit the waves are exponentially damped, the solution approaching the stationary string.

One could also consider introducing transverse and longitudinal waves on the black string at the same time. This would amount to adding to the line element (85) a term of the form, e.g.,

$$\frac{r^2a(u)\cos\theta}{r+Q_2}du^2. \tag{92}$$

In this case, we would perform all of the same transformations described above with the result that the final metric (90) is modified by the addition of

$$q^4(z)\frac{a(z)\cos\theta}{r+Q_2}\left(rd\tilde{z}+\sqrt{\frac{P_2}{P_1}}(r+P_1)dr\right)^2, \tag{93}$$

where we have rewritten $a(u)$ as $a(z)$. Now regularity of the horizon is secured by demanding that $a(z)$ is an analytic function of $\tilde{z}$ and r. The latter is easily accomplished with wave forms similar to those discussed for $q(z)^2$ above.

VI. DISCUSSION

In this work, we have investigated geometric properties of Garfinkle-Vachaspati waves, obtained by superposing gravitational oscillations on stationary solutions with null hypersurface-orthogonal Killing vectors. We have first given a detailed discussion of the adaptation of this solution-generating technique to a variety of supergravity models, relaxing some of the matter sector constraints imposed previously [17]. Then we have developed a purely geometric theorem, stating that the GV modes are completely invisible to any scalar invariants constructed from the metric and matter. The proof of our theorem does not rely on the dynamics at all. It is a consequence of the null symmetry, which must be present in order to interpret the solutions as waves, and hence it applies to any metric which can be represented in the generalized Kerr-Schild form with respect to the symmetry. Thus our theorem holds in any theory of gravity and in an arbitrary number of dimensions.

As a consequence, it is evident that one must further scrutinize wavy string geometries using noninvariant probes, such as tidal forces. Recall that we have in fact shown that none of the scalar invariants changes under the Garfinkle-Vachaspati map. In group-theoretic language, they are invariants of the orbits of the solution-generating map. Now, this could suggest that the wavy strings might contain curvature singularities identical to those of the corresponding stationary solutions belonging to the same orbit. Consider, e.g., the stationary solutions with all charges equal. Since they correspond to the extremal Reissner-Nordstrom black strings, with traversible horizons and spacelike curvature singularities at the core, one could be tempted to conjecture that all the GV wavy modes superposed on such solutions also have similar curvature singularities—for the curvature scalars are exactly the same, and in particular also blow up at the ''core.'' However, in general this is incorrect. The argu-

[10]Note that we would require $A^2<1$ in order to ensure that $p^2=1/q^2>0$, as assumed above.

ment is incomplete because it does not answer whether such singular regions can be causally connected to the physically interesting parts of the spacetime, such as the asymptotic infinity. Thus it becomes evident that the notion of singularity is a far more complex issue here: In the examples of oscillating solutions we have encountered, some of the singularities encoded in scalar invariants may not belong to the physical spacetime, because no causal geodesics can ever reach it. The simplest example of such behavior is given by the solutions discussed by Gibbons, Horowitz, and Townsend [22]. These solutions correspond to the choice of the wave degree of freedom Ψ such that $f+b(u)/r=1$ in Eq. (65). Although they belong to the same GV orbit as the stationary extremal Reissner-Nordström strings, their causal structure is dramatically different: In order to get the analytic extension across the horizon, one has to extricate the region normally associated with the ''interior'' of the black string from the physical spacetime, and replace it with a mirror image of the ''exterior.'' The resulting manifold is a completely nonsingular spacetime with a horizon, with the topology of the anti–de-Sitter horizon crossed with a sphere, dividing two asymptotically flat regions. Noncurvature singularities, manifest in the presence of geodesic incompleteness, may reemerge if one attempts to compactify some of the longitudinal directions in order to descend to lower dimensions. Nevertheless, these singularities are typically milder than the original ones which got cut out in the process of analytic extension. Therefore we can see that our theorem states that scalar invariants are identical as analytic functions of the coordinates on each GV orbit. Observers ''surfing'' on GV waves of differing geometries, however, may choose to access different regions of this ''complex'' coordinate plane as physical, and hence see different divergences. We believe that this unusual behavior is peculiar to wavy strings (or more generally wavy p-branes) and hence pathological.

With this in hindsight, we have found that most of the undulating solutions cannot be interpreted as black strings: The excitation of any localized multipole mode with $l\geqslant 1$ gives rise to a naked null singularity as opposed to an event horizon. Similarly, oscillations of growing modes with $l\geqslant 2$ result in divergences very far away from the string. To uncover these divergences, we have found a timelike geodesic for each excited multipole, which extends to infinity and reaches the null surface in a finite affine time. We have then propagated an observer along this geodesic to the vicinity of the region of geometry we wished to explore, and isolated the leading divergences of the curvature forms Lorentz transformed to the rest frame of the infalling observer. These divergences translate into infinite physical forces detectable by the observer, and hence must be interpreted as naked singularities.

One could easily generalize these conclusions to include the waves propagating in the remaining ''internal'' dimensions, which we have ignored until now because they were passive spectators for the most part of our investigation. In the framework of heterotic string theory this can be done as follows: We could take the tensor product of the string-frame solution (51) with a four-torus, lifting it to nine dimensions. This is guaranteed to be a solution of the effective action of heterotic string theory in nine dimensions, with the matter fields identical to those of the original five-dimensional solution (51). We could then view the Laplacian constraint of Eq. (18) as a linear combination of the spacetime and internal parts. From the solution (51), the Laplacian constraint in the Einstein frame in nine dimensions can be decomposed as $\vec{\nabla}_x^2\Psi+kl\vec{\nabla}_y^2\Psi=0$, with k and l defined in Eq. (53). The operators $\vec{\nabla}_x^2$ and $\vec{\nabla}_y^2$ denote the flat space Laplacians in the 3D spatial sections and the internal space, respectively. Because the internal space is a four-torus, we can expand the wave profile Ψ in the Fourier series with respect to the internal space basis, which consists of the exponentials of linear combinations of the internal coordinates $\exp(i\vec{c}\cdot\vec{y})$. The coefficients of the coordinates are integer multiples of the inverse radii of the torus, and would give rise to the mass of the transversal part of the wave profile $M^2=\Sigma_{i=1}^4 c_i^2$. In particular, the zero modes, corresponding to $c_1=\cdots=c_4=0$ would trivially reduce to the case of the massless spacetime waves we have discussed in the main part of the paper. These are the only scalar harmonics on the torus and hence we can ignore them. All the massive modes however turn out to be naked singularities. This can be seen as follows: The spacetime part of the Laplacian for each massive mode of the wave profile becomes $\vec{\nabla}_x^2\Psi_M=M^2kl\Psi_M$. Now, using the spherical symmetry of the stationary solution to implement the separation of variables $\Psi_M=R_{Ml}Y_{lm}$, we at last obtain the equation for the radial degree of freedom:

$$r^2\ddot{R}_{Ml}+2r\dot{R}_{Ml}-[l(l+1)+M^2P_1P_2+M^2(P_1+P_2)r+M^2r^2]R_{Ml}=0. \tag{94}$$

From the general theory of differential equations, we know that this equation has two classes of solutions—localized (approaching a constant as $r\to\infty$) and growing (diverging in the limit $r\to\infty$). The latter do not represent wavy strings but rather waves filling up the whole spacetime—and hence lead to divergences far away from the string, as in the case of the massless growing modes discussed in Sec. IV. This can be seen as follows: Substituting $R_{Ml}=\mathcal{R}_{Ml}/r$ in Eq. (94) and keeping only the leading terms in the limit $r\to\infty$, we find $\ddot{\mathcal{R}}_{Ml}=M^2\mathcal{R}_{Ml}$. Hence the growing modes diverge at infinity as $R_{Ml}\simeq\exp(Mr)/r$—i.e., faster than any power, and so they must produce naked singularities far away from the string for all l. The former case is more interesting, because by the same arguments as above, far away from the string, these solutions damp out as $\exp(-Mr)/r$. Thus they might be interpreted as wavy strings—except that they are singular in the limit $r\to 0$. To show this, we note that the point $r=0$ is a regular singular point of the differential equation (94). Hence all the solutions admit Laurent series representation, with the leading power being r^β. From Eq. (94) we find $\beta=[-1\pm\sqrt{1+4l(l+1)+4M^2P_1P_2}]/2$. The plus sign in the definition of β leads to finite (in fact, vanishing) limits of the radial function as $r\to 0$ and, as we know from the theory of differential equations, it must correspond to the growing solutions which are singular at infinity. The minus sign gives solutions which diverge as $R_{Ml}\simeq 1/r^{|\beta|}$ on the null surface. In particular, even when $l=0$, the wave profile diverges with the negative power $\beta=-(1+\sqrt{1+4M^2P_1P_2})/2$. As we have seen in Sec. IV, such divergences could produce infinite tides—and since now $\beta(\beta+1)=M^2P_1P_2\neq 0$, the tides we

have computed there diverge even for the monopole mode. Similarly, we can see that other localized massive multipole modes also have divergent tides on the null surface. Hence we see that all the nontrivial internal modes lead to naked singularities, which can be detected by tidal forces.

In the case of localized $l=0$ and growing $l=1$ modes, we have found a special subclass of wavy strings which are analytic on, and everywhere outside of, the null surface, and not only continuous—thus extending the earlier results of Horowitz and Marolf [9,10]. A puzzling feature of our special family of analytic black strings with nonstationary hair is obviously the quantized nature of the parameter α. The quantization of α in a technical sense is more similar to, say, the quantization of the radial modes of an electron in a hydrogen atom, where we also require that the wave function of the system is analytic — yet the obvious absence of any quantum-mechanical scales precludes the possibility that our quantization is of microscopic origin. On the other hand, it is possible that this is just an artifact of our choice of the form of the wave profile yielding analytic metric. In our opinion, this ambiguity merits further interest.

Finally, we should remark once more that our analytic examples comprise only an existence proof of time-dependent hair. They do not encompass the general $l=0$ and $l=1$ modes. Specifically, we do not cover arguably the most interesting case when the coordinate along the string is compact, which is a candidate for a hairy black hole in four dimensions. Given our demonstration above that the excitation of the passive internal dimensions leads to naked singularities, we should note here that the compact wavy strings may also be singular. A simple argument supporting this could be based on the fact that the compact wave profiles are periodic functions. Hence the exponential damping of the wave in the horizon limit, manifest in Eq. (91) and crucial for demonstrating analyticity of the metric in our examples, is absent. This actually turns out to be correct — but to prove that the compact wavy strings are indeed singular requires a careful examination of the tides beyond the leading order, and will be given in the forthcoming work of Horowitz and Yang [25]. So, while our examples show that it is possible to have wavy strings with regular horizons, and that the waves can be understood as time-dependent hair on black objects, this identification must be applied sparingly and discriminatively. In general, the wavy solutions are not hairy black strings, or decompactified black holes, because they contain regions with infinite tides, which render the solutions singular. Thus it appears unlikely that the classical wave modes alone can give the complete accounting of the microscopic black hole degrees of freedom, as proposed in [8,11,15]. The excitation of most of these modes turns the original black string into a very bright object. Hence such modes cannot be interpreted as black hole hair — in a manner of speaking, they are too curly to weave a smooth spacetime.

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