Topological invariants, instantons, and the chiral anomaly on spaces with torsion

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In a spacetime with nonvanishing torsion there can occur topologically stable configurations associated with the frame bundle which are independent of the curvature. The relevant topological invariants are integrals of local scalar densities first discussed by Nieh and Yan (NY). In four dimensions, the NY form $N=(T^a\wedge T_a - R_{ab}\wedge e^a\wedge e^b)$ is the only closed four-form invariant under local Lorentz rotations associated with the torsion of the manifold. The integral of N over a compact D-dimensional (Euclidean) manifold is shown to be a topological invariant related to the Pontryagin classes of SO$(D+1)$ and SO(D). An explicit example of a topologically nontrivial configuration carrying a nonvanishing instanton number proportional to $\int N$ is constructed. The chiral anomaly in a four-dimensional spacetime with torsion is also shown to contain a contribution proportional to N, in addition to the usual Pontryagin density related to the spacetime curvature. The violation of chiral symmetry can thus depend on the instanton number of the tangent frame bundle of the manifold. Similar invariants can be constructed in $D>4$ dimensions and the existence of the corresponding nontrivial excitations is also discussed. [S0556-2821(97)01312-X]

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I. INTRODUCTION

In the traditional approach to gravitation theory, torsion plays no significant role in the spacetime geometry. Torsion is commonly set equal to zero from the start and there seems to be no compelling experimental reason to relax this condition. In a more geometric approach, however, the affine and metric properties of the spacetime geometry are independent notions and should, therefore, be described by dynamically independent fields: the spin connection $\omega^a{}_b$ and the local frames (vielbein) e^a, respectively [1]. In the tradition of general relativity these two fields are assumed to be linked by the torsion-free condition $T^a=0$, where the torsion two-form is defined by

$$T^a=de^a+\omega^a{}_b\wedge e^b. \tag{1}$$

This expression is similar to that of the curvature two-form,

$$R^a{}_b=d\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b\,, \tag{2}$$

whose vanishing is not to be imposed *a priori*.

From these two expressions, curvature seems to be more fundamental than torsion: the definition (2) depends on the existence of the connection field alone, whereas torsion depends on both the connection and the vielbein. On the other hand, since on any smooth metric manifold a local frame (vielbein) is necessarily always defined, torsion can exist even if the connection vanishes. This implies that in a geometric theory of spacetime the local frame structure is as basic a notion as the connection and, therefore, torsion and curvature should be treated on a similar footing.

From a group-theoretic point of view, the curvature two-form is the commutator of the covariant derivative for the connection of the group of rotations on the tangent space of the manifold [SO(D) or SO$(D-1,1)$, for Euclidean or Minkowskian signature, respectively].[1] This is reflected by the fact that the curvature depends on the group connection $\omega^a{}_b$ alone. In contrast, no analogous simple geometric interpretation can be assigned to torsion. [For a discussion on this point, see Sec. II, below.] This is perhaps one reason why torsion has been perceived as less fundamental than curvature since the early days of general relativity [2]. Nevertheless, torsion appears rather naturally in the commutator of two covariant derivatives for the group of diffeomorphisms of a manifold in a coordinate basis [3],

$$[\nabla_\mu,\nabla_\nu]V^A=-T^\lambda_{\mu\nu}\nabla_\lambda V^A+R^A_{B\mu\nu}V^B, \tag{3}$$

where V^A represents any tensor (or spinor) under diffeomorphisms or under the group of tangent rotations, and R^A_B is the curvature tensor in the corresponding representation. Here curvature and torsion play quite different roles: $T^\lambda_{\mu\nu}$ is the structure function for the diffeomorphism group and $R^A_{B\mu\nu}$ is a central charge. From this expression it is clear that one can consider equally well spaces with curvature and no torsion, and ''teleparallelizable'' spaces with zero curvature and nonvanishing torsion. Both possibilities are special cases of the generic situation.

[1]Here we will assume the signature to be Euclidean. Whenever spacetime is mentioned, the appropriate Wick rotation will be assumed.

Another realm where curvature plays an important role is in the characterization of the topological structure of the manifold. It is a remarkable result of differential geometry that certain global features of a manifold are determined by some local functionals of its intrinsic geometry. The four-dimensional Pontryagin and Euler classes,

$$P_4=\frac{1}{8\pi^2}\int_{M_4}R^{ab}\wedge R_{ab}\,, \tag{4}$$

$$E_4=\frac{1}{32\pi^2}\int_{M_4}\epsilon_{abdc}R^{ab}\wedge R^{cd}, \tag{5}$$

are well-known examples. For compact manifolds in four dimensions P_4 and E_4 take integer values that label topologically distinct four-geometries. Although these topological invariants are given in terms of local functions, their values depend on the global properties of the manifold. These topological invariants are expected to be related to physical observables as, for instance, in the case of anomalies. The Pontryagin class can be defined for any compact gauge group G, on any even-dimensional compact manifold,

$$P_{2n}[G]=\frac{1}{2^{n+1}\pi^n}\int_{M_{2n}}\{\underbrace{F\wedge\cdots\wedge F}_{n}\}, \tag{6}$$

where F is the curvature two-form for the group G whose generators are normalized so that $\mathrm{Tr}\{G_aG_b\}=\delta_{ab}$, and the braces $\{\ldots\}$ indicate a particular product of traces of products of F's (see [4]). Since the curvature two-form for the manifold (R^{ab}) in the standard representation is antisymmetric, the Pontryagin form *of the manifold*, $P_D[\mathrm{SO}(D)]$ is only defined for $D=4n$. In contrast with the Pontryagin forms, the Euler form cannot be defined for a generic gauge group G.

Invariants analogous to these, constructed using the torsion tensor are less known. The lowest-dimensional torsional invariant is the four-form first discussed by Nieh and Yan (NY) [5],

$$N=T^a\wedge T_a-R_{ab}\wedge e^a\wedge e^b. \tag{7}$$

This is the only nontrivial locally exact four-form which vanishes in the absence of torsion and is clearly independent of the Pontryagin and Euler densities. In any local patch where the vierbein is well defined, N can be written as

$$N=d(e^a\wedge T_a), \tag{8}$$

and is, therefore, locally exact. More explicitly, if N and N' are the NY densities for (ω,e), and (ω',e'), where $\omega'=\omega+\lambda$, $e'=e+\zeta$, then $\Delta=N-N'$ is locally exact (a total derivative). If the deformation between ω and ω' is globally continuous, Δ is globally exact. Therefore, $\int N$ is a topological-invariant quantity in the same sense as the Pontryagin and Euler numbers. Similar invariants can be defined in higher dimensions as discussed in [6].

The three-form $e^a\wedge T_a$ is a Chern-Simons-like form that can be used as a Lagrangian for the dreibein in three dimensions. The dual of this three-form in four dimensions is also known as the totally antisymmetric part of the torsion (contorsion) and is sometimes also referred to as H torsion:

$$e^a\wedge T_a\sim\epsilon^{\mu\nu\lambda\rho}T_{\nu\lambda\rho}. \tag{9}$$

This component of the torsion tensor is the one that couples to the spin-$\frac{1}{2}$ fields [7]. This is one of the irreducible pieces of the first Bianchi identity. In a metric-affine space, the 1st Bianchi identity can be decomposed according to 16 = 9 + 6 + 1. And the ''1'' is that corresponding to Eq. (8) [8].

In the next section N is shown to be related to the Pontryagin class, shedding some light on the origin of its topological nature. Section III contains the construction of a field configuration that exhibits the relevant instanton number. In Sec. IV, the contribution of N to the chiral anomaly and the corresponding index theorem are discussed. A general discussion, in particular, about the possibility of having similar invariants in higher dimensions, are contained in Sec. V.

II. RELATION TO THE PONTRYAGIN CLASS

It seems natural to investigate the extent to which the Nieh-Yan invariant (7) is analogous to the Pontryagin and Euler invariants. In particular, it would be interesting to know whether the integral of N over a compact manifold has a discrete spectrum as is the case for E_4 and P_4.

This question can be answered by embedding the group of rotations on the tangent space, SO(4) into SO(5). This can be done quite naturally combining the spin connection and the vierbein together in a connection for SO(5) in the form [6,9,10]

$$W^{AB}=\begin{bmatrix}\omega^{ab} & \frac{1}{l}e^a\\ -\frac{1}{l}e^b & 0\end{bmatrix}, \tag{10}$$

where $a,b=1,2,\ldots,4$ $A,B=1,2,\ldots,5$. Note that the constant l with dimensions of length has been introduced to match the standard units of the connection (l^{-1}) and those of the vierbein (l^0). In the usual embedding of the Lorentz group into the (anti-) de Sitter group, l is called the radius of the Universe and is related to the cosmological constant $(|\Lambda|=l^{-2})$.

The curvature two-form constructed from W^{AB} is

$$F^{AB}=dW^{AB}+W^{AC}\wedge W^{CB}=\begin{bmatrix}R^{ab}-\frac{1}{l^2}e^a\wedge e^b & \frac{1}{l}T^a\\ -\frac{1}{l}T^b & 0\end{bmatrix}. \tag{11}$$

It is then direct to check that the Pontryagin density for SO(5) is the sum of the Pontryagin density for SO(4) and the Nieh-Yan density:

$$F^{AB}\wedge F_{AB}=R^{ab}\wedge R_{ab}+\frac{2}{l^2}[T^a\wedge T_a-R^{ab}\wedge e_a\wedge e_b]. \tag{12}$$

This shows, in particular, that

$$\frac{2}{l^2}\int_{M_4} N = P_4[\mathrm{SO}(5)] - P_4[\mathrm{SO}(4)] \tag{13}$$

is indeed a topological invariant, as it is the difference of two Pontryagin classes.

From Eq. (13) one can directly read off the spectrum of $\int N$. As is well known, the Pontryagin class of $P_{2n}[G]$ takes on integer values (the instanton number) of the corresponding homoptopy group, $\Pi_{2n-1}(G)$ (see, e.g., [4]). In the case at hand, $\Pi_3[\mathrm{SO}(5)] = \mathbf{Z}$ and $\Pi_3[\mathrm{SO}(4)] = \mathbf{Z}+\mathbf{Z}$. Thus, the integral of the Nieh-Yan invariant over a compact manifold M must be a function of three integers:

$$\int_M N = \mathrm{const}\times(z_1+z_2+z_3), \quad z_i \in \mathbf{Z}. \tag{14}$$

III. INSTANTON

It is of interest to construct an example geometry with nonvanishing $\int_M N$. As it is seen from Eq. (8), the integral Eq. (14) can be evaluated integrating of the three-form $e_a \wedge T^a$ over the boundary ∂M.

A particular example of a geometry characterized by nonvanishing $\int N$ is easily constructed using the fact that N may be nonzero even if the curvature vanishes. The simplest example occurs in $\mathbb{R}^4$, where the connection can be chosen to vanish everywhere $\omega^{ab} = 0$. Consider now a vierbein field that approaches a regular configuration as $r \to \infty$. The question is how to cover the sphere at infinity (S^3_∞) with an everywhere regular set of independent vectors.

It is a classical result on fibre bundles that S^3 is parallelizable, i.e., there exist three linearly independent globally defined vector fields over the sphere [11]. Using this fact it is possible to take one of the vierbein fields along the radius (e^r) and the other three tangent to the S^3. Defining the sphere through its embedding in $\mathbb{R}^4$, $x^2+y^2+z^2+u^2 = r^2$, we chose on its surface

$$e^r = \frac{l}{r}dr,$$

$$e^1 = \frac{l}{r^2}(ydx - xdy - udz + zdu),$$

$$e^2 = \frac{l}{r^2}(-zdx - udy + xdz + ydu), \tag{15}$$

$$e^3 = \frac{l}{r^2}(udx - zdy + ydz - xdu).$$

These fields are well defined for $r \neq 0$ and can be smoothly continued inside the sphere, for instance, rescaling it by a function that vanishes as $r \to 0$ and approaches 1 for $r \to \infty$. In any case, it is clearly impossible to do this without producing a singular point where e^a vanishes. The integral of $e_a \wedge T^a$ over a sphere of any radius is

$$\frac{1}{l^2}\int_{S^3} e_a \wedge de^a = 3\times 2\times 2\pi^2. \tag{16}$$

Thus, using Eq. (8), one concludes that the above result corresponds to the integral of the Nieh-Yan form over $\mathbb{R}^4$. The factor 3 comes from the fact that there are three independent fields summed in the integrand of Eq. (16). Each term in the sum contributes twice the area of the unit three-sphere ($2\pi^2$).

Configurations with other instanton numbers can be easily generated by simply choosing different winding numbers for each of the three tangent vectors e^i. In the example above, each of these vectors makes a complete turn around the equatorial lines defined by the planes $x = y = 0$, $x = z = 0$, and $x = u = 0$, respectively. We are thus led to conclude that, in general,

$$\frac{1}{l^2}\int_M N = 4\pi^2(z_1+z_2+z_3), \quad z_i \in \mathbf{Z}. \tag{17}$$

The instanton presented here is analogous to the one discussed by D'Auria-Regge [12]. Their's is also associated to a singularity in the vierbein structure of the manifold, but has vanishing NY number and nonzero Pontryagin and Euler numbers.

IV. CHIRAL ANOMALY

It is well known that the existence of anomalies can be attributed to the topological properties of the background where the quantum system is defined. In particular, for a massless spin-$\frac{1}{2}$ field in an external (not necessarily quantized) gauge field G, the anomaly for the conservation law of the chiral current is proportional to the Pontryagin form for the gauge group:

$$\partial_\mu\langle J^\mu_5\rangle = \frac{1}{4\pi^2}\mathrm{Tr}F\wedge F. \tag{18}$$

The question then naturally arises as to whether the torsional invariants can produce similar physically observable effects [6].

Kimura [13], Delbourgo and Salam [14], and Eguchi and Freund [15] evaluated the quantum violation of the chiral current conservation in a four-dimensional Riemannian background *without torsion*, finding it proportional to the Pontryagin density of the manifold:

$$\partial_\mu\langle J^\mu_5\rangle = \frac{1}{8\pi^2}R^{ab}\wedge R_{ab}. \tag{19}$$

This result was also supported by the computation of Alvarez-Gaumé and Witten [16], of all possible gravitational anomalies and the Atiyah-Singer index for the Dirac operator for massless fermions in a curved background, and the complete study of consistent non-Abelian anomalies on arbitrary manifolds by Bonora, Pasti, and Tonin [17].

It has been sometimes argued that the presence of torsion could not affect the chiral anomaly (see, e.g., [17–20]). This is motivated by the fact that the Pontryagin number is insensitive to the presence of torsion, as it is obvious from Eqs.

(2) and (4). This does not prove, however, that the anomaly is given by the Pontryagin class and nothing else.

As for the Atiyah-Singer index theorem, it is fairly clear that the difference between the number of left- and right-handed zero modes should not jump under any continuous deformation of the geometry. Therefore, the index could not change under adiabatic inclusion of torsion in the connection. However, nothing can be said *a priori* about the changes of the index under *discontinuous* modifications in the torsion, as it might happen if flat spacetime is replaced by one containing a topologically nontrivial configuration.

The integral of an anomaly must be a topological invariant [21] and, therefore, the assertion above would be true if there were no other independent topological invariants that could be constructed out of the torsion tensor.

Direct computations of the chiral anomaly in spaces with torsion were first done by Obukhov [7], and later by Yajima and collaborators [19,22]. These authors find a number of torsion-dependent contributions to the anomaly which are not clearly interpreted as densities of topological invariants. In a related work, Mavromatos [20] calculates the Atiyah-Singer index of the Dirac operator in the presence of curl-free H torsion. He finds a contribution which is an exact form by virtue of his assumption [curl-free H torsion amounts to assuming $d(e^a\wedge T_a)=0$] and is, therefore, dropped out.

In all the previous cases [7,19,22] (and in [20] if one does not assume $dH=0$), the NY term appears among many others. Many of these torsional pieces, including the NY term, are divergent when the regulator is removed, which was interpreted as an indication that these terms were a regulator artifact and should, therefore, be ignored.

Here we recalculate the anomaly with the Fujikawa method [23,24] and explicitly show the dependence of the anomaly on the NY four-form.

Consider a massless Dirac spinor on a curved background with torsion. The action is

$$S=\frac{i}{2}\int d^4x e\,\overline{\psi}\nabla\!\!\!\!/\,\psi+\text{H.c.}, \tag{20}$$

where the Dirac operator is given by

$$\nabla\!\!\!\!/=e_a{}^{\mu}\gamma^a\nabla_{\mu}, \tag{21}$$

here $e_a{}^{\mu}$ is the inverse of the tetrad $e^a{}_{\mu}$, γ^a are the Dirac γ matrices, and ∇_{μ} is the covariant derivative for the SO(4) connection in the appropriate representation.

This action is invariant under rigid chiral transformations

$$\psi\rightarrow e^{i\varepsilon\gamma_5}\psi, \tag{22}$$

where ε is a real constant parameter. This symmetry leads to the classical conservation law

$$\partial_{\mu}J_5^{\mu}=0, \tag{23}$$

where $J_5^{\mu}=e e_a{}^{\mu}\overline{\psi}\gamma^a\gamma_5\psi$.

The chiral anomaly is given by

$$\partial_{\mu}\langle J_5^{\mu}\rangle=\mathcal{A}(x), \tag{24}$$

where

$$\mathcal{A}(x)=2\sum_n e(x)\psi_n^{\dagger}\gamma_5\psi_n. \tag{25}$$

With the standard regularization, $\mathcal{A}$ is

$$\mathcal{A}(x)=2\lim_{y\rightarrow x}\lim_{M\rightarrow\infty}\text{Tr}\left[\gamma_5\exp\left(\frac{\nabla\!\!\!\!/^2}{M^2}\right)\right]\delta(x,y). \tag{26}$$

The square of the Dirac operator is given by

$$\nabla\!\!\!\!/^2=\nabla^{\mu}\nabla_{\mu}-e_a^{\mu}e_b^{\nu}e_c^{\lambda}J^{ab}T^c_{\mu\nu}\nabla_{\lambda}+\tfrac{1}{2}e_a^{\mu}e_b^{\nu}J^{ab}J^{cd}R_{cd\mu\nu}, \tag{27}$$

where $J_{ab}=\frac{1}{4}[\gamma_a,\gamma_b]$ is the generator of SO(4) in the spinorial representation.

The Dirac δ on a curved background is represented by

$$\delta(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{ik^{\mu}\nabla_{\mu}\Sigma(x,y)}, \tag{28}$$

where $\Sigma(x,y)$ is the geodesic biscalar [25]. Applying the operator $\exp(\nabla\!\!\!\!/^2/M^2)$ on Eq. (26), taking the limit $y\rightarrow x$, and tracing over spinor indices, one finds

$$\mathcal{A}=\frac{1}{8\pi^2}[R^{ab}\wedge R_{ab}+2M^2(T_a\wedge T^a-R_{ab}\wedge e^a\wedge e^b)]+O(M^{-2}). \tag{29}$$

The leading contribution of torsion to the anomaly $(M^2/4\pi^2)N$ diverges as the regulator is removed in agreement with the results of [7,19,22,20].[2]

A finite result would be obtained if the vierbein were rescaled as

$$e^a\rightarrow\tilde{e}^a=\frac{1}{Ml}e^a. \tag{30}$$

In that case the expression for the anomaly becomes (dropping the tildes)

$$\mathcal{A}(x)=\frac{1}{8\pi^2}\left(R^{ab}\wedge R_{ab}+\frac{2}{l^2}(T^a\wedge T_a-R_{ab}\wedge e^a\wedge e^b)\right). \tag{31}$$

It is interesting to observe that if in the earlier results of Refs. [7,19,22], one makes the same rescaling, all but one of the torsional contributions to the anomaly vanish in the limit $M\rightarrow\infty$. The remaining term is N.

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[2]In a Pauli-Villars regularization scheme, such divergent terms would be eliminated by an appropriate choice of regulator mass parameters. This scheme, however, rests on the assumption that at high energy, spacetime has Poincaré invariance, but this is not a trivial assumption in the presence of gravity. We thank G. 't Hooft for pointing this out to us.

V. DISCUSSION

A. Higher dimensions

Topological invariants associated to the spacetime torsion exist in higher dimensions whose occurrence, however, is very hard to predict for arbitrary D [6]. An obvious family of these invariants for $D=4k$ is of the form $N^k = N \wedge N \wedge \cdots N$, but there are others which do not fall into this class. For example, in 14 dimensions, the 14-form $(T_a \wedge R^a{}_b \wedge R^b{}_c \wedge R^c{}_d \wedge e^d) \wedge (T_a \wedge R^a{}_b \wedge e^b)$ is a locally exact. The number $n(D)$ of independent torsional invariants for a given dimension is as follows:

D	2	4	6	8	10	12	14
$n(D)$	0	1	0	4	0	12	1

The instanton constructed here is easily generalized for $D=8$, where there are four NY forms (the wedge product is implicitly assumed),

$$\begin{aligned} N_1 &= N^2, \\ N_2 &= (R^a{}_b R^b{}_a) N, \\ N_3 &= 4(T_a R^a{}_b e^b)(T_a e^a) + (T_a T^a)^2 - (e_a R^a{}_b e^b)^2, \\ N_4 &= T_a R^a{}_b R^b{}_c T^c - e_a R^a{}_b R^b{}_c R^c{}_d e^d. \end{aligned} \tag{32}$$

Of all these, only $(T_a T^a)^2 = d[e^a T_a T^b T_b]$ survives if the space is assumed to be curvature-free. The integration over a seven-sphere $x_1^2 + \cdots + x_8^2 = r^2$ embedded on $\mathbb{R}^8$ can be easily performed using a frame formed by one radial one-form (e^r) and seven orthonormal fields (e^i), tangent to S^7. The e^i's are generated using the canonical isomorphism between $\mathbb{R}^8$ and the octonion algebra: multiplying e^r by each of the seven generators of the algebra, and seven orthonormal fields tangent to the sphere are produced. The first one is

$$\begin{aligned} e^1 = & -x_2 dx_1 + x_1 dx_2 - x_4 dx^3 + x_3 dx^4 - x_6 dx^5 + x_5 dx^6 \\ & - x_8 dx^7 + x_7 dx^8, \end{aligned} \tag{33}$$

and the rest are similarly obtained. The integral is thus a combinatorial factor times the volume of the S^7 $(\pi^4/3)$.

In eight dimensions the integral of N^2 is not simply equal to the difference of the Pontryagin classes of SO(9) and SO(8), as one could naively expect by analogy with the case $D=4$. The Pontryagin density of SO(9) is

$$\begin{aligned} \mathrm{Tr}(F^4) - \frac{1}{2}(\mathrm{Tr}(F^2))^2 = & \mathrm{Tr}(R^4) - \tfrac{1}{2}(\mathrm{Tr}(R^2))^2 \\ & + \frac{4}{l^4}[(2T_a R^a{}_b e^b)(T_a e^a) \\ & + (e_a R^a{}_b e^b)N] + \frac{4}{l^2}[e_a R^a{}_b R^b{}_c R^c{}_d e^d \\ & + \tfrac{1}{2}\mathrm{Tr}(R^2)N - T_a R^a{}_b R^b{}_c T^c]. \end{aligned} \tag{34}$$

The first two terms are the Pontryagin form of SO(8), but the terms that depend on the torsion vanish for $R^a{}_b = 0$.

Note that a construction similar to the one for the three- and seven-spheres cannot be repeated in any other dimension because only the one-, three-, and seven-spheres admit a globally defined basis of vector fields [11]. In general, the maximum number of independent global vectors that can be defined on S^{n-1} is given by Radon's formula [26],

$$\rho_n = 2^c + 8d - 1, \tag{35}$$

where n is written as

$$n = (\text{odd integer}) 2^c 16^d, \tag{36}$$

with $c \leqslant 3$ and d positive integers. From this formula, it is clear that for all odd-dimensional spheres $\rho_n \geqslant 1$, while for even-dimensional spheres (odd n), $\rho_n = 0$.

Thus, it is only in four dimensions that the NY class can be computed in a curvature-free background.

B. Anomaly

In Sec. IV we argued that the anomaly could be made finite if one were to rescale the tetrad as $e^a_\mu \to \tilde{e}^a_\mu = (lM)^{-1} e^a_\mu$. Two remarks are in order: First, it should be stressed that this is the only rescaling that is needed to yield a finite result. Second, the Lagrangian for the tetrad field has not been discussed and, therefore, the replacement $e \to \tilde{e}$ is purely formal and can have no physical consequences as long as its dynamics is not specified.

In our analysis e is an external (classical) background field. One could view the rescaling of the vierbein as an invariance of the action, provided the Dirac field is suitably rescaled as well. This transformation was also considered by Nieh and Yan in [27]. However, in order for this invariance of the action to be interpreted as a symmetry generated by charges acting on the fields, one should include a scale-invariant Lagrangian for e.

The vierbein has units of $(\text{mass})^0$ and is, therefore, not of the same canonical dimension as the connection. If e/l is to be regarded as part of a connection of SO(5), the limit $M \to \infty$ keeping l fixed could be interpreted as a way to turn the SO(4)-invariant action (20) into that for a spinor minimally coupled to an SO(5) connection [28]. In this case, the chiral anomaly is then given by $P_4[\mathrm{SO}(5)]$, which is precisely Eq. (31).

C. Index

The Atiyah-Singer index for the Dirac operator in the absence of torsion is given by the Pontryagin number. Obviously, as P is independent of the vierbein, its invariance under continuous deformations of the geometry also allows for continuous deformations of the local frames and, in particular, for the addition of torsion. A different issue is whether the presence of torsion can affect the index of the Dirac operator through these invariants.

In [20] it is shown that there is a torsional contribution to the index although it is set equal to zero by the additional requirement of curl-free H torsion, and our result (31) agrees with that conclusion. The expression of the anomaly (31) indicates that if the index is calculated for an SO(5) connection, the result would reproduce our expression [28].

Note added. In the process of writing this article, we received a draft by Obukhov, Mielke, Budczies, and Hehl [29] where the instanton of Sec. III is reobtained in a somewhat different analysis, and our result for the anomaly (29) is also found in the heat kernel approach.

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