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Gauging $N=2$ supersymmetric nonlinear σ models in the Atiyah-Ward space-time

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We build up a class of $N=2$ supersymmetric nonlinear σ models in an $N=1$ superspace based on the Atiyah-Ward space-time of $(2+2)$-signature metric. We also discuss the gauging of isometries of the associated hyper-Kählerian target spaces and present the resulting gauge-covariant supersymmetric action functional. [S0556-2821(97)07010-0]

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I. INTRODUCTION

In the recent years much attention has been paid to the construction of new classical field models in the Atiyah-Ward space-time of $(2+2)$–signature metric [1]. As demonstrated in Refs. [2,3], this structure emerges in connection with a consistent $N=2$ superstring theory, whose underlying superconformal algebra requires a complex manifold as the relevant space-time background.

From the viewpoint of mathematics, the Atiyah-Ward space-time is also quite attractive when regarded as a four-dimensional arena in which one could introduce self-dual Yang-Mills connections [4,5]. In fact, these objects are known to play a significant role as a field-theoretical tool in the Donaldson's program on algebraic geometry [6] and, as conjectured by Ward [7], may be also of importance in the classification of lower-dimensional integrable models.

In view of these facts, it seems also interesting to build up and analyze supersymmetric Yang-Mills theories in the Atiyah-Ward space-time. Indeed, such models were first considered by Gates *et al.* in Refs. [9], where a superspace formalism adapted to the $2+2$ signature was introduced: the so-called $N=1$ superspace of Atiyah-Ward. Other related aspects in this domain were further investigated in Ref. [10]. Moreover, in Ref. [11], one was able to present a supersymmetric nonlinear σ model also in the Atiyah-Ward superspace and to couple its associated scalar superfields to a super-Yang-Mills gauge sector through the gauging of isometries of the target manifold [12–20]. Clearly, the class of theories focused here should be necessarily understood in the sense of the dimensional reduction framework used by Ward in [7]. In that scheme, one may eventually obtain new examples of integrable field models in two dimensions (see also Ref. [8]).

This is the purpose of the present work: to give a detailed account on the construction and gauging of supersymmetric σ models in the manner of Atiyah and Ward. Specifically, we will be concerned here with hyper-Kählerian σ models possessing $N=2$ supersymmetries, one of them being nonlinerarly realized and, subsequently, with the issue of performing their gauging by means of the approach developed in Ref. [15].

Our paper is organized as follows. In Sec. II we describe in a self-contained fashion all the necessary steps needed to build up the gauged $N=1$ supersymmetric σ model in $D=2+2$ dimensions (a problem already addressed in Ref. [11]) and state the essential notions on hyper-Kähler geometry which are crucial for the $N=2$ extension of the following section; Sec. III is then devoted to the study of $N=2$ supersymmetry in the $N=1$ superspace of Atiyah and Ward and to the gauging of the hyper-Kählerian σ model in the context of a certain Kählerian vector supermultiplet. In Sec. IV we interpret our results and present our conclusions.

II. THE HYPER-KÄHLERIAN σ MODEL IN SUPERSPACE

We begin the present investigation by focusing on the construction of gauged $N=1$ supersymmetric σ models in the Atiyah-Ward space-time. The notation and conventions for a superspace with base space-time possessing a $2+2$ signature are the same as in [10]. To build up the action functional for a class of Kählerian σ models we will follow here the well-known method of Zumino [21] (see Refs. [22,23] for an extensive discussion on Kähler geometry). We introduce a set of complex chiral and antichiral superfields, Φ^i and Ξ^i ($i=1,\ldots,n$), with their component field expansions written as[1]

$$\Phi^i=A^i+i\theta\psi^i+i\theta^2F^i+i\tilde{\theta}\tilde{\sigma}^\mu\theta\partial_\mu A^i+\tfrac{1}{2}\theta^2\tilde{\theta}\tilde{\sigma}^\mu\partial_\mu\psi^i-\tfrac{1}{4}\theta^2\tilde{\theta}^2\Box A^i, \tag{1}$$

$$\Xi^i=B^i+i\tilde{\theta}\tilde{\chi}^i+i\tilde{\theta}^2G^i+i\theta\sigma^\mu\tilde{\theta}\partial_\mu B^i+\tfrac{1}{2}\tilde{\theta}^2\theta\sigma^\mu\partial_\mu\tilde{\chi}^i-\tfrac{1}{4}\theta^2\tilde{\theta}^2\Box B^i, \tag{2}$$

where A^i and B^i are complex scalar fields, ψ^i and $\tilde{\chi}^i$ are Majorana-Weyl spinors and F^i and G^i are complex scalar auxiliary fields. One has to observe that, differently to the Minkowskian situation, the scalar superfields at hand do not change their chirality properties under the complex conjugation operation:

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[1] The Grassmann coordinates θ and $\tilde{\theta}$ are Majorana-Weyl spinors.

$$\widetilde{D}_{\dot\alpha}\Phi^i=\widetilde{D}_{\dot\alpha}\Phi^{*i}=0,$$
$$D_\alpha\Xi^i=D_\alpha\Xi^{*i}=0, \tag{3}$$

with

$$D_\alpha=\partial_\alpha-i\widetilde{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha},$$
$$\widetilde{D}_{\dot\alpha}=\widetilde{\partial}_{\dot\alpha}-i\theta^\alpha\widetilde{\partial}_{\dot\alpha\alpha}, \tag{4}$$

and

$$\{D_\alpha,\widetilde{D}_{\dot\alpha}\}=-2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu,\quad \{D_\alpha,D_\beta\}=\{\widetilde{D}_{\dot\alpha},\widetilde{D}_{\dot\beta}\}=0,$$
$$[D_\alpha,\partial_\mu]=[\widetilde{D}_{\dot\alpha},\partial_\mu]=0.$$

Now, one writes down a rather specific supersymmetric action to govern the dynamics of the scalar superfields. We take[2]

$$I=2\int d^4x d^2\theta d^2\widetilde{\theta}K(\Phi^i,\Xi^i;\Phi^{*i},\Xi^{*i}), \tag{5}$$

where the Kähler potential K decomposes into two conjugated pieces as below:

$$K(\Phi^i,\Xi^i;\Phi^{*i},\Xi^{*i})=H(\Phi^i,\Xi^{*i})+H^*(\Phi^{*i},\Xi^i). \tag{6}$$

The pure scalar sector stemming from the projection of Eq. (5) into component fields is given by

$$I_{\text{scalar}}=2\int d^4x\left(\frac{\partial^2K}{\partial A^i\partial B^{*j}}\partial_\mu A^i\partial^\mu B^{*j}+\frac{\partial^2K}{\partial A^{*i}\partial B^j}\partial_\mu A^{*i}\partial^\mu B^j\right). \tag{7}$$

Upon dimensional reduction and proper field truncations, I_{scalar} above will give rise to a sensible (ghost-free) scalar kinetic term in $D=1+2$ space-time dimensions (see Ref. [10]).

The possible target spaces associated with the action I in Eq. (5) do belong to a restricted class of $4n$-dimensional Kähler manifolds, their Hermitian metric tensor appearing in a four-block structure as

$$g_{\mathcal{I}\mathcal{J}}=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & g_{i\bar{\hat{j}}} \\ \mathbf{0} & \mathbf{0} & g_{\hat{i}\bar{j}} & \\ \mathbf{0} & g_{\bar{i}\hat{j}} & \mathbf{0} & \mathbf{0} \\ g_{\bar{\hat{i}}j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{8}$$

with

$$g_{i\bar{\hat{j}}}=\frac{\partial^2H}{\partial\Phi^i\partial\Xi^{*j}},\quad g_{\hat{i}\bar{j}}=\frac{\partial^2H^*}{\partial\Xi^i\partial\Phi^{*j}},\quad g_{\bar{i}\hat{j}}=\frac{\partial^2H^*}{\partial\Phi^{*i}\partial\Xi^j},$$
$$g_{\bar{\hat{i}}j}=\frac{\partial^2H}{\partial\Xi^{*i}\partial\Phi^j}, \tag{9}$$

and

$$\mathcal{I},\mathcal{J}=1,\ldots,4n\quad\text{and}\quad i,j=1,\ldots,n.$$

It is clear now that the particular form of $g_{\mathcal{I}\mathcal{J}}$ will entail a number of consequences for the geometry of our Kählerian target manifold. The most general type of Kähler transformation one can perform upon the potential K while keeping the action (5) invariant and the metric (8) unchanged is

$$K\rightarrow K'=K+\eta(\Phi)+\eta^*(\Phi^*)+\rho(\Xi)+\rho^*(\Xi^*), \tag{10}$$

with (η,η^*) and (ρ,ρ^*) standing for arbitrary chiral and antichiral functions, respectively. Hence, every isometry transformation of the target manifold will be a symmetry of Eq. (5) provided its action on K writes into a form compatible with Eq. (10). The Killing vectors $[\kappa^i_a(\Phi),\tau^i_a(\Xi),\kappa^{*i}_a(\Phi^*),\tau^{*i}_a(\Xi^*)]$ are the generators of the isometry group $\mathcal{G}$ and satisfy the usual Lie algebraic relations:

$$\kappa^i_a\kappa^j_{b,i}-\kappa^i_b\kappa^j_{a,i}=f_{ab}{}^c\kappa^j_c,\quad \kappa^{*i}_a\kappa^{*j}_{b,i}-\kappa^{*i}_b\kappa^{*j}_{a,i}=f_{ab}{}^c\kappa^{*j}_c,$$
$$\tau^i_a\tau^j_{b,i}-\tau^i_b\tau^j_{a,i}=f_{ab}{}^c\tau^j_c,\quad \tau^{*i}_a\tau^{*j}_{b,i}-\tau^{*i}_b\tau^{*j}_{a,i}=f_{ab}{}^c\tau^{*j}_c, \tag{11}$$

where $f_{ab}{}^c$ are the structure constants. A global isometry transforms the target coordinates as

$$\Phi'^i=\exp(L_{\lambda\cdot\kappa})\Phi^i,\quad \Phi'^{*i}=\exp(L_{\lambda\cdot\kappa^*})\Phi^{*i},$$
$$\Xi'^i=\exp(L_{\lambda\cdot\tau})\Xi^i,\quad \Xi'^{*i}=\exp(L_{\lambda\cdot\tau^*})\Xi^{*i}, \tag{12}$$

where λ is for a real parameter and $L_{\lambda.\kappa}$ ($L_{\lambda.\tau}$) is the Lie derivative along the vector field $\lambda.\kappa\equiv\lambda^a\kappa^i_a\partial_i$ ($\lambda.\tau\equiv\lambda^a\tau^i_a\partial_{\hat{i}}$). The set of laws above may be related to some Kähler transformation such as Eq. (10), the chiral and antichiral functions being given as

$$\eta_a(\Phi)=\partial_iH(\Phi,\Xi^*)\kappa^i_a(\Phi)+Y_a(\Phi,\Xi^*),$$
$$\rho_a(\Xi)=\partial_{\hat{i}}H^*(\Xi,\Phi^*)\tau^i_a(\Xi)-Y^*_a(\Xi,\Phi^*),$$
$$\eta^*_a(\Phi^*)=\partial_{\bar{i}}H^*(\Xi,\Phi^*)\kappa^{*i}_a(\Phi^*)+Y^*_a(\Xi,\Phi^*),$$
$$\rho^*_a(\Xi^*)=\partial_{\bar{\hat{i}}}H(\Phi,\Xi^*)\tau^{*i}_a(\Xi^*)-Y_a(\Phi,\Xi^*). \tag{13}$$

By differentiating the first and last equations in Eqs. (13) with respect to Ξ^{*j} and Φ^j, respectively, one gets

$$H_{i\bar{\hat{j}}}\kappa^i_a=-Y_{a\bar{\hat{j}}},$$
$$H_{\bar{\hat{i}}j}\tau^{*i}_a=Y_{aj}, \tag{14}$$

which, in turn, allows one to write the identity

$$\kappa^i_aY_{bi}+\tau^{*i}_bY_{a\bar{\hat{i}}}=0. \tag{15}$$

From the algebra (11), and from Eq. (13), we have

$$H_i\kappa^j_{[a}\kappa^i_{b]j}+H_{\bar{\hat{i}}}\tau^{*j}_{[a}\tau^{*\bar{\hat{i}}}_{b]\bar{\hat{j}}}=f_{ab}{}^c(\eta_c+\rho^*_c), \tag{16}$$

which, by means of Eq. (15), can be rewritten as

[2] $\int d^4xd^2\theta d^2\widetilde{\theta}\equiv\frac{1}{16}\int d^4xD^\alpha\widetilde{D}^{\dot\alpha}\widetilde{D}_{\dot\alpha}D_\alpha$.

$$\kappa^{j}_{[a}\eta_{b]j}+\tau^{*j}_{[a}\rho^{*}_{b]j}=f_{ab}{}^{c}(\eta_c+\rho^{*}_c). \tag{17}$$

From holomorphicity considerations, one may set

$$\kappa^{j}_{[a}\eta_{b]j}=f_{ab}{}^{c}\eta_c+ic_{ab},$$
$$\tau^{*j}_{[a}\rho^{*}_{b]j}=f_{ab}{}^{c}\rho^{*}_c-ic_{ab}, \tag{18}$$

where $c_{ab}=-c_{ba}$ are real constants. In the restricted case of a semisimple gauge group $\mathcal{G}$, we may remove the c_{ab}'s by simply imposing $c_{ab}=0$ (in other cases they represent an obstruction to the gauging [15]). With this restriction, one writes the variation on the Killing potential as

$$\delta Y_a=\tfrac{1}{2}\lambda^{b}(\kappa^{i}_{[b}Y_{a]i}+\tau^{*i}_{[b}Y_{a]i})=-\lambda^{b}f_{ab}{}^{c}Y_c, \tag{19}$$

where one has used Eqs. (11), (13), and (18). Now, from Eqs. (14) and (19), we obtain the complex potential Y_a:

$$Y_a=2f_{ab}{}^{c}\kappa^{i}_{d}\tau^{*j}_{c}\frac{\partial^2 H}{\partial\Phi^{i}\partial\Xi^{*j}}g^{bd}, \tag{20}$$

in which g^{bd} is the inverse Killing metric.

To proceed to the covariantization of the action (5) with respect to gauged isometries, i.e., the local version of the set of field transformations (12), one introduces a couple (Λ,Γ) of real chiral and antichiral superfield parameters, respectively [11]. The local isometry transformations are defined as

$$\Phi'=\exp(L_{\Lambda\cdot\kappa})\Phi,\quad \Xi'=\exp(L_{\Gamma\cdot\tau})\Xi. \tag{21}$$

The gauge sector is built up from the prepotential V, a real superfield transforming such as

$$\exp(L_{V'\cdot\tau})=\exp(L_{\Lambda\cdot\tau})\exp(L_{V\cdot\tau})\exp(-L_{\Gamma\cdot\tau}). \tag{22}$$

We modify then the action (5) by replacing the antichiral superfields (Ξ,Ξ^{*}) with the redefined quantities $(\tilde{\Xi},\tilde{\Xi}^{*})$ given below:

$$\tilde{\Xi}^{i}\equiv\exp(L_{V\cdot\tau})\Xi^{i},\quad \tilde{\Xi}^{*i}\equiv\exp(L_{V\cdot\tau^{*}})\Xi^{*i}. \tag{23}$$

Infinitesimally, one has the following isometry transformation laws for the superfields:

$$\delta\Phi^{i}=\Lambda^{a}\kappa^{i}_{a},\quad \delta\Phi^{*i}=\Lambda^{a}\kappa^{*i}_{a},$$
$$\delta\tilde{\Xi}^{i}=\Lambda^{a}\tau^{i}_{a},\quad \delta\tilde{\Xi}^{*i}=\Lambda^{a}\tau^{*i}_{a}. \tag{24}$$

It turns out, moreover, that the correct covariantization of Eq. (5) still demands the introduction of a complex-conjugated pair of antichiral superfields (v,v^{*}) transforming as

$$\delta v=\lambda^{a}\rho_a(\Xi),$$
$$\delta v^{*}=\lambda^{a}\rho^{*}_a(\Xi^{*}). \tag{25}$$

The isometry-covariant action functional is then taken to be

$$I_{\rm cov}=2\int d^4x d^2\theta d^2\tilde{\theta}[H(\Phi,\tilde{\Xi}^{*})+H^{*}(\Phi^{*},\tilde{\Xi})-\tilde{v}-\tilde{v}^{*}], \tag{26}$$

which, in terms of the original variables, writes as

$$I_{\rm cov}=2\int d^4x d^2\theta d^2\tilde{\theta}\left\{H(\Phi,\Xi^{*})+H^{*}(\Phi^{*},\Xi)+2\,{\rm Re}\left[\frac{e^{L}-1}{L}V^{a}Y^{*}_a(\Phi^{*},\Xi)\right]\right\}, \tag{27}$$

with $L\equiv L_{V.\tau}$.

As mentioned in the Introduction, it will be our aim hereafter to extend the construction leading to $I_{\rm cov}$ in Eq. (27) above to the more general task of analyzing the gauging of $N=2$ supersymmetric σ model in the $N=1$ superspace of Atiyah and Ward. With this purpose in mind, one is enforced here to consider the more restricted class of hyper-Kählerian σ models in order to introduce a second set of supersymmetry field transformations, following in much the same way what was envisaged already in the last decade by Bagger and Witten [24]. The Kählerian target space of our σ model can also be taken as a hyper-Kähler manifold as long as its metric tensor $g_{\mathcal{I}\mathcal{J}}$ in Eq. (8) is Hermitian with respect to a quaternionic structure $\{J^{(1)\mathcal{J}}_{\mathcal{I}},J^{(2)\mathcal{J}}_{\mathcal{I}},J^{(3)\mathcal{J}}_{\mathcal{I}}\}$. The tensors $J^{(x)\mathcal{J}}_{\mathcal{I}}$ are covariantly constant and generate the SU(2) algebra:

$$J^{(x)\mathcal{J}}_{\mathcal{I}}J^{(y)\mathcal{K}}_{\mathcal{J}}=-\delta^{xy}\delta^{\mathcal{K}}_{\mathcal{I}}+\epsilon^{xyz}J^{(z)\mathcal{K}}_{\mathcal{I}}.$$

The complex structures are parametrized here as

$$J^{(1)\mathcal{J}}_{\mathcal{I}}=\begin{pmatrix} i\delta^{j}_{i} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & i\delta^{\hat{j}}_{\hat{i}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -i\delta^{\bar{j}}_{\bar{i}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -i\delta^{\bar{\hat{j}}}_{\bar{\hat{i}}}\end{pmatrix}, \tag{28}$$

$$J^{(2)\mathcal{J}}_{\mathcal{I}}=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & J^{\bar{\hat{j}}}_{i}\\ \mathbf{0} & \mathbf{0} & J^{\bar{j}}_{\hat{i}} & \mathbf{0}\\ \mathbf{0} & J^{\hat{j}}_{\bar{i}} & \mathbf{0} & \mathbf{0}\\ J^{j}_{\bar{\hat{i}}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}, \tag{29}$$

and

$$J^{(3)\mathcal{J}}_{\mathcal{I}}=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & iJ^{\bar{\hat{j}}}_{i}\\ \mathbf{0} & \mathbf{0} & iJ^{\bar{j}}_{\hat{i}} & \mathbf{0}\\ \mathbf{0} & -iJ^{\hat{j}}_{\bar{i}} & \mathbf{0} & \mathbf{0}\\ -iJ^{j}_{\bar{\hat{i}}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}. \tag{30}$$

It is the very existence of such a quaternionic structure that enables one to introduce a nonlinearly realized supersymmetry in the theory. In fact, we shall see in the next section that the action (27) can be conveniently supplemented with new interaction terms which will render it invariant under $N=2$ supersymmetries, while preserving its covariance under the gauged isometries (24).

III. THE $N=2$ SUPERSYMMETRIC EXTENSION

In this section we analyze the $N=2$ supersymmetric extension of our gauged σ model in the Atiyah-Ward superspace. By following a reasoning similar to that of [15], one defines the second supersymmetry in terms of two sets of complex functions of the target coordinates, the potentials $\Omega^i \equiv \Omega^i(\Phi,\Xi^*)$ and $\Upsilon^i \equiv \Upsilon^i(\Xi,\Phi^*)$ $(i=1,\ldots,n)$, the field transformation laws being given by

$$\delta\Phi^i = i\tilde{D}^2(\epsilon\Omega^i), \quad \delta\Phi^{*i} = i\tilde{D}^2(\epsilon\Omega^{*i}),$$
$$\delta\Xi^i = iD^2(\zeta\Upsilon^i), \quad \delta\Xi^{*i} = iD^2(\zeta\Upsilon^{*i}), \tag{31}$$

where ζ and ϵ are real constant chiral and antichiral scalar superfields, respectively, i.e.,

$$D_\alpha\epsilon = \partial_\mu\epsilon = 0, \quad \tilde{D}_{\dot\alpha}\zeta = \partial_\mu\zeta = 0, \tag{32}$$

and, moreover,

$$\tilde{D}^2\epsilon = D^2\zeta = 0. \tag{33}$$

The on-shell closure of the algebra of transformations in Eqs. (31) imposes the following constraints on the potentials:

$$\Omega^i{}_{,\bar{\hat{i}}\hat{k}}\Upsilon^{*j}{}_{,n} + \Omega^i{}_{,\bar{j}}\Upsilon^{*j}{}_{,n\bar{\hat{k}}} = 0,$$
$$\Upsilon^i{}_{,\bar{j}\bar{k}}\Omega^{*j}{}_{,\hat{n}} + \Upsilon^i{}_{,\bar{j}}\Omega^{*j}{}_{,\hat{n}\bar{k}} = 0,$$
$$\Omega^i{}_{,\bar{j}}\Upsilon^{*j}{}_{,n} = -\delta^i_n, \quad \Upsilon^i{}_{,\bar{j}}\Omega^{*j}{}_{,\hat{n}} = -\delta^{\hat{i}}_{\hat{n}},$$
$$\Omega^i{}_{,j[\bar{k}}\Omega^j{}_{,\bar{n}]} = 0, \quad \Upsilon^i{}_{,\hat{j}[\bar{k}}\Upsilon^j{}_{,\bar{n}]} = 0,$$
$$\tilde{D}^2\Omega^i = 0, \quad D^2\Upsilon^i = 0, \tag{34}$$

with the lower indices standing for derivatives with respect to the target coordinates. Moreover, by requiring the invariance of the action (5) under Eqs. (31), we arrive at the additional conditions upon the functions Ω^i and Υ^i:

$$H_{i\hat{j}}\Omega^i{}_{,\bar{\hat{n}}} + H_{i\hat{n}}\Omega^i{}_{,\bar{\hat{j}}} = 0, \quad H^*_{\hat{i}j}\Upsilon^i{}_{,\bar{n}} + H^*_{\hat{i}n}\Upsilon^i{}_{,\bar{j}} = 0,$$
$$H_{i\hat{n}}\Omega^i{}_{,\bar{\hat{j}}\hat{k}} + H_{i\hat{j}\hat{k}}\Omega^i{}_{,\bar{\hat{n}}} = 0, \quad H^*_{\hat{i}n}\Upsilon^i{}_{,\bar{j}\bar{k}} + H^*_{\hat{i}jk}\Upsilon^i{}_{,\bar{n}} = 0,$$
$$H_{i\hat{j}}\Omega^i{}_{,\bar{\hat{n}}\bar{k}} + H_{i\hat{j}\hat{k}}\Omega^i{}_{,\bar{\hat{n}}} = 0, \quad H^*_{\hat{i}j}\Upsilon^i{}_{,\bar{n}\bar{k}} + H^*_{\hat{i}jk}\Upsilon^i{}_{,\bar{n}} = 0, \tag{35}$$

together with their complex-conjugated counterparts. At this point, by means of a careful inspection of Eqs. (34) and (35), one observes that the functions Ω^i and Υ^i are encompassing in their structure all the important features of the hyper-Kählerian geometry [15,16]. Indeed, this property can be made even more apparent if we introduce the identifications

$$J_{\bar{\hat{i}}}^{j} = \Omega^j{}_{,\bar{\hat{i}}}, \quad J_{\bar{i}}^{\hat{j}} = \Upsilon^j{}_{,\bar{i}}, \quad J_{\hat{i}}^{\bar{j}} = \Omega^{*\bar{j}}{}_{,\hat{i}}, \quad J_i^{\bar{\hat{j}}} = \Upsilon^{*j}{}_{,i}, \tag{36}$$

in the complex structures (29) and (30).

Furthermore, from the assumption of triholomorphicity of the Killing vectors with respect to the quaternionic structure, one can define the potentials $P_a^{(+)} \equiv P_a^{(+)}(\Phi,\Xi)$ and $P_a^{(-)} \equiv P_a^{(-)}(\Phi^*,\Xi^*)$ such that $P_a^{(-)} = (P_a^{(+)})^*$ and

$$k_a^i\omega_{ij}^{(+)} = -P_a^{(+)}{}_{,j}, \quad k_a^{*i}\omega_{\bar{i}\bar{j}}^{(-)} = -P_a^{(-)}{}_{,\bar{j}}, \tag{37}$$

$$\tau_a^{\hat{i}}\omega_{\hat{i}j}^{(+)} = -P_a^{(+)}{}_{,\hat{j}}, \quad \tau_a^{*\hat{i}}\omega_{\bar{\hat{i}}\bar{j}}^{(0)} = -P_a^{(-)}{}_{,\bar{\hat{j}}}, \tag{38}$$

with

$$\omega_{ij}^{(+)} = -2H_{j\bar{\hat{k}}}\Upsilon^{*k}{}_{,i}, \quad \omega_{\bar{i}\bar{j}}^{(-)} = -2H^*_{\bar{j}\hat{k}}\Upsilon^k{}_{,\hat{i}},$$
$$\omega_{\hat{i}\hat{j}}^{(+)} = -2H^*_{\hat{j}\bar{k}}\Omega^{*k}{}_{,\hat{i}}, \quad \omega_{\bar{\hat{i}}\bar{\hat{j}}}^{(-)} = -2H_{\bar{\hat{j}}\bar{k}}\Omega^k{}_{,\bar{\hat{i}}}. \tag{39}$$

From Eqs. (13) above and from the formulas expressing the chiral and antichiral functions (37) and (38), we derive some useful relations involving the Killing potentials $Y_a(\Phi,\Xi^*)$:

$$P^{(+)}_{a,j}\Omega^j{}_{,\bar{\hat{i}}} = -2Y_{a,\bar{\hat{i}}} \rightleftharpoons P^{(+)}_{a,i} = 2Y_{a,\bar{\hat{j}}}\Upsilon^{*j}{}_{,i},$$

$$P^{(+)}_{a,\hat{j}}\Upsilon^j{}_{,\bar{i}} = 2Y^*_{a,\bar{i}} \rightleftharpoons P^{(+)}_{a,\hat{i}} = -2\overline{Y^*_{a,\bar{j}}}\Omega^{*j}{}_{,\hat{i}},$$

$$P^{(-)}_{a,\bar{\hat{j}}}\Omega^{*j}{}_{,\hat{i}} = -2Y^*_{a,\hat{i}} \rightleftharpoons P^{(-)}_{a,\bar{\hat{i}}} = 2Y^*_{a,j}\Upsilon^j{}_{,\bar{\hat{i}}},$$

$$P^{(-)}_{a,\bar{j}}\Upsilon^{*j}{}_{,i} = 2Y_{a,i} \rightleftharpoons P^{(-)}_{a,\bar{i}} = -2Y_{a,j}\Omega^j{}_{,\bar{\hat{i}}}. \tag{40}$$

To obtain $P_a^{(+)}$ and $P_a^{(-)}$, one observes that the complex functions

$$U_a = P_a^{(-)} - P_a^{(+)} - 2iY_a + 2iY_a^* \tag{41}$$

do satisfy the differential equations

$$(\partial_i + i\Upsilon^{*j}_{,i}\partial_{\bar{\hat{j}}})U_a = 0, \tag{42}$$

$$(\partial_{\hat{i}} + i\Omega^{*j}_{,\hat{i}}\partial_{\bar{j}})U_a = 0, \tag{43}$$

the complex conjugated U_a^*, obeying the complexified analogues thereof. Actually, Eqs. (42) and (43) are specifying the U_a's (U_a^*'s) as holomorphic functions (antiholomorphic functions) relatively to a noncanonical complex structure [15]. From the definition given in Eq. (41), one can write

$$U + U^* = 4(-iY + iY^*) \tag{44}$$

and

$$U - U^* = 2(P^{(-)} - P^{(+)}). \tag{45}$$

Now, from the holomorphicity and the gauge transformation of Y_a,

$$\delta Y_a = -\lambda^b f_{ab}{}^c Y_c, \tag{46}$$

one arrives at

$$\delta P_a^{(+)} = -\lambda^b f_{ab}{}^c P_c^{(+)} \tag{47}$$

and

$$\delta P_a^{(-)} = -\lambda^b f_{ab}{}^c P_c^{(-)}, \tag{48}$$

where the gauge group was assumed to be semisimple, which implies the absence of obstructions in Eqs. (46)–(48) above. On the other hand, we have

$$\delta P_a^{(+)} = \lambda^b (k_b^i P_a^{(+)}{}_{,i} + \tau_b^i P_a^{(+)}{}_{,\hat{i}}), \tag{49}$$

which, by comparison with Eq. (47) and use of the first equations of Eqs. (37) and (38), gives

$$P_a^{(+)} = f_a{}^{bc}(k_c^i k_b^j \omega_{ji}^{(+)} + \tau_c^i \tau_b^j \omega_{\hat{j}\hat{i}}^{(+)}). \tag{50}$$

Through complex conjugation, one also has

$$P_a^{(-)} = f_a{}^{bc}(k_c^{*i} k_b^{*j} \omega_{\underline{j}\,\underline{i}}^{(-)} + \tau_c^{*i} \tau_b^{*j} \omega_{\underline{\hat{j}}\,\underline{\hat{i}}}^{(-)}). \tag{51}$$

We now turn to the construction of the $N=2$ supersymmetric gauge sector in the $N=1$ superspace of Atiyah and Ward. In [25], Gates succeeded in writting down a set of nonlinear supersymmetry transformations for a certain $N=2$ gauge supermultiplet in $N=1$ Minkowski superspace (see also Ref. [26]). We adopt a similar approach here: our Kählerian gauge supermultiplet consists of a chiral scalar superfield S and an antichiral scalar superfield T, together with the vector superfield V of the previous section. All the three superfields are real and take values in the adjoint representation of the isometry gauge group $\mathcal{G}$. We propose the nonlinear supersymmetry transformations on gauge superfields as below:

$$\delta S = i W^\alpha D_\alpha \zeta,$$
$$\delta T = i \widetilde{W}^{\dot\alpha} \widetilde{D}_{\dot\alpha} \epsilon,$$
$$e^{-iV}\delta e^{iV} = \epsilon e^{-iV} S e^{iV} - \zeta T, \tag{52}$$

where the real scalar superfield parameters (ϵ,ζ) are the ones appearing in the supersymmetry transformations (31) for the matter sector; the gauge superfield strengths are defined to be

$$W_\alpha \equiv i\widetilde{D}^2(e^{iV} D_\alpha e^{-iV}),$$
$$\widetilde{W}_{\dot\alpha} \equiv i D^2(e^{-iV} \widetilde{D}_{\dot\alpha} e^{iV}), \tag{53}$$

they are covariant under gauge transformations of the type

$$e^{-iV}\delta_g e^{iV} = i(e^{-iV}\Lambda e^{iV} - \Gamma). \tag{54}$$

One has also to consider gauge transformation laws for the scalar gauge superfields:

$$\delta_g S = i[\Lambda, S], \quad \delta_g T = i[\Gamma, T]. \tag{55}$$

At this stage, we are ready to present the fully gauged $N=2$ supersymmetric nonlinear σ model in terms of $N=1$ superfields of the Atiyah-Ward superspace. As stated previously, this task is accomplished by suplementing the action (27) with new interaction pieces such as to render the second supersymmetry, i.e., Eqs. (31) and (52), a further invariance of the model [15]. Our main result is

$$I_{\rm cov} = 2\int d^4x d^2\theta d^2\widetilde{\theta}\Bigg\{ H(\Phi,\Xi^*) + H^*(\Phi^*,\Xi) + 2\,{\rm Re}\Bigg[\frac{e^{\hat{\mathcal{L}}}-1}{\hat{\mathcal{L}}} V^a Y_a^*(\Phi^*,\Xi)\Bigg] - \frac{1}{2} S^a \widetilde{T}_a \Bigg\} - \frac{1}{16}\int d^4x d^2\theta \{ g_{ab} W^{a\alpha} W_\alpha^b - 4iS^a[F_a(\Phi) + F_a^*(\Phi^*)]\} - \frac{1}{16}\int d^4x d^2\widetilde{\theta}\{ g_{ab}\widetilde{W}^{a\dot\alpha}\widetilde{W}_{\dot\alpha}^b - 4iT^a[G_a(\Xi) + G_a^*(\Xi^*)]\}, \tag{56}$$

with

$$\widetilde{T} \equiv e^{iV} T e^{-iV}, \tag{57}$$

and where we have made implicit use of the splittings in the functions $P_a^{(+)}$ and $P_a^{(-)}$ in Eqs. (50) and (51):

$$P_a^{(+)} = F_a(\Phi) + G_a(\Xi), \quad P_a^{(-)} = F_a^*(\Phi^*) + G_a^*(\Xi^*). \tag{58}$$

Finally, it is straightforward to check the invariance of Eq. (56) under Eqs. (52) and the (gauge-covariant) supersymmetry transformations for the matter superfields:

$$\delta\Phi^i = i\widetilde{D}^2[\epsilon\Omega^i(\Phi, e^{2\hat{\mathcal{L}}^*}\Xi^*)],$$
$$\delta\Phi^{*i} = i\widetilde{D}^2[\epsilon\Omega^{*i}(\Phi^*, e^{2\hat{\mathcal{L}}}\Xi)],$$
$$\delta\Xi^i = iD^2[\zeta\Upsilon^i(e^{-2\mathcal{L}^*}\Phi^*, \Xi)],$$
$$\delta\Xi^{*i} = iD^2[\zeta\Upsilon^{*i}(e^{-2\mathcal{L}}\Phi, \Xi^*)], \tag{59}$$

in which

$$\mathcal{L} = V^a k_a^i \frac{\partial}{\partial\Phi^i}, \quad \hat{\mathcal{L}} = V^a \tau_a^i \frac{\partial}{\partial\Xi^i}. \tag{60}$$

Indeed, one may impose the Wess-Zumino gauge condition, i.e., $V^3=0$, and to verify the invariance of $I_{\rm cov}$ under the nonlinear supersymmetry transformations at each order in the prepotential V. It should be observed once more that because of the presence of some gauge-algebraic obstructions [15,27], the supersymmetric gauging expressed in Eq. (56) will only hold for semisimple gauge groups $\mathcal{G}$, in which case one can always determine the potentials (20), (50), and (51).

IV. CONCLUDING REMARKS

We have explicitly constructed a class of $N=2$ supersymmetric nonlinear σ models coupled to a super-Yang-Mills gauge sector in the $N=1$ superspace of Atiyah and Ward. In order to perform this gauge coupling, one makes use of a general formalism introduced by Hull *et al.* in [15], gauging the isometries of the associated (hyper-Kähler) target manifold. We observe then that, also in the Atiyah-Ward superspace, it is possible to obtain the specific potentials needed for the referred gauging of the hyper-Kählerian σ model, namely, the Killing potential (20) (which is complex here) and the so-called momentum maps (50) and (51).

The gauge-invariant supersymmetric σ model obtained in the previous section may have some interesting applications in connection with the study of gauge dynamics of supersymmetric gauge theories in lower dimensions. In fact, by suppressing one time coordinate in the action (56), one may in principle arrive at new supersymmetric field models in three Minkowskian dimensions. The latter type of theories could then be regarded as an alternative scenario for checking the consequences of the duality hypothesis of four dimensions, following in much the same way what has been proposed in the recent literature [28].

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