Problem of unitarity and quantum corrections in semiclassical quantum gravity

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Using both the Born-Oppenheimer idea and the de Broglie–Bohm interpretation of a wave function we represent in a different way the semiclassical quantum gravity from the Wheeler-DeWitt equation in an oscillating regime which can preserve completely the unitary quantum evolution of a matter field at the expense of a nonlinear gravitational field equation, but has the same asymptotic limit as the others. We apply the de Broglie–Bohm interpretation to the nonlinear gravitational field equation to develop a perturbation method to find the quantum corrections of a matter field to the gravity. The semiclassical Einstein equation with the quantum corrections is found for a minimal quantum FRW cosmological model. [S0556-2821(97)05112-6]

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I. INTRODUCTION

The problem of unitarity and back reaction of a matter field in a curved spacetime has been an important and critical issue of gravity such as black holes and the very early Universe. Although the complete solution of this problem should be found within the context of the right and consistent quantum gravity, there has been quite recently an attempt to tackle this problem from the canonical quantum gravity based on the Wheeler-DeWitt (WDW) equation [1]. In the canonical quantum gravity approach to the gravity plus matter system, the gravitational wave functions of the WDW equation are peaked along (semi)classical trajectories of the (semi)classical Einstein equation, and the matter field, which obeys a time-dependent functional Schrödinger equation, evolves quantum mechanically along each trajectory. These two equations constitute the so-called semiclassical quantum gravity. We shall distinguish this approach from the conventional field theoretic approach in the curved spacetime [2]. Recently, the semiclassical quantum gravity has been applied to a two-dimensional black hole model [3].

One can derive the semiclassical quantum gravity in many different ways, for instance, depending on whether or not one includes the back reaction of the matter field to the semiclassical Einstein equation in the lowest order of the power series expansion in terms of the inverse Planck mass. When one searches for a WKB-type wave function for the entire WDW equation and expands its total action in the inverse power series of the Planck mass, one obtains the vacuum (classical) Einstein equation and the time-dependent functional Schrödinger equation for the matter field on it [4]. On the other hand, when one adopts the Born-Oppenheimer idea by treating the gravitational field as a massive particle degree of freedom and the matter field as a light particle degree of freedom, one can separate two different mass scales and derive the semiclassical Einstein equation with the quantum expectation value of the energy-momentum tensor operator as a source of matter and the time-dependent Schrödinger equation for the matter field [5,6]. In previous papers [7], by separating and expanding the WDW equation in the inverse power series of the Planck mass, we were able to derive the semiclassical quantum gravity and to find the quantum states of the matter field in the asymptotic limit of large Planck mass. Quite recently, Bertoni *et al.* [8] have shown that the semiclassical quantum gravity represented in three different ways which are in fact equivalent to one another in the asymptotic limit $O(\hbar/m_P)$.

In the asymptotic limit of large Planck mass, the semiclassical quantum gravity is unitary in the sense of conservation of probability of the matter wave functions [7]. In a different representation, there are, however, unitarity-violating terms in the time-dependent Schrödinger equation for the matter field beyond the asymptotic limit [9]. These unitarity-violating terms come from the slowly varying amplitude of the gravitational wave function in the expansion of the WDW equation in the inverse power series of the Planck mass. The origin of these unitarity-violating terms may not, however, follow inherently from the WDW equation. Bertoni *et al.* [8] have further shown that the semiclassical quantum gravity can indeed preserve the unitarity of quantum fields provided that one can define the cosmological time defined through the Hamilton-Jacobi equation of gravitational field including the back reaction of matter field. A remark on the different representations of almost the same semiclassical quantum gravity is that the semiclassical quantum gravity in Ref. [7] is based on the complicated matrix equation of the WDW equation which incorporates actually the scattering of Cauchy data at the second or third quantized level and also considers linear combinations of the wave functions [10], in contrast with one wave function in Refs. [5,8]. So, it would be worthy to show the unitarity of quantum fields from the WDW equation using the equivalent matrix equation [6,7].

In this paper we observe that the WDW equation is not uniquely separated into the gravitational field equation and the time-dependent Schrödinger equation according to the different mass scales following only the Born-Oppenheimer idea [7], and that, especially when we apply de Broglie–Bohm interpretation [11] at the same time to the wave function of the gravitational field equation, this arbitrariness in the separation of the WDW equation in an oscillating regime can also allow the semiclassical quantum gravity which consists of a nonlinear gravitational field equation and a time-dependent Schrödinger equation that indeed does *preserve* the norm of the quantum states of matter field during the quantum evolution. As in Ref. [8], we assume no asymptotic limit to derive the semiclassical quantum gravity. However, in the asymptotic limit, it reduces identically to the semiclassical quantum gravity in previous papers [5,7]. Thus, the question of unitarity of the quantum evolution of matter fields may not be settled within the canonical quantum gravity, and should be answered by more fundamental theory. Furthermore, we shall use this semiclassical quantum gravity to find the higher order quantum corrections of the matter field to the gravity. Finally, we shall apply the semiclassical quantum gravity to the quantum Friedmann-Robertson-Walker (FRW) cosmological model coupled to a minimal scalar field.

The organization of the paper is as follows. In Sec. II we derive the nonlinear gravitational field equation and the time-dependent Schrödinger equation, and show that they preserve the unitarity of the quantum evolution of matter field. In Sec. III we develop a perturbation method for the semiclassical Einstein equation with the quantum potential and the higher order quantum corrections of matter field. In Sec. IV we apply the semiclassical quantum gravity to the quantum FRW cosmological model to find the effective energy density.

II. NONLINEAR GRAVITATIONAL FIELD EQUATION AND UNITARITY

We shall follow the Born-Oppenheimer idea to separate the WDW equation into the nonlinear gravitational field equation and the time-dependent Schrödinger equation. The canonical quantum gravity for a gravity coupled to a matter field, typically represented by a scalar field, is described by the WDW equation

$$\left[-\frac{\hbar^2}{2m_P}\nabla^2 - m_P V(h_a) + \hat{\mathbf{H}}\left(\frac{i}{\hbar}\frac{\delta}{\delta\phi},\phi,h_a\right)\right]\Psi(h_a,\phi)=0, \tag{1}$$

where h_a represents the gravitational field and ϕ the matter field. Here ∇^2 is the Laplace-Beltrami operator in superspace with the signature $(-,+,\ldots,+)$. Take the Born-Oppenheimer-type wave function for the WDW equation

$$|\Psi(h_a,\phi)\rangle = \psi(h_a)|\Phi(\phi,h_a)\rangle, \tag{2}$$

where ψ depends only on the gravitational field and Φ both on the matter field and on the gravitational field as an explicit parameter. The Born-Oppenheimer idea assumes that the quantum states of matter field belong to a Hilbert space that varies continuously on the superspace of the gravitational field. The validity of the form of wave function (2) should be checked by the unitarity of the quantum state of matter field in the semiclassical quantum gravity.

Before we delve into representing in a different way the semiclassical quantum gravity that preserves the unitarity of the quantum matter field, we point out a major similarity and a minor difference among the approaches to the semiclassical quantum gravity in Refs. [6,7] and [8]. The similarity is that, as explained in the Introduction, most of the approaches [5–8] take into account the expectation value of energy-momentum tensor as its matter source in the semiclassical Einstein equation. This is the quintessence of the Born-Oppenheimer idea originally applied to a molecular system, but in Ref. [4] the back reaction was excluded at the lowest order by expanding the total action in the inverse power series of the Planck mass.

The difference comes from the interpretation of the WDW equation (1) whether as the zero-energy Schrödinger equation for a coupled system [5,8] or as a functional wave equation describing the evolution (scattering) of Cauchy data from an initial spacelike hypersurface, usually assumed to be the early stage of the Universe, to a later spacelike hypersurface [10]. In a toy model without a matter field, the WDW equation may apparently look like the Schrödinger equation with zero energy, but with additional inhomogeneous gravitational degrees of freedom or matter field degrees of freedom, Eq. (1) has the Minkowski signature $(-,+,\ldots,+)$. The exact wave functions for the quantum FRW cosmological model minimally coupled to a scalar field with a general potential, which is to be considered in Sec. IV, are extremely difficult to find due to the nonseparability of quantum states of the scalar field from the gravitational field and have not been found yet. In a systematic way we can make use of the initial value problem for Eq. (1) formulated in a matrix form to describe the evolution (scattering) of Cauchy data from one spacelike hypersurface to another, including the evolution of any quantum state. The matrix gravitational field equations in Ref. [6] indeed incorporated the Cauchy data. The semiclassical quantum gravity in previous papers [7] was based on this exact matrix formulation for the canonical quantum gravity, and took into account not only single wave function but also a linear combination of wave functions.

Following Refs. [6,7], we may expand any quantum state with respect to a suitably chosen orthonormal basis relevant to the matter field:

$$|\Phi(\phi,h_a)\rangle = \sum_k c_k(h_a)|\Phi_k(\phi,h_a)\rangle. \tag{3}$$

Substitute Eqs. (2) and (3) into the WDW equation (1) to get the matrix equation [7]

$$\begin{aligned} &c_n(h_a)\left(-\frac{\hbar^2}{2m_P}\nabla^2 - m_P V(h_a) + \mathbf{H}_{nn}(h_a)\right)\psi(h_a) \\ &\quad+\left[-\frac{\hbar^2}{m_P}\nabla\psi(h_a)\cdot\left(\nabla c_n(h_a) - i\sum_k \mathbf{A}_{nk}(h_a)c_k(h_a)\right)\right. \\ &\quad+\psi(h_a)\left(\sum_{k\neq n}\mathbf{H}_{nk}(h_a)c_k(h_a)\right. \\ &\quad\left.\left.-\frac{\hbar^2}{2m_P}\sum_k \boldsymbol{\Omega}_{nk}(h_a)c_k(h_a)\right)\right]=0, \end{aligned} \tag{4}$$

where

$$\mathbf{H}_{nk}(h_a) := \langle\Phi_n|\hat{\mathbf{H}}|\Phi_k\rangle,$$

$$\mathbf{A}_{nk}(h_a) := i\langle\Phi_n|\nabla|\Phi_k\rangle,$$

$$\boldsymbol{\Omega}_{nk}(h_a) := \nabla^2\delta_{nk} - 2i\mathbf{A}_{nk}\cdot\nabla + \langle\Phi_n|\nabla^2|\Phi_k\rangle. \tag{5}$$

This is an intermediate step for the derivation of the semiclassical quantum gravity.

We now derive the semiclassical quantum gravity in a different way from the previous papers [7], so that it can preserve the unitarity of quantum states completely. A complex solution of the gravitational field equation in an oscillating regime can always be rewritten in the form

$$\psi(h_a) = (\psi^* \psi)^{1/2} \exp\left(\frac{i}{\hbar} S(h_a) \right). \tag{6}$$

Substituting the complex solution (6) into the matrix equation (4) and assuming $c_n \neq 0$, we obtain the two coupled equations

$$\begin{aligned}\Bigg[& -\frac{\hbar^2}{2m_P}\nabla^2 - m_P V(h_a) + \mathbf{H}_{nn}(h_a) \\ & - \frac{\hbar^2}{m_P}\frac{\nabla(\psi^*\psi)^{1/2}}{(\psi^*\psi)^{1/2}} \cdot \left(\frac{\nabla c_n(h_a)}{c_n(h_a)} \right. \\ & \left. - i \sum_k \mathbf{A}_{nk}(h_a) \frac{c_k(h_a)}{c_n(h_a)} \right) \Bigg] \psi(h_a) = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} i\frac{\hbar}{m_P}\nabla S(h_a)\cdot\nabla c_n(h_a) + \frac{\hbar}{m_P}\nabla S(h_a)\cdot \sum_k \mathbf{A}_{nk}(h_a) c_k(h_a) \\ - \sum_{k\neq n} \mathbf{H}_{nk}(h_a) c_k(h_a) + \frac{\hbar^2}{2m_P}\sum_k \mathbf{\Omega}_{nk}(h_a) c_k(h_a) = 0. \end{aligned} \tag{8}$$

Equation (7) may be interpreted as the nonlinear gravitational field equation with the quantum back reaction of the matter field. In terms of the cosmological time

$$\frac{\delta}{\delta\tau} := \frac{1}{m_P}\nabla S(h_a)\cdot\nabla, \tag{9}$$

Eq. (8) can be rewritten as

$$\begin{aligned} i\hbar\frac{\delta}{\delta\tau} c_n(h_a) = & \sum_{k\neq n} \mathbf{H}_{nk}(h_a) c_k(h_a) \\ & - \frac{\hbar}{m_P}\nabla S(h_a)\cdot\sum_k \mathbf{A}_{nk}(h_a)c_k(h_a) \\ & - \frac{\hbar^2}{2m_P}\sum_k \mathbf{\Omega}_{nk}(h_a)c_k(h_a). \end{aligned} \tag{10}$$

Equation (10) may now be interpreted as the time-dependent functional Schrödinger equation for the matter field on the spacetime determined by Eq. (7). It should be noted that there can be some arbitrariness in separating the WDW equation. Instead of $\mathbf{H}_{nn}$ in Eq. (7), one may use a quantum back reaction $\mathbf{K}_{nn}(h_a)$. Then, the nonlinear gravitational field equation (7) is replaced with $\mathbf{K}_{nn}$. We rewrite the matter field equation by a vector notation

$$i\hbar\frac{\delta}{\delta\tau}\mathbf{c}(h_a(\tau)) = \mathbf{M}(h_a(\tau))\mathbf{c}((h_a)(\tau)), \tag{11}$$

where $\mathbf{M}$ is the Hermitian matrix

$$\mathbf{M}(h_a(\tau)) = \mathbf{M}_0(h_a(\tau)) + \mathbf{M}_1(h_a(\tau)), \tag{12}$$

consisted of

$$\begin{aligned} \mathbf{M}_0(h_a(\tau)) = & \sum_{k\neq n}\mathbf{H}_{nk}(h_a(\tau)) \\ & - \frac{\hbar}{m_P}\nabla S(h_a(\tau))\cdot\sum_k \mathbf{A}_{nk}(h_a(\tau)) \\ & - \frac{\hbar^2}{2m_P}\sum_k \mathbf{\Omega}_{nk}(h_a(\tau)), \end{aligned}$$

$$\mathbf{M}_1(h_a(\tau)) = \mathbf{K}_{nn}(h_a(\tau)) - \mathbf{H}_{nn}(h_a(\tau)). \tag{13}$$

The unitarity of the quantum matter field is always fully respected:

$$\mathbf{c}^\dagger(\tau)\cdot\mathbf{c}(\tau) = 1. \tag{14}$$

A minor difference of this paper from other related works [4,5,9] is that we followed closely Ref. [10], which interpreted the WDW equation as the quantum scattering of the Cauchy data including the quantum states of matter fields. The difference from previous works [7] lies on the fact that we made use of the arbitrariness in order to separate the nonlinear gravitational field equation (7) from the WDW equation and the time-dependent Schrödinger equation (10) that preserves the unitarity of the quantum evolution of matter field. The unitarity of quantum field in the oscillating regime has been first proved by Bertoni *et al.* [8] using the semiclassical quantum gravity in Ref. [5].

III. QUANTUM CORRECTIONS

We apply the de Broglie–Bohm interpretation of wave functions to the nonlinear gravitational field equation (7). The main idea of the de Broglie–Bohm interpretation is that a time-dependent Schrödinger equation is mathematically equivalent to a pair of equations, the time-dependent Hamilton-Jacobi equation with an additional quantum potential and the equation for the conservation of probability [11]. The concept of trajectories with the quantum potential provides us with all the same physical predictions as those with the standard quantum mechanics. Thus, the de Broglie–Bohm interpretation is another way of interpreting quantum mechanics, not a kind of approximation methods such as the WKB method, even though the de Broglie–Bohm interpretation has the WKB as an asymptotic limit, when the quantum potential is negligible.

According to the de Broglie–Bohm interpretation, we separate the real and the imaginary parts of the nonlinear gravitational field equation. Thus, by substituting $f \equiv (\psi^*\psi)^{1/2}$ and equating the real and the imaginary parts of Eq. (7), we obtain

$$\frac{1}{2m_P}(\nabla S)^2 - m_P V + \mathbf{H}_{nn} - \frac{\hbar^2}{2m_P}\frac{\nabla^2 F}{F} - \frac{\hbar^2}{m_P}\mathrm{Re}(\mathbf{Q}_{nn}) = 0, \tag{15}$$

$$\frac{1}{2}\nabla^2 S + \frac{\nabla F}{F}\cdot\nabla S + \mathrm{Im}(\mathbf{Q}_{nn}) = 0, \tag{16}$$

where

$$\mathbf{Q}_{nn} = \frac{\nabla F}{F}\cdot\left(\frac{\nabla c_n}{c_n} - i\sum_k \mathbf{A}_{nk}\frac{c_k}{c_n}\right). \tag{17}$$

If there is no parametric coupling of gravity with matter, the term

$$V_{\mathrm{quant}} := -\frac{\hbar^2}{2m_P}\frac{\nabla^2 F}{F} \tag{18}$$

is the quantum potential, and Eq. (16) is the equation for the probability conservation [11]. The last terms in Eqs. (15) and (16) are the quantum back reaction of matter field to the gravity.

In the asymptotic limit $O(\hbar/m_P)$, the matter field equation becomes

$$i\hbar\frac{\delta}{\delta\tau}c_n^{(0)}(h_a) = \sum_{k\neq n}\mathbf{H}_{nk}(h_a)c_k^{(0)}(h_a) - \frac{\hbar}{m_P}\nabla S^{(0)}(h_a) \times \sum_k \mathbf{A}_{nk}(h_a)c_k^{(0)}(h_a), \tag{19}$$

and the gravitational field equation reduces to the Hamilton-Jacobi equation of the zeroth order

$$\frac{1}{m_P}(\nabla S^{(0)})^2 - m_P V + \mathbf{H}_{nn} = 0. \tag{20}$$

As an orthonormal basis, we choose the exact quantum states of the time-dependent functional Schrödinger equation

$$i\hbar\frac{\delta}{\delta\tau}\left|\Phi^{(0)}(\phi,\tau)\right\rangle = \hat{\mathbf{H}}\left(\frac{i}{\hbar}\frac{\delta}{\delta\phi},\phi,h_a(\tau)\right)\left|\Phi^{(0)}(\phi,\tau)\right\rangle. \tag{21}$$

This orthonormal basis simplifies considerably the algebra; especially, the matter field equation becomes

$$i\hbar\frac{\delta}{\delta\tau}c_n^{(0)}(h_a) = -\frac{\hbar}{m_P}\nabla S^{(0)}(h_a)\cdot\mathbf{A}_{nn}(h_a)c_n^{(0)}(h_a). \tag{22}$$

The solution is

$$c_n^{(0)} = \exp\left(i\int(\mathbf{A}_{nn})_a dh_a\right). \tag{23}$$

To find the gravitational field wave function and the quantum states of matter field, we use a perturbation method that expands the coefficient functions and the gravitational action as

$$c_n(h_a) = \sum_{p=0}^{\infty}\left(\frac{\hbar}{m_P}\right)^p c_n^{(p)}(h_a),$$

$$c_k(h_a) = \sum_{p=1}^{\infty}\left(\frac{\hbar}{m_P}\right)^p c_k^{(p)}(h_a) \quad (k\neq n), \tag{24}$$

and

$$S(h_a) = \sum_{p=0}^{\infty}\left(\frac{\hbar}{m_P}\right)^p S^{(p)}(h_a),$$

$$F(h_a) = \sum_{p=0}^{\infty}\left(\frac{\hbar}{m_P}\right)^p f^{(p)}(h_a). \tag{25}$$

The dominant contribution to the quantum state comes from c_n. The other terms $c_k, (k\neq n)$ are of the order of $O(\hbar/m_P)$. The prefactor f of the gravitational wave function is determined by

$$\frac{\nabla f^{(0)}}{f^{(0)}}\cdot\nabla S^{(0)} = -\tfrac{1}{2}\nabla^2 S^{(0)}, \tag{26}$$

whose solution is

$$f^{(0)} = \frac{1}{(\nabla S^{(0)}\cdot\nabla S^{(0)})^{1/4}}. \tag{27}$$

The first order coefficient functions of the quantum state are

$$i\frac{\delta}{\delta\tau}c_k^{(1)}(h_a) = -\tfrac{1}{2}\boldsymbol{\Omega}_{kn}(h_a)c_n^{(0)}(h_a), \tag{28}$$

where

$$\boldsymbol{\Omega}_{kn}(h_a)c_n^{(0)}(h_a) = [2\mathbf{A}_{nk}\cdot\mathbf{A}_{nn} - \mathbf{A}_{nn}\cdot\mathbf{A}_{kn} - (\mathbf{A}\cdot\mathbf{A})_{nn}] \times c_n^{(0)}(h_a). \tag{29}$$

We determine $S^{(1)}$ from

$$\frac{1}{m_P}\nabla S^{(1)}\cdot\nabla S^{(0)} = \frac{\hbar}{2}\frac{\nabla^2 f^{(0)}}{f^{(0)}}. \tag{30}$$

The procedure can be repeated to yield the higher order quantum corrections of the matter field to the semiclassical Einstein equation and the exact quantum state.

IV. QUANTUM FRW COSMOLOGICAL MODEL

We now apply the semiclassical quantum gravity developed in this paper to the frequently employed quantum FRW cosmological model. The higher order quantum corrections in Sec. III to the asymptotic semiclassical quantum gravity have not yet been found explicitly even for the FRW model, although higher order quantum corrections were found in the semiclassical quantum gravity whose lowest order equation is the vacuum Einstein equation and which does violate the unitarity at the first order [9]. Moreover, since the quantum field preserves the unitarity throughout the evolution, the application to the FRW model is expected to be particularly useful in searching for the quantum effects of matter field to the gravity.

A. Minimal scalar field

The simplest but nontrivial FRW quantum cosmological model is with a minimal scalar field (inflaton). The WDW equation is

$$\left[\frac{2\pi\hbar^2}{3m_P a}\frac{\partial^2}{\partial a^2}-\frac{3m_P}{8\pi}V(a)+\hat{\mathbf{H}}\left(\frac{i}{\hbar}\frac{\delta}{\delta\phi},\phi,a\right)\right]\Psi(a,\phi)=0, \tag{31}$$

where a is the size of the Universe, and

$$V(a)=ka-\Lambda a^3 \tag{32}$$

is the gravitational potential consisting of the three-curvature and the cosmological constant, and ϕ denotes the massive scalar field. The extended supermetric is

$$ds^2=-a\,da^2+a^3 d\phi^2, \tag{33}$$

and the rescaling $a=(3/4\pi)^{1/3}\tilde{a}$ recovers the WDW equation of the form in Eq. (1). Remembering the superspace signature $(-)$, the cosmological time is related to the gravitational action by

$$\frac{\partial}{\partial\tau}=-\frac{4\pi}{3m_P a}\frac{\partial S(a)}{\partial a}\frac{\partial}{\partial a}. \tag{34}$$

Then, we find that

$$\dot{a}(\tau)=-\frac{4\pi}{3m_P a}S', \tag{35}$$

where here and hereafter overdots and primes will denote derivative with respect to τ and a, respectively. We rewrite the nonlinear gravitational field equation as

$$\left[\frac{2\pi\hbar^2}{3m_P a}\frac{\partial^2}{\partial a^2}-\frac{3m_P}{8\pi}V(a)+\mathbf{H}_{nn}-\frac{4\pi\hbar^2}{3m_P a}\frac{F'}{F}\left(\frac{c_n'}{c_n}+i\frac{3m_P}{4\pi}\frac{a}{S'}\sum_k \mathbf{B}_{nk}\frac{c_k}{c_n}\right)\right]\psi(a)=0. \tag{36}$$

In terms of the cosmological time, the time-dependent functional Schrödinger equation is rewritten as

$$i\hbar\frac{\partial}{\partial\tau}c_n(\tau)=\sum_{k\neq n}\mathbf{H}_{nk}(\tau)c_k(\tau)-\hbar\sum_k \mathbf{B}_{nk}(\tau)c_k(\tau)-\frac{2\pi\hbar^2}{3m_P a}\sum_k \mathbf{\Omega}_{nk}(\tau)c_k(\tau), \tag{37}$$

where

$$\mathbf{H}_{nk}(\tau)=\langle\Phi_n|\hat{\mathbf{H}}|\Phi_k\rangle,$$

$$\mathbf{B}_{nk}(\tau)=i\left\langle\Phi_n\left|\frac{\partial}{\partial\tau}\right|\Phi_k\right\rangle,$$

$$\mathbf{\Omega}_{nk}(\tau)=-\frac{1}{\dot{a}^2}\left[\left(\frac{\partial^2}{\partial\tau^2}-\frac{\ddot{a}}{\dot{a}}\frac{\partial}{\partial\tau}\right)\delta_{nk}-2i\mathbf{B}_{nk}\frac{\partial}{\partial\tau}+\left\langle\Phi_n\left|\frac{\partial^2}{\partial\tau^2}-\frac{\ddot{a}}{\dot{a}}\frac{\partial}{\partial\tau}\right|\Phi_k\right\rangle\right]. \tag{38}$$

In the above equations we made use of the relations (34) and (35). Following the de Broglie–Bohm interpretation, we substitute the wave function $\psi=Fe^{(i/\hbar)S}$ and separate the real and the imaginary parts of the gravitational field equation to obtain

$$\frac{2\pi}{3m_P a}S'^2+\frac{3m_P}{8\pi}V(a)=\mathbf{H}_{nn}-\frac{4\pi\hbar^2}{3m_P a\dot{a}}\frac{F'}{F}\mathrm{Re}(\mathbf{R}_{nn})+\frac{2\pi\hbar^2}{3m_P a}\frac{F''}{F}, \tag{39}$$

and

$$\frac{F'}{F}S'+\frac{1}{2}S''=\frac{\hbar}{\dot{a}}\frac{F'}{F}\mathrm{Im}(\mathbf{R}_{nn}), \tag{40}$$

where

$$\mathbf{R}_{nn}=\frac{\dot{c}_n}{c_n}-i\sum_k \mathbf{B}_{nk}\frac{c_k}{c_n}, \tag{41}$$

and $\mathbf{Q}_{nn}=(F'/F)\mathbf{R}_{nn}$. We solve Eq. (40), and let

$$\mathbf{U}_{nn}:=\frac{F'}{F}=-\frac{1}{2}\frac{(a\dot{a})^{\cdot}}{a\dot{a}^2+(4\pi\hbar/3m_P)\mathrm{Im}(\mathbf{R}_{nn})}, \tag{42}$$

where we again used Eqs. (34) and (35). Using again Eqs. (35) and (42), we rewrite Eq. (39) as

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{a^3}V(a)=\frac{8\pi}{3m_P a^3}\left[\mathbf{H}_{nn}-\frac{4\pi\hbar^2}{3m_P a\dot{a}}\mathbf{U}_{nn}\mathrm{Re}(\mathbf{R}_{nn})+\frac{2\pi\hbar^2}{3m_P a}\left(\mathbf{U}_{nn}^2+\frac{1}{\dot{a}}\dot{\mathbf{U}}_{nn}\right)\right]. \tag{43}$$

We can interpret Eq. (43) as the semiclassical Einstein equation with the quantum back reaction of the matter field and the higher order quantum corrections from the fluctuations of matter field and geometry itself. In this sense,

$$\mathbf{T}_{nn}:=\mathbf{H}_{nn}-\frac{4\pi\hbar^2}{3m_P a\dot{a}}\mathbf{U}_{nn}\mathrm{Re}(\mathbf{R}_{nn})+\frac{2\pi\hbar^2}{3m_P a}\left(\mathbf{U}_{nn}^2+\frac{1}{\dot{a}}\dot{\mathbf{U}}_{nn}\right) \tag{44}$$

is the effective energy density from the quantum fluctuation of matter fields. Equation (37) is the time-dependent functional Schrödinger equation for the matter field on the spacetime determined by Eq. (43).

B. Massive scalar field

We work out explicitly the specific case of a massive scalar field. The Hamiltonian of the massive scalar field is

$$\hat{\mathbf{H}}\left(\frac{i}{\hbar}\frac{\delta}{\delta\phi},\phi,a\right)=-\frac{\hbar^2}{2a^3}\frac{\partial^2}{\partial\phi^2}+\frac{a^3m^2}{2}\phi^2. \tag{45}$$

In Ref. [12] a Fock space representation of quantum states satisfying the time-dependent functional Schrödinger equation was constructed in terms of the annihilation and creation operators

$$\hat{b}(\tau)=\phi_c^*(\tau)\hat{\pi}_\phi-a^3(\tau)\dot{\phi}_c^*(\tau)\hat{\phi},\quad \hat{b}^\dagger(\tau)=(\hat{b}(\tau))^\dagger, \tag{46}$$

where ϕ_c is a complex solution of the classical field equation

$$\ddot{\phi}_c(\tau)+3\frac{\dot{a}(\tau)}{a(\tau)}\dot{\phi}_c(\tau)+m^2\phi_c(\tau)=0. \tag{47}$$

The effective gauge potential is found in an operator form:

$$\mathbf{B}=\alpha(\tau)\hat{b}^\dagger(\tau)\hat{b}(\tau)+\beta(\tau)\hat{b}^2(\tau)+\beta^*(\tau)\hat{b}^{\dagger 2}(\tau), \tag{48}$$

where

$$\alpha(\tau)=\hbar a^3(\tau)[\dot{\phi}^*(\tau)\dot{\phi}(\tau)+m^2\phi^*(\tau)\phi(\tau)],$$

$$\beta(\tau)=-\frac{\hbar a^3}{2}[\dot{\phi}^2(\tau)+m^2\phi^2(\tau)]. \tag{49}$$

Since c_n is of the order 1 but $c_k(k\neq n)$ of the order $O(\hbar/m_P)$, we obtain the coefficient function

$$c_n^{(0)}(\tau)=\exp\left(i\int\mathbf{B}_{nn}(\tau)d\tau\right). \tag{50}$$

The lowest contributions to $\mathbf{R}_{nn}$ and $\mathbf{U}_{nn}$ are found

$$\mathbf{R}_{nn}^{(0)}=0,$$

$$\mathbf{U}_{nn}^{(0)}=-\frac{1}{2}\frac{(a\dot{a})^{\cdot}}{a\dot{a}^2}. \tag{51}$$

Thus, the semiclassical Einstein equation at this order becomes

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{a^3}V(a)=\frac{8\pi}{3m_Pa^3}\left[\mathbf{H}_{nn}+\frac{2\pi\hbar^2}{3m_Pa}\left(\mathbf{U}_{nn}^{(0)2}+\frac{1}{\dot{a}}\dot{\mathbf{U}}_{nn}^{(0)}\right)\right], \tag{52}$$

where

$$\mathbf{H}_{nn}=\hbar a^3(\tau)(n+\tfrac{1}{2})[\dot{\phi}^*(\tau)\dot{\phi}(\tau)+m^2\phi^*(\tau)\phi(\tau)] \tag{53}$$

is the quantum expectation value of the Hamiltonian operator. It should be remarked that at the lowest order the quantum effects enter the semiclassical Einstein equation only through the quantum potential and there is no contribution from the effective gauge potential. The contribution from the quantum potential is still of the order $O(\hbar/m_P)$, so we recover the asymptotic semiclassical Einstein equation in previous papers [7].

The first order correction can be found from the equations

$$\mathbf{\Omega}_{nn}c_n^{(0)}=\frac{1}{\dot{a}^2}\left[\sum_{m\neq n}\mathbf{B}_{nm}\mathbf{B}_{mn}\right]c_n^{(0)}, \tag{54}$$

and

$$\mathbf{\Omega}_{nk}c_n^{(0)}=-\frac{1}{\dot{a}^2}\left[2\mathbf{B}_{nk}\mathbf{B}_{nn}-(\mathbf{B}^2)_{nk}-i\left(\dot{\mathbf{B}}_{nk}+\frac{\ddot{a}}{\dot{a}}\mathbf{B}_{nk}\right)\right]c_n^{(0)}. \tag{55}$$

Integrating the first order matter field equation

$$i\frac{\partial}{\partial\tau}c_k^{(1)}=-\frac{2\pi\hbar}{3a}\mathbf{\Omega}_{kn}c_n^{(0)}, \tag{56}$$

we get

$$\mathbf{R}_{nn}^{(1)}=\frac{\dot{c}_n^{(0)}+(\hbar/m_P)\dot{c}_n^{(1)}}{c_n^{(0)}+(\hbar/m_P)c_n^{(1)}}-\sum_k\mathbf{B}_{nk}\frac{c_k^{(0)}+(\hbar/m_P)c_k^{(1)}}{c_n^{(0)}+(\hbar/m_P)c_n^{(1)}}. \tag{57}$$

By substituting Eq. (57) into Eq. (43), we obtain the semiclassical Einstein equation up to the first order in $\hbar/m_P$.

V. CONCLUSION

The unitarity of quantum field in a curved spacetime is an important issue in quantum gravity. In this paper, by applying both the Born-Oppenheimer idea and the de Broglie–Bohm interpretation, we derived in a different way the semiclassical quantum gravity which consists of the nonlinear gravitational equation and the time-dependent functional Schrödinger equation that *does* preserve the unitarity of quantum field without assuming any limit. In particular, we applied the de Broglie–Bohm interpretation to the nonlinear gravitational equation, whose real part is nothing but the Hamilton-Jacobi equation with the quantum potential and the contributions from the effective gauge potential of matter fields. In an oscillating regime of gravity, we were able to develop a perturbation method for the semiclassical Einstein equation, which includes the quantum corrections of the matter field. Finally, we applied the perturbation method to the quantum FRW cosmological model with a minimal scalar field (inflaton) and obtained the semiclassical Einstein equation (43). The right-hand side of the equation, the effective energy density, is the sum of the expectation value of the Hamiltonian operator, the higher order quantum corrections from the matter field, and the quantum potential coupled to the effective gauge potential. Since the quantum corrections have the order of $O(\hbar/m_P)$, the semiclassical Einstein equation reduces to the same asymptotic semiclassical Einstein equation obtained already in Ref. [7].

This paper does not, however, resolve completely the problem of unitarity of quantum field in quantum gravity and leaves it to a more fundamental theory. As pointed out earlier, bearing the de Broglie–Bohm interpretation in mind, we made use of the freedom in the separation of the Wheeler-DeWitt equation into the gravitational field equation and the time-dependent Schrödinger equation, and derived explicitly the semiclassical quantum gravity that does indeed *preserve* the unitarity of quantum evolution of matter field at the expense of the nonlinearity of the gravitational field equation.

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