Primordial gravitational waves from open inflation

Martin Bucher*
Department of Physics, Princeton University, Princeton, New Jersey 08544
and Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794

J. D. Cohn†
Institute for Cosmology, Department of Physics, Tufts University, Medford, Massachusetts 02155
(Received 21 January 1997)

We calculate the spectrum of gravitational waves generated during inflation in open ($\Omega_0<1$) inflationary models. In such models an initial epoch of old inflation solves the horizon and flatness problems, and during this first epoch of inflation the quantum state of the graviton field rapidly approaches the Bunch-Davies vacuum. Then the old inflation ends by the nucleation of a single bubble, inside of which there is a shortened epoch of slow-roll inflation giving $\Omega_0<1$ today. In this paper we reexpress the Bunch-Davies vacuum for the graviton field in terms of the hyperbolic modes inside the bubble and propagate these modes forward in time into the present era. We derive the expression for the contribution from these gravity waves to the cosmic microwave background anisotropy including the effect of a finite-energy difference across the bubble wall. [S0556-2821(97)03612-6]

PACS number(s): 98.80.Cq, 04.30.Db

I. INTRODUCTION

It has been well known for quite some time that gravitational waves are generated during inflation and give an observable, and sometimes substantial, contribution to the cosmic microwave background anisotropy [1–7]. For flat ($\Omega_0=1$) inflation the spectrum of gravitational waves generated and their observational consequences have been studied quite extensively. However, for open ($\Omega_0<1$) inflationary models in which our entire observable universe lies inside a single bubble [8–11], there has been no complete calculation of the gravitational waves generated. In this paper we present such a calculation.

Gravitational waves from inflation result from the stretching of quantum vacuum fluctuations of the linearized graviton field to superhorizon scales. Following a given mode of fixed comoving wave number k, one finds that at early times its physical wavelength $\lambda=a(t)(2\pi)/k$ is much smaller than the Hubble length $l_H=H^{-1}$ (i.e., the mode is well within the *horizon*). This implies that for determining a physically reasonable *vacuum state* for the mode at early times, one may ignore the expansion of the universe and match onto the usual flat Minkowski space vacuum for the graviton field. Once the correct quantum vacuum state has been determined at early times, to continue these modes to later times becomes a mathematically well-defined exercise in classical field theory, which involves propagating the *positive* frequency modes—those associated with annihilation operators of the vacuum—forward in time, through the end of inflation, and into the present epoch. As a mode crosses the horizon, its amplitude becomes frozen in. The process of generating gravitational waves during inflation is quite analogous to the process of generating scalar fluctuations during inflation. There is, however, an important difference. The amplitude of the gravitational waves does not depend on the slope of the potential; rather only the overall height of the potential, or equivalently the expansion rate H during inflation, is relevant. As a first approximation for calculating the gravitational waves, H may be regarded as fixed during the relevant epoch of inflation.

For open inflation identifying the correct initial conditions for the linearized graviton field is not as straightforward as for the flat case, because the underlying spacetime geometry is more complicated. The simplest case involves one matter field, the inflaton ϕ, with minimal coupling to gravity. More complicated models such as found in [12] have similar properties for the gravity wave calculations done here, so the simple one field model is used in the following description. (For a more detailed discussion of single-bubble inflation see [10].)

In open inflation, there is an initial epoch of old inflation during which ϕ is stuck in a false vacuum with $\phi=\phi_{\rm fv}$. During this epoch, the spacetime geometry approaches that of pure de Sitter space, characterized by an expansion rate $H_{\rm fv}$, where $H_{\rm fv}^2=(8\pi G/3)V[\phi_{\rm fv}]$. During this initial epoch of old inflation the graviton field is driven to the vacuum state. This determines the initial conditions using the same considerations as for the flat case described above. Then old inflation ends through the nucleation of a bubble, which expands roughly at the speed of light. The preferred time slicing inside the light cone of the bubble center corresponds to a spatially open universe. Inside the bubble the inflaton field first slowly rolls down a rather flat part of the potential, giving a shortened epoch of slow-roll inflation inside the bubble. Later inside the bubble, the inflaton field rolls more quickly and the usual reheating occurs, converting the vacuum energy of the inflaton field into radiation and matter.

The coordinate chart with the line element

*Current address: Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11794-3840. Electronic address: bucher@insti.physics.sunysb.edu

†Electronic address: jdc@cosmos2.phy.tufts.edu

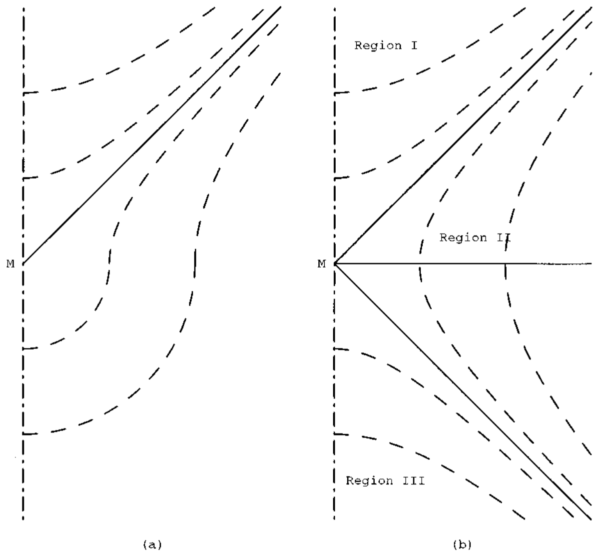

FIG. 1. (a) shows a spacetime diagram for bubble nucleation. The double-dashed vertical line to the left indicates an $r=0$ coordinate singularity. Time flows upward and the horizontal axis represents a radial coordinate. On the surfaces represented by dashed curves the inflaton field is constant. The lower portion of the diagram (with $t<0$) represents the nucleation of a critical bubble, a classically forbidden Euclidean process. For $t>0$ the bubble expands classically, at a speed approaching that of light. The classical expanding bubble evolution is SO(3,1) symmetric. In (b) the hyperbolic coordinates that maximally exploit the SO(3,1) symmetry of the expanding bubble solution are sketched. Spacetime is divided into three hyperbolic coordinate patches. The light cones separating these regions represent unphysical coordinate singularities of a character similar to that of the Schwarzschild horizon.

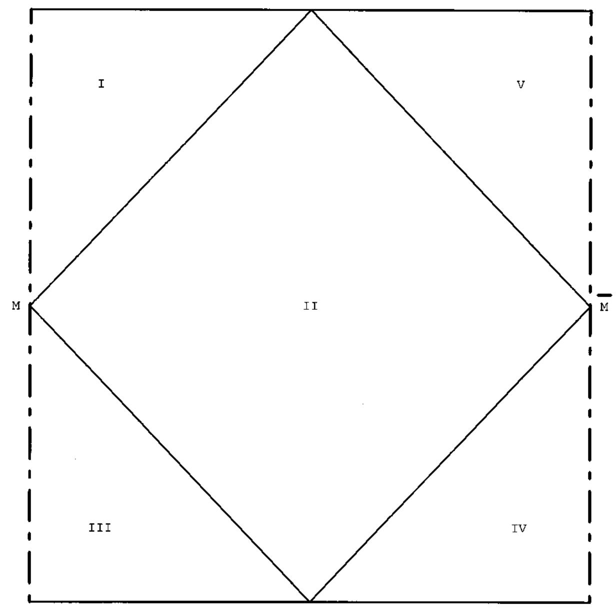

FIG. 2. A conformal diagram for all of maximally extended de Sitter space is shown here. $\bar{M}$ is the antipodal point of M. The hyperbolic coordinates that exploit the symmetry of the SO(3,1) subgroup of the full de Sitter group SO(4,1) that leaves invariant M (and $\bar{M}$ as well) divides spacetime into the five indicated coordinate patches.

$$ds^2=-dt^2+a^2(t)[d\xi^2+\sinh^2[\xi]d\Omega^2_{(2)}], \qquad (1.1)$$

describes an expanding Friedman-Robertson-Walker universe with spatially uniform negative spatial curvature. Hyperbolic open coordinates (rather than the *flat* coordinates, to be described later) are the natural coordinate choice in the presence of the bubble wall, which is why the interior of a bubble is an open universe [8,9]. For de Sitter space and these hyperbolic coordinates, $a(t)=\sinh[t]$ and

$$ds^2=-dt_h^2+\sinh^2[t_h][d\xi^2+\sinh^2[\xi]d\Omega^2_{(2)}], \qquad (1.2)$$

where $(0\leqslant t_h<+\infty)$. The hyperbolic coordinate chart has an unphysical coordinate singularity at $t=0$, and to determine initial conditions it is necessary to consider a larger region of spacetime than that covered by the open coordinates with the line element (1.1).

The bubble nucleation process underlying open inflation is sketched in Fig. 1, with the dashed lines indicating the surfaces on which the inflaton field is constant. Roughly speaking, the forward light cone of the materialization center M, which we shall call region I and which is covered by the coordinate chart just described, may be considered the bubble interior. Regions II and III cover the spacetime prior to bubble nucleation and the part of spacetime into which the bubble expands, at a speed approaching the speed of light.

For future reference, in Fig. 2, we present the conformal diagram for all of maximally extended de Sitter space. In addition to the regions already mentioned, there also exist regions IV and V, which are the past and future light cones of $\bar{M}$, the antipodal point of the apex of region I. For a discussion of the global structure of the de Sitter vacuum see [13,14].

The subject of *scalar* perturbations in open inflation has been studied extensively in recent years. Lyth and Stewart [15] and Ratra and Peebles [16] calculated the scalar perturbations in open inflation assuming conformal boundary conditions for the vacuum of the inflaton field as $t\to 0$ in region I. Bucher, Goldhaber, and Turok [10] presented the first computation of the scalar perturbations in open inflation using the Bunch-Davies vacuum for the inflaton field outside the bubble during the prior epoch of old inflation as initial conditions and propagating the scalar modes through the bubble wall into the open universe. Yamamoto, Sasaki, and Tanaka obtained essentially the same results using Euclidean methods [17,18]. Many recent calculations of scalar field perturbations, for example [12,19,20,18,21], take into account additional effects such as finite bubble size, varying bubble wall profile, fluctuations in the bubble wall, etc.

The subject of gravitational waves generated during open inflation was previously investigated by Allen and Caldwell in an unpublished manuscript [22]. Within their approximations they found an infrared divergence in the multipole moments of the cosmic microwave background anisotropy (CMB). In their computation, in order to simplify the calculation, at early times a flat spacetime geometry (i.e., that of Minkowski space) is assumed, and to improve the infrared behavior of the graviton field during this early epoch, the graviton field is given a small mass m_g, which at the end of the calculation is taken to approach zero. Later on, inside region I on a hyperboloid of constant time with respect to the region I coordinates, the spacetime geometry is taken to change discontinuously to that of de Sitter space. Subsequently, inflation inside the bubble ends, so that the scale factor $a(\eta)$ becomes governed by a radiation-matter equation of state, and the evolution of the graviton field is computed to take into account this change.

In Sec. IV of this paper we derive the Bunch-Davies vacuum for the graviton field in de Sitter space in terms of the hyperbolic region I coordinates, assuming de Sitter space everywhere and a vanishing graviton mass. The Bunch-Davies vacuum for the graviton field in de Sitter space, expressed in terms of the spatially flat coordinate slicing, is used to express the Bunch-Davies vacuum for the graviton field in de Sitter space in terms of the hyperbolic coordinates. The latter are the natural coordinates for studying perturbations in an open universe. While in principle it should be possible to compute directly the transformation between the flat and the hyperbolic modes, in practice this transformation has proven algebraically intractable. Instead, we characterize linear combinations of modes of *positive* frequency with respect to the flat coordinates (which is the same as *positive* frequency with respect to the Bunch-Davies vacuum) in terms of analytic properties in the complex plane. In this way linear combinations of hyperbolic modes of purely *positive* frequency with respect to the Bunch-Davies vacuum may be constructed without explicitly expressing these combinations in terms of the flat modes. The basic approach is somewhat analogous to the Euclidean methods used by Sasaki *et al.* for scalar modes [14], except that here, rather than using a Euclidean principle as the starting point, we determine the vacuum in terms of the flat modes, using the complex plane as a mathematical tool.

Our result for the Bunch-Davies vacuum for the graviton field essentially coincides with that found by Allen and Caldwell. There are some minor differences: whereas we obtain a mixed state directly, Allen and Caldwell obtain a pure state, which in the limit $m_g \to 0$ mocks our mixed state. This does not alter the divergence of the CMB multipole moments.

One caveat in computing the CMB moments relates to gauge fixing. In the Sachs-Wolfe formula for the CMB anisotropy the gravity waves contribute only in the integral along the line of sight—there is no contribution from the *tensor* modes from the last scattering surface. Since linearized gauge transformations are *scalar* and *vector*, one would think that the *tensor* part is gauge invariant.[1] However, gauge transformations can move the last scattering surface, and thus change the *intrinsic* contribution in the Sachs-Wolfe formula. If gravity wave modes mix with pure gauge modes, cancellations may occur [23]. Linearized gravity wave modes typically are taken to satisfy the synchronous gauge condition $\hat{t}^\mu h_{\mu\nu}=0$, where $\hat{t}^\mu$ points along the time direction of the preferred coordinate system. In de Sitter space these conditions do not coincide for the flat and for the region I hyperbolic coordinatizations. These issues have been investigated for hyperbolic coordinates in flat Minkowski space [25].

In this paper we drop the approximation of vanishing energy difference across the bubble wall, which is never exactly the case in the presence of a bubble. This requires including finite critical bubble size as well. Taking these effects into account removes the infrared divergence of the CMB multipole moments. The calculation resembles the calculations for scalar perturbations in Refs. [18,21].

The organization of this paper is as follows. Sections II and III give the multipole expansion for the pure tensor perturbations in hyperbolic space. Section II gives the evolution equation for the graviton field, which is solved by separation of variables. In Sec. III we study the properties of tensor harmonics in three-dimensional hyperbolic space, first written down explicitly by Tomita [26]. As mentioned above, in Sec. IV we identify the Bunch-Davies vacuum for the graviton field in region I in terms of the hyperbolic modes. In Sec. V we compute the effect of nonvanishing bubble size and of nonvanishing energy density difference across the bubble wall. This result is used in Sec. VI to give the tensor mode contribution to the CMB anisotropy for an open universe, and finally Sec. VII concludes. There are two appendixes containing technical details. We set $\hbar = G = 1$ throughout, and $H=1$ until Sec. V.

II. GRAVITATIONAL WAVES IN AN OPEN UNIVERSE

Gravitational waves are fluctuations about a background metric. The metric can be written as a background metric $g^B_{\mu\nu}$ plus a small perturbation:

$$g_{\mu\nu}=g^B_{\mu\nu}+\hat{h}_{\mu\nu}, \tag{2.1}$$

where we set $\hat{h}_{00}=0$, $\hat{h}_{0i}=0$, $\hat{h}_i{}^i=0$, and $\hat{h}_{ij}{}^{|j}=0$. These conditions require the *scalar* and *vector* perturbations to vanish and fix the gauge as well. The unperturbed spatial metric γ_{ij} (with line element $d\overline{s}^2 = d\xi^2+\sinh^2[\xi]d\Omega^2_{(2)}$) is used to raise and lower roman indices, and the vertical line indicates the covariant derivative induced by γ_{ij}.

For the background corresponding to the natural coordinates in the interior of the bubble center's light cone, the background metric is given in Eq. (1.2). Hyperbolic conformal time

$$\eta=\ln[\tanh[t_h/2]] \tag{2.2}$$

will be used primarily in the following, in which the line element is

$$ds^2=a^2[\eta][-d\eta^2+d\xi^2+\sinh^2[\xi]d\Omega^2_{(2)}], \tag{2.3}$$

where $-\infty<\eta<0$.

The condition that the first-order perturbation of the Ricci tensor vanishes $\delta R^{(1)}_{\mu\nu}=0$ gives the equation of motion [24]

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\hat{h}_{\mu\nu}+2R^{(B)}_{\alpha\mu\beta\nu}\hat{h}^{\alpha\beta}=0. \tag{2.4}$$

This may be rewritten as

$$[D^2_\eta-\nabla^2_{(3)}+2\mathcal{K}]\hat{h}_{ij}(\xi,\theta,\phi,\eta)=0, \tag{2.5}$$

[1] We define *scalar*, *vector*, and *tensor* here such that a vector that can be expressed as the spatial gradient of a scalar is regarded as a *scalar*, a tensor that may be expressed as spatial derivatives acting on a *vector* is regarded as a *vector*, etc.

where $\mathcal{K}$ is the spatial curvature, with $\mathcal{K}=-1$ for a hyperbolic universe, and we have used $R^{(B)}_{ijkl}=\mathcal{K}(\gamma_{ik}\gamma_{jl}-\gamma_{jk}\gamma_{il})$. Using the ansatz

$$\mathcal{T}_i^{\ j}(\mathbf{x},\eta;\zeta,P,j,m)=n(\zeta)[\mathbf{T}^{P,jm}(\xi,\theta,\phi;\zeta)]_i^{\ j}T_h(\eta;\zeta), \quad (2.6)$$

we obtain

$$[\nabla^2_{(3)}+(\zeta^2+3)]\mathbf{T}^{P,jm}_{ij}(\xi,\theta,\phi;\zeta)=0, \quad (2.7a)$$

$$\left[\partial^2_\eta+\frac{2a'}{a}\,\partial_\eta+(\zeta^2+1)\right]T_h(\eta;\zeta)=0. \quad (2.7b)$$

The normalization $n(\zeta)$ will be fixed later.

Using properties of the hyperbolic tensor harmonics that solve Eq. (2.7a), the gravitational waves in region I can be expanded as[2]

$$\hat{h}_{rs}(\eta,\xi,\theta,\phi)$$

$$=\sum_{jmP}\int_0^\infty d\zeta n(\zeta)\mathbf{T}^{P,jm}_{ij}(\xi,\theta,\phi;\zeta)T_h(\eta;\zeta)\hat{a}_{P,j,m}(\zeta)+\text{H.c.}$$

$$=\sum_{jmP}\int_0^\infty d\zeta\ \mathcal{T}^{P,jm}_{ij}(\xi,\theta,\phi,\eta;\zeta)\hat{a}_{P,j,m}(\zeta)+\text{H.c.} \quad (2.8)$$

Here the operators $\hat{a}_I(\zeta,P,j,m)$, $\hat{a}_I^\dagger(\zeta,P,j,m)$ satisfy the canonical commutation relations

$$[\hat{a}_I(\zeta,P,j,m),\hat{a}_I(\zeta',P',j',m')]$$
$$=[\hat{a}_I^\dagger(\zeta,P,j,m),\hat{a}_I^\dagger(\zeta',P',j',m')]=0$$

and

$$[\hat{a}_I(\zeta,P,j,m),\hat{a}_I^\dagger(\zeta',P',j',m')]=\delta(\zeta-\zeta)\delta_{PP'}\delta_{jj'}\delta_{mm'},$$

where ζ, $\zeta'>0$.

The spatial tensor harmonics and $n(\zeta)$ are discussed in the following section. Here the time dependence for the hyperbolic tensor modes and the flat tensor modes is found. For open de Sitter space, $a(\eta)=-1/\sinh[\eta]$. Thus for hyperbolic tensor modes Eq. (2.7b) becomes

$$T''_h-2\coth[\eta]T'_h+(\zeta^2+1)T_h=0, \quad (2.9)$$

replacing $-\nabla^2_{(3)}$ and $\mathcal{K}$ with (ζ^2+3) and -1, respectively. Transforming to the dependent variable $T=\sinh^2[\eta]\cdot F$, one may recast Eq. (2.9) into the form

$$F''+2\coth[\eta]F'+(\zeta^2+1)F-\frac{2}{\sinh^2[\xi]}F=0, \quad (2.10)$$

which is identical to the equation for the spatial hyperbolic radial functions with orbital angular momentum $l=1$. [See for example Ref. [10], Eq. (5.21).] It follows that

$$T_h(\eta;\zeta)=\{i\zeta\ \sinh[\eta]+\cosh[\eta]\}e^{-i\zeta\eta}, \quad (2.11)$$

where ζ is allowed to take both signs.

For the flat tensor modes, which will be needed in order to find the Bunch-Davies vacuum, the time evolution equation becomes

$$T''_f-\frac{2}{\eta_f}T'_f+\omega^2T_f=0, \quad (2.12)$$

where we replace $-\nabla^2$ and $\mathcal{K}$ with ω^2 and 0, respectively. With the substitution $T=\eta_f^{3/2}H$, Eq. (2.12) becomes the Bessel equation of order $\nu=\frac{3}{2}$ whose solutions $H^{(+)}_{3/2}(\omega\eta_f)$ and $H^{(-)}_{3/2}(\omega\eta_f)$ are proportional to the spherical Bessel functions $h_1^{(+)}(\omega\eta_f)$ and $h_1^{(-)}(\omega\eta_f)$ multiplied by $\eta_f^{1/2}$. Consequently,

$$T_f(\eta_f;\omega)=[1+i\omega\eta_f]e^{-i\omega\eta_f}, \quad (2.13)$$

where ω is allowed to take both signs.

III. HYPERBOLIC TENSOR HARMONICS

We now turn to computing and normalizing the hyperbolic tensor harmonics, which satisfy the equation

$$h_{ij}{}^{|k}{}_{|k}+(\zeta^2+3)h_{ij}=0. \quad (3.1)$$

The offset in (ζ^2+3) is chosen for later convenience. The pure tensor character of these modes requires that they satisfy the conditions of tracelessness

$$h_i^{\ i}=0, \quad (3.2)$$

and transversality

$$h_{ij}{}^{|j}=0. \quad (3.3)$$

Here the roman letters $(i,j=1,2,3)$ indicate spatial indices.

Since Eq. (3.1)–(3.3) are invariant under rotations and spatial inversion about $\xi=0$, multipole solutions may be classified according to their angular momentum quantum numbers $\mathbf{J}^2$, J_3, and their parity π. Parity is either *electric* with $\pi=(-)^j$ or *magnetic* with $\pi=(-)^{j+1}$, denoted by $P=E$ and $P=M$, respectively. Fixing j, m, and P, we write down the most general symmetric tensor field with these quantum numbers. This restricts the angular dependence to a few terms but does not specify the radial dependence. Imposing Eqs. (3.1)–(3.3) and solving for the ξ dependence gives a solution for each $\zeta>0$ and (j,m,P) for $j\geq 2$ unique up to an overall normalization. [There are no monopole $(j=0)$ or dipole $(j=1)$ modes.] The solution to these conditions has been found by Tomita [26].

The tensor field with electric parity has the form

[2]It should be pointed out that in addition to the continuous modes with $0\leq\zeta<\infty$ there might also exist some discrete *supercurvature* modes, as have been found for the minimally coupled scalar field in de Sitter space of mass m when $(m^2/H^2)<2$. See Refs. [14] and [27] for a discussion. The one supercurvature mode which would be expected to appear here by direct analogy has zero contribution to the CMB because of its lack of time dependence.

$$\widetilde{\mathbf{T}}^{E,jm}(\xi,\theta,\varphi;\zeta)=F_j(\xi;\zeta)(\mathbf{e}^{\xi}\otimes\mathbf{e}^{\xi})Y_{jm}(\theta,\varphi)+G_j(\xi;\zeta)\delta_{\tilde{a}\tilde{b}}(\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\tilde{b}})Y_{jm}(\theta,\varphi)+H_j(\xi;\zeta)(\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\xi}+\mathbf{e}^{\xi}\otimes\mathbf{e}^{\tilde{a}})\widetilde{\nabla}_{\tilde{a}}Y_{jm}(\theta,\varphi)+I_j(\xi;\zeta)(\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\tilde{b}})\widetilde{\nabla}_{\tilde{a}}\widetilde{\nabla}_{\tilde{b}}Y_{jm}(\theta,\varphi), \quad (3.4)$$

where $(\tilde{a},\tilde{b}=1,2)$ indicate angular indices and $\widetilde{\nabla}$ indicates an S^2 (rather than an H^3) covariant derivative. As a result, $\delta_{\tilde{a}\tilde{b}}(\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\tilde{b}})=(\mathbf{e}^{\theta}\otimes\mathbf{e}^{\theta}+\sin^2\theta\,\mathbf{e}^{\varphi}\otimes\mathbf{e}^{\varphi})$. The basis functions are $\mathbf{e}^{\xi}=d\xi$, $\mathbf{e}^{\theta}=d\theta$, and $\mathbf{e}^{\varphi}=d\varphi$. Note that this is not a *vielbein* (normalized) basis.

We use

$$\nabla^2=D_{\xi}^2+2\coth[\xi]D_{\xi}+\frac{1}{\sinh^2[\xi]}\left[D_{\theta}^2+\cot[\theta]D_{\theta}+\frac{1}{\sin^2\theta}D_{\varphi}^2\right] \quad (3.5)$$

and

$$\begin{aligned}
&D_{\xi}\mathbf{e}^{\xi}=0, && D_{\theta}\mathbf{e}^{\varphi}=-\cot[\theta]\mathbf{e}^{\varphi},\\
&D_{\xi}\mathbf{e}^{\theta}=-\coth[\xi]\mathbf{e}^{\theta}, && D_{\varphi}\mathbf{e}^{\xi}=+\sinh[\xi]\cosh[\xi]\sin^2[\theta]\mathbf{e}^{\phi},\\
&D_{\xi}\mathbf{e}^{\varphi}=-\coth[\xi]\mathbf{e}^{\varphi}, && D_{\varphi}\mathbf{e}^{\theta}=+\sin[\theta]\cos[\theta]\mathbf{e}^{\varphi},\\
&D_{\theta}\mathbf{e}^{\xi}=+\sinh[\xi]\cosh[\xi]\mathbf{e}^{\theta}, && D_{\varphi}\mathbf{e}^{\varphi}=-\cot[\theta]\mathbf{e}^{\theta}-\coth[\xi]\mathbf{e}^{\xi},\\
&D_{\theta}\mathbf{e}^{\theta}=-\coth[\xi]\mathbf{e}^{\xi}.
\end{aligned} \quad (3.6)$$

We now impose the constraints of transversality and tracelessness. For transversality, taking the divergence of $\mathbf{T}^{jm,E}$ gives

$$\nabla\cdot T_{jm}=\left[\frac{\partial F}{\partial\xi}+2\coth[\xi]F(\xi)-\frac{2\coth[\xi]}{\sinh^2[\xi]}G(\xi)-\frac{j(j+1)}{\sinh^2[\xi]}H(\xi)+\frac{j(j+1)\coth[\xi]}{\sinh^2[\xi]}I(\xi)\right]Y_{jm}(\Omega)\mathbf{e}^{\xi}+\left[\frac{G(\xi)}{\sinh^2[\xi]}+\frac{\partial H}{\partial\xi}+2\coth[\xi]H(\xi)-\frac{1}{\sinh^2[\xi]}[j(j+1)-1]I(\xi)\right]\left[\mathbf{e}^{\theta}\frac{\partial Y_{jm}}{\partial\theta}+\mathbf{e}^{\varphi}\frac{\partial Y_{jm}}{\partial\varphi}\right]=0. \quad (3.7)$$

Both terms must individually vanish. Likewise, requiring the trace to vanish gives the condition

$$T_i^i=F(\xi)+\frac{2G(\xi)}{\sinh^2[\xi]}-\frac{j(j+1)I(\xi)}{\sinh^2[\xi]}=0. \quad (3.8)$$

Thus transversality and tracelessness give

$$H_j(\xi;\zeta)=\frac{\sinh^2[\xi]}{j(j+1)}\left[\frac{\partial F_j(\xi;\zeta)}{\partial\xi}+3\coth[\xi]F_j(\xi;\zeta)\right],$$

$$I_j(\xi;\zeta)=\frac{\sinh^2[\xi]}{(j+2)(j-1)}\left[2\left(\frac{\partial H_j(\xi;\zeta)}{\partial\xi}+2\coth[\xi]H_j(\xi;\zeta)\right)-F_j(\xi;\zeta)\right], \quad (3.9)$$

$$G_j(\xi;\zeta)=\frac{1}{2}\{j(j+1)I_j(\xi;\zeta)-\sinh^2[\xi]F_j(\xi;\zeta)\}.$$

The Laplacian in Eq. (3.5) acting on the various components of $\widetilde{\mathbf{T}}^{E,jm}$ in Eq. (3.4) gives

$$\nabla^2[F(\xi)Y_{jm}(\Omega)(\mathbf{e}^{\xi}\otimes\mathbf{e}^{\xi})]=\left[\frac{\partial^2F}{\partial\xi^2}+2\coth[\xi]\frac{\partial F}{\partial\xi}-\left(\frac{j(j+1)}{\sinh^2[\xi]}+4\coth^2[\xi]\right)F(\xi)\right]Y_{jm}(\Omega)(\mathbf{e}^{\xi}\otimes\mathbf{e}^{\xi})+\{2\cosh^2[\xi]F(\xi)Y_{jm}(\Omega)\}(\mathbf{e}^{\theta}\otimes\mathbf{e}^{\theta}+\sin^2\theta\,\mathbf{e}^{\varphi}\otimes\mathbf{e}^{\varphi})+\{2\coth[\xi]F(\xi)\}\left[(\mathbf{e}^{\xi}\otimes\mathbf{e}^{\theta}+\mathbf{e}^{\theta}\otimes\mathbf{e}^{\xi})\frac{\partial Y_{jm}}{\partial\theta}+(\mathbf{e}^{\xi}\otimes\mathbf{e}^{\varphi}+\mathbf{e}^{\varphi}\otimes\mathbf{e}^{\xi})\frac{\partial Y_{jm}}{\partial\varphi}\right], \quad (3.10)$$

$$\nabla^2\{G(\xi)Y_{jm}(\Omega)(\mathbf{e}^\theta\otimes\mathbf{e}^\theta+\sin^2\theta\mathbf{e}^\varphi\otimes\mathbf{e}^\varphi)\}=\left[\frac{4\coth^2[\xi]}{\sinh^2[\xi]}\,G(\xi)Y_{jm}(\Omega)\right](\mathbf{e}^\xi\otimes\mathbf{e}^\xi)+\left[\frac{\partial^2 G}{\partial\xi^2}-2\coth[\xi]\,\frac{\partial G}{\partial\xi}-2G-\frac{j(j+1)}{\sinh^2[\xi]}\,G\right]Y_{jm}(\Omega)(\mathbf{e}^\theta\otimes\mathbf{e}^\theta+\sin^2\theta\,\mathbf{e}^\varphi\otimes\mathbf{e}^\varphi)-\frac{2\cosh[\xi]}{\sinh^3[\xi]}\,G(\xi)\left[(\mathbf{e}^\xi\otimes\mathbf{e}^\theta+\mathbf{e}^\theta\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\theta}+(\mathbf{e}^\xi\otimes\mathbf{e}^\varphi+\mathbf{e}^\varphi\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\varphi}\right], \tag{3.11}$$

$$\nabla^2\left[H(\xi)\left\{(\mathbf{e}^\xi\otimes\mathbf{e}^\theta+\mathbf{e}^\theta\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\theta}+(\mathbf{e}^\xi\otimes\mathbf{e}^\varphi+\mathbf{e}^\varphi\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\varphi}\right\}\right]=\frac{4j(j+1)\coth[\xi]}{\sinh^2[\xi]}\,Y_{jm}(\Omega)H(\xi)(\mathbf{e}^\xi\otimes\mathbf{e}^\xi)+\left[\frac{\partial^2 H}{\partial\xi^2}-2H-4\coth^2[\xi]H-\frac{j(j+1)}{\sinh^2[\xi]}\,H\right]\left[(\mathbf{e}^\xi\otimes\mathbf{e}^\theta+\mathbf{e}^\theta\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\theta}+(\mathbf{e}^\xi\otimes\mathbf{e}^\varphi+\mathbf{e}^\varphi\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\varphi}\right]+4H(\xi)\coth[\xi]\left[(\mathbf{e}^\theta\otimes\mathbf{e}^\theta)\,\frac{\partial^2 Y_{jm}}{\partial\theta^2}+(\mathbf{e}^\varphi\otimes\mathbf{e}^\varphi)\left(\frac{\partial^2 Y_{jm}}{\partial\varphi^2}+\sin\theta\,\cos\theta\frac{\partial Y_{jm}}{\partial\theta}\right)+(\mathbf{e}^\theta\otimes\mathbf{e}^\varphi+\mathbf{e}^\varphi\otimes\mathbf{e}^\theta)\left(\frac{\partial^2 Y_{jm}}{\partial\theta\partial\varphi}-\cot\theta\,\frac{\partial Y_{jm}}{\partial\varphi}\right)\right], \tag{3.12}$$

and

$$\nabla^2\left[I(\xi)\left\{(\mathbf{e}^\theta\otimes\mathbf{e}^\theta)\,\frac{\partial^2 Y_{jm}}{\partial\theta^2}+(\mathbf{e}^\varphi\otimes\mathbf{e}^\varphi)\left(\frac{\partial^2 Y_{jm}}{\partial\varphi^2}+\sin\theta\,\cos\theta\frac{\partial Y_{jm}}{\partial\theta}\right)+(\mathbf{e}^\theta\otimes\mathbf{e}^\varphi+\mathbf{e}^\varphi\otimes\mathbf{e}^\theta)\left(\frac{\partial^2 Y_{jm}}{\partial\theta\partial\varphi}-\cot\theta\,\frac{\partial Y_{jm}}{\partial\varphi}\right)\right\}\right]=\frac{-2I(\xi)j(j+1)\coth^2[\xi]}{\sinh^2[\xi]}\,Y_{jm}(\Omega)(\mathbf{e}^\xi\otimes\mathbf{e}^\xi)+\frac{2I(\xi)j(j+1)}{\sinh^2[\xi]}\,Y_{jm}(\Omega)(\mathbf{e}^\theta\otimes\mathbf{e}^\theta+\sin^2\theta\,\mathbf{e}^\varphi\otimes\mathbf{e}^\varphi)+\frac{2[j(j+1)-1]\coth[\xi]I(\xi)}{\sinh^2[\xi]}\left[(\mathbf{e}^\xi\otimes\mathbf{e}^\theta+\mathbf{e}^\theta\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\theta}+(\mathbf{e}^\xi\otimes\mathbf{e}^\varphi+\mathbf{e}^\varphi\otimes\mathbf{e}^\xi)\,\frac{\partial Y_{jm}}{\partial\varphi}\right]+\left[\frac{\partial^2 I}{\partial\xi^2}-2\coth[\xi]\,\frac{\partial}{\partial\xi}+\frac{6I(\xi)}{\sinh^2[\xi]}-2I(\xi)\coth^2[\xi]-\frac{j(j+1)}{\sinh^2[\xi]}\,I(\xi)\right]\left[(\mathbf{e}^\theta\otimes\mathbf{e}^\theta)\,\frac{\partial^2 Y_{jm}}{\partial\theta^2}+(\mathbf{e}^\varphi\otimes\mathbf{e}^\varphi)\left(\frac{\partial^2 Y_{jm}}{\partial\varphi^2}+\sin\theta\,\cos\theta\frac{\partial Y_{jm}}{\partial\theta}\right)+(\mathbf{e}^\theta\otimes\mathbf{e}^\varphi+\mathbf{e}^\varphi\otimes\mathbf{e}^\theta)\left(\frac{\partial^2 Y_{jm}}{\partial\theta\partial\varphi}-\cot\theta\,\frac{\partial Y_{jm}}{\partial\varphi}\right)\right]. \tag{3.13}$$

To solve for $F_j(\xi;\zeta)$, we take the $(\mathbf{e}^\xi\otimes\mathbf{e}^\xi)$ component of the Laplacian acting on Eq. (3.4), and after applying the substitutions in Eq. (3.9), the coefficient of $\mathbf{e}^\xi\otimes\mathbf{e}^\xi$ term in Eq. (3.1) becomes

$$\frac{\partial^2 F_j(\xi;\zeta)}{\partial\xi^2}+6\coth[\xi]\,\frac{\partial F_j(\xi;\zeta)}{\partial\xi}+\left[(\zeta^2+3)+6\coth^2[\xi]-\frac{j(j+1)}{\sinh^2[\xi]}\right]F_j(\xi;\zeta)=0. \tag{3.14}$$

With the change of variable $\phi_j(\xi;\zeta)=\sinh^2[\xi]F_j(\xi;\zeta)$, one recovers the differential equation for the *scalar* hyperbolic radial functions [see, for example, [10] Eq. (5.21)]:

$$\frac{\partial^2\phi_j(\xi;\zeta)}{\partial\xi^2}+2\coth[\xi]\,\frac{\partial\phi_j(\xi;\zeta)}{\partial\xi}+\left[(\zeta^2+1)-\frac{j(j+1)}{\sinh^2[\xi]}\right]\phi_j(\xi;\zeta)=0. \tag{3.15}$$

Consequently,

$$F_j(\xi;\zeta)=N_j(\zeta)\sinh^{j-2}[\xi]\,\frac{d^{j+1}}{d(\cosh[\xi])^{j+1}}\,\cos[\zeta\xi], \tag{3.16}$$

where the normalization $N_j(\zeta)$ is determined in the following and we have imposed regularity at the origin.

For future reference, for $j=2$,

$$F_2(\xi;\zeta)=\frac{N_2(\zeta)}{\sinh^3[\xi]}\{3\zeta^2\coth[\xi]\cos[\zeta\xi]+(\zeta^3+\zeta-3\zeta\coth^2[\xi])\sin[\zeta\xi]\}\quad(3.17)$$

and, from Eq. (3.9),

$$G_2(\xi;\zeta)=N_2(\zeta)\sinh[\xi]\times\left[\left\{\frac{\zeta^2\coth[\xi]}{4}(1-2\zeta^2-3\coth^2[\xi])\right\}\sin[\zeta\xi]+\left\{\frac{-\zeta^5-2\zeta^3-\zeta}{4}+\frac{\zeta\coth^2[\xi]}{4}\times(\zeta^2-2+3\coth^2[\xi])\right\}\cos[\zeta\xi]\right],$$

$$H_2(\xi;\zeta)=N_2(\zeta)\frac{1}{\sinh[\xi]}\left[\left\{\frac{\zeta^2}{6}(\zeta^2+4-6\coth^2[\xi])\right\}\cos[\zeta\xi]+\left\{\frac{\zeta\coth[\xi]}{2}(2\coth^2[\xi]-\zeta^2-2)\right\}\sin[\zeta\xi]\right],\quad(3.18)$$

$$I_2(\xi;\zeta)=N_2(\zeta)\sinh[\xi]\times\left[\left\{\frac{\zeta^2\coth[\xi]}{12}(-5-2\zeta^2+3\coth^2[\xi])\right\}\cos[\zeta\xi]+\left\{\frac{-\zeta^5-4\zeta^3-3\zeta}{12}+\frac{\zeta\coth^2[\xi]}{4}\times(-\coth^2[\xi]+\zeta^2+2)\right\}\sin[\zeta\xi]\right].$$

Similarly, for the magnetic parity, the tensor field must take the form

$$\mathbf{T}^{B,jm}(\xi,\theta,\varphi;\zeta)=U(\xi;\zeta)(\mathbf{e}^{\tilde a}\otimes\mathbf{e}^{\xi}+\mathbf{e}^{\xi}\otimes\mathbf{e}^{\tilde a})\mathbf{L}_{\tilde a}Y_{jm}(\theta,\varphi)+V(\xi;\zeta)(\mathbf{e}^{\tilde a}\otimes\mathbf{e}^{\tilde b}+\mathbf{e}^{\tilde b}\otimes\mathbf{e}^{\tilde a})\mathbf{L}_{\tilde a}\tilde{\nabla}_{\tilde b}Y_{jm}(\theta,\varphi).\quad(3.19)$$

The magnetic parity modes do not contribute to the CMB anisotropy because their component along the $(\mathbf{e}^{\xi}\otimes\mathbf{e}^{\xi})$ direction vanishes; therefore, we do not give their explicit form.

Normalization of the hyperbolic tensor harmonics. To normalize the tensor harmonics, we impose the condition

$$\int_0^\infty d\xi\int d\Omega\sqrt{\hat g_{(3)}}T_{ij}(\mathbf{x};\zeta,P,j,m)^*T^{ij}(\mathbf{x};\zeta',P',j',m')=\delta(\zeta-\zeta')\,\delta_{P,P'}\,\delta_{j,j'}\,\delta_{m,m'}\quad(3.20)$$

where P indicates mode parity and $\mathbf{x}=(\xi,\theta,\phi)$.

Because the tensor harmonics are eigenfunctions of a self-adjoint operator, the inner product is proportional to a δ function. Thus $N_j(\zeta)$ is determined by the coefficients of $e^{\pm i\zeta\xi}$ in the asymptotic expansion of the tensor harmonics for large ξ. (For $\xi\to0$, F_j, H_j, G_j, $I_j\to0$.) In comparing the asymptotic behaviors of F, G, H, and I, it is more meaningful to consider the rescaled quantities $\hat F_j(\xi;\zeta)=F_j(\xi;\zeta)$, $\hat G_j(\xi;\zeta)=G_j(\xi;\zeta)/\sinh^2[\xi]$, $\hat H_j(\xi;\zeta)=H_j(\xi;\zeta)/\sinh[\xi]$, and $\hat I_j(\xi;\zeta)=I_j(\xi;\zeta)/\sinh^2[\xi]$, components with respect to a normalized ''vielbein'' basis. From Eq. (3.9), $\hat F\sim\sinh^{-3}[\xi]$, $\hat H\sim\sinh^{-2}[\xi]$, and $\hat G$, $\hat I\sim\sinh^{-1}[\xi]$. For $\xi\gg\lambda$, $\hat G$ and $\hat I$ dominate, exactly as one would expect. That is, at large distances a spherical gravitational wave should locally resemble a plane gravitational wave propagating in the radial direction.

To compute the coefficient of the δ function in Eq. (3.20), we impose a boundary condition at $\xi=\xi_{\max}$ (for specificity say Dirichlet boundary conditions) and take the limit $\xi_{\max}\to\infty$. For $\xi_{\max}\gg1$, the integral in Eq. (3.20) is dominated by the G and I components. These may be approximated by their large-ξ asymptotic forms, starting with Eq. (3.16) and substituting $\sinh[\xi]\to(e^{\xi}/2)=(w/2)$. This gives

$$F_j(\xi;\zeta)\approx4N_j(\zeta)w^{-3}\left(\frac{d}{dw}\right)^{j+1}[w^{+i\zeta}+w^{-i\zeta}]=4N_j(\zeta)e^{-3\xi}[(i\zeta)_je^{+i\zeta\xi}+\text{c.c.}].\quad(3.21)$$

Here $(x)_j$ is shorthand for $x(x-1)\cdots(x-j)$. Using Eq. (3.9), we obtain

$$H_j(\xi;\zeta)\approx\frac{4N_j(\zeta)e^{-\xi}}{j(j+1)}\{(i\zeta)(i\zeta)_je^{+i\zeta\xi}+\text{c.c.}\},$$

$$I_j(\xi;\zeta)\approx\frac{2N_j(\zeta)e^{+\xi}}{j(j+1)(j-1)(j+2)}\{(i\zeta+1)(i\zeta)(i\zeta)_je^{+i\zeta\xi}+\text{c.c.}\},\quad(3.22)$$

$$G_j(\xi;\zeta)\approx\frac{N_j(\zeta)e^{+\xi}}{(j-1)(j+2)}\{(i\zeta+1)(i\zeta)(i\zeta)_je^{+i\zeta\xi}+\text{c.c.}\}.$$

Consequently, for $(\xi\gg1)$,

$$\mathbf{T}(\mathbf{x};\zeta,E,j,m)\approx\left[\frac{2N_j(\zeta)e^{\xi}}{j(j+1)(j-1)(j+2)}(i\zeta+1)(i\zeta)\times(i\zeta)_je^{+i\zeta\xi}\{\nabla_{\tilde a}\nabla_{\tilde b}Y_{jm}(\theta,\phi)+\delta_{\tilde a\tilde b}\tfrac{1}{2}j(j+1)Y_{jm}(\theta,\phi)\}(\mathbf{e}^{\tilde a}\otimes\mathbf{e}^{\tilde b})\right]+\text{c.c.}\quad(3.23)$$

Inserting this asymptotic expression into Eq. (3.20) and factoring out $\delta_{jj'}\delta_{mm'}$ gives

$$\int_0^{\xi_{\max}} d\xi\ \sinh^2[\xi]\int d\Omega\, T_{ij}(\mathbf{x};\zeta,E,j,m)T^{ij}(\mathbf{x};\zeta,E,j,m)^*$$
$$\approx |N_j(\zeta)|^2\ \frac{4\zeta^2(\zeta^2+1)[\zeta^2(\zeta^2+1^2)\cdots(\zeta^2+j^2)]}{j^2(j+1)^2(j-1)^2(j+2)^2}$$
$$\times\frac{\xi_{\max}}{2}\int d\Omega\left|\nabla_{\tilde a}\nabla_{\tilde b}Y_{jm}+\frac{1}{2}\ \delta_{\tilde a\tilde b}j(j+1)Y_{jm}\right|^2. \tag{3.24}$$

For the angular integration, note that

$$\int d\Omega(\nabla_{\tilde a}\nabla_{\tilde b}Y_{jm})^*\nabla^{\tilde a}\nabla^{\tilde b}Y_{jm}$$
$$=-\int d\Omega(\nabla^{\tilde a}\nabla_{\tilde a}\nabla_{\tilde b}Y_{jm})^*(\nabla^{\tilde b}Y_{jm})$$
$$=-\int d\Omega(\nabla_{\tilde a}\nabla_{\tilde b}\nabla^{\tilde a}Y_{jm})^*(\nabla^{\tilde b}Y_{jm})$$
$$=-\int d\Omega\left(\nabla_{\tilde b}\left\{\nabla^2+\frac{1}{2}R\right\}Y_{jm}\right)^*(\nabla^{\tilde b}Y_{jm})$$
$$=[j(j+1)][j(j+1)-1], \tag{3.25}$$

given that $g^{\tilde a\tilde c}[\nabla_{\tilde a},\nabla_{\tilde b}]\nabla_{\tilde c}f=g^{\tilde a\tilde c}R_{\tilde a\tilde b\tilde c}{}^{\tilde d}\nabla_{d}f=R_{\tilde b}{}^{\tilde d}(\nabla_{\tilde d}f)=\frac{1}{2}R\nabla_{\tilde b}f$. Since the unit two-sphere is isotropic, $R_{\tilde a\tilde b}=\frac{1}{2}\delta_{\tilde a\tilde b}R$. Also, for the two-sphere $R=2$. Thus the integral on the last line of Eq. (3.24) is equal to $[j(j+1)][\frac{1}{2}j(j+1)-1]$, and Eq. (3.24) reduces to

$$|N_j(\zeta)|^2\ \frac{4[j(j+1)][(1/2)j(j+1)-1]}{j^2(j+1)^2(j-1)^2(j+2)^2}\ \zeta^2(\zeta^2+1)$$
$$\times[\zeta^2(\zeta^2+1^2)\cdots(\zeta^2+j^2)]\ \frac{\xi_{\max}}{2}, \tag{3.26}$$

from which it follows that

$$N_j(\zeta)=\frac{1}{\sqrt{\pi}}\ \frac{\sqrt{j(j+1)(j-1)(j+2)}}{\zeta\sqrt{1+\zeta^2}\sqrt{\zeta^2(\zeta^2+1^2)\cdots(\zeta^2+j^2)}}. \tag{3.27}$$

The $\pi^{-1/2}$ comes from passing to the continuum and the phase of $N_j(\zeta)$ has been chosen to be real.

Flat tensor harmonics. The general form of the flat modes is needed to identify the Bunch-Davies vacuum in the next section. These may be regarded as the large-ζ, small-ξ limit of the hyperbolic modes, a limit in which the effects of spatial curvature disappear. We use lower case letters to denote the flat analogues of hyperbolic quantities. In particular,

$$f(r,\omega)=\overline{n_j}(\omega)r^{j-2}\left(\frac{1}{r}\ \frac{d}{dr}\right)^{j+1}\cos[\omega r]. \tag{3.28}$$

Similarly, Eq. (3.9) is modified to

$$h_j(r;\omega)=\frac{r^2}{j(j+1)}\left[\frac{\partial f_j}{\partial r}+\frac{3}{r}\ f_j\right],$$
$$i_j(r;\omega)=\frac{r^2}{(j+2)(j-1)}\left[2\ \frac{\partial h_j}{\partial r}+\frac{4}{r}\ h_j-f_j\right], \tag{3.29}$$
$$g_j(r;\omega)=\frac{1}{2}\ [j(j+1)i_j(r)-r^2f_j].$$

In particular, for $j=2$

$$f_2(r;\omega)=\left(\frac{3\omega^2}{r^3}\right)\cos[\omega r]+\left(\frac{-3\omega}{r^4}+\frac{\omega^3}{r^2}\right)\sin[\omega r],$$
$$g_2(r;\omega)=\left(\frac{-3\omega^2}{r}\right)\cos[\omega r]+\left(\frac{-\omega^5r^2}{4}-\omega^3+\frac{3\omega}{r^2}\right)\sin[\omega r], \tag{3.30}$$
$$h_2(r;\omega)=\left(\frac{-\omega^2}{2r^2}+\frac{\omega^4}{6}\right)\cos[\omega r]+\left(\frac{-\omega^3}{3r}+\frac{\omega}{2r^3}\right)\sin[\omega r],$$
$$i_2(r;\omega)=\left(\frac{-\omega^2}{2r}\right)\cos[\omega r]+\left(\frac{-\omega^5r^2}{12}-\frac{\omega^3}{6}+\frac{\omega}{2r^2}\right)\sin[\omega r].$$

In order to calculate the normalization, one needs the asymptotic behavior for large r:

$$f_j(r;\omega)\approx\overline{n_j}(\zeta)\ \frac{1}{2r^3}\ [(i\omega)^{j+1}e^{+i\omega r}+\text{c.c.}],$$
$$h_j(r;\omega)\approx\overline{n_j}(\zeta)\ \frac{1}{j(j+1)}\ \frac{1}{2r}\ [(i\omega)^{j+2}e^{+i\omega r}+\text{c.c.}], \tag{3.31}$$
$$i_j(r;\omega)\approx\overline{n_j}(\zeta)\ \frac{1}{j(j+1)(j-1)(j+2)}\times r[(i\omega)^{j+3}e^{+i\omega r}+\text{c.c.}],$$
$$g_j(r;\omega)\approx\overline{n_j}(\zeta)\ \frac{1}{2(j-1)(j+2)}\ r[(i\omega)^{j+3}e^{+i\omega r}+\text{c.c.}].$$

Again i_j and g_j dominate, and following the same steps as for the hyperbolic modes we get

$$\overline{n_j}(\omega)=\frac{\sqrt{j(j+1)(j+2)(j-1)}}{\sqrt{\pi}\omega^{j+3}}. \tag{3.32}$$

Normalization of the time-dependent mode functions. We may define the antisymmetric bilinear form

$$\langle\mathcal{U},\mathcal{V}\rangle=-\int_\Sigma d\Sigma^\mu\mathcal{U}_{\alpha\beta}(\Sigma)(i\overleftrightarrow{D}_\mu)\mathcal{V}^{\alpha\beta}(\Sigma), \tag{3.33}$$

where $\mathcal{U}$, $\mathcal{V}$ are solutions to Eq. (2.4). This product is analogous to the Klein-Gordon product for scalar field modes. Equation (2.4) insures that the current $\mathcal{U}_{AB}(\Sigma)(i\overleftrightarrow{\nabla}_\mu)\mathcal{V}^{AB}(\Sigma)$ is conserved and thus that $\langle\mathcal{U},\mathcal{V}\rangle$ is invariant under deformations of the surface Σ. In order that the modes for a spacetime M orthonormalized with respect to Eq. (3.33) are associated with operators that satisfy the customary canonical commutation relations, it is necessary to choose Σ in Eq. (3.33) so that Σ is a Cauchy surface for M. The product $\langle\ ,\ \rangle$ is the same for all Cauchy surfaces for the spacetime M. A Cauchy surface is a spacelike hypersurface which each nonspacelike curve intersects once and only once [28]. In this paper we shall consider both the case where M is just region I, in which case a surface of constant region I hyperbolic time serves as a convenient choice of Cauchy surface, and also the case where M is all of maximally extended de Sitter space, in which case the surface defined by $\tau=0$ in the region II hyperbolic coordinates serves as a convenient Cauchy surface.

For the hyperbolic modes defined in Eq. (2.6) in region I one calculates

$$\langle \mathcal{T}(\zeta,P,j,m),\mathcal{T}(\zeta',P',j',m')^*\rangle = -\int_\Sigma d^3x a^3(\eta)[\mathcal{T}^\beta{}_\alpha(\mathbf{x},\eta;\zeta,P,j,m)(i\overleftrightarrow{D}_{\tilde{0}})\mathcal{T}^\alpha{}_\beta(\mathbf{x},\eta;\zeta',P',j',m')^*]$$
$$= n^2(\zeta)2\zeta(\zeta^2+1)\,\delta(\zeta-\zeta')\,\delta_{P,P'}\,\delta_{j,j'}\,\delta_{m,m'}, \tag{3.34}$$

where $D_{\tilde{0}}=(1/a)D_\eta=-\sinh[\eta]D_\eta$. Note that Eq. (3.34) is independent of η. This may be seen by applying D_η to Eq. (3.34) and using Eq. (2.9). For the mixed tensor representation chosen above, the covariant derivative $D_{\tilde{0}}$ may be replaced with the ordinary derivative $\partial_{\tilde{0}}$. Therefore,

$$n(\zeta)=\frac{1}{\sqrt{2\zeta(\zeta^2+1)}}. \tag{3.35}$$

As the initial conditions are determined by the bubble which extends outside of region I, a proper Cauchy surface for initial conditions for an open universe extends outside of region I as well. It was shown in [14] that for some cases, inner products taken on a Cauchy surface in region II agreed with those on fixed time surfaces in region I and V, even though the latter do not make up a Cauchy surface for the whole spacetime. The norms agreed for scalar fields with sufficiently fast falloff at infinity, a condition satisfied by subcurvature modes of the Laplacian (modes with eigenvalue $\zeta^2+1\geqslant 1$). There are some differences in extending this comparison between norms taken in I and V and norms taken in II for gravity waves. When the gravity waves are continued across the light cone, into region II, time and space are interchanged and the gauge choice becomes $h_{u\mu}=0$ instead of $h_{\eta\mu}=0$. As u is a spacelike coordinate, this is not the usual gauge for metric perturbations. It may be thought of as analogous to axial gauge (where $A_3=0$) rather than Coulomb gauge (where $A_0=0$) in electrodynamics. Secondly, the inner product in region II involves a Wronskian in (the analytic continuation of) ξ, and is more complicated due to the tensor structure in ξ. As shown in Appendix B, this inner product on a Cauchy surface in region II coincides with the inner product in regions I, V, for modes which have sufficiently fast falloff at infinity. The fields with sufficiently fast falloff for gravity waves are again subcurvature modes, with $0\leqslant\zeta^2<\infty$. As the falloff for the gravity wave modes is as fast as that for the scalars in regions I, V (both go as $\sinh^{-1}[\xi]$ for large ξ), it was reasonable to expect this.

IV. IDENTIFYING THE GRAVITATIONAL WAVE BUNCH-DAVIES VACUUM

The preceding section gave the mode expansion for the linearized gravitational waves in region I hyperbolic coordinates, which are the natural coordinates for the open universe inside the bubble. In this section the initial Bunch-Davies vacuum is expressed in terms of these open hyperbolic modes. In the open universe inflationary scenario, the Bunch-Davies vacuum is a preferred quantum state for de Sitter space in the sense that it is a weak attractor: any initial quantum state for perturbations from de Sitter space, subject only to the requirement that the initial energy density be finite, approaches the Bunch-Davies vacuum to arbitrary accuracy after a sufficient amount of inflationary expansion. The convergence is weak rather than strong because the initial perturbations are not erased but rather pushed to larger and larger scales, so that for an observer able to probe only a fixed physical volume, the perturbations seem to disappear.

The Bunch-Davies vacuum is physically characterized using the flat coordinates for de Sitter space. In these coordinates, at sufficiently early times, a mode of fixed comoving wave number evolves as if it were a mode in Minkowski space. For the flat coordinates the line element is

$$ds^2=-dt_f^2+e^{2t_f}[dr_f^2+r_f^2d\Omega_{(2)}^2], \tag{4.1}$$

where $-\infty<t_f<+\infty$, or in terms of flat conformal time $\eta_f=-e^{-t_f}$

$$ds^2=\frac{1}{\eta_f^2}[-d\eta_f^2+dr_f^2+r_f^2d\Omega_{(2)}^2]. \tag{4.2}$$

Early on, when the mode is subhorizon, so that there are many oscillations within an expansion time, one identifies the mode with positive frequency asymptotic behavior with an annihilation operator of the Bunch-Davies vacuum. These are the modes that behave as $e^{-ik\eta_f}/\eta_f$ for $\eta_f\rightarrow-\infty$ where η_f is flat conformal time.

To relate the flat and hyperbolic coordinates one can embed de Sitter space in $(4+1)$-dimensional Minkowski space as described in Appendix A. Using $\sinh[t_h]=-1/\sinh[\eta]$, $\cosh[t_h]=-\coth[\eta]$, and $\coth[t_h]=\cosh[\eta]$, one gets that the relation between the coordinate systems is

$$r_f = \frac{\sinh[\xi]}{\cosh[\xi]+\cosh[\eta]},$$
$$\eta_f = \frac{\sinh[\eta]}{\cosh[\eta]+\cosh[\xi]}. \quad (4.3)$$

The flat coordinates do not cover all of de Sitter space but only half of maximally extended de Sitter space, as indicated in the conformal diagram in Fig. 3. Nevertheless the flat coordinates cover enough of de Sitter space to contain an initial value surface for all of de Sitter space. The null surface indicated by the dashed diagonal line, the boundary of the region covered by the flat coordinates, is such a surface.[3]

To identify the hyperbolic modes of *positive* frequency with respect to the Bunch-Davies vacuum, it is convenient to use the null coordinates in the flat chart

$$u_f = \eta_f + r_f,$$
$$v_f = \eta_f - r_f. \quad (4.4)$$

One has $v_f < 0$ and $|u_f| < -v_f$. We also define the region I hyperbolic null coordinates

$$U = \eta + \xi,$$
$$V = \eta - \xi, \quad (4.5)$$

where $V<0$ and $|U|<-V$. Using the above relations between (η,ξ) and (η_f, r_f) one can see that the two sets of null coordinates are related according to

$$U = \ln\left[\frac{1+u_f}{1-u_f}\right], \quad V = \ln\left[\frac{1+v_f}{1-v_f}\right]. \quad (4.6)$$

Conversely, $u_f = \tanh(U/2)$ and $v_f = \tanh(V/2)$. Thus we see that the region I hyperbolic coordinates cover only the region $-1 < v_f < 0$, $|u_f| < -v_f$ in terms of the flat coordinates.

The positive frequency modes with respect to open hyperbolic null coordinates in region I are

$$e^{-i\zeta U} = \left[\frac{1+u_f}{1-u_f}\right]^{-i\zeta}, \quad e^{-i\zeta V} = \left[\frac{1+v_f}{1-v_f}\right]^{-i\zeta}. \quad (4.7)$$

However, a full mode function for the Bunch-Davies vacuum must be specified in both regions I and V. A region I mode function must be continued into region V in order to calculate its inner product on the full space and the choice of analytic continuation distinguishes between positive and negative frequency Bunch-Davies modes. The identification of the positive frequency modes for the Bunch-Davies vacuum is determined by the analytic properties of the factors in $e^{-i\zeta U}$ and $e^{-i\zeta V}$ appearing in the hyperbolic mode functions above. The other factors in the hyperbolic mode functions contain only isolated singularities and no branch cuts, and thus will be seen to be irrelevant in making this identification. The bubble interior lies within the strip $-1 < u_f < +1$. Outside this strip one encounters a branch cut, taken here to lie near the real axis starting at $u_f = 1$, passing through $+\infty$, coming back from $-\infty$, and finally ending at $u_f = -1$, as indicated in Fig. 4. Thus $[(1+u_f)/(1-u_f)]^{i\zeta}$ has two possible continuations to the entire real line—one through the upper half plane above the branch cut and another through the lower half plane below the branch cut.

One can identify the positive frequency Bunch-Davies modes by requiring they have vanishing overlap with the flat negative frequency modes $e^{i\omega u_f}$, $(\omega > 0)$. Thus one requires

$$\int_{-\infty}^{+\infty} du_f\, e^{i\omega u_f}\left(\frac{1+u_f}{1-u_f}\right)^{i\zeta} = 0 \quad (4.8)$$

[3]Technically, because this surface is null it is not a Cauchy surface, but initial data on this surface can be continued everywhere in de Sitter space.

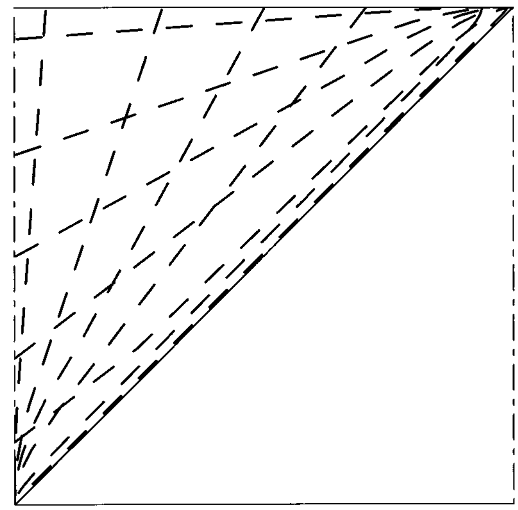

FIG. 3. The region covered by the flat coordinates is shown in a conformal diagram for all of maximally extended de Sitter space, identical to that in Fig. 2. Although the flat coordinates cover only half of maximally extended de Sitter space, the diagonal line, which is the boundary of the region covered by the flat coordinates, represents an initial value surface for all of de Sitter space.

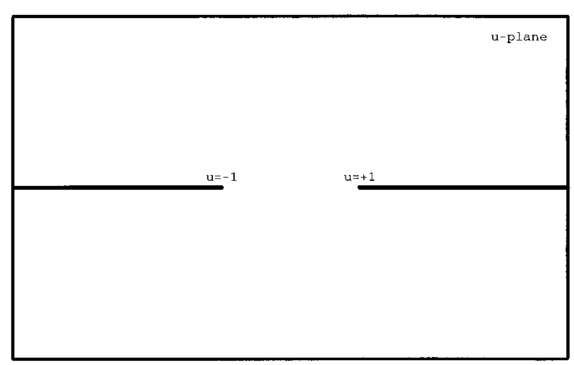

FIG. 4. The points $u_f = -1$ and $u_f = +1$ are branch points. To impose single valuedness, we take a branch cut to start at $u_f = +1$, extend to $u_f = +\infty$, come back from $u_f = -\infty$, and finally end at $u_f = -1$. The analytic continuation through the upper half plane onto the remainder of the real line corresponds to the Bunch-Davies positive frequency modes. Similarly, the continuation through the lower half plane corresponds to the Bunch-Davies negative frequency modes.

for all $\omega>0$. This is satisfied by the analytic continuation through the upper half-plane above the branch cut, as deforming the contour toward $+i\infty$ makes this integral vanish. As a result, positive frequency Bunch-Davies modes correspond to analytic continuation through the upper half plane into region V. This identification generalizes to gravitational wave hyperbolic modes and other more complicated mode functions of the form

$$R(u_f,v_f)\left(\frac{1+u_f}{1-u_f}\right)^{i\zeta}+S(u_f,v_f)\left(\frac{1+v_f}{1-v_f}\right)^{i\zeta}, \qquad (4.9)$$

where $R(u_f,v_f)$ and $S(u_f,v_f)$ are rational functions.

We shall use two sets of flat coordinates (u_f,v_f) and $(\tilde{u}_f,\tilde{v}_f)$, as indicated in Fig. 5, related by a spatial reflection symmetry of de Sitter space that maps region I into region V. It may be shown that

$$\tilde{u}_f=\frac{-1}{v_f}, \quad \tilde{v}_f=\frac{-1}{u_f}, \qquad (4.10)$$

either directly by using closed coordinates or by the analytic methods in Appendix A.

This transformation relates the flat coordinates (u_f,v_f) covering the upper left triangular wedge of de Sitter space to the coordinates $(\tilde{u}_f,\tilde{v}_f)$ covering the upper right triangular wedge, as indicated in Fig. 5. As discussed earlier, for positive frequency mode functions the analytic continuation is with positive imaginary part $(-1)=e^{i\pi}$, and so

$$\left(\frac{1+u_f}{1-u_f}\right)^{-i\zeta}=\left((-1)^{-1}\,\frac{1+\tilde{v}_f}{1-\tilde{v}_f}\right)^{i\zeta}\equiv(-1)^{-i\zeta}e^{i\zeta\tilde{V}},$$
$$\left(\frac{1+v_f}{1-v_f}\right)^{-i\zeta}=\left((-1)^{-1}\,\frac{1+\tilde{u}_f}{1-\tilde{u}_f}\right)^{i\zeta}\equiv(-1)^{-i\zeta}e^{i\zeta\tilde{U}}. \qquad (4.11)$$

As a result, positive hyperbolic frequency modes $e^{-i\zeta U}$, $e^{-i\zeta V}$ in region I become negative hyperbolic frequency modes in region V with an attentuation factor $e^{-\pi\zeta}$, and negative frequency region I hyperbolic modes become positive frequency hyperbolic modes in region V with an amplification factor $e^{\pi\zeta}$. The result of all this is that the positive frequency modes in region I are the modes proportional to $e^{iU\zeta}$, $e^{iV\zeta}$ for all values of ζ plus the continuation into region V using the upper half plane. The negative frequency modes are the same collection of modes with the opposite analytic continuation into region V.

We may normalize the modes by combining the fixed time surfaces of regions I and V. As shown in [14] for scalar subcurvature modes, and in the appendix for gravity waves, for the subcurvature modes under consideration here, the inner product is equivalent to that taken on a Cauchy surface for all of de Sitter space. Let $\mathcal{T}^{(I)}(\zeta,P,j,m)$ denote the modes with positive hyperbolic frequency in region I, as defined in Eq. (2.6), normalized using only the region I fixed time surface, and let $\mathcal{T}^{(V)}(\zeta,P,j,m)$ be the analogously defined modes in region V. It follows that in region I one has the mode expansion

$$\sum_{Pjm}\int_0^\infty\frac{d\zeta}{\sqrt{2\zeta(\zeta^2+1)}}\Bigg\{[\mathbf{T}^{Ejm}(\xi,\theta,\phi,\eta;\zeta)]^r_s\Bigg[\frac{e^{+\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;\zeta)\hat{b}_I(\zeta,P,j,m)$$
$$+\frac{e^{-\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;-\zeta)\hat{b}_V(\zeta,P,j,m)\Bigg]+\text{H.c.}\Bigg\} \qquad (4.12)$$

and similarly in region V

$$\sum_{Pjm}\int_0^\infty\frac{d\zeta}{\sqrt{2\zeta(\zeta^2+1)}}\Bigg\{[\mathbf{T}^{Ejm}(\xi,\theta,\phi,\eta;\zeta)]^r_s\Bigg[\frac{e^{+\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;\zeta)\hat{b}_V(\zeta,P,j,m)$$
$$+\frac{e^{-\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;-\zeta)\hat{b}_I(\zeta,P,j,m)\Bigg]+\text{H.c.}\Bigg\} \qquad (4.13)$$

where the annihilation operators $\hat{b}_I(\zeta,P,j,m)$ and $\hat{b}_V(\zeta,P,j,m)$ annihilate the Bunch-Davies vacuum for the graviton field and satisfy the usual commutation relations

$$[\hat{b}_A(\zeta,P,j,m),\hat{b}_B(\zeta',P',j',m')]=0,$$
$$[\hat{b}^\dagger_A(\zeta,P,j,m),\hat{b}^\dagger_B(\zeta',P',j',m')]=0, \qquad (4.14)$$
$$[\hat{b}_A(\zeta,P,j,m),\hat{b}^\dagger_B(\zeta',P',j',m')]$$
$$=\delta_{AB}\delta(\zeta-\zeta')\,\delta_{PP'}\delta_{jj'}\delta_{mm'},$$

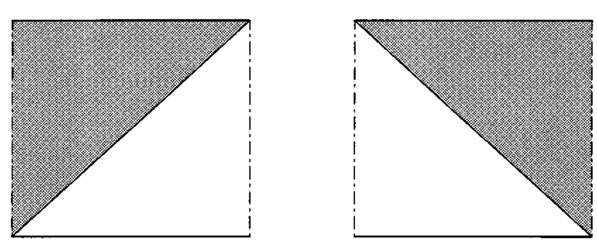

FIG. 5. Both panels are conformal diagrams for maximally extended de Sitter space. The shaded triangle in (a) shows the region covered by the flat null coordinates (u_f,v_f), while in (b) the region covered by null coordinates $(\tilde{u}_f,\tilde{v}_f)$, is shown.

where $(A,B=\mathrm{I,V})$. In the Bunch-Davies vacuum we observe a doubling in the number of modes. This reflects the existence of correlations between regions I and V—or, alternatively, between the inside and the outside of the bubble.

Another way of understanding this result for the vacuum is to note that the scalar mode functions and mixed tensor (with one raised and one lowered index, $\mathcal{T}^A_B$) mode functions considered here have the same time dependence. As the choice of vacuum depends only on the time dependence of the mode functions, the time dependence in the Bunch-Davies vacuum also has the same form for both cases.[4] For the scalars there is a discrete vacuum mode [14] and here one also is normalizable, by the same arguments.

V. GRAVITY WAVES AND A BUBBLE WITH FINITE ENERGY DIFFERENCE

In this section we take into account the effects of the nonvanishing size of the critical bubble and of the nonvanishing difference in energy density across the bubble wall. In the presence of a bubble, one starts with a Bunch-Davies vacuum with $H=H_F$ as an initial condition outside the bubble and then continues these modes across the bubble wall into the open universe, where we idealize the expansion rate to be a constant with $H=H_T$. Of course, in realistic single-bubble inflationary models H inside the bubble is neither constant, nor does it correspond to the true vacuum as the subscript T suggests. However, in order to capture the qualitative consequences of changing H across the bubble wall without getting involved in the messy details of specific models, we consider the idealization of an infinitely thin wall with constant $H=H_T$ inside for calculating the gravitational waves. Whereas earlier in the paper we set H to unity, from here on factors of H will be displayed explicitly. Given that the gravity waves and the scalars have the same time dependence, and this is the only place the dependence on the wall appears, one can carry over the results for scalar fields. In the case at hand, we have a massless field before and after tunneling and a finite energy difference across the wall. The specific relevant results are from [18,29,21] and are briefly sketched below.

As will be shown below, any bubble with $H_F \neq H_T$ has a finite radius and extends outside the light cone (region I) covered by the (η,ξ,θ,ϕ) coordinate system. By considering a curve of constant field value outside the light cone in Fig. 1(a), one sees that the matching across the wall thus has to be done in region II and its Euclidean continuation. The continuation into region II from region I is $e^{\eta}\to -ie^{-u}$, $\xi\to\tau+i\pi/2$, and so the metric in region II becomes

$$ds^2=a^2(u)[du^2-d\tau^2+\cosh^2\,\tau d\Omega^2_{(2)}]. \tag{5.1}$$

For an open universe $a(u)=1/(H\cosh[u])$. The analytic continuation of the basis functions is given in somewhat more detail in Appendix B and is straightforward. With the Euclidean continuation appropriate for tunneling, one also, before nucleation, rotates the space to Euclidean time $\tau\to i\tau_E$.

The wall has the same description in the Euclidean and Lorentzian regions of spacetime, since it only depends on u and not τ, τ_E. For a thin wall, the u dependent scale factor is [30]

$$a(u)=\begin{cases}\dfrac{1}{H_F\cosh[u]} & \text{for } u<u_R,\\[2ex] \dfrac{1}{H_T\cosh[u+\delta]} & \text{for } u>u_R,\end{cases} \tag{5.2}$$

where

$$\frac{1}{H_T\cosh[u+\delta]}=\frac{1}{H_F[\cosh[u-u_R]\cosh u_R+\sinh[u-u_R]\sqrt{\cosh^2[u_R]-(H_T/H_F)^2}]}. \tag{5.3}$$

The metrics are explicitly matched at the same value of $a(u)$, the bubble radius

$$R=\frac{1}{H_F\cosh[u_R]}=\frac{1}{H_T\cosh[u_R+\delta]}. \tag{5.4}$$

The mode functions in this background have only their u dependence changing across the wall. Requiring continuity in the unchanged (τ,θ,ϕ) dependence forces the exterior mode functions to match onto interior modes with the same value of ζ^2. To match the mode functions across this wall, we start with the initial u dependence corresponding to the analytic continuation of the time (η) dependence found in Eq. (2.11). As the exterior of the bubble has $H=H_F$, the false vacuum mode functions $e^{\pi\zeta/2}T_h(\eta\zeta)$ extend to region II as

$$A_\zeta(u)=H_F\cosh[u]e^{i\zeta u}(\tanh[u]-i\zeta) \tag{5.5}$$

up to a factor of i. These are matched across the wall onto modes of the *true* vacuum (as there is some slow roll after tunneling this is not exactly the true vacuum but is close enough for our purposes). The corresponding mode functions interior to the bubble, with $u>u_R$, $H=H_T$, are

$$B_\zeta(u+\delta)=H_T\cosh[u+\delta]e^{i\zeta(u+\delta)}(\tanh[u+\delta]-i\zeta). \tag{5.6}$$

Matching these and their first derivatives at the wall at $a(u_R)=R$ gives

[4] J.D.C. thanks K. Schleich for discussions about this.

$$A_\zeta(u)=\alpha_\zeta B_\zeta(u+\delta)+\beta_\zeta B_{-\zeta}(u+\delta), \tag{5.7}$$

where

$$\alpha_\zeta=\frac{2i\zeta-z}{2i\zeta}\,e^{-i\zeta\delta},\qquad \beta_\zeta=\frac{z}{2i\zeta}\,e^{i\zeta(2u_R+\delta)}, \tag{5.8}$$

and

$$z=\tanh[u_R]-\tanh[u_R+\delta]=\sqrt{1-(H_FR)^2}-\sqrt{1-(H_TR)^2}. \tag{5.9}$$

Note that these become trivial in the limit $\delta\to0$, that is, when the energy difference $(H_F-H_T)\to0$. From the definition of z, it appears possible that $z=0$ is possible even if $H_F\neq H_T$, via taking $R=0$. However, within the *new thin wall* approximation of [10], $R\propto\sqrt{H_F^2-H_T^2}$ with a nonzero constant of proportionality, and in the usual thin wall approximation [31] one has that the radius obeys

$$R^2=\frac{S_1^2}{c_0(H_F^2-H_T^2)+c_1S_1^2(H_F^2+H_T^2)+c_2S_1^4}, \tag{5.10}$$

where S_1 is the surface tension (related to the integral of the change in the background field through the wall) and c_0,c_1,c_2 are positive constants. Since both H_F and H_T are finite and the surface tension does not vanish, R is finite.

One actually has to go to another coordinate system in order to show that there is a ''time'' where there is only false vacuum and no bubble [30] for initial conditions. The consequence here is that the mode functions must be orthonormalized once they are continued across the bubble wall, as they do not correspond to a normalized initial mode functions in this other ''time.'' Thus the mode functions given in Eq. (5.7) are not yet properly orthonormalized. At the nucleation time (conventionally taken as the $\tau=0$ slice in region II), the whole system is rotated to a Lorentzian signature. The mode functions can then be orthonormalized in region II, or since these are subcurvature modes, one can continue them to regions I and V and orthonormalize them there. The latter was done in detail in [18,29] and the former was done in detail in [21]. Substituting the tensor rather than scalar spatial functions into these results, in region II the wave function is (again up to overall factors of i)

$$\begin{aligned}h^r_s(\xi,\theta,\phi,\eta)=\sum_{jm}\int_0^\infty &\frac{d\zeta}{\sqrt{4\zeta(\zeta^2+1)\sinh\pi\zeta}}\,[\mathbf{T}^{Ejm}(\tau+i\pi/2,\theta,\phi,\eta;\zeta)]^\tau{}_S\\ &\times\{\hat b_I(\zeta,E,j,m)\{[C(+\zeta)\alpha_\zeta+S(+\zeta)\beta_{-\zeta}]B_\zeta(u)+[C(+\zeta)\beta_\zeta+S(+\zeta)\alpha_{-\zeta}]B_{-\zeta}(u)\}\\ &+\hat b_V(\zeta,E,j,m)\{[C(-\zeta)\alpha_{-\zeta}+S(-\zeta)\beta_\zeta]B_{-\zeta}(u)+[C(-\zeta)\beta_{-\zeta}+S(-\zeta)\alpha_\zeta]B_\zeta(u)\}\}.\end{aligned} \tag{5.11}$$

As a result, the positive frequency part of the wave function inside region I for subcurvature modes is

$$\begin{aligned}h^\tau_s(\xi,\theta,\phi,\eta)=\sum_{jm}\int_0^\infty &\frac{d\zeta}{\sqrt{2\zeta(\zeta^2+1)}}\,[\mathbf{T}^{Ejm}(\xi,\theta,\phi,\eta;\zeta)]^r{}_s\Bigg\{\hat b_I(\zeta,E,j,m)\\ &\times\Bigg[\{C(+\zeta)\alpha_\zeta+S(+\zeta)\beta_{-\zeta}\}\frac{e^{+\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}T_h(\eta;+\zeta)+\{C(+\zeta)\beta_\zeta+S(+\zeta)\alpha_{-\zeta}\}\frac{e^{-\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}T_h(\eta;-\zeta)\Bigg]\\ &+\hat b_V(\zeta,E,j,m)\Bigg[\{C(-\zeta)\alpha_{-\zeta}+S(-\zeta)\beta_\zeta\}\,\frac{e^{-\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;-\zeta)\\ &+\{C(-\zeta)\beta_{-\zeta}+S(-\zeta)\alpha_\zeta\}\,\frac{e^{+\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;+\zeta)\Bigg]\Bigg\},\end{aligned} \tag{5.12}$$

where

$$D_1(\zeta)=\frac{1}{2}\,[|\alpha_\zeta|^2+|\beta_\zeta|^2+1]=D_1(-\zeta)=1+\frac{z^2}{4\zeta^2},\qquad D_2(\zeta)=\alpha_\zeta\beta_\zeta=\frac{-z}{4\zeta^2}\,(2i\zeta-z)e^{2i\zeta u_R}, \tag{5.13}$$

and

$$C(\zeta)=\sqrt{\frac{D_1}{D_1^2-|D_2|^2}}\;\sqrt{\frac{1}{2}\left(1+\sqrt{1-\frac{|D_2|^2}{D_1^2}}\right)}=\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{4\zeta^2}{z^2+4\zeta^2}}\right)}, \tag{5.14}$$

$$S(\zeta)=\frac{-D_2}{|D_2|}\sqrt{\frac{D_1}{D_1^2-|D_2|^2}}\sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{|D_2|^2}{D_1^2}}\right)}$$

$$=\frac{-D_2}{|D_2|}\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{4\zeta^2}{z^2+4\zeta^2}}\right)}.$$

The identity ${D_1}^2-|D_2|^2=D_1$ was used in the above and can be checked by substituting into the definitions.

For scalars, an analogue of the vacuum supercurvature mode in the presence of a bubble, with $\zeta=-i$, remains normalizable when matched across the wall [29]. This is true for the case here as well, but as this mode is η independent, it does not contribute to the CMB anisotropy below. Consequently, we do not consider it further here.

VI. IMPLICATIONS FOR THE CMB ANISOTROPY

Gravitational waves provide a time-dependent background for the photons in the CMB [32] (for a review see [33]). The contribution of gravitational waves to the CMB anisotropy is given by the integral in the Sachs-Wolfe formula [32]

$$\frac{\delta T_{\rm GW}(\Omega)}{T}=\frac{1}{2}\int_{\eta_e}^{\eta_0}d\eta\;\hat r^a\hat r^b\tilde h_{ab,\eta}(\xi=\eta_0-\eta,\eta)$$

$$=\frac{1}{2}\int_{\eta_e}^{\eta_0}d\eta\,\frac{\partial}{\partial\eta}\,\tilde h_{\hat\xi\hat\xi}(\xi=\eta_0-\eta,\eta). \qquad (6.1)$$

The metric $\tilde h$ is defined with the conformal factor scaled out, hence $\tilde h_{\mu\nu}=h_{\mu\nu}/a^2(\eta)$ in our notation. Here we have chosen the path to be parametrized by conformal time, where η_0 is the observing time and η_e is the last scattering time for the photon. The surface term vanishes for the tensor contribution and hence is omitted. The radial-radial component of the mode function has spatial dependence proportional to $F_j(\xi;\zeta)$, as seen in Eq. (3.4).

The Bunch-Davies positive frequency part of the temperature contrast operator is, using Eq. (5.12) from the last section,

$$\frac{\delta T^{(+)}_{\rm GW}(\Omega)}{T_{\rm CMB}}=\frac{1}{2}\sum_{jm}Y_{jm}(\Omega)\int_0^\infty d\zeta\,\frac{1}{\sqrt{2\zeta(\zeta^2+1)}}\int_{\eta_e}^{\eta_0}d\eta F_j(\xi=\eta_0-\eta;\zeta)\Bigg\{\hat b_I(\zeta,E,j,m)\Bigg[\{C(+\zeta)\alpha_\zeta+S(+\zeta)\beta_{-\zeta}\}$$
$$\times\frac{e^{+\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,\dot T_h(\eta;+\zeta)+\{C(+\zeta)\beta_\zeta+S(+\zeta)\alpha_{-\zeta}\}\,\frac{e^{-\pi\xi/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,\dot T_h(\eta;-\zeta)\Bigg]$$
$$+\hat b_V(\zeta,E,j,m)\Bigg[\{C(-\zeta)\alpha_{-\zeta}+S(-\zeta)\beta_\zeta\}\,\frac{e^{-\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,\dot T_h(\eta;-\zeta)$$
$$+\{C(-\zeta)\beta_{-\zeta}+S(-\zeta)\alpha_\zeta\}\,\frac{e^{+\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,\dot T_h(\eta;+\zeta)\Bigg]\Bigg\}, \qquad (6.2)$$

where the dots indicate a derivative with respect to conformal time. The Bunch-Davies negative frequency component is

$$\frac{\delta T^{(-)}_{\rm GW}(\Omega)}{T_{\rm CMB}}=\left[\frac{\delta T^{(+)}_{\rm GW}(\Omega)}{T_{\rm CMB}}\right]^\dagger. \qquad (6.3)$$

It is customary to expand the CMB anisotropy in terms of multipoles according to

$$\frac{\delta T_{\rm GW}(\Omega)}{T_{\rm CMB}}=\sum_{lm}a_{lm}Y_{lm}(\Omega). \qquad (6.4)$$

The statistical average of the ensemble of classical gravity waves is found by taking the corresponding quantum average, so that the two-point correlation is

$$c_l=\langle|a_{lm}|^2\rangle$$
$$=\frac{1}{8}\int_0^\infty\frac{d\zeta}{\zeta(\zeta^2+1)}\int_{\eta_e}^{\eta_0}d\eta_1\int_{\eta_e}^{\eta_0}d\eta_2F_l(\xi=\eta_0-\eta_1;\zeta)$$
$$\times F_l(\xi=\eta_0-\eta_2;\zeta)\Bigg\{\coth[\pi\zeta]{\rm Re}[\dot T_h(\eta_1;\zeta)\dot T_h(\eta_2;\zeta)^*]$$
$$+\frac{1}{D_1\,\sinh[\pi\zeta]}\,{\rm Re}[\alpha_\zeta\beta_{-\zeta}\dot T_h(\eta_1;\zeta)\dot T_h(\eta_2;\zeta)]\Bigg\} \qquad (6.5)$$

for all l. In showing this, it is useful to note that $|C(\zeta)|^2+|S(\zeta)|^2=1$ and the definitions in Eq. (5.13). Also since $D_1=\frac{1}{2}[1+|\alpha_\zeta|^2+|\beta_\zeta|^2]$ and $|\alpha_\zeta|^2-|\beta_\zeta|^2=1$,

$$\left|\frac{2\alpha_\zeta\beta_{-\zeta}}{1+|\alpha_\zeta|^2+|\beta_\zeta|^2}\right|\leqslant +1. \tag{6.6}$$

Consequently as one probes larger wave numbers ζ (through larger-l multipoles) the influence of the bubble dynamics quickly becomes negligible in a uniform manner. The dependence on Ω_0 appears in η_0, η_e, and in ζ. For an open universe, $\tanh(\eta_0/2)=\sqrt{1-\Omega_0}$ and $\sinh(\eta_0/2)=\sqrt{1+z_{\text{last scattering}}}\sinh(\eta_e/2)$ [34]. As Ω approaches one, η_0,η_e decrease as does their difference. In addition, the momentum ζ is measured in terms of the curvature scale $\zeta H_0\sqrt{1-\Omega_0}=k$. As a result, for large Ω, large values of ζ dominate the integral in terms of physical momentum and $F_l(\xi;\zeta)$ and $T_h(\eta;\zeta)$ are evaluated for small ξ,η, where they approach their flat space counterparts.

The integrand in Eq. (6.5) for the CMB multipole moments is well behaved for small ζ. This can be demonstrated with the explicit form of $F_2(\xi;\zeta)$ in Eq. (3.17). (Recall that for gravity waves the $l=0$, 1 moments vanish because the graviton is a spin-two particle.) One then notes that F_j for higher j is obtained by taking derivatives, which will not alter the leading power of ζ (although the coefficients may change). As $\zeta\to 0$, the time-dependence factor in the basis functions becomes

$$\begin{aligned}\partial_\eta T_h(\eta,\zeta)]&=\partial_\eta[(i\zeta\sinh[\eta]+\cosh[\eta])e^{-i\zeta\eta}]\\&\to H_T\sinh[\eta](1-i\zeta\eta)+O(\zeta^2).\end{aligned} \tag{6.7}$$

The first term in the curly brackets has the form

$$\begin{aligned}&\coth[\pi\zeta]\mathrm{Re}[\dot T_h(\eta_1;\zeta)\dot T_h(\eta_2;\zeta)^*]\\&\quad=\left(\frac{1}{\pi\zeta}\right)H_T^2\sinh^2[\eta][1+O(\zeta^2)]\end{aligned} \tag{6.8}$$

since $\mathrm{Re}[1-i\zeta(\eta_1-\eta_2)]=1$. For the second term in the curly brackets, for small ζ,

$$\begin{aligned}&\frac{1}{D_1\sinh[\pi\zeta]}\mathrm{Re}[\alpha_\zeta\beta_{-\zeta}\dot T_h(\eta_1;\zeta)\dot T_h(\eta_2;\zeta)]\\&\quad=\left[1+\frac{z^2}{4\zeta^2}\right]^{-1}\frac{1+O(\zeta^2)}{\pi\zeta}\\&\qquad\times\mathrm{Re}[\alpha_\zeta\beta_{-\zeta}\dot T_h(\eta_1;\zeta)\dot T_h(\eta_2;\zeta)].\end{aligned} \tag{6.9}$$

The argument in brackets on the right has the form

$$\begin{aligned}&\mathrm{Re}\left[\frac{2i\zeta-z}{2i\zeta}e^{-i\zeta(2\delta+2u_r)}\frac{-z}{2i\zeta}[1-i\zeta(\eta_1+\eta_2)\right.\\&\qquad\left.+O(\zeta^2)]\right]H_T^2\sinh^2[\eta]\\&\quad=-\frac{z^2}{4\zeta^2}[1+O(\zeta^2)]H_T^2\sinh^2[\eta]\end{aligned} \tag{6.10}$$

including the small-ζ expansion of the exponentials. Thus, after factoring out the $H_T^2\sinh^2[\eta]$ dependence, the term in curly brackets in Eq. (6.5) behaves as

$$\frac{1}{\pi\zeta}+\frac{-z^2/4\zeta^2+O(1)}{\pi\zeta(1+z^2/4\zeta^2)}\sim\zeta \tag{6.11}$$

for ζ small. In the Sachs-Wolfe integral above, we also have the factor of ζ^{-1} in the measure. Since $N_j\sim\zeta^{-2}$ one has [using Eq. (3.17) in conjunction with Eq. (3.27)] $F_j\sim$const. This implies that the integrand approaches a constant as $\zeta\to 0$, rendering the integral infrared convergent.

As (H_F-H_T) becomes small, both $\beta_{-\zeta}$ and z approach zero for fixed ζ, and the second term in curly brackets approaches zero. For vanishing z, as is found in the vacuum, Eq. (6.11) approaches $\coth[\pi\zeta]$, making the integrand appear to diverge as $\sim\zeta^{-2}$. We do not have an intuitive understanding of this limiting behavior. Vanishing z is never the case in the presence of the bubble since it implies exactly zero energy difference across the bubble wall.

In order to calculate the a_{lm}, one needs the time dependence of the wave functions from the inflationary period, through radiation domination and into the current epoch of matter domination. Matching conditions have been found by [22] and are calculated analogously to the flat [1–7] and closed [35] universe cases. Time dependence has also been considered in [36] but seemingly for a different initial vacuum.

VII. DISCUSSION

We have determined the initial condition for the graviton field in an open universe originating from a bubble inflation model and calculated the contribution from gravitational waves to the CMB anisotropy. The total observed CMB anisotropy for a given multipole is obtained by combining the gravity wave contribution calculated here with the scalar field contributions for the particular model. The effects of the bubble wall for the tensors, just as for the scalars, seems confined mostly to very large scales, corresponding to small ζ. It appears that the effects of curvature provide a larger effect for the tensors than the effects of the wall [37].

Noted added. After this work was completed we learned that Allen and Caldwell [38] and Sasaki *et al.* [39] have reached similar conclusions with respect to the influence of nonvanishing bubble size. Including the results of [39] and the unpublished version of this paper, [37] integrated the Sachs-Wolfe formula to get numerical predictions for the C_l.

ACKNOWLEDGMENTS

We would like to thank R. Brandenberger, P. Ferreira, L. Ford, A. Guth, A. Liddle, A. Linde, B. Ratra, K. Schleich, N. Turok, and A. Vilenkin for useful discussions, and especially B. Allen and R. Caldwell for useful discussions and for sharing their prior unpublished manuscript [22] with us. We thank R. Caldwell for comments on the draft. J.D.C. is grateful in particular to M. White for numerous discussions and was supported by an ONR grant. J.D.C. also thanks the University of British Columbia, the Harvard-Smithsonian Center for Astrophysics, and the Center for Particle Astrophysics, the Physics Department, and LBNL at Berkeley for hospitality in the course of this work. M.B. was supported by the David and Lucille Packard Foundation and by National Science Foundation Grant No. PHY 9309888.

APPENDIX A: RELATION BETWEEN FLAT AND HYPERBOLIC COORDINATES

For the flat coordinates the line element is (setting $H=1$)

$$ds^2=-dt_f^2+e^{2t_f}[dr_f^2+r_f^2 d\Omega_{(2)}^2], \tag{A1}$$

where $-\infty<t_f<+\infty$, or in terms of flat conformal time $\eta_f=-e^{-t_f}$

$$ds^2=\frac{1}{\eta_f^2}[-d\eta_f^2+dr_f^2+r_f^2 d\Omega_{(2)}^2]. \tag{A2}$$

To relate the flat and hyperbolic coordinates one can embed de Sitter space in (4+1)-dimensional Minkowski space (see Ref. [10], Sec. V or Ref. [28]). The Minkowski coordinates are $(\overline{w,u,x,y,z})=(\overline{w,u,r})$ and de Sitter space is defined by

$$\overline{r}^2+\overline{u}^2-\overline{w}^2=1. \tag{A3}$$

The embedding of the open hyperbolic coordinates is

$$\begin{aligned}\overline{w}&=\sinh[t_h]\cosh[\xi],\\ \overline{u}&=\cosh[t_h],\\ \overline{r}&=\sinh[t_h]\sinh[\xi].\end{aligned} \tag{A4}$$

Generally, $\overline{r}>0$, and as $0<t_h<\infty$ and $0<\xi<\infty$, so that one sees that region I hyperbolic coordinates cover the range $0\leqslant\overline{w}\leqslant\infty$, $1\leqslant\overline{u}$. The flat coordinates are embedded in (4+1)-dimensional Minkowski space according to

$$t_f=\ln[\overline{w}+\overline{u}], \quad \eta_f=\frac{-1}{\overline{w}+\overline{u}}, \quad r_f=\frac{\overline{r}}{\overline{w}+\overline{u}}. \tag{A5}$$

These cover $\overline{w}$, $\overline{u}>0$, a larger region than the hyperbolic open coordinates. As $\sinh[t_h]=-1/\sinh[\eta]$, $\cosh[t_h]=-\coth[\eta]$, and $\coth[t_h]=\cosh[\eta]$, the relation between the coordinate systems is

$$\begin{aligned}r_f&=\frac{\sinh[\xi]}{\cosh[\xi]+\cosh[\eta]},\\ \eta_f&=\frac{\sinh[\eta]}{\cosh[\eta]+\cosh[\xi]}.\end{aligned} \tag{A6}$$

The flat coordinates do not cover all of de Sitter space but only half of maximally extended de Sitter space, as indicated in the conformal diagram in Fig. 3 in the text.

The null coordinates in the flat chart are

$$\begin{aligned}u_f&=\eta_f+r_f,\\ v_f&=\eta_f-r_f.\end{aligned} \tag{A7}$$

One has $v_f<0$ and $|u_f|<|v_f|$. We also define the region I hyperbolic null coordinates

$$\begin{aligned}U&=\eta+\xi,\\ V&=\eta-\xi,\end{aligned} \tag{A8}$$

where $V<0$ and $|U|<|V|$. Using the above relations between (η,ξ) and (η_f,r_f) one can show that the two sets of null coordinates are related according to

$$U=\ln\left[\frac{1+u_f}{1-u_f}\right], \quad V=\ln\left[\frac{1+v_f}{1-v_f}\right]. \tag{A9}$$

Conversely, $u_f=\tanh(U/2)$ and $v_f=\tanh(V/2)$. Thus we see that the hyperbolic coordinates cover only the region $|u_f|$, $|v_f|\leqslant 1$, $|u_f|\leqslant|v_f|$ in terms of the flat coordinates.

In order to see the continuation into region V, consider the transformation $u\to -u$ in the embedding Minkowski space. This can be accomplished by taking $t_h=-t_V-i\pi$. In terms of conformal time in region V, this is

$$\begin{aligned}\eta_V&=\ln[\tanh(-t_h/2+i\pi/2)]\\ &=\ln(-1)+\ln[\tanh(t_h/2-i\pi/2)]=\ln(-1)-\eta_h.\end{aligned} \tag{A10}$$

Thus we have

$$\begin{aligned}\tilde{U}&=\eta_V+r_V=\ln(-1)-V,\\ \tilde{V}&=\eta_V-r_V=\ln(-1)-U,\end{aligned} \tag{A11}$$

where $\tilde{U},\tilde{V}$ are the hyperbolic open coordinates in region V. The definition of $\ln(-1)$ requires a choice of analytic continuation, which has been identified in terms of u_f, v_f, so converting to these coordinates, one has

$$\begin{aligned}u_f&=\tanh(U/2)=\tanh(-\tilde{V}/2-\pm i\pi/2)\\ &=-\coth(\tilde{V}/2)=-\tilde{v}_f^{-1},\\ v_f&=\tanh(V/2)=\tanh(-\tilde{U}/2-\pm i\pi/2)\\ &=-\coth(\tilde{U}/2)=-\tilde{u}_f^{-1}\end{aligned} \tag{A12}$$

as given in the text. This transformation relates the flat coordinates (u_f,v_f) covering the upper left triangular wedge of de Sitter space to the coordinates $(\tilde{u}_f,\tilde{v}_f)$ covering the upper right triangular wedge, as indicated in Fig. 5.

APPENDIX B: INNER PRODUCT IN REGION II

For this appendix, $H=1$. To continue into region II, $e^{\eta}\to -ie^{-u}$, $\xi\to\tau+i\pi/2$, and the metric becomes

$$ds^2=a^2(u)[du^2-d\tau^2+\cosh^2[\tau]d\Omega^2_{(2)}], \qquad \text{(B1)}$$

where $a(u)=1/\cosh[u]$. The Wronskian, with respect to $-i\nabla_\tau$ on the symmetric perturbations of the metric, gives for an inner product

$$-\int_\Sigma \frac{du\ d\Omega\ \cosh^2[\tau]}{\cosh^2[u]}\,\mathcal{U}^B_A(\tau,\Sigma;\zeta,j,m)\times(i\overleftrightarrow{\partial}_\tau)\mathcal{U}^{A*}_B(\tau,\Sigma;\zeta',j',m'). \qquad \text{(B2)}$$

The Bunch-Davies vacuum mode expansion [see Eq. (4.12)] in region I,

$$\sum_{Pjm}\int_0^\infty \frac{d\zeta}{\sqrt{2\zeta(\zeta^2+1)}}\,[\mathbf{T}^{Ejm}(\xi,\theta,\phi,\eta;\zeta)]^r_s\Bigg[\frac{e^{+\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;\zeta)\hat{b}_I(\zeta,P,j,m)+\frac{e^{-\pi\zeta/2}}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h(\eta;-\zeta)\hat{b}_V(\zeta,P,j,m)\Bigg]+\text{H.c.}, \qquad \text{(B3)}$$

becomes, in region II,

$$\sum_{PJm}\int_0^\infty d\zeta\Bigg\{\frac{e^{+\pi\zeta/2}n(\zeta)}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h[\ln(-ie^{-u});\zeta]T^B_A\left(\tau+i\,\frac{\pi}{2},\theta,\phi;\zeta,j,m\right)\hat{b}_I(\zeta,P,J,m)+\frac{e^{-\pi\zeta/2}n(\zeta)}{(e^{+\pi\zeta}-e^{-\pi\zeta})^{1/2}}\,T_h[\ln(-ie^{-u});-\zeta]\times\left[T^B_A\left(\tau+i\,\frac{\pi}{2},\theta,\phi;\zeta,j,m\right)\right]\hat{b}_V(\zeta,P,J,m)\Bigg\}+\text{H.c.}$$
$$=\sum_{PJm}\int_0^\infty d\zeta\{\mathcal{U}^B_A(u,\tau,\theta,\phi;\zeta,j,m)\hat{b}_I(\zeta,P,J,m)+\mathcal{U}^B_A(u,\tau,\theta,\phi;-\zeta,j,m)\hat{b}_V(\zeta,P,J,m)+\text{H.c.}\}. \qquad \text{(B4)}$$

Note that the complex conjugate is only taken in the u dependence in region II and both terms have the same τ dependence corresponding to positive frequency. The integral over the surface Σ in region II is an integral over the analytic continuation of time u and over (θ,ϕ). The analytic continuation of $T_h(\eta)$ multiplies the whole expression and is independent of τ. So for calculating the Wronskian, first consider only the τ dependence and the integral over θ,ϕ. The integral over u will be done subsequently. We have

$$\tilde{\mathbf{T}}^{E,jm}(\tau,\theta,\varphi;\zeta)=F_j(\tau+i\pi/2;\zeta)(\mathbf{e}^\tau\otimes\mathbf{e}^\tau)Y_{jm}(\theta,\varphi)+G_j(\tau+i\pi/2;\zeta)\delta_{\tilde{a}\tilde{b}}(\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\tilde{b}})Y_{jm}(\theta,\varphi)+H_j(\tau+i\pi/2;\zeta)(\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^\tau+\mathbf{e}^\tau\otimes\mathbf{e}^{\tilde{a}})\tilde{\nabla}_{\tilde{a}}Y_{jm}(\theta,\varphi)+I_j(\tau+i\pi/2;\zeta)(\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\tilde{b}})\tilde{\nabla}_{\tilde{a}}\tilde{\nabla}_{\tilde{b}}Y_{jm}(\theta,\varphi). \qquad \text{(B5)}$$

Thus,

$$T^\tau_\tau=-F_j(\tau+i\pi/2;\zeta)Y_{jm},$$
$$T^\tau_{\tilde{a}}=-H_j(\tau+i\pi/2;\zeta)\tilde{\nabla}_{\tilde{a}}Y_{jm},$$
$$T^{\tilde{a}}_\tau=\cosh^{-2}[\tau]H_j(\tau+i\pi/2;\zeta)\tilde{\nabla}^{\tilde{a}}Y_{jm},$$
$$T^{\tilde{a}}_{\tilde{b}}=\cosh^{-2}[\tau][I_j(\tau+i\pi/2;\zeta)\tilde{\nabla}^{\tilde{a}}\tilde{\nabla}_{\tilde{b}}Y_{jm}+\delta^{\tilde{a}}_{\tilde{b}}G_j(\tau+i\pi/2;\zeta)Y_{jm}]. \qquad \text{(B6)}$$

As we have $\nabla^2Y_{jm}=-j(j+1)Y_{jm}$ and from earlier

$$\int d\Omega(\nabla_{\tilde{a}}\nabla_{\tilde{b}}Y_{jm})^*\nabla^{\tilde{a}}\nabla^{\tilde{b}}Y_{j'm'}=-\int d\Omega(\nabla^{\tilde{a}}\nabla_{\tilde{a}}\nabla_{\tilde{b}}Y_{jm})^*(\nabla^{\tilde{b}}Y_{j'm'})$$
$$=-\int d\Omega(\nabla_{\tilde{a}}\nabla_{\tilde{b}}\nabla^{\tilde{a}}Y_{jm})^*(\nabla^{\tilde{b}}Y_{j'm'})$$
$$=-\int d\Omega\left(\nabla_{\tilde{b}}\left\{\nabla^2+\frac{1}{2}\,R\right\}Y_{jm}\right)^*(\nabla^{\tilde{b}}Y_{j'm'})$$
$$=[j(j+1)][j(j+1)-1]\delta_{jj'}\delta_{mm'}, \qquad \text{(B7)}$$

we obtain immediately that

$$\int d\Omega T^A_B \overleftrightarrow{\partial}_\tau T^{B*}_A = \{F_j \overleftrightarrow{\partial}_\tau F'_j - 2j(j+1)\cosh^{-2}[\tau] H_j \overleftrightarrow{\partial}_\tau H^*_j + \cosh^{-4}[\tau][I_j \overleftrightarrow{\partial}_\tau I^*_j (j(j+1))(j(j+1)-1) + 2G_j \overleftrightarrow{\partial}_\tau G^*_j - j(j+1) \times (G_j \overleftrightarrow{\partial}_\tau I^*_j + I_j \overleftrightarrow{\partial}_\tau G^*_j)]\} \delta_{jj'}\delta_{mm'}$$

$$= \left\{ \frac{3}{2} F_j \overleftrightarrow{\partial}_\tau F^*_j - 2j(j+1)\cosh^{-2}[\tau] H_j \overleftrightarrow{\partial}_\tau H^*_j + \frac{1}{2}\cosh^{-4}[\tau] J^2 (J^2-2) I_j \overleftrightarrow{\partial}_\tau I^*_j \right\} \delta_{jj'}\delta_{mm'}. \tag{B8}$$

The sign in front of H_j is due to the negative signature of τ. There is a factor of 2 in front of both H_j and G_j. For H_j there is both a term T_{1a} as well as a term T_{a1}, and for G_j there is a contribution from both G_2 and G_3. The minus sign in front of the last term is due to the negative sign relating ∇^2 and J^2.

After some algebra, one finds that

$$\int d\Omega T^A_B \overleftrightarrow{\partial}_\tau T^{B*}_A = \frac{2\zeta^2(1+\zeta^2)\cosh^4[\tau]}{j(j+1)(j+2)(j-1)} F_j \overleftrightarrow{\partial}_\tau F^*_j \delta_{jj'}\delta_{mm'}. \tag{B9}$$

After some work one can show that

$$F_j \overleftrightarrow{\partial}_\tau F^*_j = N_j(\zeta)^2 \frac{i \sinh[\pi\zeta]}{\cosh^6[\tau]} \zeta^3 (1+\zeta^2)(4+\zeta^2)\cdots(j^2+\zeta^2). \tag{B10}$$

Putting this all together and substituting for $N_j(\zeta)^2$, we get

$$\int d\Omega T^A_B \overleftrightarrow{\partial}_\tau T^{B*}_A = \frac{2i\zeta \sinh[\pi\zeta]}{\pi \cosh^2[\tau]} \delta_{jj'}\delta_{mm'}. \tag{B11}$$

The integral over u remains, with mode functions

$$T_h[\ln(-ie^{-u});\zeta] \frac{n(\zeta)e^{\pi\zeta/2}}{\sqrt{(e^{+\pi\zeta}-e^{-\pi\zeta})}} = (i\zeta - \tanh[u])e^{i\zeta u} \frac{\cosh[u]n(\zeta)}{\sqrt{(e^{\pi\zeta}-e^{-\pi\zeta})}}, \tag{B12}$$

so the last integral, including the rest of the measure, becomes

$$\int du \frac{\cosh^2[\tau]}{\cosh^2[u]} \frac{\cosh[u](i\zeta-\tanh[u])n(\zeta)}{\sqrt{2\sinh[\pi\zeta]}} \frac{\cosh[u](-i\zeta'-\tanh[u])n(\zeta')}{\sqrt{2\sinh[\pi\zeta']}} e^{i(\zeta-\zeta')u} = \frac{2\pi\cosh^2[\tau]}{4\zeta\sinh[\pi\zeta]} \delta(\xi-\xi'). \tag{B13}$$

As a result, we have

$$-\int_\Sigma du \cosh^{-2}[u]\cosh^2[\tau] \mathcal{U}^{\ B}_{A}(u,\Sigma;\zeta,j,m)(i\overleftrightarrow{\partial}_\tau)\mathcal{U}^{\ A*}_{B}(u,\Sigma;\zeta',j',m') = \delta(\zeta-\zeta')\delta_{jj'}\delta_{mm'} \tag{B14}$$

which agrees with Eq. (3.34).

---

[1] L. Grishchuk, Lett. Nuovo Cimento **12**, 60 (1975); **12**, 432(E) (1975); L. Grishchuk, Zh. Eksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys. JETP **40**, 409 (1975)].

[2] A. Starobinsky, Phys. Lett. **91B**, 99 (1980); A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 719 (1979) [JETP Lett. **30**, 682 (1979)].

[3] L. Abbott and M. Wise, Nucl. Phys. **B244**, 541 (1984); L. Abbott and M. Wise, Phys. Lett. **135B**, 279 (1984).

[4] L. Abbott and R. Schaefer, Astrophys. J. **308**, 462 (1986).

[5] L. Abbott and D. Harari, Nucl. Phys. **B264**, 487 (1986).

[6] M. White, Phys. Rev. D **46**, 4198 (1992); B. Allen and S. Koranda, *ibid.* **50**, 3713 (1994).

[7] L. Krauss and M. White, Phys. Rev. Lett. **69**, 869 (1992); R. Crittenden, J. R. Bond, R. Davis, G. Efstathiou, and P. Steinhardt, *ibid.* **71**, 324 (1993).

[8] J. R. Gott III, Nature (London) **295**, 304 (1982); J. R. Gott III and T. Statler, Phys. Lett. **136B**, 157 (1984); J. R. Gott III, in *Inner Space/Outer Space*, edited by E. W. Kolb *et al.* (University of Chicago Press, Chicago, 1986).

[9] S. Coleman and F. de Luccia, Phys. Rev. D **21**, 3305 (1980).

[10] M. Bucher, A. S. Goldhaber, and N. Turok, Phys. Rev. D **52**, 3314 (1995); M. Bucher and N. Turok, *ibid.* **52**, 5538 (1995).

[11] M. Sasaki, T. Tanaka, K. Yamamoto, and J. Yokoyama, Prog. Theor. Phys. **90**, 1019 (1993); Phys. Lett. B **317**, 510 (1993).

[12] A. Linde, Phys. Lett. B **351**, 99 (1995); A. Linde and A. Mezhlumian, Phys. Rev. D **52**, 6789 (1995); L. Amendola, C. Baccigalupi, and F. Occhionero, *ibid.* **54**, 4760 (1996); A. Green and A. Liddle, *ibid.* **55**, 609 (1997); J. Garcia-Bellido and A. Liddle, *ibid.* **55**, 4603 (1997).

[13] B. Allen, Phys. Rev. D **32**, 3136 (1985).

[14] M. Sasaki, T. Tanaka, and K. Yamamoto, Phys. Rev. D **51**, 2979 (1995).
[15] D. Lyth and E. Stewart, Phys. Lett. B **252**, 336 (1990).
[16] B. Ratra and P. J. E. Peebles, Phys. Rev. D **52**, 1837 (1995); B. Ratra and P. J. E. Peebles, Astrophys. J. Lett. **432**, L5 (1994).
[17] K. Yamamoto, T. Tanaka, and M. Sasaki, Phys. Rev. D **51**, 2968 (1995).
[18] K. Yamamoto, T. Tanaka, and M. Sasaki, Phys. Rev. D **54**, 5031 (1996).
[19] T. Hamazaki, M. Sasaki, T. Tanaka, and K. Yamamoto, Phys. Rev. D **53**, 2045 (1996).
[20] J. Garriga and A. Vilenkin, Phys. Rev. D **44**, 1007 (1991); **45**, 3469 (1992); second reference of Ref. [12] and Ref. [19]; J. Garriga, Phys. Rev. D **54**, 4764 (1996); J. Garcia-Bellido, *ibid.* **54**, 2473 (1996).
[21] J. D. Cohn, Phys. Rev. D **54**, 7215 (1996).
[22] B. Allen and R. Caldwell (unpublished).
[23] R. Caldwell (private communication).
[24] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984); V. Mukhanov, H. Feldman, and R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[25] T. Tanaka and M. Sasaki, Phys. Rev. D **55**, 6061 (1997); R. Caldwell (private communication).
[26] K. Tomita, Prog. Theor. Phys. **68**, 310 (1982).
[27] D. Lyth and A. Woszczyna, Phys. Rev. D **52**, 3338 (1995).
[28] S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[29] M. Sasaki and T. Tanaka, Phys. Rev. D **54**, 4705 (1996).
[30] T. Tanaka and M. Sasaki, Phys. Rev. D **50**, 6444 (1994).
[31] S. Parke, Phys. Lett. **121B**, 313 (1983).
[32] R. Sachs and A. Wolfe, Astrophys. J. **147**, 73 (1967).
[33] M. White, D. Scott, and J. Silk, Annu. Rev. Astron. Astrophys. **32**, 319 (1994).
[34] J. Garcia-Bellido, A. R. Liddle, D. H. Lyth, and D. Wands, Phys. Rev. D **52**, 6750 (1995); K. Yamamoto and E. F. Bunn, Astrophys. J. **464**, 8 (1996).
[35] B. Allen, R. Caldwell, and S. Koranda, Phys. Rev. D **51**, 1553 (1995).
[36] M. R. de Garcia Maia and J. A. S. Lima, Phys. Rev. D **54**, 6111 (1996).
[37] W. Hu and M. White, Report No. astro-ph/9701210 (unpublished).
[38] B. Allen and R. Caldwell (unpublished).
[39] T. Tanaka and M. Sasaki, Prog. Theor. Phys. (to be published).