# Matching conditions and Higgs boson mass upper bounds reexamined

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Matching conditions relate couplings to particle masses. We discuss the importance of one-loop matching conditions in the Higgs boson and top quark sector as well as the choice of the matching scale. We argue for matching scales $\mu_{0,t} \simeq m_t$ and $\mu_{0,H} \simeq \max\{m_t, M_H\}$. Using these results, the two-loop Higgs boson mass upper bounds are reanalyzed. Previous results for $\Lambda \approx$ few TeV are found to be too stringent, and a recent update is found to be wrong. For $\Lambda = 10^{19}$ GeV we find $M_H < 180 \pm 4 \pm 5$ GeV, the first error indicating the theoretical uncertainty and the second error reflecting the experimental uncertainty due to $m_t = 175 \pm 6$ GeV. Hence a Higgs boson mass of about 160–170 GeV certainly satisfies both upper and lower Higgs boson mass bounds for cutoff scales up to $\Lambda = 10^{19}$ GeV if $m_t = 175$ GeV. For such Higgs boson and top quark masses the renormalization group behavior of the minimal standard model does not require new physics to set in before the Planck scale. [S0556-2821(97)01509-9]

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## I. INTRODUCTION

The Higgs sector of the minimal standard model (SM) is usually considered to be an effective theory. The possible triviality problem connected to the underlying $\phi^4$ theory [1] can be avoided if new physics appears at some high energy $\Lambda$. Depending on the specific value of $\Lambda$, an upper bound on the mass $M_H$ of the SM Higgs boson can be derived [2–4]. This upper bound is connected to the unsatisfactory high energy behavior of the Higgs quartic self-coupling $\lambda$ if $M_H$ is large. It manifests itself in the (one-loop) Landau pole [5] when using a perturbative approach or in large cutoff effects when performing lattice calculations [6–9].

Previous work [3,4] extensively investigated the dependence of the $M_H$ upper bound on the top quark mass $m_t$. The discovery of the top quark and the steadily improving mass determination of $m_t$ allow for an update of this work, leading to the question of which uncertainties remain in the theoretical prediction of the $M_H$ upper bound. Using the perturbative approach up to two loops, we investigate the sensitivity of the $M_H$ upper bound with regard to various cutoff criteria, the inclusion of matching corrections, and the choice of the matching scale $\mu_0$. We show that choosing $M_Z$ as the matching scale, taken in some earlier analyses, is unreasonable and can lead to unreliable predictions. This result is also important for higher-order calculations other than two-loop Higgs boson mass upper bounds.

Taking into account the results from lattice calculations [7–9], we investigate the dependence of the $M_H$ upper bound on the cutoff condition. In the case of a low cutoff scale $\Lambda \simeq 10^3$–$10^4$ GeV, we find the sum of theoretical uncertainties to be large, $\sim 200$ GeV. Earlier work underestimated these uncertainties and gave too stringent bounds. At a high cutoff scale of $\Lambda \simeq 10^{15}$–$10^{19}$ GeV, the theoretical uncertainties are much smaller, $\sim 10$ GeV. Taking the present-day value of the top quark mass of $m_t = 175 \pm 6$ GeV [10] and a cutoff scale of $\Lambda = 10^{19}$ GeV, we find an upper bound $M_H < 180 \pm 4 \pm 5$ GeV, where the first error estimates the theoretical uncertainty and the second error indicates the remaining top quark mass dependence. Comparing our results with recent vacuum stability bounds [11–13], we find that a Higgs boson mass of about 160–170 GeV satisfies both upper and lower Higgs boson mass bounds for cutoff scales up to $\Lambda = 10^{19}$ GeV if $m_t = 175$ GeV. For such Higgs boson and top masses the renormalization group behavior of the minimal standard model does not require new physics to set in before the Planck scale. The recent update of Higgs boson mass upper bounds in [14] is wrong.

In Sec. II we review the scale dependence of the SM matching conditions. We argue that the most reasonable choice of the matching scale for the Higgs boson quartic coupling is $\mu_{0,H} \simeq \max\{m_t, M_H\}$ and that the top quark Yukawa coupling should be fixed at $\mu_{0,t} \simeq m_t$. In particular, the use of the scale $M_Z$ leads to unreliable results in the case of the Higgs boson coupling. Using these observations, we calculate the SM Higgs boson mass upper bounds at the two-loop level in Sec. III.

## II. MATCHING CONDITIONS

We start with a detailed look at the so-called matching conditions in the SM: the relations between the physical masses and the corresponding running couplings. This part of our paper is therefore not specific to the calculation of Higgs boson mass upper bounds but has further applications.

The modified minimal subtraction ($\overline{\text{MS}}$) Higgs boson

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quartic coupling $\overline{\lambda}$ and top quark Yukawa coupling $\overline{g_t}$ are related to $M_H$ and $m_t$ using the matching conditions

$$\overline{\lambda}(\mu_0)=\frac{M_H^2}{2v^2}[1+\delta_H(\mu_0)], \tag{2.1}$$

$$\overline{g_t}(\mu_0)=\frac{\sqrt{2}m_t}{v}[1+\delta_t(\mu_0)], \tag{2.2}$$

where $v=(\sqrt{2}G_F)^{-1/2}\approx 246$ GeV. The definitions of the corresponding tree-level couplings are obtained by dropping the matching corrections $\delta$, thus fixing our notation. The use of two-loop renormalization group (RG) equations in connection with $\overline{\rm MS}$ couplings requires one-loop expressions for the corrections $\delta_H(\mu_0)$ and $\delta_t(\mu_0)$; they are given in [15] and [16], respectively.

In the case of the electroweak gauge couplings, one-loop matching corrections have also been calculated [17,18]. However, it is customary to extract the $\overline{\rm MS}$ gauge couplings directly using $\overline{\rm MS}$ definitions for experimental observables. The measured values for the $\overline{\rm MS}$ electroweak mixing angle and QED coupling fix the $\overline{\rm MS}$ electroweak couplings at the scale $\mu_0=M_Z=91.187$ GeV [19]:

$$\overline{\alpha}^{-1}(M_Z)=4\pi\frac{\overline{g^2}(M_Z)+\overline{g'^2}(M_Z)}{\overline{g^2}(M_Z)\overline{g'^2}(M_Z)}=127.90, \tag{2.3}$$

$$\sin^2\theta_W^{\overline{\rm MS}}(M_Z)=\frac{\overline{g'^2}(M_Z)}{\overline{g^2}(M_Z)+\overline{g'^2}(M_Z)}=0.2315. \tag{2.4}$$

The $\overline{\rm MS}$ electroweak couplings are obtained as $\overline{g}(M_Z)=0.651\ldots$ and $\overline{g'}(M_Z)=0.357\ldots$.

For comparison, it is nevertheless interesting to define gauge sector matching conditions in analogy to Eqs. (2.1) and (2.2), that is, using gauge boson masses and matching corrections: $\overline{g^2}(M_Z)\equiv(4M_W^2/v^2)(1+\delta_W)$ and $\overline{g^2}(M_Z)+\overline{g'^2}(M_Z)\equiv(4M_Z^2/v^2)(1+\delta_Z)$. Taking $M_W=80.35$ GeV and $M_Z$ as above one obtains $\delta_W\approx-0.4\%$ and $\delta_Z\approx 0.7\%$. As we will see below, the one-loop matching corrections $\delta_t$ and especially $\delta_H$ are significantly larger.

In the following we examine in detail the interesting structure of the matching corrections $\delta_H(\mu_0)$ and $\delta_t(\mu_0)$ as a function of $\mu_0$ and $M_H$. For $\delta_H$, the heavy top quark mass of $m_t=175$ GeV changes drastically the original discussion[1] presented in [15], except for $M_H\gg m_t$.

Using the result of [15], the correction $\delta_H(\mu_0)$ can be rewritten as

$$\delta_H(\mu_0)=\frac{2v^2}{M_H^2}\frac{1}{32\pi^2v^4}\{h_0(\mu_0)+M_H^2h_1(\mu_0)+M_H^4h_2(\mu_0)\}, \tag{2.5}$$

with

$$h_0(\mu_0)=-24m_t^4\ln\frac{\mu_0^2}{m_t^2}+6M_Z^4\ln\frac{\mu_0^2}{M_Z^2}+12M_W^4\ln\frac{\mu_0^2}{M_W^2}+c_0, \tag{2.6}$$

$$h_1(\mu_0)=12m_t^2\ln\frac{\mu_0^2}{m_t^2}-6M_Z^2\ln\frac{\mu_0^2}{M_Z^2}-12M_W^2\ln\frac{\mu_0^2}{M_W^2}+c_1, \tag{2.7}$$

$$h_2(\mu_0)=\frac{9}{2}\ln\frac{\mu_0^2}{M_H^2}+\frac{1}{2}\ln\frac{\mu_0^2}{M_Z^2}+\ln\frac{\mu_0^2}{M_W^2}+c_2. \tag{2.8}$$

The constants $c_i$ are independent of $\mu_0$. For $m_t=175$ GeV and 75 GeV $<M_H<$ 570 GeV their total contribution to $\delta_H$ is in magnitude less than 0.02, though some individual terms can exceed 0.05. Depending on the choice of $\mu_0$, the logarithmic terms in Eqs. (2.6)–(2.8) can yield a much larger correction. In Fig. 1(a) we show the one-loop result of $\delta_H$ as a function of $\mu_0$ and $M_H$ for $m_t=175$ GeV. We find that the matching correction $\delta_H$ can be in magnitude larger than 25% for various regions in the parameter space $(\mu_0,M_H)$, even exceeding 100%. Clearly the matching correction should be taken into account and the choice of the matching scale $\mu_0$ is important: Some choices are more appropriate than others.

To discuss the dependence of $\delta_H$ on $\mu_0$ we consider its derivative

$$\begin{aligned}\frac{d\delta_H(\mu)}{d\mu}&=\frac{1}{\mu}\frac{2v^2}{M_H^2}\frac{1}{8\pi^2v^4}[3M_H^4-3M_H^2(M_Z^2+2M_W^2-2m_t^2)\\&\quad+3M_Z^4+6M_W^4-12m_t^4]\\&\equiv\frac{1}{\mu}\frac{2v^2}{M_H^2}\beta_\lambda,\end{aligned} \tag{2.9}$$

where $\beta_\lambda$ is the one-loop $\beta$ function of the coupling $\lambda$ expressed in terms of the different physical masses rather than in terms of the various $\overline{\rm MS}$ couplings (which is consistent at one-loop order). For $m_t=175$ GeV, $\beta_\lambda$ equals zero if $M_H\simeq 208$ GeV. Taking $M_H$ to be different from this value, $\beta_\lambda$ quickly becomes large. If $M_H\ll 208$ GeV, the $m_t^4$ contribution dominates and $\beta_\lambda\ll 0$; if $M_H\gg 208$ GeV, the $M_H^4$ contribution causes $\beta_\lambda\gg 0$. Correspondingly, the magnitude of $\delta_H$ is insensitive to the choice of $\mu_0$ only for a small range of $M_H$ values; see Fig. 1.

Natural choices of $\mu_0$ in Eq. (2.1) are the various masses appearing in the logarithms in Eqs. (2.6)–(2.8): $M_H$, $m_t$, or $M_Z$. Since the impact of the choice of $\mu_0$ is connected to the value of $M_H$, we consider three cases.

(1) $M_H\ll m_t$ ($M_H\simeq$70–100 GeV). This is the range where $\beta_\lambda\ll 0$ due to the dominant $m_t^4$ term in Eq. (2.9). Such a large contribution to $\beta_\lambda$ is possible for low values of $M_H$ because there is no symmetry in the scalar sector which imposes $\beta_\lambda$ to go to 0 for $\lambda\to 0$. This is in contrast to the $\beta$ functions of the gauge and Yukawa sectors. Consequently, the coefficients of the logarithmic terms in $\delta_H$ can be large for small values of $M_H$, actually going to infinity as $M_H\to 0$. (In contrast, the coefficients in the matching corrections of the nonscalar sectors vanish or approach a finite constant if the corresponding particle mass goes to zero.) Indeed, the $m_t^4$ term in $\beta_\lambda$ gives rise to the large coefficient

[1]The analysis of [15] is based on a value $m_t$=40 GeV.

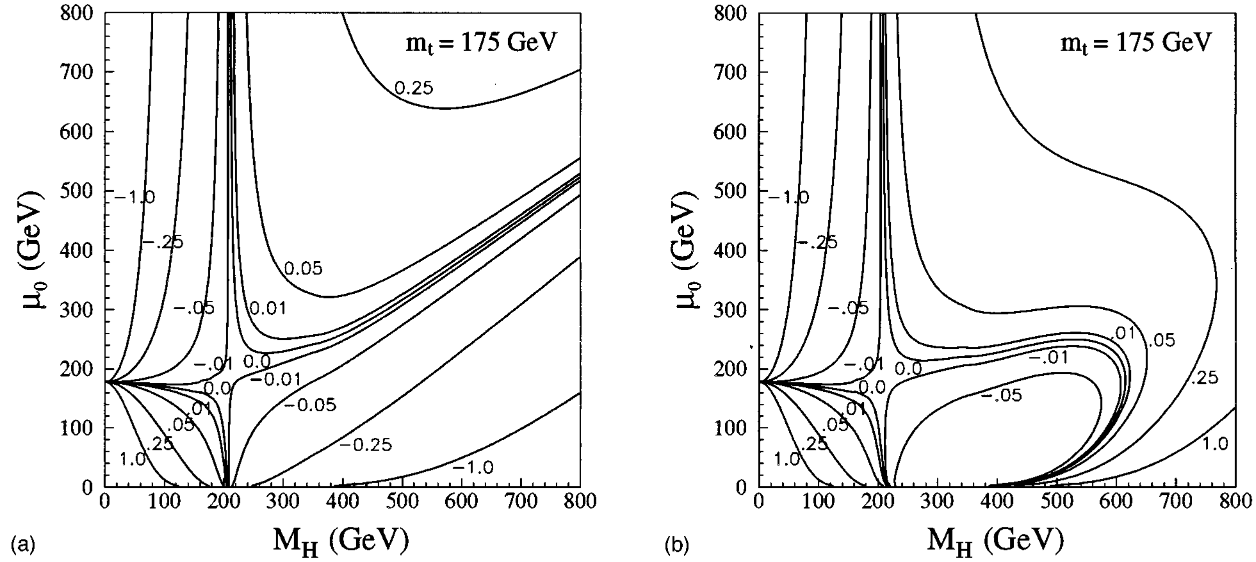


FIG. 1. (a) Values of $\mu_0$ and $M_H$ for which the one-loop Higgs boson matching correction $\delta_H(\mu_0)$, Eq. (2.1), equals the values indicated next to the various contour lines. The top quark mass is taken to be 175 GeV. (b) Same plot, but the leading two-loop heavy-Higgs-boson corrections [23] have been added.

$m_t^4/M_H^2$ which multiplies $\ln(m_t^2/\mu_0^2)$. (The overall factor $1/M_H^2$ is present because $\delta_H$ is the ratio of the loop contribution to the lowest order contribution to $\lambda$, the latter being proportional to $M_H^2$.) Consequently, if $M_H$ is small, then $\delta_H$ is small only if $\mu_0$ is chosen close to $m_t$, not $M_H$. For example, if $M_H = 70$ GeV and $m_t = 175$ GeV, we find $\delta_H(M_H) \simeq 80\%$ whereas $\delta_H(m_t) \simeq -0.7\%$. The dominance of the $\ln(\mu_0^2/m_t^2)$ term indicates that the top-quark mass scale is the correct scale of reference for low values of $M_H$. Interestingly, even if the top-quark one-loop correction to $\delta_H$ is large, perturbation theory is still applicable: $\delta_H$ is formally the product of a series in powers of $g_t$ and $\lambda$, with an overall factor $1/M_H^2$. The higher-order terms contributing to $\delta_H$ are expected to be small in the same way in which the two-loop term of $\beta_\lambda$ [20,21] is smaller than the one-loop contribution[2] to $\beta_\lambda$.

(2) $M_H \simeq (0.8\text{–}1.7) m_t$. Taking $m_t = 175$ GeV, the function $\beta_\lambda$ features a zero in this Higgs boson mass range, indicating that both Higgs boson and top quark contributions have similar weight. Both $\mu_0 = m_t$ and $\mu_0 = M_H$ are acceptable choices. In fact, we find the Higgs boson matching corrections to satisfy $|\delta_H| < 5\%$ for a large range of $\mu_0$ around $\mu_0 \simeq M_H \simeq m_t$. This property remains true if the top quark mass has a value somewhat different from $m_t = 175$ GeV. Choosing $\mu_0 = \max\{m_t, M_H\}$, a variation of 160 GeV $< m_t <$ 190 GeV results in $-1.1\% < \delta_H(m_t) < -1.0\%$ if $M_H = 140$ GeV and $2.4\% < \delta_H(M_H) < 3.6\%$ if $M_H = 300$ GeV.

(3) $M_H \gg m_t$. Such a value of $M_H$ causes a large and positive value of $\beta_\lambda$. The leading logarithmic contribution to $\delta_H$ is the $M_H^2 \ln(\mu_0^2/M_H^2)$ term which can be suppressed choosing $\mu_0 \simeq M_H$. Yet the other terms, including $\ln(\mu_0^2/m_t^2)$, are viable for $\mu_0 = M_H$. For example, $M_H = 570$ GeV results in $\delta_H(M_H) \simeq 20\%$. For larger $M_H$ the matching correction approaches the heavy-Higgs-boson result

$$\delta_H = \frac{M_H^2}{32\pi^2 v^2}\left(12\ln\frac{\mu_0^2}{M_H^2} + 25 - 3\pi\sqrt{3}\right). \tag{2.10}$$

A possible choice, used in [15], would be $\mu_0 \simeq 0.7 M_H$ such that $\delta_H(\mu_0) \simeq 0$. This approach, however, fails at two loops since the two-loop heavy-Higgs-boson terms are sizable [23]. Adding these two-loop contributions to the full one-loop result of $\delta_H$, we show the resulting $\mu_0$ dependence in Fig. 1(b). A satisfactory perturbative behavior is obtained for $\mu_0 = M_H$ if $M_H \lesssim 800$ GeV. The choices $\mu_0 = m_t$ or $M_Z$ are inappropriate since they lead to unreliable perturbative predictions for even smaller values of $M_H$. In particular, the choice $\mu_0 = M_Z$ leads to $\delta_H < -1.0$ for $M_H > 690$ GeV, resulting in an unphysical *negative* $\overline{\text{MS}}$ Higgs boson coupling.

Summarizing our results for the three different Higgs boson mass scenarios described above, we find the scale $\mu_0 \simeq \max\{m_t, M_H\}$ to be the appropriate Higgs boson matching scale for $m_t \simeq 175$ GeV. The calculation of the $M_H$ upper bound (see Sec. II) is an example how physical quantities are sensitive to the choice of $\mu_0$.

Next we consider the matching correction $\delta_t(\mu_0)$ entering Eq. (2.2). It has been given at one loop in [16], with the dominant QCD correction given earlier in [24] and the Yukawa corrections in [25]. The result can be written as

$$\delta_t(\mu_0) = \left(-4\frac{\alpha_s}{4\pi} - \frac{4}{3}\frac{\alpha}{4\pi} + \frac{9}{4}\frac{m_t^2}{16\pi^2 v^2}\right)\ln\frac{\mu_0^2}{m_t^2} + c_t, \tag{2.11}$$

[2]The simultaneous largeness and perturbativity of the top quark contribution in the scalar sector could be the origin of the symmetry breaking of SU(2) $\times$ U(1). A recent model [22] using this approach yields $M_H \simeq 80\text{–}100$ GeV.

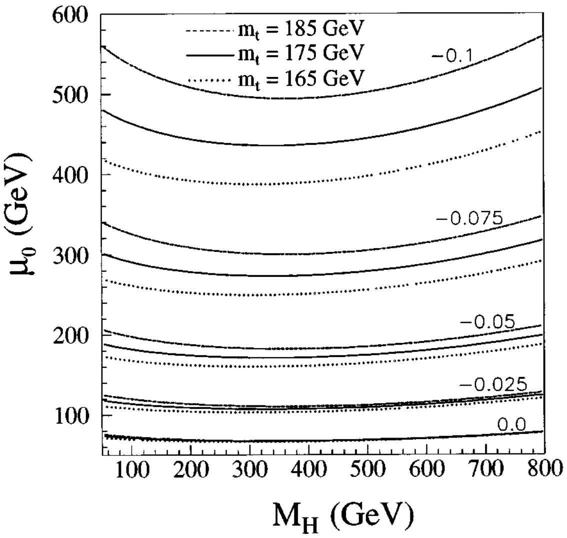


FIG. 2. Values of $\mu_0$ and $M_H$ for which top quark matching correction $\delta_t(\mu_0)$, Eq. (2.2), equals the values indicated next to the various contour lines. Results are shown using $m_t=165$ GeV (dotted line), 175 GeV (solid line), and 185 GeV (dashed line).

where $c_t$ is independent of $\mu_0$ and can be evaluated using the results[3] in [16]. Taking $\alpha_s(M_Z)=0.118$ [26], we find $-0.052<c_t<-0.042$ for a top quark mass of 150 GeV $<m_t<$ 200 GeV and a Higgs boson mass of 50 GeV $<M_H<$ 600 GeV. The correction due to $c_t$ is therefore in magnitude larger than the sum of the $\mu_0$-independent contributions $c_i$ to $\delta_H$. The largeness of $c_t$ is mostly due to the QCD correction. In contrast, there is no one-loop QCD correction contributing to $\delta_H$.

Since the CERN $e^+e^-$ collider LEP I provides the result for $\alpha_s$ at scale $M_Z$, it seems plausible to use a matching scale $\mu_0=M_Z$. This yields $\delta_t(M_Z)\simeq-2\%$ as can be seen in Fig. 2. Looking at the logarithm appearing in Eq. (2.11), however, the adequate choice is $\mu_0\simeq m_t$: No other particle mass enters the $\mu_0$-dependent logarithms. With this choice we immediately obtain $\delta_t(m_t)=c_t\simeq5\%$. Here the difference in taking $\alpha_s(M_Z)$ vs $\alpha_s(m_t)$ amounts to higher-order corrections which are suppressed.

The present-day experimental result of $m_t=175\pm6$ GeV [10] leads to $\pm3.4\%$ uncertainty in the tree-level result of $g_t$. Comparing with the results above, we find the one-loop matching correction $\delta_t$ to be of equal importance. This concludes our review of the matching conditions.

### III. HIGGS BOSON MASS UPPER BOUNDS

The triviality problem of the SM is completely fixed by the $\beta$ functions of the theory. The functions $\beta_i$ for all SM couplings have been calculated in the $\overline{\mathrm{MS}}$ scheme up to two loops [20,21,27]. At the one-loop level, a heavy Higgs boson particle gives rise to a positive function $\beta_\lambda$, causing the running Higgs quartic coupling $\lambda(\mu)$ to permanently increase as $\mu$ increases. At some value $\mu=\Lambda_L$, the position of the one-loop Landau pole [5], the Higgs boson running coupling becomes infinite: Perturbation theory has ceased to be meaningful long before.

At the two-loop level, a heavy Higgs boson mass causes $\lambda(\mu)$ to approach an ultraviolet (metastable) fixed point. This fixed point is almost entirely determined by the leading Higgs boson coupling contributions to $\beta_\lambda$ at two loops:

$$\beta_\lambda=24\frac{\lambda^2}{(16\pi^2)^2}-312\frac{\lambda^3}{(16\pi^2)^3}. \tag{3.1}$$

The resulting fixed point value, corresponding to $\beta_\lambda=0$, is

$$\lambda_{\mathrm{FP}}=12.1\ldots . \tag{3.2}$$

Increasing the scale $\mu$ even further, the growing value of the running top quark coupling can no longer be neglected and changes $\beta_\lambda$, hence modifying the above fixed point behavior. Since perturbation theory is already meaningless even before $\lambda(\mu)$ reaches $\lambda_{\mathrm{FP}}$, we are not concerned about the details of the $\lambda(\mu)$ behavior beyond the metastable fixed point.

At three loops, only the leading contribution to $\beta_\lambda$ is known [23,28,29]. It causes the running Higgs boson coupling to again have a Landau singularity. Since the complete set of three-loop SM $\beta$ functions and the corresponding two-loop matching conditions is not yet available, we restrict our present analysis to two-loop $\beta$ functions.

To obtain $M_H$ upper bounds from the RG evolution of $\lambda(\mu)$ to some embedding scale $\mu=\Lambda$, one has to choose a cutoff value for $\lambda(\Lambda)$. We denote this cutoff condition by $\lambda_c(\Lambda)$. At one loop, the standard choice is to require that $\lambda(\mu)$ avoid the Landau singularity for $\mu<\Lambda$. This corresponds to $\lambda_c(\Lambda)=\infty$. At two loops, the running Higgs boson coupling remains finite and $\lambda(\mu)\to\lambda_{\mathrm{FP}}$ as $\mu$ increases. The perturbative approximation, however, fails long before reaching the fixed point. Therefore we examine two different two-loop cutoff conditions

$$\lambda_c(\Lambda)=\lambda_{\mathrm{FP}}/4 \quad \text{and} \quad \lambda_c(\Lambda)=\lambda_{\mathrm{FP}}/2. \tag{3.3}$$

The first choice corresponds to a two-loop correction of 25% to the one-loop $\beta$ function $\beta_\lambda$; see Eq. (3.1). Perturbation theory is expected to be reliable for such a value of $\lambda(\Lambda)$ [30]. The second choice causes a 50% correction, and its value is comparable with upper bounds on $\lambda(\Lambda)$ which can be obtained from lattice calculations [7–9]. In addition, it is also relatively close to the upper bound of the perturbative regime [30].

Choosing four different embedding scales $\Lambda=10^3$, $10^6$, $10^{10}$, and $10^{16}$ GeV, we give in Fig. 3 the different values of $\lambda(M_Z)$ which lead to the corresponding cutoff conditions $\lambda_c(\Lambda)$ when evolving all SM couplings from $M_Z$ to $\Lambda$. The one-loop result in Fig. 3 with $\lambda_c(\Lambda)=\infty$ agrees with the result obtained by Lindner [3] when setting $\lambda(M_Z)=M_H^2/2v^2$ and $g_t(M_Z)=\sqrt{2}m_t/v$, and taking into account the updated experimental input for the various

[3]There is a misprint in Table I of [16]: The term $6.90\times10^{-3}$ should have the opposite sign.

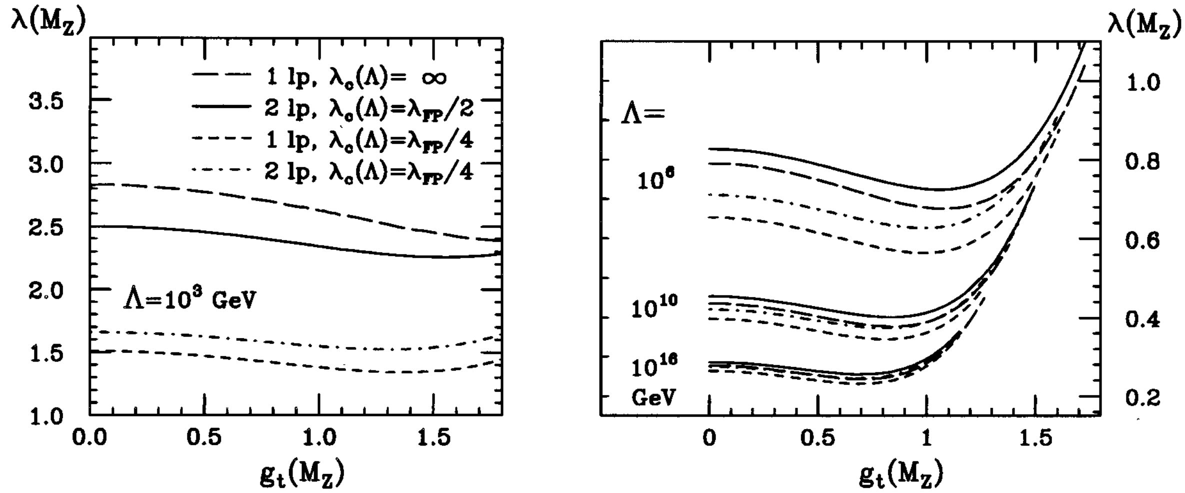


FIG. 3. Choosing either one-loop or two-loop RG evolution and various cutoff conditions $\lambda_c(\Lambda)$, the maximally allowed value of $\lambda(M_Z)$ is given as a function of $g_t(M_Z)$. The cutoff condition $\lambda_c(\Lambda)$ is imposed at scales $\Lambda=10^3$ GeV (left plot) and $\Lambda=10^6,10^{10},10^{16}$ GeV (right plot).

couplings[4] at $\mu_0=M_Z$. The recent results in [14] which question the Lindner results at all scales $\Lambda$ are incorrect.[5]

Taking instead the value $\lambda_c(\Lambda)=\lambda_{\rm FP}/4$ at one loop, we find a value of $\lambda(M_Z)$ for which perturbation theory is definitely reliable when evolving all SM couplings to $\Lambda$. For $\Lambda=10^{16}$ GeV the one-loop perturbative upper bound on $\lambda(M_Z)$ is only slightly less than the nonperturbative value obtained using the Landau pole criterion, indicating the insensitivity of the upper bound to the cutoff condition. For $\Lambda=10^3$ GeV, however, the perturbative upper bound is about 50% less than the Landau pole bound, a sign for a strong dependence on the cutoff condition $\lambda_c(\Lambda)$.

Going to two loops, the perturbative bound corresponding to $\lambda_c(\Lambda)=\lambda_{\rm FP}/4$ differs from the corresponding one-loop result by less than 12%: Perturbation theory indeed seems applicable. The maximal upper bound as modeled by $\lambda_c(\Lambda)=\lambda_{\rm FP}/2$ gives upper bounds on $\lambda(M_Z)$ which are of the order of the one-loop Landau pole bounds.

We conclude that our two-loop cutoff conditions are suitable for representing two scenarios: $\lambda_c(\Lambda)=\lambda_{\rm FP}/4$ corresponds to a perturbatively reliable Higgs boson sector at embedding scale $\Lambda$, and the condition $\lambda_c(\Lambda)=\lambda_{\rm FP}/2$ is at the verge of being nonperturbative.

The procedure for obtaining an $M_H$ upper bound from the bound on $\lambda(M_Z)$ is as follows. The couplings $\lambda(M_Z)$ and $g_t(M_Z)$ in Fig. 3 are $\overline{\rm MS}$ couplings at $\mu=M_Z$. The $\overline{\rm MS}$ gauge couplings are fixed at $M_Z$ using Eqs. (2.3) and (2.4), and $\alpha_s(M_Z)=0.118$. The matching scale for the top quark coupling is taken according to our previous discussion (Sec. I) as $\mu_{0,t}\equiv m_t$, and we take $m_t=175$ GeV. (Taking $\mu_{0,t}=M_Z$ has little effect on the final numerical results.) The matching scale for the Higgs boson coupling is chosen to be $\mu_{0,H}\equiv M_H$ as argued above. With these settings, we evolve $\lambda(M_Z)$ and all other SM couplings from $M_Z$ to some value $\mu_{0,H}$ such that Eq. (2.1) is solved for some value $M_H$ with $\mu_{0,H}=M_H$. Subsequent evolution to $\mu_{0,t}$ checks the top quark matching condition, Eq. (2.2), using $m_t=175$ GeV and the value of $M_H$ found in the previous step. If the top quark matching condition is not satisfied, we iterate our procedure, starting at scale $M_Z$ with a different value of $\overline{g_t}(M_Z)$. Eventually, we find a final solution for $M_H$ which is consistent with both matching conditions. To investigate the importance of the one-loop matching corrections, we repeat the above procedure taking the matching corrections $\delta_H$ and $\delta_t$ to be zero.

In Fig. 4 we show the resulting two-loop upper bound on $M_H$ with and without the use of matching corrections, fixing the cutoff condition as $\lambda_c(\Lambda)=\lambda_{\rm FP}/2$. Using the choice $\mu_{0,H}=M_H$, the comparison of the solid line (with matching corrections) and long-dashed line (without matching corrections) allows for a conservative estimate of higher-order corrections. We find that the difference of the two results can exceed 100 GeV at a small embedding scale $\Lambda$, but reduces to less than about 6 GeV at a large scale.

In addition to the preferred choice $\mu_{0,H}=M_H$, we also give results when using $\mu_{0,H}=M_Z$. For a large embedding scale $\Lambda$ (resulting in small values of $M_H$), the two different choices of $\mu_{0,H}$ give similar results. For a small scale $\Lambda$, the difference is significant (Fig. 4, dotted line). This was already anticipated in a one-loop study of pure $\phi^4$ theory which underlies the SM Higgs boson sector [31]. However, the inclusion of matching corrections (short-dashed curve) shows that the scale choice $\mu_{0,H}=M_Z$ is completely inadequate for large values of $M_H$ as indicated by the largeness of the corrections compared to the choice $\mu_{0,H}=M_H$. Even more strikingly, values $\Lambda<2\times10^4$ GeV [which lead to bounds $\lambda(M_Z)>1.2$ when using $\lambda_c(\Lambda)=\lambda_{\rm FP}/2$] have no so-

[4]In the case of $\Lambda=10^3$ GeV, for which Lindner [3] only gives a qualitative estimate, we find a slightly higher upper bound on $\lambda(M_Z)$.

[5]For large scale $\Lambda$, the errors in [14] seem to be partially connected to the errorneous use of $10^n$ instead of $e^n$ in all equations and figures where $\Lambda$ is specified. This replacement, however, still does not correct all their results.

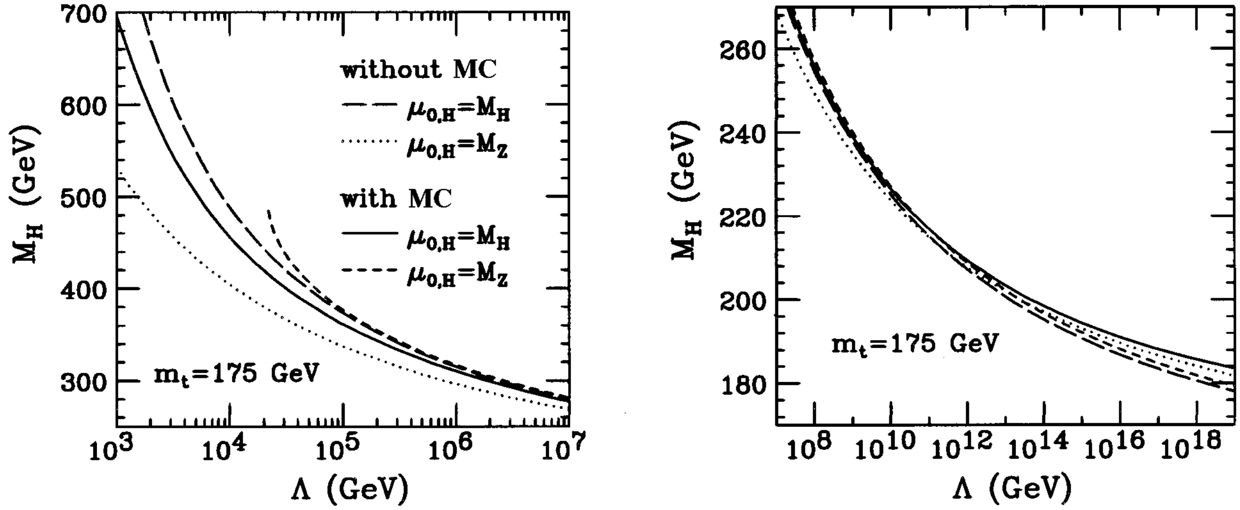


FIG. 4. Choosing two-loop RG evolution and cutoff condition $\lambda_c(\Lambda)=\lambda_{FP}/2$, the upper bound on $M_H$ is calculated. The running Higgs boson and top quark Yukawa couplings, $\lambda(\mu)$ and $g_t(\mu)$, are fixed by the physical masses $M_H$ and $m_t$ using matching conditions with and without one-loop matching corrections. In addition, the Higgs boson matching scale is varied to be $\mu_{0,H}=M_H$ and $M_Z$. The top quark mass is fixed at $m_t=175$ GeV, and $\mu_{0,t}=m_t$. The left plot shows the result for small values of $\Lambda$, and the right plot extends up to values of $\Lambda=10^{19}$ GeV.

lution in $M_H$ which satisfies the $\overline{\text{MS}}$ matching condition. This is due to the fact that the choice $\mu_{0,H}=M_Z$ restricts the $\overline{\text{MS}}$ coupling to a maximal value $\lambda(M_Z)=1.2$ which is obtained for $M_H \approx 495$ GeV. We will only consider the $\mu_{0,H}=M_H$ results when determining the final $M_H$ upper bounds.

The results of Fig. 4 can also be compared with the two-loop results of [4]. There no matching corrections have been included, and $\mu_{0,H}=M_Z$ is used. The cutoff condition $\lambda_c(\Lambda)$ is determined as a turning point in the two-loop calculation rather than a fixed value. This procedure yields larger two-loop values of $\lambda_c(\Lambda)$ than used here. The resulting $M_H$ bounds are therefore larger than our corresponding result with $\mu_{0,H}=M_Z$ and no matching corrections, but with $\lambda_c(\Lambda)=\lambda_{FP}/2$.

In Fig. 5 we analyze the dependence of the upper $M_H$ bound on $m_t$. Varying $m_t$ in the range 150–200 GeV, the bound on $M_H$ changes less than 40 GeV for the largest embedding scale considered, $\Lambda=10^{19}$ GeV. The latest experimental result [10] $m_t=175\pm 6$ GeV reduces this uncertainty to less than 5 GeV at the $1\sigma$ level. For embedding scales $\Lambda<10^{10}$ GeV the uncertainty due to $m_t$ can then entirely be neglected compared to the theoretical uncertainties connected to the cutoff condition and higher-order corrections. The uncertainty in the QCD coupling, $\alpha_s(M_Z)=0.118\pm 0.003$ [26], causes a shift of less than 1 GeV in the $M_H$ upper bound, with the maximal effect at $\Lambda=10^{19}$ GeV.

In summary, we have discussed the uncertainties in the $M_H$ upper bound due to the choice of the cutoff condition (Fig. 3), the importance of one-loop matching corrections and the choice of the matching scale $\mu_{0,H}$ (Fig. 4), and the top quark mass dependence (Fig. 5). Fixing the top quark mass to be 175 GeV, using two-loop $\beta$ functions, and appropriately choosing the matching scale to be $\mu_{0,H}=M_H$, we find the sum of all theoretical uncertainties to be represented by the upper solid area indicated in Fig. 6. They are obtained by choosing $\mu_{0,H}=M_H$ and using matching conditions with and without one-loop matching corrections. The cutoff condition is varied between $\lambda_c(\Lambda)=\lambda_{FP}/4$ and $\lambda_{FP}/2$. The lower edge of the solid area indicates a value of $M_H$ for which perturbation theory is certainly reliable up to scale $\Lambda$; in particular, the triviality problem of the standard model is clearly avoided for such values of $\Lambda$ and $M_H$. The upper edge of the solid area can be used to estimate the scale $\Lambda(M_H)$ at which the standard model ceases to be meaningful

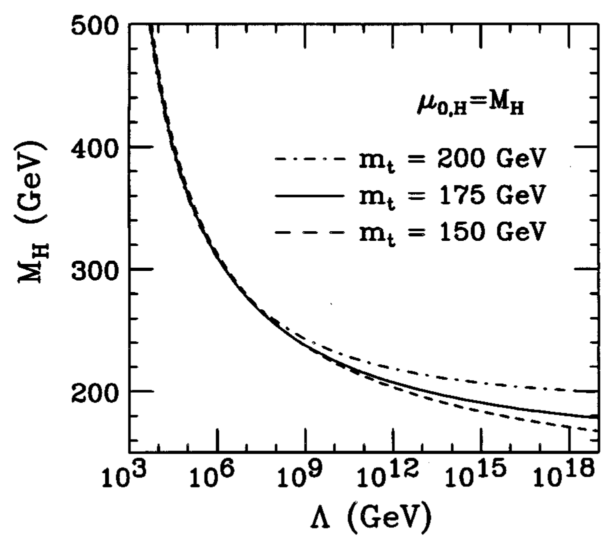


FIG. 5. The dependence of the upper $M_H$ bound on the top quark mass. The $\overline{\text{MS}}$ matching conditions with $\mu_{0,H}=M_H$ and $\mu_{0,t}=m_t$ are used in connection with two-loop RG evolution and cutoff condition $\lambda_c(\Lambda)=\lambda_{FP}/2$. For low values of the embedding scale $\Lambda$, the $M_H$ upper bound is insensitive to the exact value of $m_t$. For large embedding scales there is a larger $m_t$ dependence. Without matching corrections (not shown), the top quark mass dependence is qualitatively the same.

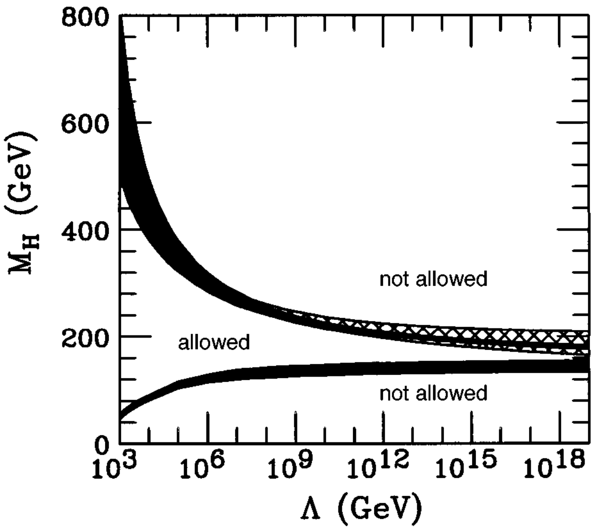


FIG. 6. Summary of the uncertainties connected to the bounds on $M_H$. The upper solid area indicates the sum of theoretical uncertainties in the $M_H$ upper bound when keeping $m_t=175$ GeV fixed. The cross-hatched area shows the additional uncertainty when varying $m_t$ from 150 to 200 GeV. The upper edge corresponds to Higgs boson masses for which the SM Higgs boson sector ceases to be meaningful at scale $\Lambda$ (see text), and the lower edge indicates a value of $M_H$ for which perturbation theory is certainly expected to be reliable at scale $\Lambda$. The lower solid area represents the theoretical uncertaintites in the $M_H$ lower bounds derived from stability requirements [11–13] using $m_t=175$ GeV and $\alpha_s=0.118$.

as an effective theory. Although the perturbative approach does not allow for extraction of absolute upper bounds, the consideration of lattice calculations in $\phi^4$ theory seems to reinforce or even tighten the upper bounds presented here [7,9,8,30]. For low values of $\Lambda$, the one-loop Landau pole bounds of [3] are found to be near the perturbative lower edge of the upper solid area in Fig. 6. The additional experimental uncertainty due to the top quark mass is represented by the cross hatched area in Fig. 6, generously varying the top quark mass from 150 GeV to 200 GeV. The present-day $1\sigma$ result of $m_t=175\pm6$ GeV is sufficient to make it the smallest source of error except for large values of the embedding scale $\Lambda$. In particular, we find

$$M_H<180\pm4\pm5\,\mathrm{GeV} \quad \text{if} \quad \Lambda=10^{19}\,\mathrm{GeV}, \tag{3.4}$$

the first error indicating the theoretical uncertainty, the second error reflecting the $m_t$ dependence.[6]

For comparison, we also give the lower bounds on $M_H$ from stability conditions on the SM Higgs effective potential. At a large scale $\Lambda$, the stability bound is well approximated by requiring the Higgs boson running coupling to remain positive: $\lambda(\Lambda)>0$. Such an analysis has been carried out at the two-loop level including matching corrections [11], and they agree within the theoretical errors with a more careful treatment of the one-loop effective potential [12]. The discrepancy at scales $\Lambda<10$ TeV has been resolved recently [13], and we use the latter results. Fixing $m_t=175$ GeV and $\alpha_s(M_Z)=0.118$ we show the lower bound in Fig. 6 (lower solid area), with the solid area indicating the theoretical uncertainty. At large $\Lambda$, the theoretical error is estimated by using $\mu_{0,H}=m_t$ and comparing the results with and without matching corrections, and at low $\Lambda$ the theoretical error is $\pm5$ GeV according to [13]. The variation $m_t=175\pm25$ GeV yields a much larger uncertainty in the $M_H$ lower bound than in the $M_H$ upper bound and is not shown.

Looking at Fig. 6 we see that a Higgs boson mass of about 160–170 GeV certainly satisfies both upper and lower Higgs boson mass bounds for cutoff scales up to $\Lambda=10^{19}$ GeV if $m_t=175$ GeV. For such Higgs boson and top quark masses the renormalization group behavior of the minimal standard model does not require new physics to set in before the Planck scale.

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[6]The very recent result [32] of $M_H<174$ GeV for $\Lambda=10^{19}$ GeV is lower than our lowest result due to the use of the smaller cutoff condition $\lambda_c(\Lambda)=5/3<\lambda_{\mathrm{FP}}/4\approx3$ (our notation).

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