Behavior of Einstein-Rosen waves at null infinity

Abhay Ashtekar
Center for Gravitational Physics and Geometry, Department of Physics, Penn State, University Park, Pennsylvania 16802

Jiří Bičák
Department of Theoretical Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

Bernd G. Schmidt
Max-Planck-Institut für Gravitationsphysik, Schlaatzweg 1, 14473 Potsdam, Germany

(Received 1 August 1996)

The asymptotic behavior of Einstein-Rosen waves at null infinity in four dimensions is investigated in *all* directions by exploiting the relation between the four-dimensional space-time and the three-dimensional symmetry reduction thereof. Somewhat surprisingly, the behavior in a generic direction is *better* than that in directions orthogonal to the symmetry axis. The geometric origin of this difference can be understood most clearly from the three-dimensional perspective. [S0556-2821(97)01902-4]

PACS number(s): 04.20.Ha, 04.20.Jb, 04.30.Nk

I. INTRODUCTION

Although the literature of Einstein-Rosen waves is quite rich (see, e.g., the references listed in the companion paper [1]) it appears that there is only one article that discusses the asymptotic behavior of these waves at infinity in four-dimensions: the paper by Stachel [2] written already in the sixties. Moreover, even in this work, Stachel deals solely with the directions *orthogonal* to the axis of symmetry, i.e., to the $\partial/\partial z$-Killing field. The purpose of this note is to analyze the asymptotic structure in *all* directions.

Since these space-times admit a translational Killing field, one would expect them not to be asymptotically flat. This is precisely what Stachel discovered in directions orthogonal to the symmetry axis. Somewhat surprisingly, however, we will find that the falloff is much better in generic directions. Indeed, if one restricts oneself to the ''averaged'' time-symmetric situation (which in particular occurs in the time symmetric case; for details, see Sec. III), one finds that, in all other directions, curvature peels normally and a regular null infinity, I, exists. In fact, all radiation is concentrated along the two generators in which the null geodesics orthogonal to the symmetry axis meet I. In other directions, there is curvature but no radiation. If one goes beyond the ''averaged'' time symmetric case, the behavior is not as nice; I may have a logarithmic character [3,4]. That is, the metric does admit Bondi-type expansions but in terms of $r^{-j}\ln^i r$. Nonetheless, even this behavior is better than the one encountered in the directions orthogonal to the symmetry axis.

The key idea behind our analysis is to exploit the relation between four-dimensional Einstein-Rosen waves and the associated three-dimensional geometry on the manifold of orbits of the translational Killing field. Since the translational Killing field has been ''factored out'' in the passage to three dimensions, the three-dimensional space-time is asymptotically flat at null infinity [1] and, as we will see, also admits a regular timelike infinity. To analyze the behavior of the four-dimensional metric, we can draw on this three-dimensional information. We will find that the behavior at I in generic directions in four dimensions is dictated by the behavior of various fields at *timelike* infinity in three dimensions. Since this three-dimensional timelike infinity is regular, the behavior in generic directions in four dimensions is better than what one might naively expect.

The plan of the paper is as follows. In Sec. II, we shall present the three-dimensional structure. In Sec. III, we will use this structure to investigate four-dimensional null infinity. The Appendix spells out the relation between three- and four-dimensional curvatures.

II. THREE-DIMENSIONAL DESCRIPTION

This section is divided into three parts. In the first, we briefly recall the symmetry reduction procedure and apply it to obtain the three-dimensional equations governing Einstein-Rosen waves. (For details, see [1].) This procedure reduces the task of finding a four-dimensional Einstein-Rosen wave to that of finding a solution to the wave equation on three-dimensional *Minkowski* space. In the second part, we analyze the asymptotic behavior of these solutions to the wave equation at timelike infinity of the three-dimensional Minkowski space. In the third part, we combine the results of the first two to analyze the asymptotic behavior of the three-dimensional metric associated with Einstein-Rosen waves at its timelike infinity. We show that this timelike infinity is regular. Although this result is not needed directly for our main result, it is included because it complements the general analysis of three-dimensional null infinity presented in [1].

A. Symmetry reduction

Recall first that the metric of a vacuum space-time with two commuting, hypersurface orthogonal spacelike Killing vectors can always be written locally as [5]

$$ds^2 = e^{2\psi}dz^2 + e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi}d\phi^2, \qquad (2.1)$$

where ρ and t (the ''Weyl canonical coordinates'') are defined invariantly and $\psi=\psi(t,\rho)$, $\gamma=\gamma(t,\rho)$. (Here, some of the field equations have been used.) Einstein-Rosen waves have cylindrical symmetry; the Killing field $\partial/\partial z$ is translational and $\partial/\partial \phi$ is rotational and keeps a timelike axis fixed. Then the coordinates used in Eq. (2.1) are unique up to a translation $t \mapsto t+a$.

The 3-manifold is obtained by quotienting the four-dimensional space-time by the orbits of the $\partial/\partial z$-Killing field and is thus coordinatized by t, ρ, and ϕ. The four-metric naturally induces a three-metric $d\overline{\sigma^2}$ on this manifold and the four-dimensional Einstein's equations can be expressed on the 3-manifold as a system of coupled equations involving the induced three-metric and the norm of the Killing field $\partial/\partial z$, which, from the three-dimensional perspective can be regarded as a (scalar) matter field. It is well known (see [1,6,7]), however, that the field equations simplify considerably if we rescale the induced three-metric $d\overline{\sigma^2}$ by $\exp(2\psi)$, the square of the norm of the Killing field; i.e., in terms of the three-metric,

$$d\sigma^2 = g_{ab}dx^a dx^b = e^{2\gamma}(-dt^2 + d\rho^2) + \rho^2 d\phi^2. \quad (2.2)$$

The four-dimensional vacuum equations are then equivalent to the set [cf. Eqs. (2.12)–(2.15) in the preceding paper [1]]

$$\gamma'' - \ddot{\gamma} + \rho^{-1}\gamma' = 2\dot{\psi}^2, \quad (2.3)$$

$$-\gamma'' + \ddot{\gamma} + \rho^{-1}\gamma' = 2\psi'^2, \quad (2.4)$$

$$\rho^{-1}\dot{\gamma} = 2\dot{\psi}\psi', \quad (2.5)$$

and

$$-\ddot{\psi} + \psi'' + \rho^{-1}\psi' = 0, \quad (2.6)$$

on the 3-manifold, where the dot and the prime denote derivatives with respect to t and ρ, respectively. The last equation is the wave equation for the nonflat three-metric (2.2) *as well as for the flat metric obtained by setting* $\gamma = 0$. This is a key simplification for it implies that the equation satisfied by the matter source ψ decouples from the Eqs. (2.3)–(2.5) satisfied by the metric. Furthermore, these latter equations reduce simply to

$$\gamma' = \rho(\dot{\psi}^2 + \psi'^2), \quad (2.7)$$

$$\dot{\gamma} = 2\rho\dot{\psi}\psi'. \quad (2.8)$$

Thus, we can first solve for the axisymmetric wave equation (2.6) for ψ *on Minkowski space* and then solve Eqs. (2.7) and (2.8) for γ—the only unknown metric coefficient—by quadratures. [Note that Eqs. (2.7) and (2.8) are compatible because their integrability condition is precisely Eq. (2.6).]

B. Asymptotic behavior of scalar waves

In this subsection we will focus on the axisymmetric wave equation in three-dimensional Minkowski space and analyze the behavior of its solutions ψ near timelike infinity of Minkowski space. (For behavior at null infinity, see [1].)

We begin with an observation. The "method of descent" from the Kirchhoff formula in four dimensions gives the following representation of the solution of the wave equation in three dimensions, in terms of Cauchy data $\Psi_0 = \psi(t=0,x,y), \Psi_1 = \psi_{,t}(t=0,x,y)$:

$$\psi(t,x,y) = \frac{1}{2\pi}\frac{\partial}{\partial t}\int\int_{S(t)} \frac{\Psi_0(x',y')dx'dy'}{[t^2-(x-x')^2-(y-y')^2]^{1/2}} + \frac{1}{2\pi}\int\int_{S(t)} \frac{\Psi_1(x',y')dx'dy'}{[t^2-(x-x')^2-(y-y')^2]^{1/2}}, \quad (2.9)$$

where S is the disk

$$(x-x')^2 + (y-y')^2 \leq t^2$$

in the initial Cauchy surface (see, e.g., [8]). We will assume that the Cauchy data are axially symmetric and of compact support.

In the preceding paper [1] [see Eq. (2.23)] we have shown that on each null hypersurface $u = t-\rho = \text{const}$ the solution (2.9) can be expanded in the form

$$\psi(u,\rho) = \frac{1}{\sqrt{\rho}}\left(f_0(u) + \sum_{k=1}^{\infty}\frac{f_k(u)}{\rho^k}\right), \quad (2.10)$$

where the coefficients in this expansion are determined by integrals over the Cauchy data. This is the behavior of ψ at null infinity I.

Let us now investigate the behavior of the solution (2.9) near timelike infinity i^+ of the three-dimensional Minkowski space. Setting

$$t = U + \kappa\rho, \quad \kappa > 1, \quad (2.11)$$

we wish to find ψ for $\rho \to \infty$ with U and κ fixed. For large enough ρ the region of integration is contained in the cone. Hence, we have to perform the derivative in Eq. (2.9) only in the integrand. We obtain

$$\begin{aligned} 2\pi\psi(t,\rho) = &-\frac{\kappa\rho + U}{[(\kappa^2-1)\rho^2]^{3/2}}\int_0^{\infty}\int_0^{2\pi}\Psi_0\rho' d\rho' d\phi' \\ &\times\left[1 + \frac{2(\kappa U + \rho'\cos\phi')}{\kappa^2-1}\frac{1}{\rho} + \frac{U^2-\rho'^2}{\kappa^2-1}\frac{1}{\rho^2}\right]^{-3/2} \\ &+ \frac{1}{[(\kappa^2-1)\rho^2]^{1/2}}\int_0^{\infty}\int_0^{2\pi}\Psi_1\rho' d\rho' d\phi' \\ &\times\left[1 + \frac{2(\kappa U + \rho'\cos\phi')}{\kappa^2-1}\frac{1}{\rho} + \frac{U^2-\rho'^2}{\kappa^2-1}\frac{1}{\rho^2}\right]^{-1/2}. \end{aligned} \quad (2.12)$$

The integrand can again be expanded in ρ^{-1} (or t^{-1}), but the leading term is ρ^{-1}. By contrast, at *null* infinity of the three-dimensional space-time, ψ falls off only as $\rho^{-1/2}$ [see Eq. (2.10) and [1] for details]. We will see that it is this difference that makes the behavior of the four-metric along generic directions better than that along directions orthogonal to the symmetry axis.

The explicit expressions of the first few terms in the expansion of ψ is given by

$$\psi = \frac{L}{(\kappa^2-1)^{3/2}}\left[-\frac{\kappa}{\rho^2}+\frac{(2\kappa^2+1)U}{\kappa^2-1}\frac{1}{\rho^3}+O(\rho^{-4})\right] + \frac{J}{(\kappa^2-1)^{1/2}}\left[\frac{1}{\rho}-\frac{\kappa U}{\kappa^2-1}\frac{1}{\rho^2}+O(\rho^{-3})\right], \tag{2.13}$$

where

$$L=\int_0^\infty \Psi_0(\rho')\rho' d\rho', \quad J=\int_0^\infty \Psi_1(\rho')\rho' d\rho'. \tag{2.14}$$

By expressing ρ in terms of t using Eq. (2.11), we may rewrite Eq. (2.13) as a series in t^{-1}:

$$\psi = \frac{L}{(\kappa^2-1)^{3/2}}\left[-\frac{\kappa^3}{t^2}+\frac{3\kappa^3 U}{\kappa^2-1}\frac{1}{t^3}+O(t^{-4})\right] + \frac{J}{(\kappa^2-1)^{1/2}}\left[\frac{\kappa}{t}-\frac{\kappa U}{\kappa^2-1}\frac{1}{t^2}+O(t^{-3})\right]. \tag{2.15}$$

The last formula is meaningful also for $\rho=0$ in the limit $\kappa\to\infty$:

$$\psi = \frac{-L}{t^2}+\frac{J}{t}+O(t^{-3}). \tag{2.16}$$

The same result can be obtained from Eq. (2.9) directly. This concludes our discussion of the asymptotic behavior of ψ near timelike infinity $i^{\pm}$.

We will conclude this subsection with three remarks.

First, the explicit representation (2.9) of the solution in terms of Cauchy data allows us to make the interesting observation that the solution is actually *analytic* in its spacetime dependence for all points for which the data are within the past null cone. To show that all solutions with data of compact support are also analytic in a neighborhood of future timelike infinity i^+, we have to use conformal rescaling techniques. Let

$$d\sigma^2 = -dt^2+dx^2+dy^2 \tag{2.17}$$

be the metric of three-dimensional Minkowski space. The conformal factor

$$\Omega = (t^2-x^2-y^2)^{-1} \tag{2.18}$$

defines, by the rescaling $d\tilde{\sigma}^2=\Omega^2 d\sigma^2$, again a flat spacetime

$$d\tilde{\sigma}^2=\Omega^2 d\sigma^2 = -d\tilde{t}^2+d\tilde{x}^2+d\tilde{y}^2, \tag{2.19}$$

where the coordinates $\tilde{t},\tilde{x},\tilde{y}$ are defined by the relations (inversion)

$$\tilde{t}=\frac{t}{t^2-x^2-y^2}, \quad \tilde{x}=\frac{x}{t^2-x^2-y^2}, \quad \tilde{y}=\frac{y}{t^2-x^2-y^2}. \tag{2.20}$$

The three-dimensional scalar wave equation has the following behavior under this conformal rescaling:

$$\nabla^2\psi=0 \Rightarrow \tilde{\nabla}^2\tilde{\psi}=0, \quad \tilde{\psi}=\Omega^{-1/2}\psi. \tag{2.21}$$

From the above consideration we know that a solution ψ with data of compact support is analytic for points within and on the future light cone of the point $\tilde{t}=a, \tilde{x}=\tilde{y}=0$, where the value of a is dictated by the support of the data. Moreover the series (2.10) is also analytic in $v=u+2\rho$ because of the converging expansion in ρ^{-1}. Hence, after the inversion we have a solution $\tilde{\psi}$ which is analytic on the extended null cone. Therefore, it is analytic in a domain which includes a neighborhood of i^+.

The second remark concerns the asymptotic behavior of ψ, regarded as a solution to the wave equation *in four dimensions*. More precisely, let us set

$$F(t,x,y,z)=\psi(t,x,y); \tag{2.22}$$

F is independent of z. How does this solution behave at null infinity of four-dimensional Minkowski space? The null geodesics in a surface $z=\text{const}$ are also null geodesics in 4-space and $F=\psi$ along these curves. Now, a solution of the four-dimensional wave equation is well behaved at null infinity if it falls off as r^{-1} (where r is the standard radial coordinate). Since the field ψ falls off only as $\rho^{-1/2}$ at null infinity in three dimensions [1], the solution F fails to define a finite radiation field at null infinity in these directions. For null lines *not* contained in $z=\text{const}$ surfaces, on the other hand, the situation is entirely different. Because such null lines project onto *timelike* lines in $z=\text{const}$ surfaces, the falloff behavior is *much better* and from Eq. (2.15) we obtain the r^{-1} decay, necessary for the radiation field to exist. Thus, in terms of a four-dimensional conformal rescaling, the rescaled field of F will be well defined on four-dimensional null infinity *except for* the two null generators determined by the $\partial/\partial z$-Killing vector. We will see in Sec. III that this behavior is the key to the understanding of the asymptotics of four-dimensional axisymmetric space-times with a further $\partial/\partial z$-Killing vector.

Finally, we wish to point out that the main results obtained in this section continue to hold also for general data of compact support which are not necessarily axisymmetric. In particular, asymptotic forms like Eq. (2.13) and (2.15) hold where, however, the coefficients depend on ϕ. The assumption of compact support can also be weakened to allow data which decay near spatial infinity sufficiently rapidly so that we still obtain solutions smooth at null and timelike infinities. This is the case, for example, with the Weber-Wheeler-Bonnor pulse discussed in the following section.

C. Asymptotic behavior of the metric

We now combine the results of the previous two subsections. Recall from Eq. (2.2) that the three-dimensional metric $d\sigma^2$ has a single unknown coefficient, $\gamma(t,\rho)$, which is determined by the solution $\psi(t,\rho)$ to the wave equation in Minkowski space (obtained simply by setting $\gamma=0$). The asymptotic behavior of $\psi(t,\rho)$, therefore, determines that of the three-metric.

At null infinity I, the asymptotic behavior (2.10) of ψ implies that γ has the form [see Eq. (2.32) in [1]]

$$\gamma=\gamma_0-2\int_{-\infty}^{u} du(\dot{f}_0(u))^2-\sum_{k=0}^{\infty}\frac{h_k(u)}{(k+1)\rho^{k+1}}. \tag{2.23}$$

We now wish to determine the metric at i^+. In the last subsection we found the asymptotic form of ψ at i^+, more specifically, at $\rho\to\infty$ (or $t\to\infty$ with $U=t-\kappa\rho$, $\kappa>1$, fixed [see Eqs. (2.11)–(2.16)]. In order to get the asymptotic forms of γ, we first express the field equations (2.7) and (2.8) for γ in terms of U and ρ:

$$\gamma_{,U}=2\rho\psi_{,U}(\psi_{,\rho}-\kappa\psi_{,U}), \tag{2.24}$$

$$\gamma_{,\rho}=\rho[\psi_{,\rho}^2+(1-\kappa^2)\psi_{,U}^2]. \tag{2.25}$$

Substituting for ψ from Eqs. (2.13) and (2.14), and integrating Eqs. (2.24) and (2.25), we obtain

$$\gamma=\frac{L^2}{4(\kappa^2-1)^4}\left[\frac{8\kappa^2+1}{\rho^4}-\frac{24\kappa(1+2\kappa^2)}{\kappa^2-1}\frac{U}{\rho^5}+O(\rho^{-6})\right]+\frac{J^2}{2(\kappa^2-1)^2}\left[\frac{1}{\rho^2}-\frac{4\kappa}{\kappa^2-1}\frac{U}{\rho^3}+O(\rho^{-4})\right]. \tag{2.26}$$

Note that we set the integration constant equal to zero. This is because we can go to i^+ along the center $\rho=0$. More precisely, since we required the regularity of the solution at $\rho=0$, we have to set $\gamma=0$ there and, as a consequence of the field equations for γ in (t,ρ) coordinates [cf. Eq. (2.8)], γ at $\rho=0$ cannot change with time.

By techniques developed, e.g., in [5] it can be now shown that the space-time has a smooth timelike infinity. The analyticity of ψ at timelike infinity shown in the last subsection and the field equations imply that the metric ''rescaled by inversion'' is analytic at i^+. In what follows, however, we will not use this result; the falloff properties (2.15) and (2.26) of ψ and γ will suffice.

III. NULL INFINITY IN FOUR DIMENSIONS

We can now return to the four-metric (2.1) and analyze its behavior at null infinity. In the main part of this section, we will consider those Einstein-Rosen waves for which the Cauchy data for ψ in the three-dimensional picture are (smooth and) of compact support. In the four-dimensional picture, these solutions correspond to *pulses* of Einstein-Rosen waves.

A. Formulation of the problem

Let us begin by summarizing the behavior in the directions perpendicular to the axis of symmetry. In these directions, the falloff of ψ is the same as in our three-dimensional treatment of null infinity [see Eq. (2.23) in [1] or Eq. (2.10)]. However, from the four-dimensional perspective, ψ is not a matter field but a metric coefficient [see Eq. (2.1)] and the $1/\sqrt{\rho}$ falloff of ψ is too slow for null infinity to exist in the sense of Penrose [9]. What is the situation with respect to curvature? In the Appendix, we use the three-dimensional results to compute the four-dimensional Riemann tensor for these space-times. We find that, in null directions perpendicular to the $\partial/\partial z$-Killing field, the tensor decays only as $1/\sqrt{\rho}$, the behavior that Stachel first discovered in his direct four-dimensional treatment [2]. [See the complex components of the Riemann tensor with respect to the null tetrad given by his Eqs. (A4)–(A6), or just ${}^{(4)}R_{3030}$, given in our Eq. (A5)]. As one would suspect from the behavior of the metric coefficients, the curvature does not peel properly in these null directions. Thus, although we have asymptotic flatness at null infinity of the *three-dimensional* space-time [1], the four-metric fails to be asymptotically flat in null directions perpendicular to the axis of rotation (i.e., along four-dimensional null lines whose projections approach null infinity in three dimensions).

In the rest of the section, we will discuss the falloff in the *remaining* null directions. We will find that, contrary to what one might have expected at first, the asymptotic behavior is much *better*. If $J=0$, the space average of the time derivative Ψ_1 of ψ vanishes at $t=0$. In this case, we will say that the solution satisfies the *averaged time-symmetry condition.* (Note that, by the wave equation, if this condition is satisfied initially, it is satisfied on all $t=$const slices subsequently.) We will see that, in this case, the falloff in fact satisfies the Bondi-Penrose [9,10] conditions and null infinity is smooth in these directions. Even for a generic data, null infinity exists, but may have a ''logarithmic behavior''; the conformally rescaled metric is continuous, but need not be differentiable there. Note that even this behavior is better than the one in directions orthogonal to the symmetry axis. The reason, in a nutshell, is that the falloff of various fields along a generic null direction in 4 dimensions is dictated by the falloff of that field along a *timelike* direction in the three-dimensional treatment and, as we saw in Sec. II, fields decay more rapidly at the three-dimensional timelike infinity than at the three-dimensional null infinity.

To see this point in detail, let us begin with the Einstein-Rosen metric [cf. Eq. (2.1)],

$$ds^2=e^{2\psi}dz^2+e^{2(\gamma-\psi)}(-dt^2+d\rho^2)+\rho^2e^{-2\psi}d\phi^2, \tag{3.1}$$

where $\psi=\psi(t,\rho)$, $\gamma=\gamma(t,\rho)$. If we pass from coordinates (ρ,z,ϕ) to spherical coordinates (r,θ,ϕ), so that $\rho=r\sin\theta, z=r\cos\theta, \phi=\phi$, and introduce flat-space retarded time $U=t-r$, we obtain Eq. (3.1) in the form

$$\begin{aligned}ds^2=&-e^{2(\gamma-\psi)}dU^2-2e^{2(\gamma-\psi)}dU\,dr\\&+(e^{2\psi}-e^{2(\gamma-\psi)})\cos^2\theta\,dr^2\\&+(e^{2(\gamma-\psi)}\cos^2\theta+e^{2\psi}\sin^2\theta)r^2d\theta^2\\&+(e^{2(\gamma-\psi)}-e^{2\psi})2r\sin\theta\cos\theta\,dr\,d\theta\\&+r^2\sin^2\theta e^{-2\psi}d\phi^2.\end{aligned} \tag{3.2}$$

Since we are considering waves with initial data of compact support in the (ρ,ϕ) plane, we can use the results of Sec. II directly. Recall that one approaches i^+ in three dimensions, fixing $U=t-\kappa\rho, \kappa=\text{const}>1$ [cf. Eq. (2.11)]. In the four-dimensional picture, this corresponds precisely to approaching the *null infinity* of the flat metric defined by t,r,θ,ϕ coordinates, along $\theta=$const, $\phi=$const, $U=t-r=$const, if we set $\kappa=1/\sin\theta$. The expansions of ψ and γ, corresponding to Eqs. (2.13) and (2.26), thus have the forms

$$\psi = \frac{L}{\cos^3\theta}\left[-\frac{1}{r^2} + \frac{(2+\sin^2\theta)U}{\cos^2\theta}\frac{1}{r^3} + O\left(\frac{1}{r^4}\right)\right] + \frac{J}{\cos\theta}\left[\frac{1}{r} - \frac{U}{\cos^2\theta}\frac{1}{r^2} + O\left(\frac{1}{r^3}\right)\right], \tag{3.3}$$

and

$$\gamma = \frac{1}{4}\frac{L^2\sin^2\theta}{\cos^8\theta}\left[(8+\sin^2\theta)\frac{1}{r^4} - \frac{24(2+\sin^2\theta)U}{\cos^2\theta}\frac{1}{r^5} + O\left(\frac{1}{r^6}\right)\right] + \frac{1}{2}\frac{J^2\sin^2\theta}{\cos^4\theta}\left[\frac{1}{r^2} - \frac{4U}{\cos^2\theta}\frac{1}{r^3} + O\left(\frac{1}{r^4}\right)\right], \tag{3.4}$$

provided we stay away from $\theta = \pi/2$, i.e., directions perpendicular to the axis. Our task now is to cast the four-metric in a Bondi form and show that the metric coefficients have the standard falloff.

We will carry out this task in the next two subsections. We will consider the cases $J=0$, $L\neq 0$, and $J\neq 0$, $L=0$ separately; since we are interested only in the leading terms, the cross terms $\sim LJ$ are not relevant. The expansions (3.3) and (3.4) show that the falloff of ψ is slower than that of γ. Hence, in the construction of the Bondi system, we can focus primarily on ψ.

B. Averaged time-symmetric case; $J=0$

If $J=0$, several simplifications occur. First, keeping just the first term in ψ in the expansion (3.3) and substituting into Eq. (3.2), we find the asymptotic form of the metric to read

$$\begin{aligned} ds^2 = & -\left[1 + \frac{2L}{\cos^3\theta}\frac{1}{r^2} + \cdots\right]dU^2 \\ & -2\left[1 + \frac{2L}{\cos^3\theta}\frac{1}{r^2} + \cdots\right]dU\,dr - \left[\frac{4L}{\cos\theta}\frac{1}{r^2} + \cdots\right]dr^2 \\ & +\left[1 + \frac{2L}{\cos^3\theta}(\cos^2\theta - \sin^2\theta)\frac{1}{r^2} + \cdots\right]r^2 d\theta^2 \\ & +\left[\frac{8L}{\cos^2\theta} + \cdots\right]\sin\theta\, dr\, d\theta \\ & +\left[1 + \frac{2L}{\cos^3\theta}\frac{1}{r^2} + \cdots\right]r^2\sin^2\theta\, d\phi^2. \end{aligned} \tag{3.5}$$

In order to bring the metric to Bondi's form, we will use the method developed in [11] to analyze space-times with a boost-rotation symmetry. What we need is a coordinate system $\bar U, \bar r, \bar\theta, \bar\phi = \phi$ such that

$$g_{\bar U\bar U} = 1 + O(\bar r^{-1}), \quad g_{\bar U\bar r} = 1 + O(\bar r^{-1}),$$
$$g_{\bar U\bar\theta} = O(1), \quad g_{\bar\theta\bar\theta} = \bar r^2 + O(\bar r), \quad g_{\bar\phi\bar\phi} = \bar r^2\sin^2\bar\theta, \tag{3.6}$$

and, to all orders,

$$g_{\bar r\bar r} = g_{\bar r\bar\theta} = 0, \quad g_{\bar\theta\bar\theta}g_{\bar\phi\bar\phi} = \bar r^4\sin^2\bar\theta. \tag{3.7}$$

Let us suppose the transformation leading to this form may be expanded in powers of $\bar r^{-1}$:

$$U = \pi^0(\bar U,\bar\theta) + \pi^1(\bar U,\bar\theta)\bar r^{-1} + \pi^2(\bar U,\bar\theta)\bar r^{-2} + \cdots,$$
$$r = q(\bar U,\bar\theta)\bar r + \sigma^0(\bar U,\bar\theta) + \sigma^1(\bar U,\bar\theta)\bar r^{-1} + \cdots,$$
$$\theta = \tau^0(\bar U,\bar\theta) + \tau^1(\bar U,\bar\theta)\bar r^{-1} + \tau^2(\bar U,\bar\theta)\bar r^{-2} + \cdots. \tag{3.8}$$

The requirements (3.6) and (3.7) restrict the undetermined functions π, q, σ, τ. From the leading terms of $g_{\bar U\bar U}$, $g_{\bar U\bar\theta}$, we first find that $q_{,\bar U} = \tau^0_{,\bar U} = 0$. The required form of $g_{\bar\theta\bar\theta}$ and $g_{\bar\phi\bar\phi}$ in the leading terms implies $(q)^2\tau^0_{,\bar\theta} = 1$, $(q)^2 = \sin^2\bar\theta/\sin^2\tau^0$. This can be solved for q and τ^0 explicitly; however, further we assume $q = \tau^0_{,\bar\theta} = 1$ since the other choices just correspond to coordinate systems connected by boosts along the symmetry axis [10,11]. Then the requirement on the leading order term in $g_{\bar U\bar r}$ implies that also $\pi^0_{,\bar U} = 1$. The falloff conditions (3.6) are thus satisfied.

The conditions $g_{\bar\theta\bar\theta}g_{\bar\phi\bar\phi} = \bar r^4\sin^2\bar\theta + O(\bar r^2)$ and $g_{\bar r\bar\theta} = O(\bar r^{-1})$ lead to $\sigma^0 = \tau^1 = 0$. It remains only to satisfy the requirements (3.7).

The conditions $g_{\bar r\bar r} = 0$ [to order $O(\bar r^{-2})$], $g_{\bar r\bar\theta} = 0$ [to $O(\bar r^{-1})$] and $g_{\bar\theta\bar\theta}g_{\bar\phi\bar\phi} = \bar r^4\sin^2\bar\theta$ [to $O(\bar r^2)$] determine the functions π^1, $\pi^2, \ldots,$ $\tau^2, \tau^3 \ldots,$ and $\sigma^1, \sigma^2, \ldots$. More specifically, the vanishing of $g_{\bar r\bar r}$ to $\sim\bar r^{-2}$ implies $\pi^1 = 2L/\cos\bar\theta$, $g_{\bar r\bar\theta} = 0$ to $\sim\bar r^{-1}$ leads to $\tau^2 = -(1/2)\pi^1_{,\bar\theta} + 2L\sin\bar\theta/\cos^2\bar\theta$, and $g_{\bar\theta\bar\theta}g_{\bar\phi\bar\phi} = 0$ in order $\bar r^2$ gives $\sigma^1 = -[L/\cos\bar\theta + (1/2)\tau^2_{,\bar\theta}]/\sin\bar\theta - (1/2)\tau^2\cot\bar\theta$. To determine the higher-order functions π, τ, and σ, we have, of course, to consider also the function γ in the metric (3.2). Calculations then become lengthy. Nonetheless, they can be performed and one can thus demonstrate the existence of the Bondi expansion for averaged time-symmetric waves. This establishes the existence of a smooth null infinity in all directions except those perpendicular to the axis of the symmetry.

Now, in axisymmetric space-times, when a spacelike Killing field with circular orbits exists, there is a reduction of the asymptotic symmetry group even if a "global" I does not exist, i.e., even if I does not admit spherical cross sections. Furthermore, in this case, the Bondi news function has a local meaning [12]. One can, therefore, try to find it in the present case. In Bondi's coordinates the news function is given by $c_{,\bar U}$, where the function $c(\bar U,\bar\theta)$ enters, for example, the expansion of $g_{\bar\phi\bar\phi} = \bar r^2\sin^2\bar\theta + 2c\bar r + O(1)$. Starting from our metric (3.2), and using the transformation (3.8) with the functions π, σ, τ, q found above, we obtain $c=0$. Hence the *news function vanishes*. In fact, this could be anticipated since $\psi\sim r^{-2}$ at I—we are here in the region in which the tails of cylindrical pulses decay, and there is no radiation field at null infinity [13]. Thus, in these space-times, the radiation field is all focused in the direction of the two "singular generators" of I singled out by the axis (or, the $\partial/\partial z$-Killing field). Along these generators, the Bondi-Penrose radiation field diverges and asymptotic flatness is lost. In other directions, there is smooth curvature, but no flux of energy.

We conclude this subsection with a remark. In their analysis of isometries compatible with gravitational radiation, Bičák and Schmidt [13] consider axisymmetric space-times, assume Bondi's expansion for all $\phi\in[0,2\pi)$ and $\theta\in(\theta_0,\theta_1)$ and conclude that cylindrical symmetry is not permissible. This assertion may seem to contradict the conclusion we just reached for the time symmetric Einstein-Rosen waves. Note, however, that the interval of permitted θ's in the assertion of [13] contains $\theta=\pi/2$, i.e., the directions perpendicular to the axis of symmetry, while in the present case, Bondi's expansion fails to hold in that direction. Thus, there is in fact no contradiction. In fact, the results obtained in the present work are fully compatible with those of [13]; Bičák and Schmidt conclude below their Eq. (52) that, if the function $c=$ vanishes, the second Killing vector field (in addition to the axial one) can generate either a time translation or the translation along the axis of rotation.

C. Case when $J\neq 0, L=0$

In this case, Eq. (3.3) tells us that the leading order behavior of ψ is different: one obtains $\psi\sim J/r\cos\theta$. Consequently, transformation (3.8) does not now lead to a Bondi system; in particular, it does not remove the "offending" term in $g_{rr}\sim r^{-1}$. Nevertheless, since the leading term in the metric does not depend on time and is $O(r^{-1})$, typical for static Weyl metrics, we can attempt to find the required Bondi system by mimicking the procedure adopted in [10]. Let us assume a transformation of the form

$$U=\overline{U}+\pi(r,\overline{\theta}),$$
$$\theta=\overline{\theta}+\tau^1(\overline{\theta})r^{-1}+\cdots. \tag{3.9}$$

Keeping then just the first term in ψ in the expansion (3.3) with $L=0$, and writing the asymptotic form of the metric analogously to Eq. (3.5), we find that the crucial term $\sim r^{-1}$ in $\overline{g_{rr}}$ will vanish if

$$-(\pi_{,r})^2-2\pi_{,r}+\frac{4J}{r}\cos\theta=0. \tag{3.10}$$

Solving in the leading order for π, we obtain

$$\pi(r,\overline{\theta})=2J\cos\overline{\theta}\ln r+\cdots. \tag{3.11}$$

In this way we can achieve at least $\overline{g_{rr}}\sim O(r^{-2})$. However, with the transformation (3.9) there is no way to satisfy the requirement $\overline{g_{r\theta}}=O(1)$. We must admit a logarithmic term also in the transformation of θ which, in turn, requires another logarithmic term in the transformation of U. By assuming expansions in $r^{-j}\ln^i r$, we find, after some effort, that a suitable transformation reads

$$U=\overline{U}+\left(\frac{2J}{\cos\theta}\cos^2\overline{\theta}\right)\ln r-(2J^2\sin^2\overline{\theta})\frac{\ln^2 r}{r},$$
$$\theta=\overline{\theta}+(2J\sin\overline{\theta})\frac{\ln r}{r}. \tag{3.12}$$

[Notice that in the leading order $(2J/\cos\theta)\cos^2\overline{\theta}=2J\cos\overline{\theta}$ is in agreement with Eq. (3.11).]

Now, transforming the metric (3.2), with ψ and γ given by Eqs. (3.3) and (3.4) (with $J\neq 0, L=0$), via Eq. (3.12), we obtain the metric in the form

$$\begin{aligned}ds^2=&-\left[1-\frac{2J}{\cos\overline{\theta}}\frac{1}{r}+O\left(\frac{\ln r}{r^2}\right)\right]d\overline{U}^2\\&-\left[1-\frac{2J\sin^2\overline{\theta}}{\cos\theta}\frac{1}{r}+O\left(\frac{\ln^2 r}{r^2}\right)\right]2d\overline{U}dr\\&+\left[4J\frac{\ln r}{r}+O\left(\frac{\ln^2 r}{r^2}\right)\right]r\sin\overline{\theta}\,d\overline{U}\,d\overline{\theta}+\left[O\left(\frac{1}{r^2}\right)\right]dr^2\\&-\left[4J\frac{1}{r}+O\left(\frac{\ln^2 r}{r^2}\right)\right]r\sin\overline{\theta}\,dr\,d\overline{\theta}\\&+\left[1+O\left(\frac{\ln^2 r}{r}\right)\right]r^2d\overline{\theta}^2+\left[1+O\left(\frac{\ln^2 r}{r}\right)\right]r^2\sin^2\overline{\theta}\,d\phi^2.\end{aligned} \tag{3.13}$$

Bondi *et al.* [10] applied a similar procedure to the Weyl metrics. In contrast to their result, however, we did not quite succeed in bringing our metric to the standard Bondi form. The reason is that, unlike the Weyl metric, in our case, the leading "offending" terms—proportional to r^{-1}—are θ dependent. [In the case of the transformation of the Weyl metric to Bondi's form —cf. [10]—we have $\pi=2m\ln r+\ldots$, m being the mass. Assuming $m=m(\theta)$ in the Weyl metric (and thus violating the field equations), one can make sure that I still exists, but the space-time is only "logarithmically" asymptotically flat.] By introducing $\tilde{l}=r^{-1}$, $\tilde{U}=U$, $\tilde{\theta}=\theta$, $\tilde{\phi}=\phi$, and rescaling the metric (3.13) by the conformal factor $\Omega=\tilde{l}$, we obtain

$$\begin{aligned}d\tilde{s}^2=\Omega^2ds^2=&-\left(1-\frac{2J}{\cos\tilde{\theta}}\tilde{l}+O(\tilde{l}^2\ln\tilde{l})\right)\tilde{l}^2d\tilde{U}^2\\&+\left(1-\frac{2J\sin^2\tilde{\theta}}{\cos\tilde{\theta}}\tilde{l}+O(\tilde{l}^2\ln^2\tilde{l})\right)2d\tilde{U}\,d\tilde{l}\\&+[-4J(\sin\tilde{\theta})\tilde{l}\ln\tilde{l}+O(\tilde{l}^2\ln^2\tilde{l})]\tilde{l}\,d\tilde{U}\,d\tilde{l}\\&+O(1)d\tilde{l}^2+[4J\sin\tilde{\theta}+O(\tilde{l}\ln^2\tilde{l})]d\tilde{l}\,d\tilde{\theta}\\&+[1+O(\tilde{l}\ln^2\tilde{l})]d\tilde{\theta}^2\\&+[1+O(\tilde{l}\ln^2\tilde{l})]\sin^2\tilde{\theta}\,d\tilde{\phi}^2.\end{aligned} \tag{3.14}$$

Thus, the metric is well behaved as $\tilde{l}\to 0$, i.e., at $\tilde{l}=0$ I does exist. The metric is continuous on I. However, it is not differentiable. Thus, it appears that there is a key difference in the asymptotic behavior in the averaged time-symmetric case and in the general case. In the general case, I appears to have a "logarithmic character" [3,4]. (A word of caution is in order: It is possible that the differentiability can be improved by continuing the transformation (3.12) into higher-order terms.)

To conclude, we wish to point out that, although we obtained the asymptotic forms for ψ and γ [Eqs. (3.3) and (3.4)] assuming that the waves have Cauchy data of compact (ρ,ϕ) support, the forms themselves hold in more general cases as well. An interesting example is provided by the Weber-Wheeler-Bonnor time-symmetric pulse solution [14,15]. [The pulse is formed by a linear superposition of monochromatic waves with a cutoff in the frequency space : $\psi(t,r)=2C\int_0^\infty e^{-a\omega}J_0(\omega\rho)\cos\omega t\, d\omega$, where J_0 is the Bessel function and the constant a is an approximate measure of the width of the pulse. It appears [16] that no other integral containing the Bessel function can be expressed in a closed form, which apparently makes the Weber-Wheeler-Bonnor pulse ''unique'' among nonsingular pulse-type solutions of the wave equation in (2+1) dimensions.] In this case, we have

$$\psi=\sqrt{2}C\left\{\frac{[(a^2+\rho^2-t^2)^2+4a^2t^2]^{1/2}+a^2+\rho^2-t^2}{(a^2+\rho^2-t^2)^2+4a^2t^2}\right\}^{1/2},$$

$$a=\text{const} \quad (3.15)$$

and

$$\gamma=\frac{1}{2}C^2\left\{\frac{1}{a^2}-\frac{2\rho^2[(a^2+\rho^2-t^2)^2-4a^2t^2]}{[(a^2+\rho^2-t^2)^2+4a^2t^2]^2}+\frac{1}{a^2}\frac{\rho^2-a^2-t^2}{[(a^2+\rho^2-t^2)^2+4a^2t^2]^{1/2}}\right\}. \quad (3.16)$$

At $t=0$, the Cauchy data for ψ are $\psi=C(a^2+\rho^2)^{-1/2}$ and $\psi_{,t}=0$. Nevertheless, expressing the asymptotic forms of ψ and γ at $U=t-r=$const, $\theta=$const, $\phi=$const, we find after somewhat lengthy calculations (or by using MATHEMATICA), that ψ and γ have asymptotically *exactly* the form (3.3) and (3.4) with $J=0$ and $L=-2Ca$. Therefore, in the directions not perpendicular to the symmetry axis, these waves do admit a smooth I.

Similarly, the asymptotic forms of ψ and γ with $J\neq 0$ may hold even though the Cauchy data are not of compact support. A simple prototype, discussed by Carmeli [17], for example, has

$$\psi=\frac{1}{2\pi}\frac{f_0}{\sqrt{t^2-\rho^2}},\quad \gamma=\frac{1}{8\pi^2}\frac{f_0^2\rho^2}{(t^2-\rho^2)},\quad f_0=\text{const}. \quad (3.17)$$

This wave is singular at $t^2=\rho^2$, but it represents the late time behavior of the solution given by

$$\psi=\frac{1}{2\pi}\int_{-\infty}^{\tau}\frac{f(t')dt'}{[(t-t')^2-\rho^2]^{1/2}},\quad \tau=t-\rho, \quad (3.18)$$

where $f(t)\neq 0$ only for $0<t<T$; $f_0=\int_0^T f(t')dt'$. With this wave we find ψ and γ to behave (at $U=t-r=$const, $\theta=$const, $\phi=$const, $r\to\infty$) according to Eqs. (3.3) and (3.4) with $J=f_0/2\pi$. The falloff is now slower, but a ''logarithmic'' null infinity I does exist.

ACKNOWLEDGMENTS

A.A. and J.B. would like to thank the Albert Einstein Institut for its kind hospitality. A.A. was supported in part by the NSF Grants No. 93-96246 and No. 95-14240, by the Eberly Research Fund of Penn State University, and by the Erwin Schrödinger International Institute for Mathematical Sciences. J.B. was supported in part by Grant Nos. GACR-202/96/0206 and No. GAUK-230/1996 of the Czech Republic and the Charles University, and by the U.S.-Czech Science and Technology Grant No. 92067.

APPENDIX A: RELATION BETWEEN RIEMANN TENSORS IN THREE AND FOUR DIMENSIONS

The Einstein-Rosen metric (2.1) in coordinates $x^0=u=t-\rho$, $x^1=\rho$, $x^2=\phi$, $x^3=z$ becomes

$$ds^2=e^{2(\gamma-\psi)}(-du^2-2\,du\,d\rho)+\rho^2e^{-2\psi}d\phi^2+e^{2\psi}dz^2. \quad (A1)$$

Assuming the expansion (2.10) for ψ, we know that γ can be written in the form (2.23), and in principle the Riemann (Weyl) tensor of the vacuum (3+1)-dimensional space-time and its asymptotic behavior can be obtained from Eq. (A1).

However, it is possible to use directly the ''reduction formulas'' for the calculation of the Riemann tensor of spaces which admit an Abelian isometry group [7]. In the coordinates $(x^\mu)=(u,\rho,\phi,z)$ in which the Killing trajectories orthogonal to the hypersurfaces $z=$const are just $u=$const, $\rho=$const, $\phi=$const, the four-dimensional components of the Riemann tensor are given by the relations [see Eq. (2.3.4) of [7] where, however, the Riemann tensor with the opposite sign is used]

$${}^{(4)}R_{abcd}=\overline{R_{abcd}},\quad {}^{(4)}R_{3abc}=0,$$

$${}^{(4)}R_{3a3b}=-VV_{||ab},\quad V=e^{\psi}, \quad (A2)$$

where the ''|'' denotes the covariant derivative with respect to the metric $\overline{g_{ab}}$ given by Eq. (A1) with $z=$const. The covariant derivatives $V_{\|ab}$ are given in terms of the Christoffel symbols listed by Stachel [2] in his Eq. (A1) in the Appendix in (u,ρ,ϕ,z) coordinates. [In Stachel's list of Γ's the following symbols are missing: $\Gamma^{\phi}_{\rho\phi}=\rho^{-1}-\psi_{,\rho}$, $\Gamma^{\rho}_{\rho\rho}=2(\gamma_{,\rho}-\psi_{,\rho})$, $\Gamma^{z}_{\rho z}=\psi_{,\rho}$. Notice also that his $x^2=z$, $x^3=\phi$, while here we put $x^3=z$. He treats waves with both polarizations so we must put his $\chi=0$ when comparing his results with ours.] Using these,

$${}^{(4)}R_{3030}=-e^{2\psi}[\psi_{,uu}+3\psi_{,u}^2-2\psi_{,u}\psi_{,\rho}+\psi_{,\rho}^2+\gamma_{,\rho}(\psi_{,u}-\psi_{,\rho})+\gamma_{,u}(\psi_{,\rho}-2\psi_{,u})],$$

$${}^{(4)}R_{3131}=-e^{2\psi}(\psi_{,\rho\rho}+3\psi_{,\rho}^2-2\gamma_{,\rho}\psi_{,\rho}),$$

$${}^{(4)}R_{3232}=-e^{2\psi-2\gamma}\rho^2[2\psi_{,u}\psi_{,\rho}-\psi_{,\rho}^2+\rho^{-1}(\psi_{,\rho}-\psi_{,u})],$$

$${}^{(4)}R_{3031}=-e^{2\psi}(\psi_{,u\rho}+\psi_{,u}\psi_{,\rho}-\gamma_{,\rho}\psi_{,\rho}+\psi_{,\rho}^2). \quad (A3)$$

These are the nonvanishing components ${}^{(4)}R_{3a3b}$ in the coordinates (u,ρ,ϕ,z). Transforming them back to the coordinates (t,ρ,ϕ,z) we find—after projecting them on the orthonormal tetrad used by Stachel—precisely his components R_{0202}, R_{1212}, R_{2323}, and R_{2021} given in his Eqs. (A3). (They have the opposite signs because Stachel uses the signature $+---$.)

The components $\overline{R_{abcd}}$ (formed from the metric $\overline{g_{ab}}$) can be expressed in terms of our (2+1)-dimensional Riemann tensor given in Appendix A of [1]. This is formed from the three-metric $g_{ab}=e^{2\psi}\overline{g_{ab}}$; hence, we use the behavior of the three-dimensional Riemann tensor under conformal rescalings [see, e.g., [7], Eq. (2.4.6)]. We find

$$\overline{R_{abjk}}=2e^{2\psi}R_{abjk}-\frac{1}{2}e^{-4\psi}(g_{j[a}V_{b]k}-g_{k[a}V_{b]j}), \tag{A4}$$

where

$$V_{ik}=2e^{2\psi}[(g^{lm}\psi_{,l}\psi_{,m})g_{ik}-2\psi_{,i}\psi_{,k}-2\psi_{;ik}]. \tag{A5}$$

Here the semicolon denotes the covariant derivative with respect to the three-metric g_{ab} (see Appendix A of [1] for the Christoffel symbols). The nonvanishing quantities V_{ik} turn out to be

$$V_{00}=2e^{2\psi}[2\psi_{,u}\psi_{,\rho}-\psi_{,\rho}^2-2\psi_{,u}^2-2\psi_{,uu}+2\psi_{,u}(2\gamma_{,u}-\gamma_{,\rho}) +2\psi_{,\rho}(\gamma_{,\rho}-\gamma_{,u})]$$

$$V_{01}=V_{10}=2e^{2\psi}(-\psi_{,\rho}^2-2\psi_{,u\rho}+2\gamma_{,\rho}\psi_{,\rho}),$$

$$V_{11}=4e^{2\psi}(-\psi_{,\rho\rho}-\psi_{,\rho}^2+2\gamma_{,\rho}\psi_{,\rho}),$$

$$V_{22}=2e^{2\psi-2\gamma}\rho[\rho(\psi_{,\rho}^2-2\psi_{,\rho}\psi_{,u})+2(\psi_{,u}-\psi_{,\rho})]. \tag{A6}$$

By substituting these expressions into Eq. (3.3) and using the components R_{abcd} from Appendix A of [1], we find R_{abjk} — and thus also ${}^{(4)}R_{abjk}$—in the coordinates (u,ρ,ϕ,z). By transforming them to (t,ρ,ϕ,z), we exactly recover Stachel's expressions given in his Eq. (A3).

[1] A. Ashtekar, J. Bičák, and B. G. Schmidt, preceding paper, Phys. Rev. D **55**, 669 (1997).
[2] J. Stachel, J. Math. Phys. (N.Y.) **7**, 1321 (1966).
[3] J. Winicour, Found. Phys. **15**, 605 (1985).
[4] L. Andersson and P. T. Chruściel, Phys. Rev. Lett. **70**, 2829 (1993); P. T. Chruściel, M. MacCallum, and D. Singleton, Philos. Trans. R. Soc. London **350**, 113 (1995).
[5] J. Bičák and B. G. Schmidt, Phys. Rev. D **40**, 1827 (1989).
[6] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).
[7] P. Jordan, J. Ehlers, and W. Kundt, Akad. Wiss. Lit. Mainz Abh. Math. Naturwiss. Kl. Nr 2 (1960).
[8] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[9] R. Penrose, Phys. Rev. Lett. **10**, 66 (1963); Proc. R. Soc. London **A284**, 159 (1965).
[10] H. Bondi, M. van der Burg, and A. Metzner, Proc. R. Soc. London **A269**, 21 (1962).
[11] J. Bičák, Proc. R. Soc. London **A302**, 201 (1968).
[12] J. Bičák and B. G. Schmidt, Class. Quantum Grav. **6**, 1547 (1989).
[13] J. Bičák and B. G. Schmidt, J. Math. Phys. (N.Y.) **25**, 600 (1984).
[14] J. Weber and J. Wheeler, Rev. Mod. Phys. **29**, 509 (1957).
[15] W. B. Bonnor, J. Math. Mech. **6**, 203 (1957).
[16] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
[17] M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (Wiley, New York, 1982).