PHYSICAL REVIEW D VOLUME 55, NUMBER 2 15 JANUARY 1997

Asymptotic structure of symmetry-reduced general relativity

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(Received 1 August 1996)

Gravitational waves with a space-translation Killing field are considered. Because of the symmetry, the four-dimensional Einstein vacuum equations are equivalent to the three-dimensional Einstein equations with certain matter sources. This interplay between four- and three-dimensional general relativity can be exploited effectively to analyze issues pertaining to four dimensions in terms of the three-dimensional structures. An example is provided by the asymptotic structure at null infinity: While these space-times fail to be asymptotically flat in four dimensions, they can admit a regular completion at null infinity in three dimensions. This completion is used to analyze the asymptotic symmetries, introduce the analogue of the four-dimensional Bondi energy momentum, and write down a flux formula. The analysis is also of interest from a purely three-dimensional perspective because it pertains to a diffeomorphism-invariant three-dimensional field theory with *local* degrees of freedom, i.e., to a midisuperspace. Furthermore, because of certain peculiarities of three dimensions, the description of null infinity has a number of features that are quite surprising because they do not arise in the Bondi-Penrose description in four dimensions. [S0556-2821(97)01802-X]

PACS number(s): 04.20.Ha, 04.30.Nk, 04.60.Kz

I. INTRODUCTION

Einstein-Rosen waves are among the simplest nonstationary solutions to the vacuum Einstein equations (see, e.g., [1]). Not surprisingly, therefore, they have been used in a number of different contexts: investigation of energy loss due to gravity waves [2], asymptotic structure of radiative space-times [3], quasilocal mass [4], the issue of time in canonical gravity [5], and quantum gravity in a simplified but field theoretically interesting context of midisuperspaces [5,6]. These solutions admit two Killing fields, both hypersurface orthogonal, of which one is rotational $\partial/\partial\phi$, and the other translational $\partial/\partial z$, along the axis of symmetry. (In certain applications, the orbits of the Killing field $\partial/\partial z$ are compactified, i.e., are taken to be circles. Our analysis will allow this possibility.) When the hypersurface orthogonality condition is removed, we obtain the cylindrical gravitational waves with *two* polarization modes. These have also been used to explore a number of issues, ranging from the study of Hamiltonian densities [7] and numerical analysis of interacting pulses [8] to the issue of cosmic censorship [9].

The presence of a translational Killing field, however, makes the analysis of the asymptotic structure of these space-times quite difficult: they fail to be asymptotically flat either at spatial or null infinity. Consequently, one cannot use the standard techniques to define asymptotic symmetries or construct the analogues of the Arnowitt-Deser-Misner (ADM) or Bondi energy momenta. Therefore, until recently, conserved quantities for these space-times—such as the C energy [2,7]—were constructed by exploiting the local field equations, without direct reference to asymptotics. It is not *a priori* clear, therefore, that the quantities have the physical interpretation that has been ascribed to them.

What is of physical interest are the values of conserved quantities *per unit length* along the axis of symmetry, i.e., along the integral curves of $\partial/\partial z$; because of the translational symmetry, the total conserved quantities in such a space-time would be clearly infinite. A natural strategy then is to go to the manifold of orbits of the $\partial/\partial z$-Killing field. Since this three-dimensional space-time does not have a translational symmetry, one would expect it to be asymptotically flat in an appropriate sense. Hence, it should be possible to analyze its asymptotic structure unambiguously. In this paper, we will adopt this approach to explore the symmetries and physical fields at null infinity. A similar analysis of spatial infinity was performed recently [10] in the context of the phase space formulation of general relativity. Somewhat surprisingly, it turned out that the C energy is *not* the generator of the time translation which is a unit at infinity; it does not, therefore, represent the Hamiltonian, or the physical energy (per unit z length) in the space-time. The physical Hamiltonian turns out to be a *nonpolynomial* function of the C energy. In the present paper, we will see that the same is true of the analogue of Bondi energy at null infinity.

Thus, the purpose of this paper is to develop a framework to discuss the asymptotic structure at null infinity for three-dimensional space-times. The underlying theory is general relativity coupled to matter fields satisfying appropriate fall-off conditions. The conditions on matter are satisfied, in particular, by the fields that arise from a symmetry reduction of a large class of four-dimensional vacuum space-times admitting a space translation Killing field $\partial/\partial z$. Therefore, we will, in particular, provide a framework for analyzing the behavior of the gravitational field near null infinity of such space-times. Note that these specific applications of our framework are themselves generalizations of cylindrical waves since they need not admit an axial Killing field $\partial/\partial\phi$. Our analysis is also useful in a completely different context; that of quantum gravity. This class of space-times also provides interesting midisuperspace for quantum gravity and our results set the stage for its asymptotic quantization and the corresponding S-matrix theory.

The plan of the paper is as follows. In Sec. II, we will analyze the asymptotic structure of the Einstein-Rosen waves from a three-dimensional perspective. This analysis will motivate our general definition of asymptotic flatness in Sec. III and also provide an intuitive understanding of the main results. In Sec. III, we introduce the notion of asymptotic flatness at null infinity in three space-time dimensions and analyze the structure of asymptotic fields. In Sec. IV, we discuss asymptotic symmetries and in Sec. V, conserved quantities. While the general methods adopted are suggested by the standard Bondi-Penrose treatment of null infinity in four-dimensional general relativity, there are a number of surprises as well. First, in three dimensions, the physical metric g_{ab} is *flat* outside sources. Consequently, there are physically interesting solutions to the constraints which lead to space-times which are flat near spatial infinity i^0; the energy momentum at i^0 is coded, not in local fields such as the curvature, but in a globally defined deficit angle. This simplifies the task of specifying boundary conditions as one approaches i^0 along null infinity I. On the other hand, there are also a number of new complications. In four dimensions, the stationary and the radiative space-times satisfy the same boundary conditions at null infinity. This is not the case in three dimensions. Hence, while dealing with radiative solutions, we cannot draw on our intuition from the stationary case. Second, in four dimensions, up to a supertranslation freedom—which corresponds to terms $O(1/r)$—there is a fixed Minkowskian metric at infinity. In three dimensions, this is not the case; the Minkowski metric η_{ab} to which a physical metric approaches varies even in the leading order, depending on the radiative content of the physical space-time. Consequently, the symmetry group is larger than what one might expect from one's experience in four dimensions. Furthermore, while one can canonically single out the translational subgroup of the Bondi-Matzner-Sachs (BMS) group in four dimensions, now the task becomes subtle; in many ways it is analogous to the task of singling out a preferred Poincaré subgroup of the BMS group. This in turn makes the task of defining the analogue of Bondi energy much more difficult. These differences make the analysis nontrivial and hence interesting.

Some detailed calculations are relegated to the Appendixes. Using Bondi-type coordinates, the asymptotic behavior of curvature tensors of Einstein-Rosen waves is analyzed in the three-dimensional framework in Appendix A. Appendix B considers static cylindrical solutions whose asymptotics, as mentioned above, is quite different from that of the radiative space-times analyzed in the main body of the paper.

It should be emphasized that while part of the motivation for our results comes from the symmetry reduction of four-dimensional general relativity, the main analysis itself refers to three-dimensional gravity coupled to *arbitrary* matter fields (satisfying suitable falloff conditions) which need not arise from a symmetry reduction. Nonetheless, the framework has numerous applications to the four-dimensional theory. For example, in the accompanying paper [11], we will use the results of this paper to study the behavior of Einstein-Rosen waves at null infinity of the *four-dimensional* space-times.

In this paper, the symbol I will generally stand for I^+ or I^-. In the few cases where a specific choice has to be made, our discussion will refer to I^+.

II. EINSTEIN-ROSEN WAVES: ASYMPTOTICS IN THREE DIMENSIONS

This section is divided into three parts. In the first, we recall the symmetry reduction procedure and apply it to obtain the three-dimensional equations governing Einstein-Rosen waves. (See, e.g., [1] for a similar reduction for stationary space-times.) This procedure reduces the task of finding a four-dimensional Einstein-Rosen wave to that of finding a solution to the wave equation on three-dimensional *Minkowski* space. In the second part, we analyze the asymptotic behavior (at null infinity) of these solutions to the wave equation. In the third part, we combine the results of the first two to analyze the asymptotic behavior of space-time metrics. We will find that there is a large class of Einstein-Rosen waves which admits a smooth null infinity I as well as a smooth timelike infinity $i^{\pm}$. (As one might expect, the spacelike infinity i^0 has a conical defect.) These waves provide an important class of examples of the more general framework presented in Sec. III.

A. Symmetry reduction

Let us begin with a slightly more general context, that of vacuum space-times which admit a spacelike, hypersurface orthogonal Killing vector $\partial/\partial z$. These space-times can be described conveniently in coordinates adapted to the symmetry:

$$ds^2 = V^2(x)dz^2 + \overline{g_{ab}}(x)dx^a dx^b, \quad a,b,\ldots = 0,1,2, \tag{2.1}$$

where $x \equiv x^a$ and $\overline{g_{ab}}$ is a three-metric metric with Lorentz signature. As in the more familiar case of static space-times [1], the field equations are

$$\overline{R_{ab}} - V^{-1}\overline{\nabla_a}\,\overline{\nabla_b}V = 0 \tag{2.2}$$

and

$$\overline{g^{ab}}\,\overline{\nabla_a}\,\overline{\nabla_b}V = 0, \tag{2.3}$$

where $\overline{\nabla}$ and $\overline{R_{ab}}$ are the derivative operator and the Ricci tensor of $\overline{g_{ab}}$. These equations can be simplified if one uses a metric in the three-space which is rescaled by the norm of the Killing vector and writes the norm of the Killing vector as an exponential [12,1]. Then, Eqs. (2.1)–(2.3) become

$$ds^2 = e^{2\psi(x)}dz^2 + e^{-2\psi(x)}g_{ab}(x)dx^a dx^b, \tag{2.4}$$

$$R_{ab} - 2\nabla_a\psi\nabla_b\psi = 0, \tag{2.5}$$

and

$$g^{ab}\nabla_a\nabla_b\psi = 0, \tag{2.6}$$

where ∇ denotes the derivative with respect to the metric g_{ab}.

These equations can be reinterpreted purely in a three-dimensional context. To see this, consider Einstein's equations in three dimensions with a scalar field Φ as source:

$$R_{ab}-\tfrac{1}{2}Rg_{ab}=8\pi GT_{ab}$$
$$=8\pi G[\nabla_a\Phi\nabla_b\Phi-\tfrac{1}{2}(\nabla_c\Phi\nabla^c\Phi)g_{ab}], \quad (2.7)$$

$$g^{ab}\nabla_a\Phi\nabla_b\Phi=0. \quad (2.8)$$

Since the trace of Eq. (2.7) gives $R=8\pi G\nabla^c\Phi\nabla_c\Phi$, Eq. (2.7) is equivalent to

$$R_{ab}=8\pi G\nabla_a\Phi\nabla_b\Phi. \quad (2.9)$$

Now, with $\Phi=\psi/\sqrt{4\pi G}$ we obtain Eqs. (2.5) and (2.6). Thus, the four-dimensional vacuum gravity is equivalent to the three-dimensional gravity coupled to a scalar field. Recall that in three dimensions, there is no gravitational radiation. Hence, the local degrees of freedom are all contained in the scalar field. One therefore expects that the Cauchy data for the scalar field will suffice to determine the solution. For data which fall off appropriately, we thus expect the three-dimensional Lorentzian geometry to be asymptotically flat in the sense of Penrose [13], i.e., to admit a two-dimensional boundary representing null infinity.

Let us now turn to the Einstein-Rosen waves by assuming that there is further spacelike, hypersurface orthogonal Killing vector $\partial/\partial\phi$ which commutes with $\partial/\partial z$. Then, as is well known, the equations simplify drastically. Hence, a complete global analysis can be carried out easily. Recall first that the metric of a vacuum space-time with two commuting, hypersurface orthogonal spacelike Killing vectors can always be written locally as [14]

$$ds^2=e^{2\psi}dz^2+e^{2(\gamma-\psi)}(-dt^2+d\rho^2)+\rho^2e^{-2\psi}d\phi^2, \quad (2.10)$$

where ρ and t (the ''Weyl canonical coordinates'') are defined invariantly and $\psi=\psi(t,\rho)$, $\gamma=\gamma(t,\rho)$. (Here, some of the field equations have been used.) Hence, the three-metric g is given by

$$d\sigma^2=g_{ab}dx^adx^b=e^{2\gamma}(-dt^2+d\rho^2)+\rho^2d\phi^2. \quad (2.11)$$

Let us now assume that $\partial/\partial\phi$ is a rotational field in the three-space which keeps a timelike axis fixed. Then the coordinates used in Eq. (2.10) are unique up to a translation $t\rightarrow t+a$. (Note, incidentally, that ''trapped circles'' are excluded by the field equations [9].)

The field equations (2.5) and (2.6) now become

$$R_{tt}=\gamma''-\ddot{\gamma}+\rho^{-1}\gamma'=2\dot{\psi}^2, \quad (2.12)$$

$$R_{\rho\rho}=-\gamma''+\ddot{\gamma}+\rho^{-1}\gamma'=2\psi'^2, \quad (2.13)$$

$$R_{t\rho}=\rho^{-1}\dot{\gamma}=2\dot{\psi}\psi', \quad (2.14)$$

and

$$-\ddot{\psi}+\psi''+\rho^{-1}\psi'=0, \quad (2.15)$$

where the overdot and the prime denote derivatives with respect to t and ρ, respectively. The last equation is the wave equation for the nonflat three-metric (2.11) *as well as for the flat metric obtained by setting* $\gamma=0$. This is a key simplification for it implies that the equation satisfied by the matter source ψ decouples from Eqs. (2.12)–(2.14) satisfied by the metric. These equations reduce simply to

$$\gamma'=\rho(\dot{\psi}^2+\psi'^2), \quad (2.16)$$

$$\dot{\gamma}=2\rho\dot{\psi}\psi'. \quad (2.17)$$

Thus, we can first solve for the axisymmetric wave equation (2.15) for ψ on Minkowski space and then solve Eqs. (2.16) and (2.17) for γ—the only unknown metric coefficient—by quadratures. [Note that Eqs. (2.16) and (2.17) are compatible because their integrability condition is precisely Eq. (2.15).]

B. Asymptotic behavior of scalar waves

In this subsection we will focus on the axisymmetric wave equation in three-dimensional Minkowski space and analyze the asymptotic behavior of its solutions ψ.

We begin with an observation. The ''method of descent'' from the Kirchhoff formula in four dimensions gives the following representation of the solution of the wave equation in three dimensions, in terms of Cauchy data $\Psi_0=\psi(t=0,x,y),\Psi_1=\psi_{,t}(t=0,x,y)$:

$$\psi(t,x,y)=\frac{1}{2\pi}\frac{\partial}{\partial t}\int\int_{S(t)}\frac{\Psi_0(x',y')dx'dy'}{[t^2-(x-x')^2-(y-y')^2]^{1/2}}$$
$$+\frac{1}{2\pi}\int\int_{S(t)}\frac{\Psi_1(x',y')dx'dy'}{[t^2-(x-x')^2-(y-y')^2]^{1/2}}, \quad (2.18)$$

where S is the disk

$$(x-x')^2+(y-y')^2\leqslant t^2 \quad (2.19)$$

in the initial Cauchy surface (see, e.g., [15]). We will assume that the Cauchy data are axially symmetric and of compact support.

Let us first investigate the behavior of the solution at future null infinity I. Let ρ,ϕ be polar coordinates in the plane and introduce the retarded time coordinate

$$u=t-\rho \quad (2.20)$$

to explore the falloff along the constant u null hypersurfaces. Because of axisymmetry, we may put $y=0$ without loss of generality. The integration region becomes

$$(\rho-x')^2+y'^2\leqslant(u+\rho)^2. \quad (2.21)$$

Let us rewrite the integrands of Eq. (2.18) as

$$\frac{\Psi(x',y')dx'dy'}{[2\rho(u+x')+u^2-x'^2-y'^2]^{1/2}} = \frac{1}{(2\rho)^{1/2}}\frac{\Psi(x',y')dx'dy'}{(u+x')^{1/2}}\left(1+\frac{u^2-x'^2-y'^2}{2(u+x')}\frac{1}{\rho}\right)^{-1/2}. \tag{2.22}$$

For large ρ, Eq. (2.22) admits a power series expansion in ρ^{-1} which converges absolutely and uniformly. Hence, we can exchange the integration in Eq. (2.18) with the summation and we can also perform the differentiation $\partial/\partial u$ term by term. Therefore, on each null hypersurface $u=\text{const}$ one can obtain an expansion of the form

$$\psi(u,\rho)=\frac{1}{\sqrt{\rho}}\left(f_0(u)+\sum_{k=1}^{\infty}\frac{f_k(u)}{\rho^k}\right). \tag{2.23}$$

The coefficients in this expansion are determined by integrals over the Cauchy data. These functions are particularly interesting for u so large that the support of the data is completely in the interior of the past cone. One finds

$$f_0(u)=\frac{1}{2\sqrt{2}\pi}\int_0^{\infty}\int_0^{2\pi}\rho' d\rho' d\phi'\left[-\frac{1}{2}\frac{\Psi_0}{(u+\rho'\cos\phi')^{3/2}}+\frac{\Psi_1}{(u+\rho'\cos\phi')^{1/2}}\right]. \tag{2.24}$$

Note that the coefficient is analytic in $u^{-1/2}$, and at $u \gg \rho_0$, ρ_0 being the radius of the disk in which the data are nonzero, we obtain

$$f_0(u)=\frac{k_0}{u^{3/2}}+\frac{k_1}{u^{1/2}}+\cdots, \tag{2.25}$$

where k_0,k_1 are constants which are determined by the data. If the solution happens to be time symmetric, so that Ψ_1 vanishes, we find $f_0 \sim u^{-3/2}$ for large u. This concludes our discussion of the asymptotic behavior along $u=\text{const}$ surfaces.

Finally, we wish to point out that the main results obtained in this section continue to hold also for general data of compact support which are not necessarily axisymmetric. In particular, one obtains an expansion such as Eq. (2.23) where the coefficients now depend on both u and ϕ, and asymptotic forms such as Eq. (2.25). The assumption of compact support can also be weakened to allow data which decay near spatial infinity sufficiently rapidly so that we still obtain solutions smooth at null infinity. (This is, in particular, the case for the Weber-Wheeler pulse considered in the accompanying paper [11].)

C. Asymptotic behavior of the metric

We now combine the results of the previous two subsections. Recall from Eq. (2.11) that the three-dimensional metric g_{ab} has a single unknown coefficient $\gamma(t,\rho)$ which is determined by the solution $\psi(t,\rho)$ to the wave equation in Minkowski space (obtained simply by setting $\gamma=0$). The asymptotic behavior of $\psi(t,\rho)$, therefore, determines that of the metric g_{ab}.

Let us begin by expressing g_{ab} in Bondi-type coordinates $(u=t-\rho,\rho,\phi)$. Then, Eq. (2.11) yields

$$d\sigma^2=e^{2\gamma}(-du^2-2du\,d\rho)+\rho^2 d\phi^2; \tag{2.26}$$

the Einstein equations take the form

$$\gamma_{,u}=2\rho\psi_{,u}(\psi_{,\rho}-\psi_{,u}), \tag{2.27}$$

$$\gamma_{,\rho}=\rho(\psi_{,\rho})^2, \tag{2.28}$$

and the wave equation on ψ becomes

$$-2\psi_{,u\rho}+\psi_{,\rho\rho}+\rho^{-1}(\psi_{,\rho}-\psi_{,u})=0. \tag{2.29}$$

The asymptotic form of $\psi(t,\rho)$ is given by the expansion (2.23). Since we can differentiate Eq. (2.23) term by term, the field equations (2.27) and (2.28) imply

$$\gamma_{,u}=-2[\dot{f}_0(u)]^2+\sum_{k=1}^{\infty}\frac{g_k(u)}{\rho^k}, \tag{2.30}$$

$$\gamma_{,\rho}=\sum_{k=0}^{\infty}\frac{h_k(u)}{\rho^{k+2}}, \tag{2.31}$$

where the functions f_k,h_k are products of the functions $f_0,f_k,\dot{f}_0,\dot{f}_k$. Integrating Eq. (2.31) and fixing the arbitrary function of u in the result using Eq. (2.30), we obtain

$$\gamma=\gamma_0-2\int_{-\infty}^{u}[\dot{f}_0(u)]^2du-\sum_{k=1}^{\infty}\frac{h_k(u)}{(k+1)\rho^{k+1}}. \tag{2.32}$$

Thus, γ also admits an expansion in ρ^{-1} where the coefficients depend smoothly on u.

It is now straightforward to show that the space-time admits a smooth future null infinity I. Setting $\tilde{\rho}=\rho^{-1}, \tilde{u}=u, \tilde{\phi}=\phi$, and rescaling g_{ab} by a conformal factor $\Omega=\tilde{\rho}$, we obtain

$$d\tilde{\sigma}^2=\Omega^2 d\sigma^2=e^{2\tilde{\gamma}}(-\tilde{\rho}^2 d\tilde{u}^2+2d\tilde{u}d\tilde{\rho})+d\tilde{\phi}^2, \tag{2.33}$$

where $\tilde{\gamma}(\tilde{u},\tilde{\rho})=\gamma(u,\tilde{\rho}^{-1})$. Because of Eq. (2.32), $\tilde{\gamma}$ has a smooth extension through $\tilde{\rho}=0$. Therefore, $\tilde{g}_{ab}$ is smooth across the surface $\tilde{\rho}=0$. This surface is the future null infinity I.

Using the expansion (2.23) of ψ near null infinity, various curvature tensors can be expanded in powers of ρ^{-1}. More precisely, a suitable null triad can be chosen which is parallel propagated along $u=\text{const}$, $\phi=\text{const}$ curves. The resulting triad components of the Riemann tensor and the Bach tensor are given in Appendix A. The (conformally invariant) Bach tensor is finite *but nonvanishing* at null infinity. This is to be contrasted with the Bondi-Penrose description of null infinity in asymptotically flat, four-dimensional space-times, where the (conformally invariant) Weyl tensor vanishes. In this sense, while in the standard four-dimensional treatments the metric is conformally flat *at* null infinity, in a three-dimensional treatment, it will not be so in general. This is one of the new complications that one encounters.

To understand the meaning of the constant γ_0 let us consider the solution on the Cauchy surface $t=0$. Equation

(2.16) implies that we can determine γ by a ρ integration from the center. If we insist on regularity at $\rho=0$, we have

$$\gamma(t=0,\rho)=\int_0^\rho \rho(\dot{\psi}^2+\psi'^2)d\rho. \tag{2.34}$$

Hence, for data of compact support, γ_0 is a positive constant whose value is determined by the initial data for ψ:

$$\gamma_0=\gamma(t=0,\rho=\infty)=\int_0^\infty \rho(\dot{\psi}^2+\psi'^2)d\rho. \tag{2.35}$$

This way the constant γ_0 in Eq. (2.32) is uniquely determined for solutions which are regular at $\rho=0$. Its value is given by the total energy of the scalar field ψ computed using the Minkowski metric (obtained from g_{ab} by setting $\gamma=0$).

On a constant t surface, for a point outside the support of the data, we have $\gamma=\gamma_0$, a constant. Hence, outside the support of the data, the three-metric on the Cauchy surface is flat. For any nontrivial data, however, γ_0 is strictly positive, whence the metric has a ''conical singularity'' at spatial infinity: the metric there is given by

$$d\sigma^2=e^{2\gamma_0}(-dt^2+d\rho^2)+\rho^2 d\phi^2. \tag{2.36}$$

Notice that a conical singularity can also be seen near null infinity in this physical metric because the change of the proper circumference of a circle with proper radial distance is different from the case of asymptotically Minkowskian space.

Finally, using Eq. (2.32), we find that, as one approaches I (i.e., $\rho\to\infty$, u=const), we have

$$\gamma(u,\infty)=\gamma_0-2\int_{-\infty}^{u}[\dot{f}_0(u)]^2du. \tag{2.37}$$

Now, a detailed examination [11] of the behavior of the scalar field ψ near timelike infinity i^+ reveals that the space-time is smooth at i^+ and that γ vanishes there. Hence, we obtain the simple result

$$\gamma_0=2\int_{-\infty}^{+\infty}[\dot{f}_0(u)]^2du. \tag{2.38}$$

Thus, there, is a precise sense in which the conical singularity, present at spacelike infinity, is ''radiated out'' and a smooth (in fact, analytic) timelike infinity ''remains.'' We will see that, modulo some important subtleties, Eq. (2.37) plays the role of the Bondi mass-loss formula [16].

III. NULL INFINITY IN THREE DIMENSIONS: GENERAL FRAMEWORK

In this section, we will develop a general framework to analyze the asymptotic structure of the gravitational and matter fields at null infinity in three dimensions along the lines introduced by Penrose in four dimensions. As a special case, when the matter field is chosen to be the massless Klein-Gordon field, we will recover a three-dimensional description of null infinity of generalized cylindrical waves (i.e., of four-dimensional vacuum space-times with one space translation). It turns out that the choice of the fall-off conditions on matter fields is rather subtle in three dimensions. Fortunately, the analysis of the Einstein-Rosen waves presented in Sec. II provides guidelines that restrict the available choices quite effectively.

In Sec. III A, we specify the boundary conditions and discuss some of their immediate consequences. In Sec. III B, we extract the important asymptotic fields and discuss the equations they satisfy at null infinity. Section III C contains an example which, so to say, lies at the opposite extreme from the Einstein-Rosen waves: the simplest solution corresponding to a static point particle in three-dimensions. This example is tailored to bring out certain subtleties which in turn play an important role in the subsequent sections.

A. Boundary conditions

A three-dimensional space-time (M,g_{ab}) will be said to be *asymptotically flat at null infinity* if there exists a manifold $\tilde{M}$ with boundary I which is topologically $S^1\times R$, equipped with a smooth metric $\tilde{g}_{ab}$ such that (i) there is a diffeomorphism between $\tilde{M}-I$ and M (with which we will identify the interior of $\tilde{M}$ and M), (ii) there exists a smooth function Ω on $\tilde{M}$ such that, at I, we have $\Omega=0$, $\nabla_a\Omega\neq0$, and on M, we have $\tilde{g}_{ab}=\Omega^2 g_{ab}$, (iii) if T_{ab} denotes the stress-energy of matter fields on the physical space-time (M,g_{ab}), then ΩT_{ab} admits a smooth limit to I which is *trace-free*, and the limit to I of $\Omega^{-1}T_{ab}\tilde{n}^a\tilde{V}^b$ vanishes, where $\tilde{V}^a$ is any smooth vector field on $\tilde{M}$ which is tangential to I and $\tilde{n}^a=\tilde{g}^{ab}\tilde{\nabla}_b\Omega$, and (iv) if Ω is so chosen that $\tilde{\nabla}^a\tilde{\nabla}_a\Omega=0$ on I, then the vector field $\tilde{n}^a$ is complete on I.

Conditions (i), (ii), and (iv) are the familiar ones from four dimensions and have the following implications. First, since Ω vanishes at I, points of I can be thought of as lying at infinity with respect to the physical metric. Second, since the gradient of Ω is nonzero at I, Ω ''falls off as $1/\rho$.'' Finally, we know that I has the topology $S^1\times R$ and condition (iv) ensures that it is as ''complete in the R direction'' as it is in Minkowski space.

The subtle part is the fall-off conditions on stress energy; these are *substantially weaker* than those in the standard four-dimensional treatment. For instance, in four dimensions, if we use Maxwell fields as sources, then because of conformal invariance, if F_{ab} solves Maxwell's equations on the physical space-time (M,g_{ab}), then $\tilde{F}_{ab}:=F_{ab}$ satisfies them on the completed space-time $(\tilde{M},\tilde{g}_{ab})$. Hence, $\tilde{F}_{ab}$ admits a smooth limit to I. This immediately implies that $\Omega^{-2}T_{ab}$ also admits a smooth limit, where T_{ab} is the stress-energy tensor of F_{ab} in the physical space-time. In the case of a scalar field source, the falloff is effectively the same although the argument is more subtle (see p. 41 in [17]). In three dimensions, on the other hand, we are asking only that ΩT_{ab} admit a limit (although, as noted above, the asymptotic falloff of Ω is the same in three and four dimensions). This is because a stronger condition will have ruled out the cylindrical waves discussed in Sec. II. To see this, consider smooth scalar fields ψ with initial data of compact support. Then, if we set $\tilde{\psi}=\Omega^{-1/2}\psi$, we have the identity

$$\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\psi}-\tfrac{1}{8}\tilde{R}\tilde{\psi}=\Omega^{-5/2}(g^{ab}\nabla_a\nabla_b\psi-\tfrac{1}{8}R\psi),$$

where R and $\tilde{R}$ are the scalar curvatures of g_{ab} and $\tilde{g}_{ab}$, respectively. Hence, $\tilde{\psi}$ is well behaved on I which implies that

$$2\Omega T_{ab} \equiv 2\Omega^2(\tilde{\nabla}_a\tilde{\psi})(\tilde{\nabla}_b\tilde{\psi}) + 2\Omega\tilde{\psi}\tilde{n}_{(a}\tilde{\nabla}_{b}\tilde{\psi} + \tfrac{1}{2}\tilde{\psi}^2\tilde{n}_a\tilde{n}_b - \tfrac{1}{2}\tilde{g}_{ab}[\Omega^2\tilde{\nabla}^m\tilde{\psi}\tilde{\nabla}_m\tilde{\psi} + \Omega\tilde{\psi}\tilde{n}^m\tilde{\nabla}_m\tilde{\psi} + \tilde{n}^m\tilde{n}_m\tilde{\psi}^2] \quad (3.1)$$

admits a well-defined, nonzero limit at I satisfying the conditions of our definition. Hence, stronger falloff requirements on T_{ab} would have made the framework uninteresting. We will see that this weak fall-off is responsible for a number of surprises in the three-dimensional theory. Could we have imposed even weaker falloff conditions? The requirement of smoothness on $\tilde{g}_{ab}$, Ω, and ΩT_{ab} can be substantially weakened: All our analysis will go through if $\tilde{g}_{ab}$ and Ω are only C^3, and ΩT_{ab} only C^1 at I. On the other hand, we will see that the condition on the trace of ΩT_{ab} is necessary to endow I with interesting structure. We will see that the vanishing of the limit of $\Omega^{-1}T_{ab}\tilde{n}^a\tilde{V}^b$ is necessary to ensure that the energy and supermomentum fluxes of matter across (finite patches of) I are finite.

Let us now examine the structure available at the boundary I.

As in four dimensions, it is convenient to work entirely with the tilde fields which are smooth at I. Let us set

$$\tilde{L}_{ab} = \Omega(R_{ab} - \tfrac{1}{4}Rg_{ab}) =: \Omega S_{ab}$$

and lower and raise its indices with $\tilde{g}_{ab}$ and its inverse. $\tilde{L}_{ab}$ carries the same information as the stress-energy tensor T_{ab} of matter and our conditions on T_{ab} ensure that $\tilde{L}_{ab}$ is smooth at I. Set

$$\overline{f} = \Omega^{-1}\tilde{n}^a\tilde{n}_a .$$

Then, using the expression $R_{abcd} = 2(S_{a[c}g_{d]b} - S_{b[c}g_{d]a})$ of the Riemann tensor in three dimensions, the formula expressing the relation between curvature tensors of g_{ab} and $\tilde{g}_{ab}$ reduces to

$$\Omega\tilde{S}_{ab} + \tilde{\nabla}_a\tilde{n}_b - \tfrac{1}{2}\overline{f}\tilde{g}_{ab} = \tilde{L}_{ab}, \quad (3.2)$$

where $\tilde{S}_{ab} = (\tilde{R}_{ab} - \tfrac{1}{4}\tilde{R}\tilde{g}_{ab})$. This is the basic field equation in the tilde variables. Since all other fields which feature in it are known to be smooth at I, it follows that $\overline{f}$ is also smooth. This implies, in particular, that $\tilde{n}^a$ is null. Since $\tilde{n}_a = \tilde{\nabla}_a\Omega$ is the normal field to I, we conclude that *I is a null surface*.

Next, we note that there is a considerable freedom in the choice of the conformal factor Ω. Indeed, if $(\tilde{M}, \tilde{g}_{ab} = \Omega^2 g_{ab})$ is an allowable completion, so is $(\tilde{M}, \Omega'^2 g_{ab})$ where $\Omega' = \omega\Omega$ for any smooth, nowhere-vanishing function ω on $\tilde{M}$. Now, under the conformal transformation $\Omega \mapsto \Omega' = \omega\Omega$, we have

$$\tilde{\nabla}'_a\tilde{n}'^a \cong \omega^{-1}\tilde{\nabla}_a\tilde{n}^a + 3\omega^{-2}\mathcal{L}_{\tilde{n}}\omega,$$

where, from now on, $\cong$ will stand for *''equals at the points of I to.''* Hence, by using an appropriate ω, we can always make $\tilde{n}'^a$ divergence-free. Such a choice will be referred to as a *divergence-free conformal frame*. This frame is, however, not unique. The restricted gauge freedom is given by

$$\Omega \mapsto \omega\Omega, \text{where} \quad \mathcal{L}_{\tilde{n}}\omega \cong 0. \quad (3.3)$$

Now, condition (iv) in our definition requires that, in any divergence-free conformal frame, the vector field $\tilde{n}^a$ be complete on I. Suppose it is so in one divergence-free conformal frame Ω. Let Ω' correspond to another divergence-free frame. Then, $\Omega' = \omega\Omega$, with ω smooth, nowhere vanishing and satisfying $\mathcal{L}_{\tilde{n}}\omega \cong 0$. The last equation implies that $\tilde{n}'^a$ is complete on I if and only if $\tilde{n}^a$ is complete there. Hence, we need to verify (iv) in just one divergence-free conformal frame. *In what follows, we will work only in divergence-free conformal frames.*

Next, taking the trace of Eq. (3.1) and using the fact that $\tilde{L}$ vanishes on I we conclude that, in any divergence-free frame, $\overline{f}$ vanishes on I, whence

$$\tilde{f} := \Omega^{-1}\overline{f}$$

admits a smooth limit there. The field $\tilde{f}$ will play an important role. Finally, it is easy to check that in any divergence-free conformal frame, we have

$$\tilde{n}^b\tilde{\nabla}_b\tilde{n}_a \cong 0 \quad \text{and} \quad \tilde{L}_{ab}\tilde{n}^b \cong 0. \quad (3.4)$$

Thus, in particular, as in four dimensions, I is ruled by null geodesics. The space $\mathcal{B}$ of orbits of $\tilde{n}^a$—the ''base space'' of I—is diffeomorphic to S^1. The second equation and the trace-free character of $\tilde{L}_{ab}$ imply that, *on I*, $\tilde{L}_{ab}$ has the form

$$\tilde{L}_{ab} \cong \tilde{L}_{(a}\tilde{n}_{b)} \quad \text{with} \quad \tilde{L}_a\tilde{n}^a \cong 0, \quad (3.5)$$

for some smooth co-vector field $\tilde{L}_a$. Hence, the pullback to I of $\tilde{L}_{ab}$ vanishes which in turn implies, via Eq. (3.1), that the pullback to I of $\tilde{\nabla}_a\tilde{n}_b$ also vanishes. Hence, if we denote by $\tilde{q}_{ab}$ the pullback of $\tilde{g}_{ab}$, we have

$$\mathcal{L}_{\tilde{n}}\tilde{q}_{ab} \cong 0. \quad (3.6)$$

Since I is null, it follows that

$$\tilde{q}_{ab}\tilde{n}^b \cong 0 . \quad (3.7)$$

Thus, $\tilde{q}_{ab}$ is the pullback to I of a positive definite metric on the manifold of orbits $\mathcal{B}$ of the vector field $\tilde{n}^a$. By construction, $\mathcal{B}$ is a one-dimensional manifold with topology of S^1. Hence, there exists a one-form $\tilde{m}_a$ on I such that

$$\tilde{q}_{ab} = \tilde{m}_a\tilde{m}_b . \quad (3.8)$$

[In cylindrical waves, $\tilde{m}_a$ is the pullback to I of $\tilde{\nabla}_a\phi$ and $\tilde{n}^a$ equals $\exp(-2\tilde{\gamma})(\partial/\partial u)$ on I.] Under a conformal rescaling $\Omega \mapsto \omega\Omega$ (from one divergence-free frame to another), we have

$$\tilde{m}_a \mapsto \omega\tilde{m}_a \quad \tilde{n}^a \mapsto \omega^{-1}\tilde{n}^a. \quad (3.9)$$

The pairs $(\tilde{m}_a, \tilde{n}^a)$ [or, equivalently, $(\tilde{q}_{ab}, \tilde{n}^a)$] are the kinematical fields which are ''universal'' to I: In any asymptotically flat space-time, we obtain the same collection of pairs. This situation is analogous to that in four dimensions where pairs $(\tilde{q}_{ab},\tilde{n}^a)$ constitute the universal kinematic structure. However, whereas the four-metric evaluated *at I* has no dynamical content, in the present case, the three-metric *at I* does carry dynamical content and varies from one space-time to another.

B. Asymptotic fields

The pairs $(\tilde{q}_{ab},\tilde{n}^a)$ on I represent the leading or the ''zeroth order'' structure at I. The next, in the hierarchy, is an intrinsic derivative operator. Let $\tilde{K}_b$ be a smooth co-vector field on $\tilde{M}$, and $\underline{\tilde{K}_b}$, its pullback to I. Define:

$$\tilde{D}_a\underline{\tilde{K}_b} := \underline{\tilde{\nabla}_a\tilde{K}_b}, \tag{3.10}$$

where the underbar on right-hand side denotes the pullback to I. [Since $\underline{\tilde{K}_b}=\underline{\tilde{K}'_b}$ if and only if $\tilde{K}'_b=\tilde{K}_b+\tilde{h}\tilde{n}_b+\Omega\tilde{W}_b$ for some smooth $\tilde{h}$ and $\tilde{W}_b$, $\tilde{D}$ is a well-defined operator if and only if the pullback to I of $\tilde{\nabla}_a(\tilde{h}\tilde{n}_b+\Omega\tilde{W}_b)$ vanishes. It is easy to check that it does.] In four dimensions, the two radiative degrees of freedom of the gravitational field are coded in this intrinsic derivative operator [18]. In three dimensions, on the other hand, there is no ''pure'' gravitational radiation. Hence, one would expect that the derivative operator $\tilde{D}$ has no invariant physical content. This is indeed the case.

To see this, note first that given any vector field $\tilde{V}^a$ tangential to I we have

$$\tilde{V}^a\tilde{D}_a\tilde{q}_{ab}\cong 0 \quad \text{and} \quad \tilde{V}^a\tilde{D}_a\tilde{n}^b\cong\tilde{V}^a\tilde{L}_a^b,$$

where, in the second equation, we have used Eq. (3.6). Now, for a zero rest mass scalar field (i.e., for four-dimensional Einstein-Rosen waves), $\tilde{L}_{ab}\cong\frac{1}{2}\tilde{\psi}^2\tilde{n}_a\tilde{n}_b$, whence $\tilde{V}^a\tilde{D}_a\tilde{n}^a\cong 0$. Hence, the difference between any two permissible derivative operators on I is given by

$$(\tilde{D}'_a-\tilde{D}_a)\tilde{K}_b\cong\tilde{C}^c_{ab}\tilde{K}_c \quad \text{with} \quad \tilde{C}^c_{ab}=\tilde{\Sigma}_{ab}\tilde{n}^c,$$

where $\tilde{K}_b$ is any covector field on I and $\tilde{\Sigma}_{ab}$, a symmetric tensor field on I, transverse to $\tilde{n}^a$, $\tilde{\Sigma}_{ab}\tilde{n}^a\cong 0$. Thus, $\tilde{\Sigma}_{ab}\cong g\tilde{m}_a\tilde{m}_b$ for some function g on I. Now, if we make a conformal transformation $\Omega\mapsto\Omega'=(1+\omega_1\Omega)\Omega$ the derivative operator $\tilde{D}$ changes through: $(\tilde{D}'_a-\tilde{D}_a)K_b=\omega_1\tilde{m}_a\tilde{m}_b\tilde{n}^cK_c$, *even though the transformation leaves* $\tilde{m}_a$ *and* $\tilde{n}^a$ *invariant*. Thus, as in four dimensions, the ''trace part'' of $\tilde{\Sigma}_{ab}$ is ''pure gauge.'' Now, in four dimensions, the degrees of freedom of the gravitational field reside in the trace-free part of $\tilde{\Sigma}_{ab}$ [18]. For the three-dimensional description of Einstein-Rosen waves, by contrast, since $\tilde{\Sigma}_{ab}$ is itself pure trace, the trace-free part vanishes identically, reflecting the absence of pure gravitational degrees of freedom.

In four dimensions, the Bondi news—which dictate the fluxes of energy momentum carried away by gravity waves—is coded in the curvature of $\tilde{D}$. By contrast, in the general three-dimensional case (i.e., without restriction on the form of matter sources), we can always make the curvature vanish by going to an appropriate conformal frame. To see this, recall first that, since I is two dimensional, the full curvature of any connection is determined by a scalar. For connections under consideration, we have $2\tilde{D}_{[a}\tilde{D}_{b]}K_c=\tilde{R}\tilde{\epsilon}_{ab}\tilde{m}_c\tilde{n}^dK_d$, where $\tilde{\epsilon}_{ab}$ is the obvious alternating tensor on I. (Thus, $\tilde{\epsilon}_{ab}=2\tilde{l}_{[a}\tilde{m}_{b]}$, where $\tilde{l}_a$ is a null co-vector field on I satisfying $\tilde{l}_a\tilde{n}^a=1$.) Under conformal rescalings $\Omega\mapsto\Omega'=(1+\omega_1\Omega)\Omega$, we have $\tilde{R}\mapsto\tilde{R}'=\tilde{R}+\mathcal{L}_{\tilde{n}}\omega_1$. Thus, by choosing an appropriate ω_1, we can always set $\tilde{R}'=0$. There is no invariant physical information in the curvature of the derivative operator $\tilde{D}$ intrinsic to I.

Let us, therefore, examine the curvature of the full three-dimensional connection $\tilde{\nabla}$. Using Eq. (3.2) and the Bianchi identity of the rescaled metric $\tilde{g}_{ab}$, we have

$$2\tilde{S}_{ab}\tilde{n}^a+\tilde{\nabla}_b(\Omega\tilde{f})=\tilde{\nabla}^a\tilde{L}_{ab}-\tilde{\nabla}_b\tilde{L}, \tag{3.11}$$

where $\tilde{L}=\tilde{g}^{ab}\tilde{L}_{ab}$. The Bianchi identity for the physical metric g_{ab} implies that the right-hand side of Eq. (3.11) is given by $2\Omega^{-1}\tilde{L}_{ab}\tilde{n}^a$. Hence, combining the two, we have

$$2\tilde{S}_{ab}\tilde{n}^a+\Omega\tilde{\nabla}_b\tilde{f}+\tilde{f}\tilde{n}_b=2\Omega^{-1}\tilde{L}_{ab}\tilde{n}^a. \tag{3.12}$$

These, together with Eq. (3.1), are the basic equations that govern the asymptotic dynamics.

Our assumptions on the stress-energy tensor imply that $\Omega^{-1}\tilde{L}_{ab}\tilde{n}^a\tilde{V}^b$ vanishes on I for any vector $\tilde{V}^a$ tangential to I. Equation (3.12) now implies $\tilde{S}_{ab}\tilde{n}^a\tilde{V}^b\cong 0$. Hence, the pullback $\underline{S}_{ab}$ to I of $\tilde{S}_{ab}$ has the form

$$\underline{S}_{ab}=\underline{S}\tilde{m}_a\tilde{m}_b.$$

Similarly, since $\tilde{L}_{ab}$ is trace-free on I and since $\tilde{L}_{ab}\tilde{n}^b$ vanishes there [cf. Eqs. (3.4) and (3.5)], the pullback $\underline{L}_{ab}$ of $\Omega^{-1}L_{ab}$ to I exists and has the form

$$\underline{L}_{ab}=\underline{L}\tilde{m}_a\tilde{m}_b.$$

The field

$$\tilde{B}:=\underline{S}-\underline{L} \tag{3.13}$$

will play an important role in what follows.

The Bach tensor $\tilde{B}_{abc}$—vanishing of which is a necessary and sufficient condition for conformal flatness in three dimensions—is given by

$$\tilde{B}_{abc}=2\tilde{\nabla}_{[b}\tilde{S}_{c]a}=2\Omega^{-1}(\tilde{\nabla}_{[b}\tilde{L}_{c]a}-\Omega^{-1}\tilde{n}^m\tilde{g}_{a[b}\tilde{L}_{c]m}). \tag{3.14}$$

Thus, the Bach tensor is nonzero only in the presence of matter. Note that, in general, it does not vanish even at I. This is in striking contrast with the situation in four dimensions where the Weyl tensor of the rescaled metric *does* vanish at I. We will see that the fact that in three dimensions we do not have conformal flatness even *at I* makes the discussion of asymptotic symmetries much more difficult. Transvecting the Bach tensor with $\tilde{n}^a$ and pulling the result back to I, we obtain

$$\underline{\tilde{n}^a\tilde{B}_{abc}}\cong-\mathcal{L}_{\tilde{n}}\underline{S}_{bc}\cong-\mathcal{L}_{\tilde{n}}\underline{L}_{bc}-(\lim_{\mapsto I}\Omega^{-2}\tilde{n}^m\tilde{n}^n\tilde{L}_{mn})\tilde{q}_{bc}. \tag{3.15}$$

Since the last term in this equation has the form of the flux of ''matter energy'' across I [it equals $2(\mathcal{L}_{\tilde n}\tilde\psi)^2$ in the case of Einstein-Rosen waves, cf. Eq. (3.1)], it is tempting to interpret this equation as the analogue of the local Bondi conservation law on I in four dimensions. Let us rewrite this equation in a more convenient form:

$$\tilde D_{[a}(\underline{S}-\underline{L})\tilde m_{b]} = \frac{1}{2}\lim_{\mapsto I}[\Omega^{-2}(\underline{L}_{mn}\tilde n^m \tilde n^n)\tilde\epsilon_{ab}]. \quad (3.16)$$

Then, it is tempting to regard the one-form $\tilde B \tilde m_a \cong (\underline{S}-\underline{L})\tilde m_a$ as the analogue of the four-dimensional ''Bondi mass aspect.'' Let us, therefore, study its conformal properties. Under a rescaling $\Omega \mapsto \Omega' = \omega\Omega$, we have

$$\tilde B \tilde m_a \mapsto \tilde B' \tilde m'_a = [\omega^{-1}\tilde B - \omega^{-2}\tilde m^a \tilde m^b \tilde D_a \tilde D_b \omega + \tfrac{3}{2}\,\omega^{-3}(\tilde m^a \tilde D_a \omega)^2]\tilde m_a, \quad (3.17)$$

where $\tilde m^a$ is a vector field tangential to I satisfying $\tilde m^a \tilde m_a = 1$. Note that the transformation law involves only the values of ω *on* I; unlike in the transformation law for $\tilde R$, discussed above, the field ω_1 (which measures the first derivative of ω off I) never enters. This transformation law will play an important role in the next two sections.

Finally, we note an identity which enables us to express, at I, the quantity $\tilde B$ constructed from the curvatures of $\tilde g_{ab}$ and g_{ab} in terms of the metric coefficients. To see this, recall first that the derivative operator $\tilde D$ within I is obtained by ''pulling back'' the space-time derivative operator $\tilde\nabla$ to I. Hence one can express the curvature $\tilde R$ of $\tilde D$ in terms of the curvature $\tilde S_{ab}$ of $\tilde \nabla$. Using the Bianchi identity (3.10) to express some of the components of $\tilde S_{ab}$ in terms of matter fields, we obtain

$$\tilde B \cong \underline{S} - \underline{L} \cong \tilde f - \tilde R. \quad (3.18)$$

Thus, in a conformal frame in which $\tilde R$ is zero, the analogue $\tilde B$ of the Bondi-mass aspect can be computed directly from the metric coefficient $\tilde f = \Omega^{-2}\tilde g_{ab}\tilde n^a \tilde n^b$. For the Einstein-Rosen waves, for example, it is straightforward to check that the completion given in Sec. II satisfies the condition $\tilde R = 0$ and by inspection $\tilde f$ is given by $\exp(-2\tilde\gamma)$. Thus, in practice, Eq. (3.18) often provides an easy way to calculate $\tilde B$. Finally, note that, under conformal rescalings $\Omega \mapsto (1+\omega_1\Omega)\Omega$, both $\tilde f$ and $\tilde R$ transform nontrivially. However, the combination $\tilde f - \tilde R$ remains unchanged.

C. Point particle

In this subsection, we will consider the simplest point-particle solution to three-dimensional gravity and, using the results obtained in the last two subsections, study its behavior at null infinity.

In an obvious coordinate system adapted to the world line of the point particle, the physical space-time metric g_{ab} is given by [19]

$$d\sigma^2 = -dt^2 + r^{-8GM}(dr^2 + r^2 d\phi^2),$$

where $-\infty < t < \infty, 0 < r < \infty$, and $0 \leqslant \phi < 2\pi$. The particle has mass M and ''resides'' at the origin. Since the stress-energy tensor vanishes everywhere outside the $r=0$ world line (which is excised from the space-time), the metric is flat outside the origin. We can transform it in a manifestly flat form by setting

$$\rho = \frac{r^\alpha}{\alpha}, \overline{\phi} = |\alpha|\phi, \quad \text{where} \quad \alpha = 1 - 4GM.$$

(Note that $\overline{\phi}$ now ranges in $[0, 2\pi|\alpha|)$.) In terms of these coordinates, the metric is given by

$$d\sigma^2 = -dt^2 + d\rho^2 + \rho^2 d\overline{\phi}^2. \quad (3.19)$$

Although the metric is manifestly flat, it fails to be globally Minkowskian because of the range of $\overline{\phi}$; there is a conical singularity at the origin and the resulting deficit angle measures the mass.

It is straightforward to conformally complete this space-time to satisfy our definition of asymptotic flatness. Setting $u = t - \rho$ and $\Omega = 1/\rho$, the rescaled metric $\tilde g_{ab}$ is given by

$$d\tilde\sigma^2 := \Omega^2 d\sigma^2 = -\Omega^2 du^2 + 2du\,d\Omega + d\overline{\phi}^2. \quad (3.20)$$

It is trivial to check that the completion satisfies all our conditions and that the conformal frame is divergence-free. The kinematic fields are given by $\tilde n^a \equiv \partial/\partial u$ and $\tilde m_a = \tilde D_a \overline{\phi}$. By inspection $\tilde f = 1$ and a simple calculation shows that $\tilde R = 0$. Thus, $\tilde B = 1/2$; it carries no information about mass. This information is hidden in the deficit angle: Integrating $\tilde m_a$ on the base space $\mathcal{B}$, we have

$$\oint_{\mathcal{B}} \tilde m_a dS^a = 2\pi\alpha = 2\pi(1 - 4GM).$$

In four dimensions, one often insists that the conformal frame be such that the metric on the base space be a unit two-sphere metric. These are the Bondi conformal frames. The obvious analogue in three dimensions is to ask that the frame be such that the length of the base space be equal to 2π, the length of a unit circle. (Although this restriction is very weak, it seems to be the only viable analogue of the Bondi restriction in four dimensions.) The completion we gave above does not satisfy this condition. However, it is trivial to rectify this situation through a (constant) conformal rescaling. Set $\Omega' = (1/\alpha)\Omega$. Then,

$$d\tilde\sigma'^2 = -\Omega'^2 du^2 + \frac{2}{\alpha} du\,d\Omega' + d\phi^2, \quad (3.21)$$

where $\phi = (1/|\alpha|)\overline{\phi}$ ranges over $[0, 2\pi)$; the base space $\mathcal{B}$ is a circle of length 2π as required. Since we have performed a *constant* rescaling, we have $\tilde R' = 0$. However, $\tilde f$ does change: $\tilde f' = \alpha^2$. Thus, in the ''Bondi-type'' frame, mass resides in $\tilde B$: Since $\tilde B = \frac{1}{2}\alpha^2$ in this frame, the mass is given by

$$M = \frac{1}{4G}(1 - \sqrt{2\tilde B}). \quad (3.22)$$

Thus, our expectation of the last subsection that $\tilde{B}$ would be the analogue of the Bondi-mass aspect is correct. However, to arrive at this interpretation, we must use a properly normalized (''Bondi-like'') conformal frame. This point will be important in Sec. V.

We will conclude this discussion with two remarks.

The metric considered in this subsection is stationary and so it is appropriate to compare the situation we encountered with that in four-dimensional stationary space-times. In both cases, the stationary Killing field selects a preferred rest frame at I (which, in our example, is given by the time translation $\partial/\partial u$). However, in four dimensions, one can find *asymptotic* Killing fields corresponding to space translations as well. In the present case, on the other hand, due to the conical singularity, globally defined space-translation vector fields fail to exist *even asymptotically* (unless $M=0$ in which case the deficit angle vanishes). For example, we can introduce Cartesian coordinates $t,\overline{x},\overline{y}$ corresponding to t,ρ,ϕ. Then, $\overline{X}^a \equiv \partial/\partial\overline{x}$ and $\overline{Y}^a \equiv \partial/\partial\overline{y}$ *are* local Killing fields. However, the chart itself fails to be globally defined and so do the vector fields. Another strategy is suggested by what happens in Minkowski space-time. In any of its standard completions space translations are represented by the vector fields $(\cos\phi)\tilde{n}^a$ and $(\sin\phi)\tilde{n}^a$ on I. In the ''Bondi-like'' conformal frame introduced above these vector fields are globally defined at null infinity of our point-particle space-time as well. However, now they fail to be Killing fields even asymptotically.

The second remark is that the stationary space-time we considered here is a very special solution. Generic stationary solutions in three-dimensional general relativity have a logarithmic behavior near infinity and, therefore, fail to satisfy our definition of asymptotic flatness at null infinity. [See Appendix B. Our point-particle solution corresponds essentially to the special case $C=0$ in Eqs. (B2) and (B3).] This is another key difference between three and four dimensions.

IV. ASYMPTOTIC SYMMETRIES

In four dimensions, the asymptotic symmetry group at null infinity is given by the BMS group [13,16,17,20]. Its structure is the same as that of the Poincaré group in that it is a semidirect product of an Abelian group with the Lorentz group. The Abelian group, however, is *infinite* dimensional; it is the additive group of functions on a two-sphere (the base space of I) with conformal weight $+1$. It is called the group of supertranslations. The four-dimensional group of translations can be invariantly singled out. However, unless additional conditions are imposed (near i^0 or i^+), the BMS group does not admit a preferred Lorentz or Poincaré subgroup. This enlargement from the ten-dimensional Poincaré group to the infinite-dimensional BMS group is brought about because, in the presence of gravitational radiation, one cannot single out a preferred Minkowski metric even at infinity; one can only single out a family of Minkowskian metrics and they are related by super translations.

In this section, we will examine the asymptotic symmetry group in three dimensions. One's first impulse is to expect that the situation would be completely analogous to that in four dimensions since the ''universal structure'' available at I in the two cases is essentially the same. It turns out, however, that because the space-time metric is dynamical even at infinity—i.e., because in general the physical metric does not approach a Minkowskian metric even to the leading order—the group of asymptotic symmetries is now enlarged even further. Furthermore, now it is not possible to single out even the group of translations without additional conditions.

This section is divided into two parts. The first discusses the asymptotic symmetry group and the second introduces additional conditions to single out translations.

A. Asymptotic symmetry group

Let us begin by recalling the universal structure, i.e., the structure at infinity that is common to all asymptotically flat space-times. As usual, the asymptotic symmetries will then be required to preserve this structure.

Given *any* space-time satisfying our definition of asymptotic flatness and *any* conformal completion thereof, its null infinity I is a two-manifold, topologically $S^1\times R$. It is ruled by a (divergence-free) null vector field $\tilde{n}^a$ and its intrinsic, degenerate metric $\tilde{q}_{ab}$ satisfies

$$\tilde{q}_{ab}\tilde{V}^b \cong 0 \quad \text{if and only if} \quad \tilde{V}^b \propto \tilde{n}^b, \tag{4.1}$$

where $\tilde{V}^b$ is an arbitrary vector field on I. The ''base space'' $\mathcal{B}$ of I, i.e., the space of integral curves of $\tilde{n}^a$ on I, has the topology of S^1. As in four dimensions, the intrinsic metric $\tilde{q}_{ab}$ on I is the pullback to I of a metric $\overline{q}_{ab}$ on $\mathcal{B}$; that is, $\mathcal{L}_{\tilde{n}}\tilde{q}_{ab}=0$. Next, we have the conformal freedom given in Eq. (3.3). Thus, I is equipped with an equivalence class of pairs $(\tilde{q}_{ab},\tilde{n}^a)$ satisfying Eqs. (4.1) and (3.6), where two are considered as equivalent if they differ by a conformal rescaling: $(\tilde{q}_{ab},\tilde{n}^a)\approx(\omega^2\tilde{q}_{ab},\omega^{-1}\tilde{n}^a)$, with $\mathcal{L}_{\tilde{n}}\omega=0$. This structure is completely analogous to that at null infinity of four-dimensional asymptotically flat space-times.

As we already saw, in three dimensions, a further simplification occurs: in any conformal frame, I admits a unique covector field $\tilde{m}_a$ such that $\tilde{q}_{ab}=\tilde{m}_a\tilde{m}_b$. Hence, in the universal structure, we can replace $\tilde{q}_{ab}$ by $\tilde{m}_a$. Thus, I is equipped with equivalence classes of pairs $(\tilde{m}_a,\tilde{n}^a)$, satisfying

$$\tilde{m}_a\tilde{n}^a \cong 0 \quad \text{and} \quad \mathcal{L}_{\tilde{n}}\tilde{m}_a \cong 0, \tag{4.2}$$

where $(\tilde{m}_a,\tilde{n}^a)\approx(\omega\tilde{m}_a,\omega^{-1}\tilde{n}^a)$ for any nowhere-vanishing smooth function ω on I satisfying $\mathcal{L}_{\tilde{n}}\omega=0$. Note that the second of Eqs. (4.2) implies that $\tilde{m}_a$ is the pullback to I of a covector field $\overline{m}_a$ on the base space $\mathcal{B}$.

The asymptotic symmetry group $\mathcal{G}$ is the subgroup of the diffeomorphism group of I which preserves this structure. An infinitesimal asymptotic symmetry is, therefore, a vector field $\tilde{\xi}^a$ on I satisfying

$$\mathcal{L}_{\tilde{\xi}}\tilde{m}_a \cong \tilde{\alpha}\tilde{m}_a \quad \text{and} \quad \mathcal{L}_{\tilde{\xi}}\tilde{n}^a \cong -\tilde{\alpha}\tilde{n}^a, \tag{4.3}$$

for some smooth function $\tilde{\alpha}$ (which depends on $\tilde{\xi}^a$) satisfying $\mathcal{L}_{\tilde{n}}\tilde{\alpha}\cong 0$. Equations (4.3) ensure that the one-parameter family of diffeomorphisms generated by $\tilde{\xi}^a$ preserves the ''ruling'' of I by the integral curves of its null normal, its divergence-free character, and maps pair $(\tilde{m}_a,\tilde{n}^a)$ to an equivalent one, thereby preserving each equivalence class. It is easy to check that vector fields satisfying Eqs. (4.3) form a

Lie algebra which we will denote by $\mathcal{LG}$. This is the Lie algebra of infinitesimal asymptotic symmetries.

To unravel the structure of $\mathcal{LG}$, we will proceed as in four dimensions. Let $\mathcal{LS}$ denote the subspace of $\mathcal{LG}$ spanned by vector fields of the type $\tilde{\xi}^a \cong \tilde{h}\tilde{n}^a$. Elements of $\mathcal{LS}$ will be called infinitesimal *supertranslations*. Equations (4.3) imply

$$\mathcal{L}_{\tilde{n}}\tilde{h} \cong 0, \quad \mathcal{L}_{\tilde{h}\tilde{n}}\tilde{m}_a = 0, \text{ and } \mathcal{L}_{\tilde{h}\tilde{n}}\tilde{n}^a = 0\,. \tag{4.4}$$

Thus, for any supertranslation, $\tilde{h}$ is the pullback to I of $\overline{h}$ on the base space $\mathcal{B}$ and the action of the supertranslation leaves each pair $(\overline{\tilde{m}_a}, \overline{\tilde{n}^a})$ individually invariant. Furthermore, given any $\tilde{\xi}^a \in \mathcal{LG}$ and any $\tilde{h}\tilde{n}^a \in \mathcal{LS}$, we have

$$[\tilde{\xi}, \tilde{h}\tilde{n}]^a = (\mathcal{L}_{\tilde{\xi}}\tilde{h} - \tilde{\alpha})\tilde{n}^a. \tag{4.5}$$

Thus, $\mathcal{LS}$ is a Lie ideal of $\mathcal{LG}$.

To unravel the structure of $\mathcal{LG}$, let us examine the quotient $\mathcal{LG}/\mathcal{LS}$. Let $[\tilde{\xi}^a]$ denote the element of the quotient defined by $\tilde{\xi}^a$; $[\tilde{\xi}^a]$ is thus an equivalence class of vector fields on I satisfying Eqs. (4.3), where two are regarded as equivalent if they differ by a supertranslation. The second of Eqs. (4.3) implies that every $\tilde{\xi}^a$ in $\mathcal{LG}$ admits an unambiguous projection $\overline{\xi^a}$ to the base space $\mathcal{B}$. The equivalence relation implies that all vector fields $\tilde{\xi}^a$ in $[\tilde{\xi}^a]$ project to the same field $\overline{\xi^a}$ on $\mathcal{B}$ and that $[\tilde{\xi}^a]$ is completely characterized by $\overline{\xi^a}$. What conditions does $\overline{\xi^a}$ have to satisfy? The only restriction comes from the first equation in Eqs. (4.3): $\overline{\xi^a}$ must satisfy $\mathcal{L}_{\overline{\xi}}\overline{m_a} = \overline{\alpha m_a}$ for some $\overline{\alpha}$ on $\mathcal{B}$. However, since $\mathcal{B}$ is *one* dimensional, this is no restriction at all. Thus, $\overline{\xi^a}$ can be *any* smooth vector field on the circle $\mathcal{B}$. $\mathcal{LG}/\mathcal{LS}$ is thus the Lie algebra of all smooth diffeomorphisms on S^1. [In four dimensions, by contrast, the first of Eqs (4.3) is very restrictive since the base space is a two-sphere; $\overline{\xi^a}$ has to be a conformal Killing field on $(S^2, \overline{q_{ab}})$. The Lie algebra of these conformal Killing fields is just six dimensional and is isomorphic to the Lie algebra of the Lorentz group in four dimensions.]

These results imply that the group $\mathcal{G}$ of asymptotic symmetries has the structure of a semidirect product. The normal subgroup $\mathcal{S}$ is the Abelian group of supertranslations. Given a conformal frame, each infinitesimal supertranslation $\tilde{\xi}^a = \tilde{h}\tilde{n}^a$ is characterized by a function $\tilde{h}$. If we change the conformal frame, $\tilde{g}_{ab} \mapsto \tilde{g}'_{ab} = \omega^2 \tilde{g}_{ab}$, we have $\tilde{n}^a \mapsto \tilde{n}'^a = \omega^{-1}\tilde{n}^a$ and hence $\tilde{h} \mapsto \tilde{h}' = \omega \tilde{h}$. Thus, each supertranslation is characterized by a conformally weighted function on the circle $\mathcal{B}$; the supertranslation subgroup $\mathcal{S}$ is isomorphic with the additive group of smooth functions on a circle with a unit conformal weight. The quotient $\mathcal{G}/\mathcal{S}$ of $\mathcal{G}$ by the supertranslation subgroup $\mathcal{S}$ is the group Diff(S^1) of diffeomorphisms on a circle. In the semidirect product, Diff(S^1) acts in the obvious way on the additive group of conformally weighted functions on S^1.

We will conclude this subsection with some remarks.

(1) In the light of the above discussion, let us reexamine the conditions on the stress-energy tensor in our definition of asymptotic flatness. In Sec. III A we pointed out that the conditions are considerably weaker than those normally imposed in four dimensions and argued that imposition of stronger conditions would deprive the framework of interesting examples. Could we have imposed even weaker conditions? Note that, if ΩT_{ab} fails to admit a well-defined limit to I, we could not even have concluded that I is a null hypersurface [see Eq. (3.2)]. What about the condition on the trace? In the absence of this condition, the pullback of $\tilde{L}_{ab}$ to I would not have vanished. This then would have implied $\mathcal{L}_{\tilde{n}}\tilde{q}_{ab} \cong (4/3)\tilde{L}\tilde{q}_{ab} \neq 0$. Consequently, the asymptotic symmetry group would have borne little resemblance to the BMS group [13,16,17,20] that arises in four dimensions. Thus, the specific conditions we used in the definition strike a balance: they are weak enough to admit interesting examples and yet strong enough to yield interesting structure at I.

(2) The semidirect product structure of the asymptotic symmetry group is the same as that of the BMS group. The supertranslation group is also the natural analogue of the supertranslation subgroup of the BMS group. The quotient, however, is quite different: while it is the Lorentz group in the four-dimensional case, it is now an *infinite-dimensional* group, Diff(S^1). Recall, however, that in the corresponding analysis in four dimensions, the base space of I is a two-sphere. S^2 admits a unique conformal structure and the Lorentz group arises as its conformal group. In the present case, the base space $\mathcal{B}$ is topologically S^1 and the quotient of $\mathcal{G}$ by the supertranslation subgroup is the conformal group of S^1. (Recall that $\overline{\xi^a}$ has to satisfy $\mathcal{L}_{\overline{\xi}}\overline{q_{ab}} = 2\overline{\alpha q_{ab}}$ since $\overline{q_{ab}} = \overline{m_a m_b}$.) It just happens that, since S^1 is one dimensional, *every* diffeomorphism of S^1 maps $\overline{q_{ab}}$ to a conformally related metric. This is the origin of the enlargement.

(3) Can one understand this enlargement from a more intuitive standpoint? Recall that the symmetry group is enlarged when the boundary conditions are weakened. Thus, it is the weaker conditions on the falloff of stress energy—and hence on the curvature of the physical metric—that is responsible for the enlargement of the group. This can be seen in the explicit asymptotic form of the metric of Einstein-Rosen waves that we encountered in Sec. II C,

$$d\sigma^2 = e^{2\gamma}(-du^2 - 2du\,d\rho) + \rho^2 d\phi^2, \tag{4.6}$$

where $\gamma \cong \gamma(u)$ is a dynamical field on I, sensitive to the radiation. If $\gamma = 0$, we obtain Minkowski space. The radiative space-times that result when $\gamma = 0$ thus differ from the radiation-free Minkowski space already to the *leading* order at null infinity. In four dimensions, by contrast, the leading order behavior of the physical metric has no dynamical content; the components of the metric carrying physical information fall as $1/r$. It is this difference that is responsible for the tremendous enlargement of the asymptotic symmetry group.

Let us analyze this point further. Suppose, in four dimensions, we consider metrics whose form is suggested by Eq. (4.6):

$$ds^2 = e^{2\gamma}(-du^2 - 2du\,dr) + r^2 d\Sigma^2, \tag{4.7}$$

where $\gamma = \gamma(u, r, \theta, \phi)$ has a well-defined limit as r tends to infinity along constant u, θ, ϕ curves, and $d\Sigma^2$ denotes the two-sphere metric. Now, the situation is similar to that encountered in the Einstein-Rosen waves: metrics with different radiative content differ already to leading order. Nonetheless, setting $\Omega = 1/r$, it is easy to carry out a conformal completion of this metric and verify that it admits a smooth I. However, the problem is that the *curvature of this metric fails to fall off sufficiently rapidly for the stress-energy tensor to have the falloff normally required in four dimensions*. Hence, this metric fails to be asymptotically flat in the usual four-dimensional sense. In three dimensions, on the other hand, to obtain an interesting framework, we are forced to admit the analogous metrics (4.6).

B. Translations

In four dimensions, one can single out translations from the BMS group in a number of ways. Somewhat surprisingly, it turns out that every one of those techniques fails in thre dimensions. We will first illustrate this point and then show that one can introduce additional conditions to single out translations. As one might expect from our discussion of Sec. III C, the situation is subtle even after introduction of the stronger conditions.

Among various characterizations of the translation subgroup of the BMS group, the one that is conceptually simplest and aesthetically most pleasing is given by group theory [20]: Translations form the unique four-dimensional *normal* subgroup of the BMS group. In three dimensions, however, the asymptotic symmetry group is much larger; the quotient of $\mathcal{G}$ by supertranslations is now Diff(S^1)—the *full* diffeomorphism group of a circle—rather than the (finite-dimensional) Lorentz group. Consequently, $\mathcal{G}$ does not admit *any* finite-dimensional normal subgroup. Thus, the most obvious four-dimensional strategy is not applicable.

In four dimensions, another method of singling out translations is to use the notion of ''conformal-Killing transport'' [21]. The conformal-Killing data at any point of I corresponding to translations are integrable because the Weyl tensor (of the tilde metric) vanishes there identically. In three dimensions, the analogous condition would be vanishing of the Bach tensor. Unfortunately, as we saw in Sec. III B, in the presence of matter fields the Bach tensor fails to vanish at I. (The explicit expression of the Bach tensor in the case of Einstein-Rosen waves is given in Appendix A.) This in turn makes the conformal-Killing transport of data that would have corresponded to translations nonintegrable on I and the strategy fails.

Finally, a third method of selecting translations in four dimensions is to go to a Bondi conformal frame, i.e., one in which the metric $\underline{q}_{ab}$ on the base space is the unit two-sphere metric and consider the four-parameter family of supertranslations $\tilde{\xi}^a = \tilde{h}\tilde{n}^a$, where $\tilde{h}$ is any linear combination of the $\ell = 0,1$ spherical harmonics. There is only a three-parameter family of Bondi frames and the conformal factor that relates them is highly constrained. As a result, if $\tilde{h}$ is a linear combination of the $\ell = 0,1$ spherical harmonics in one Bondi frame, it is so in *all* Bondi frames [20]. The construction thus selects precisely a four-parameter subgroup of the supertranslation group $\mathcal{S}$. This strategy fails in three dimensions because the base space is now S^1 and the notion of a ''unit S^1 metric'' fails to have the rigidity that the unit two-sphere metrics enjoy. Indeed, as we already remarked in Sec. III C, the only nontrivial analogue of the Bondi frame condition is to require that the conformal frame be such that the length of the base space $\mathcal{B}$ be 2π and there is an *infinite*-dimensional freedom in the choice of such frames. Consequently, we cannot select a three-dimensional space of translations in this manner.

Thus, to select translations, we need to impose additional conditions. To be viable, they should select the standard, three-dimensional translation group in Minkowski space-time. However, as we saw in the point-particle space-time, asymptotic space translations do not exist globally near I if $M \neq 0$. (This is also the case for Einstein-Rosen waves.) Hence, one would expect that, when the total (ADM-type) mass is nonzero, the conditions should select only a time translation. Thus, the conditions have to be subtle enough to achieve both these goals at once. Fortunately, such conditions do exist and are, furthermore, satisfied by a large class of examples.

A space-time (M, g_{ab}) will be said to be *strongly asymptotically flat at null infinity* if it satisfies the boundary conditions of Sec. III.A. and admits a conformal completion in which

$$\tilde{B} \equiv \underline{S} - \underline{L} \equiv \frac{1}{2}\tilde{f} - \tilde{R} \;\rightarrow\; \frac{k}{2} \geqslant 0 \quad \text{as one approaches } i^0 \text{ along } I, \tag{4.8}$$

where k is a constant. Note that if the space-time is axisymmetric, $\tilde{B}$ automatically approaches a constant: if Ω is chosen to be rotationally symmetric, $\tilde{B}$ would also be rotationally symmetric everywhere on I and hence, in particular, its limit to i^0 along I will be angle independent as required. (We will see in Sec. V that the positivity of k ensures that the ADM-type energy is well defined.) Thus, the additional condition is satisfied in a large class of examples, including the Einstein-Rosen waves and our point-particle space-time.

Note that if the last condition is satisfied in a given conformal frame, we can rescale the conformal factor by a *constant* and obtain another conformal frame in which it is also satisfied. We can eliminate this trivial freedom by a normalization condition. A conformal frame will be said to be of *Bondi-type* if $\tilde{B}$ satisfies Eq. (4.8) *and* if $\oint_{\mathcal{B}} \tilde{m}_a dS^a = 2\pi$. A natural question is the following: How many Bondi-type conformal frames does a strongly asymptotically flat space-time admit? We will show that Minkowski space admits precisely a two-parameter family of them and the freedom corresponds precisely to that of choosing a unit timelike vector (i.e., a rest frame). This is completely analogous to the freedom in the choice of Bondi frames in four dimensions. If the ADM-type mass is nonzero, however, the Bondi-type frame will turn out to be generically unique (unlike Bondi frames in four dimensions).

To establish these results, let us fix a strongly asymptotically flat space-time and two Bondi-type completions thereof in which $\tilde{B}$ tends, respectively, to $k/2$ and $k'/2$ for some constants k and k'. (In Minkowski space-time, it turns out that $k = k' = 1$.) Let us suppose that the two conformal frames are related by $\Omega = \alpha\Omega'$, i.e., $\tilde{g}_{ab} = \alpha^2 \tilde{g}'_{ab}$ Then, the transformation property (3.17) of $\tilde{B}$ implies

$$\frac{k'}{2} = \frac{k}{2}\alpha^2 + \alpha\tilde{\partial}^2\alpha - \frac{1}{2}(\tilde{\partial}\alpha)^2, \tag{4.9}$$

where $\tilde{\partial} \equiv \tilde{m}^a \tilde{D}_a \equiv \partial/\partial\phi$. The question now is: How many (smooth) solutions does Eq. (4.9) admit? The equation is nonlinear and rather complicated. However, if we take its $\tilde{\partial}$ derivative we are left with a linear equation:

$$\tilde{\partial}[(\tilde{\partial}^2\alpha)+k\alpha]=0\,. \tag{4.10}$$

This has regular solutions only if $k=n^2$ for an integer n (recall that, in a Bondi frame, the range of ϕ on $\mathcal{B}$ is in $[0,2\pi)$). Similarly, interchanging the role of primed and unprimed frames, we conclude that $k'=n'^2$ for some integer n'. Finally, the fact that the length of $\mathcal{B}$ in *both* conformal frames is 2π implies that $n'=n$. Thus, unless $k=k'=n^2$, Eq. (4.9) does not admit a regular solution. Thus, unless $k=n^2$, the Bondi-type conformal frame is, in fact, unique. In this generic case, we have a preferred time translation subgroup of $\mathcal{G}$ generated by $\tilde{\xi}^a=\tilde{n}^a$. In the point-particle example, this is precisely the time translation selected by the rest frame of the particle. In Einstein-Rosen waves, it turns out to be the one selected by the total Hamiltonian of the system [10].

If $k=n^2$, the reduced equation (4.10) clearly admits a two-parameter family of solutions: In terms of the angular coordinate ϕ on $\mathcal{B}$ (with $\tilde{m}_a=\tilde{D}_a\phi$), these are given by

$$\alpha=A+B\cos n\phi+C\sin n\phi \quad \text{with} \quad -A^2+B^2+C^2=-1\,. \tag{4.11}$$

It is straightforward to check that they also satisfy the full equation (4.9).

In the obvious completion of Minkowski space-time (obtained by setting $M=0$ in the point-particle example or $\psi=0$ in Einstein-Rosen waves), we have $\tilde{f}=1$ and $\tilde{R}=0$, whence $\tilde{B}=1/2$. This corresponds to the case $n=1$. Thus, Minkowski space-time does admit Bondi-type conformal frames and the constant k is precisely 1 (i.e., we cannot obtain any other value by going from one Bondi-type frame to another). There is precisely a two-parameter family of Bondi-type frames related by a conformal factor α of Eq. (4.11) (with $n=1$). Fix any one of these and consider the three-parameter family of supertranslations of the form $\tilde{h}\tilde{n}^a$ where

$$\tilde{h}=(a+b\cos\phi+c\sin\phi). \tag{4.12}$$

Using Eq. (4.11) (with $n=1$), one can check that this three-dimensional space of these supertranslations is left invariant if we replace one Bondi-type frame by another. Following the (third) strategy (mentioned above) used in four dimensions, one can call this the translation subgroup of the asymptotic symmetry group. This label is indeed appropriate: It is easy to check that the restrictions to I of any translational Killing field of Minkowski space has precisely this form. Thus, if $n=1$, the procedure does select for us a three-dimensional translation subgroup of $\mathcal{G}$.

It turns out, however, that if $n=1$, the deficit angle at spatial infinity vanishes and we, therefore, have zero ADM-type energy. By three-dimensional positive energy theorem [10], the only physically interesting space-time in which this can occur is the Minkowski space-time. If $k>1$, we have a surplus angle at spatial infinity and the ADM-type energy is now negative. We will, therefore, ignore the $n>1$ cases from now on (although they do display interesting mathematical structures; see Appendix B).

To summarize, strongly asymptotically flat space-times generically admit a preferred Bondi-type frame and a preferred time translation. In the exceptional cases, where $k=n^2$, we obtain a three-parameter family of Bondi-type frames. However, the only physically interesting exceptional case is Minkowski space-time where $n=1$.

V. CONSERVED QUANTITIES

This section is divided into two parts. In the first, we introduce the notion of energy at a retarded instant of time and of fluxes of energy and, in the second, we discuss supermomenta. Again, while the general ideas are similar to those introduced by Bondi, Sachs, and Penrose in four dimensions, there are also some important differences.

Perhaps, the most striking difference is the following. Consider generic, strongly asymptotically flat space-times. As we saw, in this case, there is a preferred Bondi-type frame and a preferred translation subgroup of the asymptotic symmetry group. However, as the example of Einstein-Rosen waves illustrates, because the space-time metric is dynamical even at infinity, the vector field $\tilde{n}^a$ (or a constant multiple thereof) in the Bondi-type frame is *not* the extension to I of a *unit* time translation in the space-time. If the initial data of the scalar field are of compact support, space-time is flat in a neighborhood of i^0 and a constant multiple of $\tilde{n}^a$—namely, $\exp(\tilde{\gamma}_0)\tilde{n}^a$—coincides with the extension to I of the unit time translation near i^0. However, in the region of I with nontrivial radiation, the restriction of the unit time translation is given by $\exp[\tilde{\gamma}(u)]\tilde{n}^a$; the rescaling involved is u dependent whence the vector field is not even a supertranslation. Energy, on the other hand, is associated with unit time translations. Hence, energy at null infinity is not directly associated with any component of supermomentum and a new strategy is needed to define it.

A. Energy

The strategy we will adopt is to capture the notion of energy through the appropriate deficit angle. We will first begin with motivation, then write down the general expression of energy, and finally verify that it has the expected physical properties.

Let us begin with an axisymmetric, strongly asymptotically flat space-time and consider its Bondi-type completion with an axisymmetric conformal factor. (Thus, $\oint_{\mathcal{B}} \tilde{m}_a dS^a=2\pi$.) Fix a cross section C_0 of I to which the rotational Killing field is tangential. Because of axi symmetry of the construction, the field $\tilde{B}$ is constant on C_0, say $\tilde{B}|_{C_0}=k_0/2$. If this were a cross section of I of the point-particle space-time, it follows from our discussion of Sec. III C [cf. Eq. (3.22)] that we would associate with it energy

$$E=\frac{1}{4G}(1-\sqrt{k_0}). \tag{5.1}$$

(Thus, in particular, if $k_0=1$ as in Minkowski space-time, we would have $E=0$.)

By inspection, we can generalize this expression to arbitrary cross sections of null infinity of general—i.e., nonaxisymmetric—space-times. Given any strongly asymptotically flat space-time, a Bondi-type conformal frame and a cross-section C of I, we will set

$$E[C]:=\frac{1}{8\pi G}\oint_C(1-\sqrt{2\tilde{B}})\tilde{m}_a dS^a. \tag{5.2}$$

The appearance of the square root is rather unusual and seems at first alarming: the formula would not be meaningful if $\tilde{B}$ were to become negative. Note, however, that, by assumption of strong asymptotic flatness, the limit $k/2$ of $\tilde{B}$ to i^0 is positive. Furthermore, since $\mathcal{L}_{\tilde{n}}\tilde{B}=\lim_{\mapsto I}\Omega^{-2}\tilde{L}_{cd}\tilde{n}^c\tilde{n}^d$ and since the right-hand side is positive definite if the matter sources satisfy local-energy conditions, $\tilde{B}$ remains positive on I. Thus, $E[C]$ is bounded above by $1/4G$ which is also the upper bound of the total Hamiltonian at spatial infinity [10].

Let us now verify various properties of this quantity which provide a strong support in favor of its interpretation as energy.

First, let us suppose that we are in Minkowski space-time. Then, in *any* Bondi-type frame, we have $\tilde{B}=1/2$ everywhere on I. Hence, on any cross section, the energy vanishes.

Next, let us consider the point-mass space-time with positive M. Then, from Sec. IV B we know that there is a unique Bondi-type frame and in this frame, $2\tilde{B}=(1-4GM)^2$ whence, on *any* cross-section C, we obtain $E[C]=M$. This is, of course, not surprising since our general definition was motivated by the point-mass example. However, the result is not trivial because we are now allowing arbitrary cross sections, not necessarily tangential to the rotational Killing field.

Consider Einstein-Rosen waves. In the nontrivial case when the scalar field ψ is nonzero, the Bondi-type frame is unique. In this frame, $2\tilde{B}=\exp[-2\tilde{\gamma}(u)]$. Hence,

$$E[C]=\frac{1}{8\pi G}\oint_C(1-e^{-\tilde{\gamma}(u)})d\phi.$$

In the limit to i^0 [or, in the past of the support of $\tilde{\psi}(u)$ on I], we have $E\mapsto(1/4G)[1-\exp(-\tilde{\gamma}_0)]$. This is *precisely* the value of the total Hamiltonian at spatial infinity—the generator of unit time translations near i^0. This result is highly nontrivial because the Hamiltonian is defined [10] through *entirely* different techniques using the symplectic framework based on Cauchy slices. In the limit to i^+, we know from Sec. II C that $\tilde{\gamma}(u)$ tends to zero. Hence, $E[C]$ tends to zero. This behavior of $E[C]$ is also physically correct because i^+ is regular in these space-times. We wish to emphasize that these two constraints—agreement with the known expressions both at i^0 and i^+ of Einstein-Rosen waves—on the viable expression of energy are strong. Hence, the fact that there exists a *general* expression for $E[C]$ involving only fields defined *locally* on the cross section C which reduces to the correct limits at both ends of I of the Einstein-Rosen waves is quite nontrivial.

What about the flux of energy? If a cross section C_1 is in the future of a cross-section C_2, from Eqs. (3.16) and (5.2), we have

$$\begin{aligned}E[C_1]-E[C_2]&=\frac{1}{8\pi G}\int_\Delta \tilde{D}_{[a}(1-\sqrt{2\tilde{B}})\tilde{m}_{b]}dS^{ab}\\ &=-\frac{1}{16\pi G}\int_\Delta(2\tilde{B})^{-1/2}\lim_{\mapsto I}(\Omega^{-2}\tilde{L}_{mn}\tilde{n}^m\tilde{n}^n)\\ &\quad\times\tilde{\epsilon}_{ab}dS^{ab},\end{aligned} \tag{5.3}$$

where Δ is the portion of I bounded by C_1 and C_2. [The limit in the integrand is well defined because of our conditions on the stress-energy tensor. For the Einstein-Rosen waves, it is $(\mathcal{L}_{\tilde{n}}\tilde{\psi})^2$; see Eq. (3.1).] If the matter sources satisfy local energy conditions, the integrand in the second expression is positive definite. Thus, $E[C_1]\leqslant E[C_2]$, the equality holding if and only if there is no flux of matter through the region Δ. As one would expect, radiation through I carries positive energy. The appearance of $1/\sqrt{2\tilde{B}}$ in the integrand is not alarming because, as remarked above, for the class of space-times under consideration, $\tilde{B}$ is guaranteed to be positive on I in Bondi-type frames.

In the case when the source is a zero rest-mass scalar field, we can make the energy flux more explicit: $\lim_{\mapsto I}(\Omega^{-2}\tilde{L}_{mn}\tilde{n}^m\tilde{n}^n)=2(\mathcal{L}_{\tilde{n}}\tilde{\psi})^2$. Hence, for Einstein-Rosen waves, Eq. (5.3) reduces to

$$E[C_1]-E[C_2]=-\frac{1}{8\pi G}\int_\Delta e^{\tilde{\gamma}(u)}(\mathcal{L}_{\tilde{n}}\tilde{\psi})^2\tilde{\epsilon}_{ab}dS^{ab}. \tag{5.4}$$

In the limit in which the cut $[C_2]$ tends to i^0, $E[C_2]$ reduces to the gravitational Hamiltonian [10]. Hence, on any cut, $E[C]$ is given by the difference between the total Hamiltonian and the energy that is radiated out up until that cut. Finally, note that, because of the appearance of $\exp[\tilde{\gamma}(u)]$ in the integrand, this expression of energy flux is more complicated than the flux formula (2.37) for $\gamma(u)$, i.e., the flux formula for Thorne's C energy [2]. This is, however, to be expected: Even at spatial infinity, the total Hamiltonian is $(1/4G)[1-\exp(-\tilde{\gamma}_0)]$ while the C energy is just $(1/4G)\tilde{\gamma}_0$. In the weak field limit the two agree. But in strong fields, they are quite different. In particular, the total Hamiltonian and $E[C]$ are bounded above by $1/4G$ while the C energy is unbounded above.

We saw that, in the case of Einstein-Rosen waves, our expression (5.2) of energy reduces to the total Hamiltonian in the limit as the cross section approaches i^0. We expect that this result holds much more generally: It should hold in any space-time which is strongly asymptotically flat at null infinity and also satisfies the boundary conditions at spatial infinity needed in the Hamiltonian formulation [10]. That is, broadly speaking, we expect the agreement to hold if the space-time is sufficiently well behaved to have a well-defined total Hamiltonian *and* a well-defined limit of Eq. (5.2) to i^0. It is easy to provide strong plausibility arguments for this conjecture since both quantities measure the deficit angle at i^0. However, more detailed analysis are needed to establish this result conclusively.

B. Supermomentum

We will conclude the main paper by introducing a notion of supermomentum. For reasons indicated in the beginning of this section, however, these quantities are not related to the energy in a simple way. They are given primarily for completeness. As in four dimensions [18], in a suitable Hamiltonian formulation based on null infinity, they may be the generators of canonical transformations induced by supertranslations.

Recall first that, in four dimensions, supermomentum arises as a linear map from the space of supertranslations to real translations and is expressible in any conformal frame. The basic fields that enter are constructed from the asymptotic curvature of the rescaled metric (and matter sources). However, in order to ''remove irrelevant conformal factor terms,'' one also has to introduce a kinematic field [17] with appropriate conformal properties. The situation in three dimensions is rather similar.

Let us begin by introducing the analogue $\tilde{\rho}$ of the kinematical field. Set $\tilde{\rho}=1/2$ in any Bondi-type conformal frame and transform it to any other frame via the following law: if $\Omega=\alpha\Omega'$, then

$$\tilde{\rho}'=\alpha^2\tilde{\rho}+\alpha\tilde{\partial}^2\alpha-\tfrac{1}{2}(\tilde{\partial}\alpha)^2, \tag{5.5}$$

where, as before, $\tilde{\partial}\equiv\tilde{m}^a\tilde{D}_{\underline{a}}$. Hence, the field $\tilde{\rho}-\tilde{B}$ transforms rather simply: $(\tilde{\rho}'-\tilde{B}')=\alpha^2(\tilde{\rho}-\tilde{B})$ [see Eq. (3.17)]. As in four dimensions, the field ρ serves two purposes: it removes the unwanted, inhomogeneous terms in the transformation properties of $\tilde{B}$ and it removes the ''purely kinematical'' part of $\tilde{B}$ in the Bondi-type frames.

We can now define the supermomentum. Fix any conformal completion of the physical space-time (not necessarily of a Bondi-type). The value of the supermomentum on a supertranslation $\tilde{T}\tilde{n}^a$, evaluated at a cross-section C of I will be

$$P_{\tilde{T}}[C]=\frac{1}{8\pi G}\oint_C(\tilde{\rho}-\tilde{B})\tilde{T}\tilde{m}_a dS^a. \tag{5.6}$$

Under a conformal transformation, $\Omega\mapsto\Omega'=\alpha^{-1}\Omega$, we have $\tilde{T}'=\alpha^{-1}\tilde{T}$ and $\tilde{m}'_a=\alpha^{-1}\tilde{m}_a$. Hence, the one-form integrand remains unchanged. Thus, as needed, the expression of supermomentum is conformally invariant; i.e., it is well defined.

Let us note its basic properties. First, by inspection, the map defined by the supermomentum P from supertranslations to real translations is linear. Second, in Minkowski space-time, $\tilde{\rho}=\tilde{B}$ in any conformal frame. Hence, the value of supermomentum vanishes identically on *any* cross section. Finally, since $\mathcal{L}_{\tilde{n}}\tilde{\rho}=0$, we have

$$\mathcal{L}_{\tilde{n}}[(\tilde{\rho}-\tilde{\mathcal{B}})\tilde{T}\tilde{m}_a]=-\lim_{\mapsto I}(\Omega^{-2}\tilde{L}_{mn}\tilde{n}^m\tilde{n}^n)\tilde{T}\tilde{m}_a. \tag{5.7}$$

Therefore, as in the case of energy, the flux of the component of the supermomentum along any timelike supertranslation (i.e., one in which $\tilde{T}>0$) is positive.

VI. DISCUSSION

In this paper, we developed the general framework to analyze the asymptotic structure of space-time at null infinity in three space-time dimensions. We did not have to restrict ourselves to any specific type of matter fields. However, if the matter sources are chosen to be a triplet of scalar fields constituting a nonlinear [SO(2,1)] σ model, the space-times under considerations can be thought of as arising from symmetry reduction of four-dimensional generalized cylindrical waves, i.e., vacuum solutions to the four-dimensional Einstein equations with one space-translation isometry. If the source consists of a single zero rest-mass scalar field, the translation Killing field in four dimensions is hypersurface orthogonal. Finally, if there is, in addition, a rotational Killing field, the space-times are symmetry reductions of the four-dimensional Einstein-Rosen waves.

The general strategy we adopted was to follow the procedures developed by Bondi and Penrose in four dimensions. However, we found that due to several peculiarities associated with three dimensions, those procedures have to be modified significantly. A number of unexpected difficulties arise and the final framework has several surprising features. This is in contrast with the situation in higher dimensions where the framework is likely to be very similar to that in four dimensions.

The new features can be summarized as follows. First, in three dimensions, the space-time metric is flat in any open region where stress energy vanishes and thus we are forced to consider gravity coupled with matter. To accommodate physically interesting cases, we have to allow matter fields such that the falloff of the stress-energy tensor at null infinity is significantly weaker than that in four dimensions. This, in turn, means that the metric is dynamical even at infinity; it does not approach a Minkowskian metric even in leading order. In fact, physically interesting information, such as the energy and energy fluxes, is coded in these leading order, dynamical terms. As a result, the asymptotic symmetry group $\mathcal{G}$ is enlarged quite significantly. Like the BMS group in four dimensions, it admits an infinite-dimensional normal subgroup $\mathcal{S}$ of supertranslations. The structure of this subgroup is completely analogous to that of its counterpart in four dimensions. However, the quotient $\mathcal{G}/\mathcal{S}$ is *significantly* larger. While in four dimensions the quotient is the six-dimensional Lorentz group, now it is the infinite-dimensional group Diff(S^1) of diffeomorphisms of a circle. Furthermore, whereas the BMS group admits a preferred (four-dimensional) group of translations, $\mathcal{G}$ does not. To select translations, one has to impose additional conditions, which in some ways are analogous to the conditions needed in four dimensions to extract a preferred Poincaré subgroup of the BMS group. We imposed these by demanding that there should exist a conformal frame in which the field $\tilde{B}$ tends to a constant as one approaches i^0 along I. This condition is automatically satisfied in axisymmetric space-times. We saw that, in a generic situation, it selects a unique conformal frame (up to constant rescalings which can be removed by a normalization condition) and we can then select a preferred time translation in $\mathcal{S}$. If the past limit of the I energy is zero, it selects a two-parameter family of frames—the analogues of Bondi frames in four dimensions. In this case, we can select a three-dimensional subgroup of translations from $\mathcal{S}$. Finally, given any cross section C of I, we associated with it energy $E[C]$ as well as a supermomentum $P_{\widetilde{T}}[C]$. The former is a scalar and has several properties that one would expect energy to have. The latter is a linear map from the space of supertranslations to real translations and may arise, in an appropriate Hamiltonian formulation based on I, as the generator of canonical transformations corresponding to supertranslations.

These results refer to three-dimensional general relativity coupled to arbitrary matter fields. However, as noted above, if the matter fields are chosen appropriately, we can regard the three-dimensional system as arising from a symmetry reduction of four-dimensional vacuum general relativity by a space-translation Killing field. (One can also consider four-dimensional general relativity coupled to suitable matter. Then, one acquires additional matter fields in three dimensions.) In this case, the energy $E[C]$ (or the supermomentum $P_{\widetilde{T}}[C]$) associated with a cross section C of three-dimensional null infinity represents the energy (or supermomenmtum) per unit length (along the symmetry axis) in four dimensions. Thus, the three-dimensional results have direct applications to four-dimensional general relativity as well. In addition, as we will see in the companion paper [11], the analysis of the asymptotic behavior of fields in three dimensions can also be used to shed light on the structure of null infinity in four-dimensions.

There are a number of technical issues that remain open. First, as indicated in Sec. V A it is desirable to find the precise conditions under which the past limit of $E[C]$ yields the total Hamiltonian [10]. A second important issue is that of positivity of $E[C]$. For the total Hamiltonian, this was established [10] using a suitable modification of Witten's spinorial argument in four dimensions. Can this argument be further modified to show positivity of $E[C]$? If space-time admits a regular i^+, the limit of $E[C]$ as C tends to i^+ vanishes. Since the flux is positive, this implies that $E[C]$ is positive on every cross section. However, in the general case, it is not *a priori* clear that in the Bondi-type frame, $\widetilde{B}$ will not exceed 1/2 making $E[C]$ negative on some cross section. Next, in the case when the matter fields admit initial data of compact support, space-time is flat near i^0. In this case, it should be possible to select a preferred one-parameter subgroup of rotations in $\mathcal{G}$ and define angular momentum. Finally, in the case when i^+ is regular, one would expect that, as in Minkowski space, there exists a two-parameter family of Bondi-type conformal frames in which $\widetilde{B}$ tends to a constant at i^+. It is not *a priori* clear whether the Bondi-type frame selected by the behavior of $\widetilde{B}$ at i^0 is included in the family selected at i^+. If the space-time is axisymmetric, the answer is in the affirmative. It would be interesting to investigate what happens in the general case.

The present framework provides a natural point of departure for constructing an S-matrix theory both classically and, especially, quantum mechanically. Three-dimensional quantum gravity without matter fields can be solved exactly but the solution is trivial in the asymptotically flat case. When we bring in matter, we have a genuine field theory which is diffeomorphism invariant. If the matter fields are suitably restricted, the theories are equivalent to the reduction of four-dimensional general relativity (or of ten-dimensional string theories). Quantization of such theories should shed considerable light on the conceptual problems of nonperturbative quantum gravity. As a first step towards quantization, one might use ideas from the asymptotic quantization scheme introduced in four dimensions [22]. Since the Lorentz subgroups are now replaced by the Diff(S^1) subgroups of $\mathcal{G}$ and since Diff(S^1) admits interesting representations (with nonzero central charges), the asymptotic quantum states would now have interesting, nontrivial sectors. Second, this quantization would also lead to ''fuzzing'' of space-time points along the lines of Ref. [23]. To see this, recall first that the light cone of each space-time point gives rise to a ''cut'' of I (which, in general, is quite complicated). Thus, given I and these light cone cuts, one can ''recover'' space-time points in an operational way. Now, in a number of cases with scalar field sources—including of course the Einstein-Rosen waves—one expects the initial-value problem based on I to be well posed and the classical S matrix to be well behaved. In such cases, it should be possible to express the light cone cuts on I directly in terms of the data of the scalar field on I. Now, in the quantum theory, the scalar field on I is promoted to an operator-valued distribution and, given any quantum state, one only has a probability distribution for the scalar field to assume various values. This immediately implies that one would also have only probability distributions for light cone cuts, i.e., for points of space-time. This approach may well lead one to a noncommutative picture of space-time geometry.

ACKNOWLEDGMENTS

A.A. and J.B. would like to thank the Max-Planck-Institute for Gravitational Physics for its kind hospitality. A.A. was supported in part by the NSF Grant Nos. 93-96246 and 95-14240, by the Eberly Research Fund of Penn State University, and by the Erwin Schrödinger International Institute for Mathematical Sciences, Vienna. J.B. was supported in part by the Grant Nos. GACR–202/96/0206 and GAUK–230/1996 of the Czech Republic and the Charles University, and by the U.S.-Czech Science and Technology Grant No. 92067.

APPENDIX A: RIEMANN AND BACH TENSORS

In this appendix we will provide the behavior of the Riemann and Bach tensors at null infinity in the (2+1)-dimensional description of Einstein-Rosen waves.

Assume the metric to be given in Bondi-type coordinates $(x^0,x^1,x^2)=(u,\rho,\phi)$ as in Eq. (2.26). The Christoffel symbols are

$$\Gamma^0_{00}=2\gamma_{,u}-\gamma_{,\rho}, \quad \Gamma^0_{22}=\rho e^{-2\gamma},$$
$$\Gamma^1_{00}=\gamma_{,\rho}-\gamma_{,u}, \quad \Gamma^1_{01}=\gamma_{,\rho}, \Gamma^1_{11}=2\gamma_{,\rho},$$
$$\Gamma^1_{22}=-\rho e^{-2\gamma}, \quad \Gamma^2_{12}=\rho^{-1}. \tag{A1}$$

The Riemann tensor ($R^i_{jkl}=\Gamma^i_{jl,k}-\cdots$) reads

$$R_{0101}=e^{2\gamma}(\gamma_{,\rho\rho}-2\gamma_{,u\rho}), \quad R_{0202}=\rho(\gamma_{,\rho}-\gamma_{,u}),$$
$$R_{0212}=\rho\gamma_{,\rho}, \quad R_{1212}=2\rho\gamma_{,\rho}. \tag{A2}$$

In a general (2+1)-dimensional space-time the Riemann tensor has the form

$$R_{ijkl}=2(S_{i[k}g_{jl]}-S_{j[k}g_{il]}), \tag{A3}$$

where

$$S_{ik}=R_{ik}-\tfrac{1}{4}g_{ik}R. \tag{A4}$$

It has six independent components given by the symmetric tensor S_{ik}. In the case of the rotation symmetry only the following components are nonvanishing:

$$S_{00}=\tfrac{1}{2}\gamma_{,\rho\rho}-\gamma_{,u\rho}+\rho^{-1}(\gamma_{,\rho}-\gamma_{,u}),$$

$$S_{01}=S_{10}=\tfrac{1}{2}\gamma_{,\rho\rho}-\gamma_{,u\rho}+\rho^{-1}\gamma_{,\rho},$$

$$S_{11}=2\rho^{-1}\gamma_{,\rho},$$

$$S_{22}=\rho^2e^{-2\gamma}(\tfrac{1}{2}\gamma_{,\rho\rho}-\gamma_{,u\rho}). \tag{A5}$$

The role of the Weyl tensor in three dimensions is played by the conformally invariant Bach tensor (see, e.g., [1]):

$$B_{ijk}=S_{ik;j}-S_{ij;k}. \tag{A6}$$

The Bach tensor satisfies $B_{ijk}=-B_{ikj}$ and $B_{[ijk]cykl}=0$, and it thus has five independent components. In the rotation-symmetric case the Bach tensor writes ($\delta=\gamma_{,\rho}-2\gamma_{,u}$)

$$B_{001}=\tfrac{1}{2}(\delta_{,u}-\delta_{,\rho})_{,\rho}+(\delta+\gamma_{,u})(\delta_{,\rho}+\rho^{-2})-\rho^{-1}\delta_{,\rho},$$

$$B_{101}=-\tfrac{1}{2}\delta_{,\rho\rho}+(\delta+2\gamma_{,u})(\delta_{,\rho}+\rho^{-2})-\rho^{-1}\delta_{,\rho},$$

$$B_{202}=\rho^2e^{-2\gamma}(B_{001}-B_{101}),\quad B_{212}=-\rho^2e^{-2\gamma}B_{101}. \tag{A7}$$

Let us choose the real null triad

$$l^i=(0,e^{-2\gamma},0),\quad n^i=(1,-\tfrac{1}{2},0),\quad m^i=(0,0,\rho^{-1}). \tag{A8}$$

It is easy to see that it is parallel propagated along $u=$const, $\phi=$const, and it satisfies

$$l_in^i=-1,\quad m_im^i=1, l_il^i=l_im^i=n_in^i=n_im^i=0. \tag{A9}$$

Further, let us introduce six real triad components of the Riemann tensor or, equivalently, of the tensor S_{ik} given by Eq. (A4) as

$$S_1=R_{ijkl}l^im^jl^km^l=S_{ik}l^il^k,$$

$$S_2=R_{ijkl}l^in^jl^km^l=S_{ik}l^im^k,$$

$$S_3=R_{ijkl}(\tfrac{1}{2}l^in^jl^kn^l-m^in^jm^kl^l)=S_{ik}m^im^k,$$

$$S_4=\tfrac{1}{2}R_{ijkl}l^in^jl^kn^l=S_{ik}l^in^k,$$

$$S_5=R_{ijkl}n^il^jn^km^l=S_{ik}n^im^k,$$

$$S_6=R_{ijkl}m^in^jm^kn^l=S_{ik}n^in^k. \tag{A10}$$

Under the rotation symmetry we find

$$S_1=2\rho^{-1}\gamma_{,\rho}e^{-4\gamma},\quad S_2=0,$$

$$S_3=S_4=e^{-2\gamma}(\tfrac{1}{2}\gamma_{,\rho\rho}-\gamma_{,u\rho}),\quad S_5=0$$

$$S_6=\rho^{-1}(\gamma_{,\rho}-\gamma_{,u}). \tag{A11}$$

Assume now the scalar field admits an expansion (2.23). The field equations (2.27) and (2.28) imply

$$\gamma_{,u}=-2\dot f_0^2-\frac{1}{2}f_0\dot f_0\frac{1}{\rho}+\cdots,$$

$$\gamma_{,\rho}=\frac{1}{4}f_0^2\frac{1}{\rho^2}+\cdots. \tag{A12}$$

The Riemann tensor (A11) has then the asymptotic form

$$S_1=\frac{1}{2}e^{-4\gamma_\infty}f_0^2\frac{1}{\rho^3}+O\left(\frac{1}{\rho^4}\right),$$

$$S_3=S_4=\frac{1}{2}e^{-2\gamma_\infty}f_0\dot f_0\frac{1}{\rho^2}+O\left(\frac{1}{\rho^3}\right),$$

$$S_6=2\dot f_0^2\frac{1}{\rho}+O\left(\frac{1}{\rho^2}\right), \tag{A13}$$

where $\gamma_\infty=\lim_{\rho\to\infty}\gamma(u,\rho)$.

Finally, define the five real triad components (scalars) of the Bach tensor:

$$B_1=B_{ijk}l^in^jm^k,\quad B_2=B_{ijk}l^il^jm^k,\quad B_3=B_{ijk}n^in^jm^k,$$

$$B_4=B_{ijk}m^im^jl^k,\quad B_5=B_{ijk}m^im^jn^k. \tag{A14}$$

Under the rotation symmetry we find only the last two scalars nonvanishing. Their asymptotic behavior is

$$B_4=-\frac{1}{4}e^{-4\gamma_\infty}[6(f_0f_1)\dot{}+f_0^3\dot f_0]\frac{1}{\rho^4}+O\left(\frac{1}{\rho^5}\right),$$

$$B_5=\frac{1}{2}e^{-2\gamma_\infty}[f_0\ddot f_0-3\dot f_0^2+4f_0\dot f_0^3]\frac{1}{\rho^2}+O\left(\frac{1}{\rho^3}\right). \tag{A15}$$

Now, the Bach tensor is conformally invariant and it is of interest to see precisely its form at null infinity in the unphysical space-time. Putting $\tilde\rho=\rho^{-1}$, $\tilde u=u$, $\tilde\phi=\phi$, and using again $\Omega=\tilde\rho$ as in Eq. (2.33), we introduce the null triad in the unphysical space by $\tilde l=\Omega^{-2}l, \tilde n=n, \tilde m=\Omega^{-1}m$, so that in the coordinates $(\tilde u,\tilde\rho,\tilde\phi)$ we have

$$\tilde l^i=(0,-e^{-2\tilde\gamma},0),\quad \tilde n^i=(1,\tfrac{1}{2}\tilde\rho^2,0),\quad \tilde m^i=(0,0,1). \tag{A16}$$

(Note that the vector $\tilde n^i$ is null everywhere. Outside I, it is not related in any simple way to the vector field $\tilde n^a:=\tilde g^{ab}\tilde\nabla\Omega$ used in the main text.)

Using then $\tilde B_{ijk}=B_{ijk}$ we arrive at the following form of the Bach tensor at null infinity I :

$$\widetilde{B}_4=\widetilde{B}_{ijk}\widetilde{m}^i\widetilde{m}^j\widetilde{l}^k=-\tfrac{1}{4}e^{-4\widetilde{\gamma}_0}[6(\widetilde{f}_0\widetilde{f}_1)_{,\widetilde{u}}+\widetilde{f}_0^3\widetilde{f}_{0,\widetilde{u}}]+O(\widetilde{\rho}),$$

$$\widetilde{B}_5=\widetilde{B}_{ijk}\widetilde{m}^i\widetilde{m}^j\widetilde{n}^k=\tfrac{1}{2}e^{-2\widetilde{\gamma}_0}[\widetilde{f}_0\widetilde{f}_{0,\widetilde{u}\widetilde{u}}-3\widetilde{f}_{0,\widetilde{u}}^2+4\widetilde{f}_0\widetilde{f}_{0,\widetilde{u}}^3]+O(\widetilde{\rho}), \tag{A17}$$

where $\widetilde{\gamma}_0=\widetilde{\gamma}(\widetilde{u},\widetilde{\rho}=0)=\gamma_\infty$, $\widetilde{f}_0(\widetilde{u})=f_0(u)$, $\widetilde{f}_1(\widetilde{u})=f_1(u)$. Hence, the Bach tensor is finite and nonvanishing at null infinity in general.

APPENDIX B: ASYMPTOTICS FOR STATIC CYLINDERS IN THREE DIMENSIONS

Starting from the four-dimensional Einstein-Rosen metric,

$$ds^2=e^{2\gamma-2\psi}(-dt^2+d\rho^2)+e^{2\psi}dz^2+\rho^2e^{-2\psi}d\phi^2, \tag{B1}$$

Marder [24] gives four-dimensional static solution representing the field outside a static cylinder in the form

$$\psi=-C(1-C)^{-1}\ln\rho-(1-C)\ln D, \tag{B2}$$

$$\gamma=C^2(1-C)^{-2}\ln\rho-(1-2C)\ln D, \tag{B3}$$

where C and D are constants which can be determined, by matching the solution to an interior one, in terms of mass and pressure distribution inside the cylinder. For mass M per unit length of the cylinder small, Levi-Cività and others suggest that $C=2M$; Thorne's C energy [2] leads to the same results as long as the internal pressure of the cylinder is much smaller than its energy density.

The simplest models of the static cylinders employ thin shells. By studying the exterior and flat interior metric of an infinite static cylindrical shell, Stachel [25] found the constants C and D to be related to the internal structure of the cylinder in a simple way. Denoting the radius of the shell by ρ_0, and introducing Stachel's notation, a and A^+, for the constants determining the external metric, we find Marder's constants C and D to be given by

$$C=\frac{a}{a-1},$$

$$\ln D=\frac{1-a}{1+a}(a^2\ln\rho_0+\ln A^+), \tag{B4}$$

so that

$$\gamma=a^2\ln\frac{\rho}{\rho_0}-\ln A^+, \tag{B5}$$

$$\psi=a\ln\frac{\rho}{\rho_0}+b. \tag{B6}$$

An additive constant b in ψ can be removed by a rescaling $\rho\to\xi\widetilde{\rho},t\to\xi\widetilde{t},z\to\xi^{-1}\widetilde{z},\psi\to\widetilde{\psi}+\ln\xi,\gamma\to\widetilde{\gamma}$, $\xi=$const, which leaves the metric (B1) invariant.

Let S_{ab} be the surface stress-energy tensor of the shell. Then, Stachel's equations (1.7a)–(1.7c) determine the surface energy density, $\sigma=S_t^t$ and the surface pressures, $p_z=-S_z^z$, $p_\phi=-S_\phi^\phi$ in terms of the constants a and A^+ as follows:

$$\sigma=\frac{1-A^+}{\rho_0},$$

$$p_z=\frac{A^+(a-1)^2-1}{\rho_0},\quad p_\phi=\frac{a^2A^+}{\rho_0}. \tag{B7}$$

The dominant energy condition, $\sigma\geqslant0$, $|p_z|,|p_\phi|\leqslant\sigma$, requires

$$1-A^+\geqslant0,\quad -\left[\frac{1-A^+}{A^+}\right]^{1/2}<a\leqslant0. \tag{B8}$$

Choosing $a=0$, $0<A^+<1$, we obtain the cylinders with

$$\sigma=\frac{1-A^+}{\rho_0}=-p_z,\quad p_\phi=0, \tag{B9}$$

generating the exterior fields as straight cosmic strings: locally flat but conical, with a positive deficit angle given by $2\pi(1-A^+)$. Curiously, if we admit a negative mass density such that

$$A^+=1+n,\quad n=1,2,\ldots, \tag{B10}$$

and thus

$$\sigma=-\frac{n}{\rho_0}=-p_z, \tag{B11}$$

the exterior space is some covering space of a part of Minkowski space. Indeed, it is easy to see that with $\gamma=-\ln(1+n)$, $\psi=$const, the metric (B1) can be converted to a flat metric with $\phi\in[0,2\pi(n+1)]$. The holonomy group of such a space is the same as that of a part of Minkowski space so that vectors transported parallelly around closed curves coincide with the original [cf. also [27] and [28] which find no ''gravitational Aharonov-Bohm effect'' in the cases corresponding to A^+ given by Eq. (B10)]. The Lie algebra of Killing fields does not differ from that of a part of Minkowski space. However, the geometry (determined by the metric itself, rather than by the connection) is different. With the original coordinate $\phi\in[0,2\pi)$ it reads (after rescaling t)

$$ds^2=-dt^2+\frac{1}{(n+1)^2}d\rho^2+\rho^2d\phi^2+dz^2.$$

Considering surfaces $t=$const, $z=$const, and comparing the proper lengths, $2\pi\rho_1$ and $2\pi\rho_2$, of the two circles with radii ρ_1 and ρ_2, with their proper ''orthogonal distance,'' $(n+1)^{-1}(\rho_2-\rho_1)$, the result differs from that in Minkowski space. This (anti)conical character of space-time can be observed also at infinity after performing an inversion using Cartesian coordinates [cf. Eqs. (2.19) of [11]]. This, of course, is true for any (anti)conical space with $A^+\neq1$.

In any case, the asymptotic gravitational field describing static cylinders is determined by two parameters, rather than one, describing the asymptotic field of cylindrical waves considered in the main text. (Relatively recently, Bondi [26] examined quasistatically changing cylindrical systems and concluded that there is no conservation of these parameters because of gravitational induction transferring energy parallel to the axis.)

The (2+1)-dimensional metric corresponding to Eq. (B1) is [cf. Eq. (2.11)]

$$d\sigma^2 = e^{2\gamma}(-dt^2 + d\rho^2) + \rho^2 d\phi^2. \tag{B12}$$

Introducing $u = t - \rho$ and writing γ in the form

$$\gamma = a^2 \ln\rho + B, \quad a^2 \geqslant 0, \quad B \text{ constants}, \tag{B13}$$

we get

$$d\sigma^2 = \rho^{2a^2} e^{2B}(-du^2 - 2du\,d\rho) + \rho^2 d\phi^2. \tag{B14}$$

Now, we go over to the unphysical three-dimensional space-time with coordinates

$$\tilde{u} = u, \quad \tilde{\rho} = \rho^{2a^2-1}, \quad \tilde{\phi} = \phi \tag{B15}$$

by a conformal transformation with the conformal factor

$$\Omega = \tilde{\rho}^{-1/(2a^2-1)}. \tag{B16}$$

The metric of the unphysical space-time then reads

$$d\tilde{\sigma}^2 = \Omega^2 d\sigma^2 = e^{2B}[-\tilde{\rho}^{2(a^2-1)/(2a^2-1)} d\tilde{u}^2 - 2(2a^2-1)^{-1} d\tilde{u}\, d\tilde{\rho}] + d\tilde{\phi}^2. \tag{B17}$$

Assume $a^2 < \frac{1}{2}$. This includes cases when mass per unit length of the cylinder is small because then constant $C \ll 1$ and $0 < a^2 = C^2(1-C)^{-2} \ll 1$. Transformation (B15) shows that $\rho \to \infty$ implies $\tilde{\rho} \to 0$, and Eq. (B16) implies $\Omega = 0$ at $\tilde{\rho} = 0$. The metric (B17) becomes degenerate here. The conformal completion of the space-time with a given $a^2 < \frac{1}{2}$ can thus be constructed, with infinity being at $\Omega = 0$. However, Eq. (B16) yields $\nabla\Omega = 0$ at $\tilde{\rho} = 0$. Therefore, the asymptotics for static cylinders is completely different from a standard conformal completion of an asymptotically flat space-time. In special cases of locally flat but conical space-times, the asymptotics in (3+1)-dimensional context is analyzed in [29].

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