PHYSICAL REVIEW D VOLUME 55, NUMBER 10 15 MAY 1997

Conformal symmetry and the chiral anomaly

Joshua Erlich
Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Daniel Z. Freedman
Department of Mathematics and Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 27 November 1996)

Two-loop contributions to the anomalous correlation function $\langle J_\mu(x)J_\nu(y)J_\rho(z)\rangle$ of three chiral currents are calculated by a method based on the conformal properties of massless field theories. The method was previously applied to virtual photon diagrams in quantum electrodynamics, and it is extended here to diagrams with scalars and chiral spinors in the Abelian Higgs model and in the $SU(3)\times SU(2)\times U(1)$ standard model. In each case there are nonvanishing contributions to the gauge current correlator from self-energy insertions, vertex insertions, and nonplanar diagrams, but their sum exactly vanishes. The two-loop contribution to the anomaly, therefore, also vanishes, in agreement with the Adler-Bardeen theorem. An application of the method to the correlator $\langle R_\mu(x)R_\nu(y)K_\rho(z)\rangle$ of the R and Konishi axial vector currents in supersymmetric gauge theories which was reported by Anselmi *et al.* is discussed here. The net two-loop contribution to this correlator also vanishes. [S0556-2821(97)00410-4]

PACS number(s): 11.30.Rd, 12.15.Lk

I. INTRODUCTION

The chiral anomaly discovered long ago by Adler [1] and Bell and Jackiw [2] is a seminal concept of quantum field theory. The absence of radiative corrections to the one-loop anomaly is of central importance in applications to neutral pion decay, to the structure of fermion families in the standard model, to mathematical contact between gauge theory and the Atiyah-Singer index theorem [3], and many other questions. One might have thought that this matter was settled by the early work of Adler and Bardeen [4] which involved regularizations of the theory, by the general renormalization group argument [5] for anomalies of global currents, or by Becchi-Rouet-Stora (BRS) cohomology arguments [6] for gauge current anomalies. Yet there is much literature which disputes the common wisdom [7–10]. Further a certain level of suspicion of general theorems has proved to be healthy for theoretical physics, not necessarily because proofs can be wrong, but because inappropriate assumptions can be made in the hypotheses. For example, the particular order of the operations of regularization and computing the axial vector divergence which was used in [4] can be questioned.

Thus explicit calculations of possible radiative corrections to the anomaly in chiral gauge theories are illuminating. A violation of the Adler-Bardeen theorem in the standard model would be particularly significant because it would call to question one of its most attractive features, namely, that the one-loop anomaly cancellation between quarks and leptons occurs so naturally and is sufficient to make the theory consistent. We therefore study two-loop contributions to the gauge correlator $\langle J_\mu(x)J_\nu(y)J_\rho(z)\rangle$ in the Abelian Higgs model,[1] which is a simplified form of the standard model, and then extend the treatment to the full glory of the $SU(3)\times SU(2)\times U(1)$ standard model, where there are four independent possibly anomalous correlators to be checked. In all cases the net sum of self-energy plus vertex insertions plus nonplanar diagrams vanishes. So the full two-loop current correlators vanish, and their would be anomalous divergences vanish, thus validating by explicit calculation the conventional wisdom concerning radiative corrections to the chiral anomaly.

In our method the current correlator is calculated directly in Euclidean position space using a simplifying change of variables suggested by the conformal properties of the correlator to perform the internal integrations. Conformal symmetry also explains why the net two-loop correlator $\langle J_\mu(x)J_\nu(y)J_\rho(z)\rangle$ vanishes, when one might have expected only the vanishing of its divergence $(\partial/\partial x_\mu)\langle J_\mu(x)J_\nu(y)J_\rho(z)\rangle$. The two-loop correlator is conformally covariant for massless internal lines, and one can show that for any conformally covariant contribution the abnormal parity part of the third rank tensor correlator vanishes if and only if its divergences vanish. This method was previously used by Baker and Johnson [12] to compute the two-loop vector and axial vector vertex functions in massless quantum electrodynamics. The ideas of the more comprehensive position space method of differential renormalization [13,14] also play a role, but the specific two-loop calculations re-

[1]Our investigation was motivated by papers of Cheng and Li in which a nonvanishing two-loop anomaly was obtained in this theory. A subtle error has recently been found, and there is now agreement on the vanishing of the anomaly [11].

quired to test the anomaly in this method do not require regularization or cutoff.

The basic ideas of the method are described in Sec. II. The gauge-covariant derivative in the Abelian Higgs model is

$$D_\mu\psi=[\partial_\mu+ig(\alpha L+\beta R)A_\mu]\psi, \tag{1.1}$$

where L and R are chiral projectors. We begin calculations in Sec. III at the point $\beta=-\alpha=1/2$, i.e., pure axial coupling, because this is the point at which one can choose a gauge in which the one-loop fermion vertex function and self-energy are finite. This eliminates all subdivergences in the two-loop current correlator graphs. The modifications required to handle all values of α,β are described in Sec. IV. Because of parity nonconservation, there is no true finite gauge for the vertex and self-energy functions, but we show that there is an effective finite gauge in which the two-loop vertex and self-energy insertion contributions to $\langle J_\mu(z)J_\nu(x)J_\rho(y)\rangle$ have no subdivergences. In Sec. V the method is extended to the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ standard model. We assume the usual couplings for which the one-loop anomalies cancel. There are then no genuinely new graphs to compute, but the effective finite gauge mechanism is more complicated than before. In supersymmetric gauge theories there are two formally conserved axial currents: the R-charge current $R_\mu(x)$ and the Konishi current $K_\mu(x)$. The correlator $\langle R_\mu(x)R_\nu(y)K_\rho(z)\rangle$ was calculated by the present methodology as part of a recent study [15] of the operator product expansions (OPE's) of the superconformal algebra. Details were not discussed in [15], and they are briefly presented in Sec. VI below.

II. METHOD

Although conformal symmetry is concretely used in our work largely to motivate a change of variables which simplifies the required two-loop Feynman integrals, we believe that it is useful to explain the method from a more fundamental standpoint. It is well known (see, for example, [16]) that the conformal group of Euclidean field theory is O(5,1), and that all transformations which are continuously connected to the identity are obtained by combining rotations and translations with the basic conformal inversion

$$x_\mu=\frac{x'_\mu}{x'^2},$$

$$\frac{\partial x_\mu}{\partial x'_\nu}=x^2\left(\delta_{\mu\nu}-\frac{2x_\mu x_\nu}{x^2}\right)\equiv x^2 J_{\mu\nu}(x). \tag{2.1}$$

The Jacobian tensor $J_{\mu\nu}(x)=J_{\mu\nu}(x')$, which is an improper orthogonal matrix, will be very useful for us. Because $\mathrm{Det}\,J=-1$, the inversion is a discrete operation [16], similar to parity, and not an element of the continuous component of O(5,1) which contains the identity.

The Euclidean action of the massless U(1) Higgs model is

$$S=\int d^4x\left[\frac{1}{4}F_{\mu\nu}^2+D_\mu\bar\phi D_\mu\phi+\bar\psi\gamma_\mu D_\mu\psi -f\bar\psi(L\phi+R\bar\phi)\psi-\frac{\lambda}{4}(\bar\phi\phi)^2\right], \tag{2.2}$$

$$D_\mu\phi=(\partial_\mu+igA_\mu)\phi,$$

$$D_\mu\psi=[\partial_\mu+igA_\mu(\alpha L+\beta R)]\psi,$$

$$\beta-\alpha=1,\quad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,\quad L=\tfrac{1}{2}(1-\gamma_5),$$

$$R=\tfrac{1}{2}(1+\gamma_5)$$

It is invariant under conformal transformations in the continuous component of O(5,1), but not necessarily under inversion since that question is related [17] to invariance under discrete symmetries. For the special choice $\beta=-\alpha=1/2$, where we have a parity-conserving theory with pure axial gauge coupling, invariance holds under the transformations

$$\phi(x)\to\phi'(x)=x'^2\overline{\phi(x')}, \tag{2.3}$$

$$\psi(x)\to\psi'(x)=x'^2\gamma_5 \not{x}'\,\psi(x'),$$

$$\bar\psi(x)\to\bar\psi'(x)=x'^2\bar\psi\not{x}'\gamma_5, \tag{2.4}$$

$$A_\mu(x)\to A'_\mu(x)=-x'^2J_{\mu\nu}(x')A_\nu(x'),$$

as can be verified with diligence and the help of the relations

$$d^4x=\frac{d^4x'}{x'^8},\quad \not{x}'\gamma_\mu\not{x}'=-x'^2J_{\mu\nu}(x')\gamma_\nu. \tag{2.5}$$

Inversion invariance does not hold in the general chiral theory, and it is not required for our application.

It is important that correlation functions are constructed from Feynman rules in which the vertex factors and propagators have simple inversion properties. In particular the scalar and spinor propagators transform as

$$\Delta(x-y)=\frac{1}{4\pi^2}\frac{1}{(x-y)^2}=\frac{1}{4\pi^2}\frac{x'^2y'^2}{(x'-y')^2},$$

$$S(x-y)=-\not\partial\Delta(x-y)=\frac{1}{2\pi^2}\frac{\not{x}-\not{y}}{(x-y)^2}$$

$$=-\frac{1}{2\pi^2}x'^2y'^2\not{x}'\frac{(\not{x}'-\not{y}')}{(x'-y')^4}\not{y}'. \tag{2.6}$$

The gauge field propagator is another story [12]. In the usual family of covariant gauges one has

$$\Delta_{\mu\nu}=\frac{1}{4\pi^2}\left[\frac{\delta_{\mu\nu}}{(x-y)^2}-\frac{1}{2}\Gamma\frac{J_{\mu\nu}(x-y)}{(x-y)^2}\right], \tag{2.7}$$

where $\Gamma=0$ is the Feynman gauge and $\Gamma=1$ is the Landau gauge. Only the second term transforms as expected under inversion, since

$$J_{\mu\nu}(x-y)=J_{\mu\rho}(x')J_{\rho\sigma}(x'-y')J_{\sigma\nu}(y'). \tag{2.8}$$

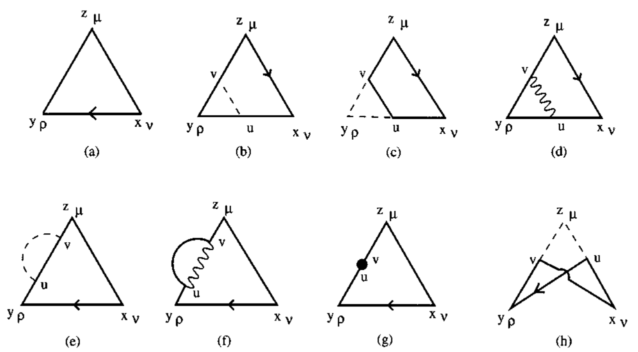

FIG. 1. One- and two-loop contributions to the anomaly in the Abelian Higgs theory. There is a chiral current at each corner of the triangle. Solid lines are fermions, dashed lines scalars, and wavy lines gauge fields. The solid circle in (g) is a local self-energy renormalization. Not shown are the same two-loop diagrams rotated $\pm 120°$ and diagrams with the opposite direction of fermion flow for all but the nonplanar diagram (h), and it is understood that in (b), (c), (e), and (h) both directions of Higgs propagation are included. The coordinate and Lorentz indices correspond to the integrals in Sec. III, and we refer to (b)–(h) with and without these indices (by their general topology) in the text.

The full propagator transforms properly only after a gauge transformation is performed [12], and this complicates applications to amplitudes with virtual photons.

It is well known that conformal symmetry restricts the tensorial form of two- and three-point correlation functions and frequently determines these tensors uniquely up to a constant multiple. (For recent discussions, see [16,18].) Inversion symmetry is sufficient to determine these restrictions, and the inversion property of a vector current of dimension 3 is

$$J_\mu(x) \to J'_\mu(x') = x'^6 J_{\mu\nu}(x') J_\nu(x'). \tag{2.9}$$

We are primarily interested in the abnormal parity part of the correlator $\langle J_\mu(x) J_\nu(y) J_\rho(z)\rangle$ of three conserved currents, and it is known that there is [17] a unique conserved rank-3 tensor function with the inversion property required by Eq. (2.9). The specific form is given, up to a multiplicative constant, by the lowest order massless fermion axial triangle amplitude [Fig. 1(a)]:

$$\begin{aligned} A_{\mu\nu\rho}(z,x,y) &= (-)\mathrm{Tr}\gamma_\mu\gamma_5 S(z-y)\gamma_\rho S(y-x)\gamma_\nu S(x-z) \\ &= \frac{1}{(2\pi^2)^3}\mathrm{Tr}\Bigg\{\gamma_5\gamma_\mu \frac{\not z - \not y}{(z-y)^4}\gamma_\rho \frac{\not y - \not x}{(y-x)^4}\gamma_\nu \\ &\quad \times \frac{\not x - \not z}{(x-z)^4}\Bigg\}, \end{aligned} \tag{2.10}$$

in which the $(-)$ is the usual factor for a closed fermion loop. The conformal properties can be readily verified using Eqs. (2.5) and (2.6).

For separated points this function obeys all desiderata. It is fully Bose symmetric and conserved on all three indices. The expected anomaly is a local violation of the conservation Ward identities which arises because the differentiation of singular functions is involved. There are several ways [13,19,20] to obtain the anomaly from this x-space viewpoint. One way [13], which we now summarize, is to recognize that the amplitude (2.10) is too singular at short distance to have a well-defined Fourier transform. One then regulates which entails the introduction of several independent mass scales, but the regulated form after the gamma matrix trace depends only on the ratios of these scales. The regulated amplitude is well defined, and one can check the Ward identities, which take the expected form

$$\frac{\partial}{\partial z^\mu} A_{\mu\nu\rho}(z,x,y) = a_z \varepsilon_{\nu\rho\lambda\sigma}\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial y_\sigma}\delta(x-z)\,\delta(y-z), \tag{2.11}$$

with similar expressions for the divergences with respect to x_ν and y_ρ. The anomaly coefficients a_z, a_x, a_y depend on the ratio of mass scales, and there is no choice of scales which makes all coefficients vanish. Specifically the sum $a_x + a_y + a_z = -1/4\pi^2$ is independent of the scales. There is a choice of scales which makes $a_x = a_y = a_z = -1/12\pi^2$ which is the Bose symmetric choice relevant for the gauge current correlation function in the U(1) Higgs model (2.2), and another choice to make $a_x = a_y = 0$ which is appropriate for the correlator of one axial and two vector currents.

A tenet of the space-time approach to renormalization is that the intrinsic ambiguity of a primitively divergent amplitude is an ultralocal distribution consistent with dimension and symmetry requirements. This corresponds to the p-space ambiguity of polynomials in external momenta. In this light the ambiguous part of the tensor amplitude (2.10) is

$$\begin{aligned} \Delta A_{\mu\nu\rho} &= \varepsilon_{\mu\nu\rho\sigma}\left[b_1\left(\frac{\partial}{\partial x_\sigma} - \frac{\partial}{\partial y_\sigma}\right) + b_2\left(\frac{\partial}{\partial y_\sigma} - \frac{\partial}{\partial z_\sigma}\right)\right] \\ &\quad \times \delta(x-z)\,\delta(y-z), \end{aligned} \tag{2.12}$$

where b_1 and b_2 are arbitrary constants. The mass scale dependence of the regulated amplitude is exactly of this form, and its Fourier transform is just the shift ambiguity due to choice of loop momenta in the traditional approach to the anomaly [21]. The nugget of this discussion of the space-time approach to the lowest order axial anomaly is that the well defined amplitude (2.10) for separated points determines the fact that there is an anomaly of specific strength. The choice of regularization or calculational procedure for the Fourier transform is just a redistribution of the anomaly between the three parameters a_x, a_y, a_z, which does not affect their sum.

We now return to the discussion of conformal symmetry and its role in the elucidation of possible radiative corrections to the anomaly. One may question this role because of the common lore that the introduction of a scale required to handle the divergences of perturbation theory spoils expected conformal properties. In general this is true, but the two-loop anomaly diagrams of Higgs models, which are drawn in Fig. 1, are exceptional. Any primitively divergent amplitude is exceptional when studied in x space for separated points, since the internal integrals converge without regularization. The nonplanar diagrams of Fig. 1(h) are primitives. Of course there are many other diagrams which contain subdivergent vertex and self-energy corrections, and these require a regularization scale. However, for the specific choice $\beta = -\alpha = 1/2$ which corresponds to pure axial coupling for the fermion, it is quite easy to see that there is a unique choice of gauge-fixing parameter Γ which makes the one-loop self-energy finite. Since the vertex and self-energy corrections are related by a Ward identity, each vertex correction is also finite in the same gauge; specifically, the sum of the three contributing Feynman diagrams is ultraviolet finite. In this ''finite gauge'' the integrals in the sum of three vertex insertion diagrams [Figs. 1(b), 1(c), 1(d)] at each corner of the two-loop triangle converge. The same statement holds for the integrals in the sum of the two self-energy insertion diagrams [Figs. 1(e), 1(f)] on each leg of the triangle.

The photon propagator (2.7) is not conformal covariant, and we will discuss this complication in the next section. We will show there that the diagrams for which this difficulty occurs are already covered by previous work [12,22]. In the remaining diagrams the photon propagator may be replaced by the inversion covariant second term of Eq. (2.7) and a finite gauge can be chosen. We then have the situation that each two-loop Feynman diagram we need to compute is constructed with inversion covariant propagators and vertices, and the sums of the self-energy diagrams on each leg and vertex diagrams at each corner are convergent. Then each of the three nonplanar diagrams and the summed self-energy or vertex insertion diagrams at each leg or corner of the triangle is a conformal covariant contribution to the current correlation function. Each of these amplitudes must be a multiple of the unique conformal tensor $A_{\mu\nu\rho}$ of Eq. (2.10), and we will show that the sum of the separate contributions to the net two-loop correlator vanishes. Further, we will show that the conformal inversion can be used as a transformation of the integration variables which makes the calculation of the eight-dimensional integrals easy and also gives an explicit verification of the conformal properties we have discussed above.

III. U(1) MODEL FOR $\beta = -\alpha = 1/2$

Euclidean correlation functions for the theory are constructed using the propagators of Eqs. (2.6) and (2.7), the vertex rules which can be read from the action (2.2), and the instruction to integrate $\int d^4u$ over each internal vertex of a diagram.

To illustrate the way conformal symmetry is used in our work, we first study the nonplanar graph of Fig. 1(h). Its amplitude is conformal covariant since no issues of subdivergences and gauge choice arise. The idea is to use the inversion $u_\alpha = u'_\alpha/u'^2$ and $v_\alpha = v'_\alpha/v'^2$ as a change of variable in the internal integrals. In order to use the simple conformal properties of the propagators (2.6) we must also refer the external points to their inverted images, e.g., $x_\mu = x'_\mu/x'^2$. If this is done for a generic configuration of x,y,z, there is nothing to be gained because the same integral is obtained in the u',v' variables. However, if we use translation symmetry to place one point at 0, say, $z=0$, it then turns out that the propagators attached to that point drop out of the integral, essentially because the inverted point is at ∞, and the integrals simplify.

After summing over both directions of Higgs field propagation and elementary manipulation of chiral factors L and R, the amplitude for the graph [Fig. 1(g)] can be written as

$$N_{\mu\nu\rho}(0,x,y) = \frac{ig^3f^2}{512\pi^{12}}\int d^4u\,d^4v\left(\frac{v_\mu}{u^2v^4} - \frac{u_\mu}{u^4v^2}\right) \times \mathrm{Tr}\,\gamma_5\gamma_\rho \frac{\not{y}-\not{u}}{(y-u)^4}\frac{\not{u}-\not{x}}{(u-x)^4} \times \gamma_\nu \frac{\not{x}-\not{v}}{(x-v)^4}\frac{\not{v}-\not{y}}{(v-y)^4}. \tag{3.1}$$

The integration variables u,v each appear in three denominators. This is not necessarily fatal, and indeed the u and v integrals can be evaluated in closed form using Feynman parameters [23]. However, we will see that the conformal inversion leads to simpler integrals. The change of variables outlined in the previous paragraph can be made with the help of (2.5) and (2.6) and the Higgs current transformation

$$\frac{v_\mu}{u^2v^4} - \frac{u_\mu}{u^4v^2} = u'^2v'^2(v'_\mu - u'_\mu). \tag{3.2}$$

Spinor propagator ''side factors,'' e.g., $\not{x}', \not{u}'$, etc., all collapse within the trace, and the Jacobian $(u'v')^{-8}$ cancels with factors in the numerator, giving the result

$$N_{\mu\nu\rho}(0,x,y) = \frac{ig^3f^2}{512\pi^{12}}x'^6y'^6J_{\nu\nu'}(x')J_{\rho\rho'}(y')\tilde{N}_{\mu\nu'\rho'},$$

$$\tilde{N}_{\mu\nu'\rho'} = \int d^4u'\,d^4v'(v'_\mu - u'_\mu) \times \mathrm{Tr}\left[\gamma_5\gamma_{\rho'}\frac{\not{y}'-\not{u}'}{(y'-u')^4}\frac{\not{u}'-\not{x}'}{(u'-x')^4}\gamma_{\nu'} \times \frac{\not{x}'-\not{v}'}{(x'-v')^4}\frac{\not{v}'-\not{y}'}{(v'-y')^4}\right]. \tag{3.3}$$

We see the expected transformation factors for the currents at x and y times an integral in which u' and v' each appear in only two denominators. Such convergent tensorial convolution integrals can be done by several methods. We have used Gegenbauer polynomial methods [24], and the results are tabulated in the Appendix. When these results are used and substituted within the trace, one finds the final amplitude

$$N_{\mu\nu\rho}(0,x,y) = -\frac{ig^3f^2}{512\pi^8}x'^6y'^6J_{\nu\nu'}(x')J_{\rho\rho'}(y') \times \frac{\mathrm{Tr}\,\gamma_5\gamma_\mu\gamma_{\rho'}\gamma_{\nu'}(\not{x}'-\not{y}')}{(x'-y')^4}. \tag{3.4}$$

The result above may be compared with the amplitude of the one-loop triangle graph [Fig. 1(a)] (with one direction of charge flow),

$$B_{\mu\nu\rho}(z,x,y) = \frac{ig^3}{8}A_{\mu\nu\rho}(z,x,y), \tag{3.5}$$

where $A_{\mu\nu\rho}$ is given in Eq. (2.10). At $z=0$, and referred to inverted points x',y', this reads

$$B_{\mu\nu\rho}(0,x,y)=\frac{ig^3}{8(8\pi^6)}x'^6y'^6J_{\nu\nu'}(x')J_{\rho\rho'}(y')\times\frac{\mathrm{Tr}\,\gamma_5\gamma_\mu\gamma_{\rho'}\gamma_{\nu'}(\not{x}'-\not{y}')}{(x'-y')^4}. \tag{3.6}$$

One may now observe that the nonplanar amplitude is just a numerical multiple of the unique conformal tensor (2.10) as discussed in Sec. II. The result may be written

$$N_{\mu\nu\rho}(0,x,y)=-\frac{f^2}{8\pi^2}B_{\mu\nu\rho}(0,x,y). \tag{3.7}$$

The nonplanar graphs with scalar vertices at x and y must give the same result by triangular symmetry. However, our method of evaluation of the amplitude has singled out the point $z=0$. Therefore a check on the result can be determined by applying the inversion to the amplitudes for the $\pm 120°$ rotated diagrams with $z=0$. The integral in inverted variables involves a different set of convolution integrals, and we have checked that it gives the same result (3.7).

We now discuss, following [12], the finite gauge mechanism for the one-loop self-energy and vertex corrections which are ingredients of our study of the two-loop anomaly. After a little algebra the sum of the Higgs and photon self-energy graphs can be written as

$$\Sigma(v-u)=\frac{1}{8\pi^4}\left[f^2+\frac{1}{2}g^2(1-\Gamma)\right]\frac{\not{v}-\not{u}}{(v-u)^6}+a\not{\partial}\delta^4(v-u). \tag{3.8}$$

The first term is the separated point part of the amplitude which is completely determined by the Feynman rules. It is a singular function of $v-u$ whose Fourier transform is linearly divergent. By choosing the gauge $\Gamma=1+2f^2/g^2$, the amplitude is made finite. It vanishes for separated points, but there is a possible local term, the second term in Eq. (3.8), which is left ambiguous by the Feynman rules. The finite constant a will be determined by the Ward identity.

The amplitudes of the three vertex subgraphs in the diagrams [Figs. 1(b), 1(c), 1(d)] are

$$V_\rho^{(1)}(y,v,u)=\frac{-igf^2}{32\pi^6}\gamma_5\frac{1}{(v-u)^2}\frac{\not{v}-\not{y}}{(v-y)^4}\gamma_\rho\frac{(\not{y}-\not{u})}{(y-u)^4}, \tag{3.9}$$

$$V_\rho^{(2)}(y,v,u)=-\frac{igf^2}{32\pi^6}\gamma_5\frac{1}{(y-u)^2}\frac{\overleftrightarrow{\partial}}{\partial y_p}\frac{1}{(y-u)^2}\frac{\not{v}-\not{u}}{(v-u)^4}, \tag{3.10}$$

$$V_\rho^{(3)}(y,v,u)=\frac{ig^3}{128\pi^6}\gamma_5\gamma_\alpha\frac{\not{v}-\not{y}}{(v-y)^4}\gamma_\mu\frac{\not{y}-\not{u}}{(y-u)^4}\gamma_\beta\times\left[\frac{\delta_{\alpha\beta}(1-\frac{1}{2}\Gamma)}{(u-v)^2}+\frac{\Gamma(u-v)_\alpha u(-v)_\beta}{(u-v)^4}\right]. \tag{3.11}$$

Each contribution has a logarithmic divergent Fourier transform, and we consider the Fourier transform at zero fermion momentum to study finiteness properties; that is, we consider the integrals $\int d^4u d^4v V_\rho(y,v,u)$. Let us examine first the single integral $\int d^4v V_\rho(y,v,u)$ which can be simplified by taking the point $y=0$. We will discuss these integrals in some detail because the same integrals will occur in the vertex insertion diagrams of the two-loop current correlator.

We see that the required integrals are again convergent convolution integrals. Using the tabulation in the Appendix, we obtain

$$\int d^4v V_\rho^{(1)}(0,v,u)=\frac{igf^2}{32\pi^4}\frac{\gamma_5\not{u}\gamma_\rho\not{u}}{u^6}=-\frac{igf^2}{32\pi^4}\gamma_5\gamma_\sigma\left(\frac{\delta_{\sigma\rho}}{u^4}-\frac{2u_\sigma u_\rho}{u^6}\right), \tag{3.12}$$

$$\int d^4v V_\rho^{(2)}(0,v,u)=\frac{igf^2}{32\pi^4}\frac{\gamma_5\gamma_\rho}{u^4}, \tag{3.13}$$

$$\begin{aligned}\int d^4v V_\rho^{(3)}(0,v,u)&=\frac{ig^3}{128\pi^4}\frac{\gamma_5\gamma_\alpha\gamma_\sigma\gamma_\rho\not{u}\gamma_\beta}{u^4}\times\left[\left(1-\frac{1}{4}\Gamma\right)\frac{\delta_{\alpha\beta}u_\sigma}{u^2}-\frac{1}{4}\Gamma\left(\frac{\delta_{\alpha\sigma}u_\beta+\delta_{\beta\sigma}u_\alpha}{u^2}-2\frac{u_\alpha u_\beta u_\sigma}{u^4}\right)\right]\\&=\frac{ig^3}{128\pi^4}\frac{\gamma_5\gamma_\alpha}{u^6}[(u^2\delta_{\alpha\rho}-4u_{\alpha\rho})+(1-\Gamma)\delta_{\alpha\mu}u^2].\end{aligned} \tag{3.14}$$

Consider next the second integration $\int d^4u\int d^4v V_\rho(0,v,u)$ which gives the zero momentum vertex function. The $\delta_{\alpha\rho}/u^4$ terms give the expected logarithmic divergence, but the integrals over the traceless tensor $(u^2\delta_{\alpha\rho}-4u_\alpha u_\rho)/u^6$ converge by symmetric integration. One sees that the sum of the divergent contributions from Eqs. (3.12), (3.13), and (3.14) is proportional to

$$2f^2-4f^2-g^2(1-\Gamma), \tag{3.15}$$

and therefore vanishes in the same gauge that makes the self-energy finite. Henceforth we will use this gauge.

The Ward identity for the theory may be derived by standard functional methods or obtained directly from the vertex amplitudes (3.9)–(3.11) using the relation

$$\Box\frac{1}{(y-u)^2}=-4\pi^2\delta^4(y-u). \tag{3.16}$$

The result is

$$\frac{\partial}{\partial y_\rho}V_\rho(y,v,u)=-i\tfrac{1}{2}g\gamma_5[\delta^4(y-v)-\delta^4(y-u)]\Sigma(v-u). \tag{3.17}$$

We wish to determine the constant a in the self-energy (3.8). This would appear in Eq. (3.17) as the coefficient of a very singular distribution, and so we integrate with respect to the smooth test function 1 and use the integrated Ward identity

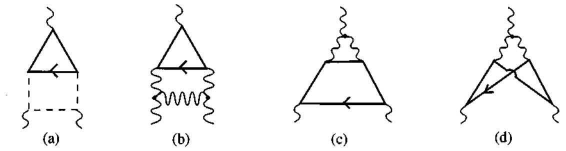

FIG. 2. Vanishing contributions to the three-current correlators. Diagrams (b), (c), and (d) are only present in correlators with non-Abelian gauge currents, and are discussed in Sec. V B.

$$\frac{\partial}{\partial y_\rho}\int d^4v\, V_\rho(y,v,0) = -i\tfrac{1}{2}g\gamma_5\Sigma(y). \tag{3.18}$$

In the finite gauge the only contributions to the integral in Eq. (3.18) come from the traceless tensor structures in Eqs. (3.12) and (3.14), and we have

$$\int d^4v\, V_\rho(y,v,0) = \frac{ig}{16\pi^4}(f^2-\tfrac{1}{2}g^2)\gamma_5\gamma_\sigma\left(\frac{y_\sigma y_\rho}{y^6}-\frac{\delta_{\sigma\rho}}{4y^4}\right). \tag{3.19}$$

The singular tensor can be expressed as the traceless form

$$\frac{y_\sigma y_\rho}{y^6}-\frac{\delta_{\sigma\rho}}{4y^4} = \frac{1}{8}\left(\partial_\sigma\partial_\rho - \frac{1}{4}\delta_{\sigma\rho}\Box\right)\frac{1}{y^2}, \tag{3.20}$$

which is a well-defined distribution, and its divergence is easily obtained:

$$\frac{\partial}{\partial y_\rho}\left(\frac{y_\sigma y_\rho}{y^6}-\frac{\delta_{\sigma\rho}}{4y^4}\right) = -\frac{3\pi^2}{8}\partial_\sigma\delta^4(y). \tag{3.21}$$

Combining Eqs. (3.18)–(3.21) one finds the self-energy in the finite gauge

$$\Sigma(y) = \frac{3}{64\pi^2}(f^2-\tfrac{1}{2}g^2)\not{\partial}\delta^4(y). \tag{3.22}$$

It is this result for $\Sigma(v-u)$ which is to be used to evaluate the self-energy insertion contributions [Fig. 1(g)] to the two-loop anomalous current correlation function. Because Eq. (3.22) is purely local, the integral $\int d^4u d^4v$ required for the graph of Fig. 1(g) is trivial. Specifically the $\delta^4(v-u)$ in Eq. (3.22) can be used to do the u integration, and $\not{\partial}_v$ acts on the resulting propagator giving a second δ function which can be used to do the v integration. The result is that the sum of the self-energy insertion graphs [Figs. 1(e), 1(f), 1(g)] is a multiple of the one-loop amplitude

$$\Sigma_{\mu\nu\rho}(z,x,y) = \frac{3}{64\pi^2}\left(f^2-\frac{1}{2}g^2\right)B_{\mu\nu\rho}(z,x,y). \tag{3.23}$$

Let us now discuss the diagrams which remain to be evaluated. The diagrams [Fig. 2(a)] with three virtual boson lines vanish trivially because the fermion trace vanishes. Next come the vertex insertion diagrams. It is convenient to view each virtual photon diagram as the sum of two graphs, one with the photon propagator in the Landau gauge $\Gamma=1$, and the second with inversion covariant pure gauge propagator

$$\tilde{\Delta}_{\mu\nu}(u-v) = -\frac{1}{8\pi^2}\frac{\gamma f^2}{g^2}\frac{1}{(u-v)^2}J_{\mu\nu}(u-v), \quad \gamma=2. \tag{3.24}$$

The Landau gauge graphs give order g^5 contributions to the two-loop $\langle J_\mu J_\nu J_\rho\rangle$ correlator, and the remainder gives an order g^3f^2 contribution. We now discuss these separately. The Landau gauge propagator is conformal covariant only after regauging by adding a gradient term given in [12]. So the sum of all virtual photon diagrams is conformal covariant, but individual virtual photon vertex insertion diagrams [Fig. 1(d)] are not, and are not simplified by the simple inversion discussed at the beginning of this section. Nevertheless, the photon vertex contributions to the two-loop $\langle A_\mu V_\nu V_\rho\rangle$ correlator in quantum electrodynamics were calculated by related but more complicated techniques in [12]. The net contribution to the correlator of vertex and self-energy insertions was found to vanish there, thus verifying the Adler-Bardeen theorem through two-loop order in QED. It is quite easy to see that, after clearing γ_5 factors, all Landau gauge virtual photon diagrams of Fig. 1(d) in the axial-coupled Abelian Higgs model are a uniform factor of 1/8 times the same graphs in QED. So it is fortunate that the work of [12] can be taken over to our case with the immediate result that the net contribution of Landau gauge virtual photon graphs to the gauge current correlator $\langle J_\mu J_\nu J_\rho\rangle$ vanishes. Specifically the sum of the Landau gauge vertex insertions and the order g^5 part of the self-energy insertions [second term in Eq. (3.23)] vanishes.

We now study the order g^3f^2 vertex insertion contributions to the three-point current correlator. These include virtual Higgs diagrams plus a virtual photon diagram with the propagator (3.24). Each of these graphs has a conformal covariant integrand, and so the inversion technique can be applied. The amplitude for the diagram shown in Fig. 1(b) is, with $z=0$,

$$V^{(1)}_{\mu\nu\rho}(0,x,y) = \frac{ig^3f^2}{512\pi^{12}}\int\frac{d^4u\,d^4v}{(u-v)^2} \times \mathrm{Tr}\left[\gamma_5\frac{\not{v}-\not{y}}{(v-y)^4}\gamma_\rho\frac{\not{y}-\not{u}}{(y-u)^4} \times\frac{\not{u}-\not{x}}{(u-x)^4}\gamma_\nu\frac{\not{x}}{x^4}\gamma_\mu\frac{\not{v}}{v^4}\right]. \tag{3.25}$$

The inversion may be performed; Jacobian factors again cancel and the trace simplifies, giving

$$V^{(1)}_{\mu\nu\rho} = \frac{ig^3f^2}{512\pi^{12}}x'^6y'^6J_{\nu\nu'}(x')J_{\rho\rho'}(y')\tilde{V}^{(1)}_{\mu\nu'\rho'},$$

$$\tilde{V}^{(1)}_{\mu\nu'\rho'} = \int\frac{d^4u'\,d^4v'}{(u'-v')^2}\times\mathrm{Tr}\left[\gamma_5\frac{\not{v}'-\not{y}'}{(v'-y')^4}\gamma_{\rho'}\frac{\not{u}'-\not{x}'}{(u'-x')^4}\gamma_{\nu'}\gamma_\mu\right]. \tag{3.26}$$

In both Eqs. (3.25) and (3.26) the first three factors in the integrals are exactly those of the vertex function

$V_\rho^{(1)}(y,v,u)$. The major difference between Eqs. (3.25) and (3.26) is that the $\not{x}/x^4$ and $\not{v}/v^4$ propagators have disappeared. The variable v' now appears in two denominators, and the v' integral is exactly the integral (3.12) with $u \to u'-y'$. We can thus write

$$\tilde{V}^{(1)}_{\mu\nu'\rho'} = -2\pi^2 \int d^4u' \left(\frac{(u'-y')_\sigma (u'-y')_{\rho'}}{(u'-y')^6} - \frac{1}{2}\frac{\delta_{\sigma\rho'}}{(u'-y')^4} \right) \frac{(u'-x')_\lambda}{(u'-x')^4} \operatorname{Tr}\gamma_5\gamma_\sigma\gamma_\lambda\gamma_{\nu'}\gamma_\mu . \tag{3.27}$$

The u' integral diverges logarithmically as $u' \to y'$, reflecting the logarithmic divergence of individual vertex diagrams. However, we must now add the contributions of the diagrams of Figs. 1(c), 1(d) in which the vertex parts of $V^{(2)}$ and $V^{(3)}$ of Eqs. (3.10) and (3.11) will appear [the latter with modified photon propagator Eq. (3.24)]. After the inversion process, one finds that the $\not{x}/x^4$ and $\not{v}/v^4$ propagators disappear, so that the v' integrals are again those of Eqs. (3.13) and (3.14). We now consider the sum of $V_\rho^{(1)}, V_\rho^{(2)}$ and gauge-modified $V_\rho^{(3)}$ insertions in the two-loop diagrams. It follows by inspection of Eqs. (3.12) and (3.13) that the net effect of the sum is to reduce the coefficient of the $\delta_{\sigma\rho'}$ term in Eq. (3.27) by a factor of 2, giving a traceless tensor in those indices, so that the remaining convolution integral in u' is convergent and may be read from Eq. (A10). The result for the net sum of vertex insertions at point y of the two-loop triangle is

$$\tilde{V}_{\mu\nu'\rho'} = \frac{\pi^4}{4} \frac{\operatorname{Tr}\gamma_5\gamma_\mu\gamma_{\rho'}\gamma_{\nu'}(\not{x}'-\not{y}')}{(x'-y')^4}. \tag{3.28}$$

When combined with the prefactors in Eq. (3.26) and expressed as a multiple of the one-loop amplitude one obtains

$$V_{\mu\nu\rho}(0,x,y) = \frac{f^2}{64\pi^2} B_{\mu\nu\rho}(0,x,y). \tag{3.29}$$

The vertex insertions at points x and z must give the same contribution. Again, we have studied the insertions at point z using a conformal inversion at $z \to 0$. A considerably more difficult set of integrals results in the inverted variables, but the final result agrees with Eq. (3.29).

The various contributions to the order $g^3 f^2$ amplitude must now be combined with careful attention to combinatorics. There is a factor of 3 from the triangular symmetry, and a factor of 2 for opposite directions of fermion charge flow for self-energy and vertex insertions, but not for the nonplanar diagrams. [From Fig. 1(h) we can see that the exchange $x \leftrightarrow y$ produces a topologically equivalent diagram.] Therefore our results (3.7), (3.23) [with $(f^2 - 1/2g^2) \to f^2$], and (3.29) must be added with weights

$$3N_{\mu\nu\rho} + 6(V_{\mu\nu\rho} + \Sigma_{\mu\nu\rho}) = -\frac{3f^2}{8\pi^2}\left(1 - \frac{1}{4} - \frac{3}{4}\right) B_{\mu\nu\rho}, \tag{3.30}$$

showing that the net order $g^3 f^2$ contribution to the gauge current correlation function $\langle J_\mu(z) J_\nu(x) J_\rho(y) \rangle$ vanishes.

It will be useful for our treatment of more general chiral gauge theories to give an alternative discussion of the integrals in the vertex insertion graphs. We have seen that after conformal inversion the $\int d^4v'$ of the three different vertex subgraphs can be read directly from Eqs. (3.12)–(3.14). These expressions show that the dependence on the remaining variable u (which is transformed to $u'-y'$ in the two-loop graphs) is a superposition of an "s-wave" $\delta_{\sigma\mu}/u^4$ and a "d-wave" $(u_\sigma u_\mu - 1/4\delta_{\sigma\mu}u^2)/u^6$ tensor form. For the pure gauge propagator (3.24), only the vertex diagram $V_\mu^{(1)}$ has a d wave, and the s waves of $\int d^4v V_\mu^{(1)}, \int d^4v V_\mu^{(2)}$, $\int d^4v V_\mu^{(3)}$ are in the ratio of $1:-2:1/2\gamma$. The final integral $\int d^4u'$ in the sum of the three graphs diverges unless the net s-wave amplitude cancels, and this selects the value $\gamma=2$ as the gauge parameter which makes the vertex insertion subgraphs finite.

IV. GENERAL U(1) MODEL

The action of this model has already been given in Eq. (2.2), and the Feynman rules differ from the special case treated in Sec. III only in the gauge vertex factors which now carry the chiral factors $-i\gamma_\mu(\alpha L + \beta R)$. We find it convenient to use the two coupling parameters β and α and impose the relation $\beta - \alpha = 1$ required for gauge invariance selectively as necessary.

The major technical problem of the more general model is that there is no true finite gauge due to the chiral gauge couplings. This is immediately clear from the one-loop self-energy amplitude

$$\hat{\Sigma}(v-u) = \frac{1}{8\pi^2}[f^2 + 2g^2(\alpha^2 R + \beta^2 L)(1-\Gamma)] \times \frac{\not{v}-\not{u}}{(v-u)^6} + (aL + bR)\not{\partial}_v \delta^4(v-u), \tag{4.1}$$

where, as in Eq. (3.8), we have included a possible local term. One sees that for $f \neq 0$ and $\alpha^2 \neq \beta^2$ there is no value of the gauge parameter Γ which eliminates the ultraviolet singular $1/(v-u)^5$ factor. Nevertheless, we will see that there is an effective finite gauge which makes the abnormal parity part of self-energy and vertex insertions on each line or each corner of the two-loop triangle finite.

Let us consider first all order $g^3 f^2$ contributions to the current correlator including all virtual Higgs graphs plus virtual photon graphs with a pure gauge propagator similar to Eq. (3.24) but with γ a numerical factor to be determined. It is easy to obtain the amplitude of the nonplanar graph. Clearing the chiral factors and comparing with the previous case (3.3), one sees that we now have

$$\hat{N}_{\mu\nu\rho}(0,x,y) = -4\alpha\beta N_{\mu\nu\rho}, \tag{4.2}$$

where the caret denotes the amplitude in the more general model.

For a given direction of fermion charge flow each vertex or self-energy insertion graph contains both normal and abnormal parity amplitudes. It follows from Furry's theorem that the normal parity part cancels and the abnormal parity part doubles in the sum of the two graphs with opposite charge flow, and so we can restrict our attention to the abnormal parity parts. We let $\hat{V}^{(i)}_{\mu\nu\rho}(z,x,y)$ for $i=1,2,3$ denote the abnormal parity part of the two-loop vertex graphs with vertex subgraph $V^{(i)}_{\rho}(y,v,u)$ inserted at one corner. The subgraph amplitudes are given in Eqs. (3.9)–(3.11) for $\beta=-\alpha=1/2$.

One can again manipulate chiral factors and compare with the previous case to find

$$\begin{aligned}\hat{V}^{(1)}_{\mu\nu\rho}(z,x,y)&=-4\alpha\beta V^{(1)}_{\mu\nu\rho}(z,x,y),\\ \hat{V}^{(2)}_{\mu\nu\rho}(z,x,y)&=2(\alpha^2+\beta^2)V^{(2)}_{\mu\nu\rho}(z,x,y),\\ \hat{V}^{(3)}_{\mu\nu\rho}(z,x,y)&=16(\beta^5-\alpha^5)V^{(3)}_{\mu\nu\rho}(z,x,y).\end{aligned}\tag{4.3}$$

The relation $\beta-\alpha=1$ has been used in the first equality.

We now recall our discussion at the end of Sec. III of the integrals which occur in the vertex insertion graphs after the conformal inversion is implemented. The integral $\int d^4v'$ gave the sum of d-wave and s-wave tensors in $u'-y'$ for $V^{(1)}$ and pure s waves for $V^{(2)}$ and $V^{(3)}$ with s waves occurring in the ratio $1:-2:1/2\gamma$. The gauge parameter γ must be chosen so that the net sum of the s waves vanishes. To implement this condition we must now weight the coefficients in Eq. (4.3) by 1, -2, and $1/2\gamma$, thus obtaining

$$\begin{aligned}&-4(\alpha^2+\alpha\beta+\beta^2)+8\gamma(\beta^5-\alpha^5)=0,\\ &\gamma=\frac{(\beta^3-\alpha^3)}{2(\beta^5-\alpha^5)},\end{aligned}\tag{4.4}$$

for the choice of gauge-fixing parameter which makes the sum of order g^3f^2 vertex insertion subgraphs at each corner of the triangle finite. For this choice the residual finite contribution to $\langle J_\mu(z)J_\nu(x)J_\rho(y)\rangle$ comes just from the d-wave tensor of $V^{(1)}$, and can be directly read from Eq. (4.3) as

$$\hat{V}_{\mu\nu\rho}(z,x,y)=\hat{V}^{(1)}_{\mu\nu\rho}+\hat{V}^{(2)}_{\mu\nu\rho}+\hat{V}^{(3)}_{\mu\nu\rho}=-4\alpha\beta V_{\mu\nu\rho},\tag{4.5}$$

where $V_{\mu\nu\rho}$ is given in Eq. (3.29).

Finally we must consider the self-energy insertion graphs. We must check that they are finite in the same gauge as the vertex insertion diagrams, and we must determine the contribution of possible local terms in $\Sigma(u-v)$, i.e., a or b in Eq. (4.1). To check finiteness we consider the insertion of $\Sigma(v-u)$ of Eq. (4.1) into the two-loop graph [Figs. 1(e), 1(f), 1(g)]. We move all chiral factors in the graph to the clockwise side of the inserted $\Sigma(v-u)$. This gives

$$\begin{aligned}(\alpha^3R+\beta^3L)\Sigma(v-u)=\frac{1}{8\pi^2}[&f^2(\alpha^3R+\beta^3L)\\ &-2g^2\Gamma(\alpha^5R+\beta^5L)]\frac{\not{v}-\not{u}}{(v-u)^6}\\ &+(\alpha^3bR+\beta^3aL)\not{\partial}_v\delta^4(v-u).\end{aligned}\tag{4.6}$$

The sum over graphs with opposite direction of fermion charge flow selects the abnormal parity part: namely,

$$\frac{1}{16\pi^2}[f^2(\alpha^3-\beta^3)-2g^2\Gamma(\alpha^5-\beta^5)]\gamma_5\frac{\not{v}-\not{u}}{(v-u)^6}+\tfrac{1}{2}(\alpha^3b-\beta^3a)\gamma_5\not{\partial}_v\delta^4(v-u).\tag{4.7}$$

This effective self-energy must be multiplied by propagators for adjacent fermions and integrated $\int d^4v\,d^4u$. The integral diverges unless the gauge parameter is chosen so that the singular $1/(v-u)^5$ term in Eq. (4.7) cancels. It is a relief, but hardly a surprise, to see that cancellation occurs for the value of γ given in Eq. (4.4), which also makes vertex contributions finite.

We now see that, in the effective finite gauge, the abnormal parity part of the self-energy insertion [Figs. 1(e), 1(f), 1(g)] involves only the local part of $\Sigma(v-u)$ [Fig. 1(g)] given by the second term in Eq. (4.7). The singularities from the diagrams shown in Fig. 1(e) and Fig. 1(f) cancel. The local term will now be obtained from the Ward identity

$$\frac{\partial}{\partial y_\rho}\hat{V}_\rho(y,v,u)=ig(\alpha R+\beta L)[\delta^4(y-v)-\delta^4(y-u)]\times\hat{\Sigma}(v-u),\tag{4.8}$$

obtained by direct differentiation of the vertex graphs $\hat{V}^{(i)}_\rho$ of the model with general couplings. The integrated form, which is the generalization of Eq. (3.18), is

$$\frac{\partial}{\partial y_\rho}\int d^4v\,\hat{V}_\rho(y,v,0)=ig(\alpha R+\beta L)\hat{\Sigma}(y).\tag{4.9}$$

We now note that in the environment of the larger two-loop graphs [Fig. 1(b), 1(c), 1(d)], all vertices at the y corner of the triangle acquire the factor $\alpha^2R+\beta^2L$ obtained by moving the $\alpha L+\beta R$ projectors at the z and x corners to the clockwise side of the point v. We are thus specifically interested in the abnormal parity part of the effective Ward identity

$$(\alpha^2R+\beta^2L)\frac{\partial}{\partial y_\rho}\int d^4v\,\hat{V}_\rho(y,v,0)=ig(\alpha^3R+\beta^3L)\hat{\Sigma}(y),\tag{4.10}$$

and we observe that the chiral factor on the right side is exactly that of the effective self-energy insertion in Eq. (4.6).

In the effective finite gauge (4.4) the integral in Eq. (4.10) involves only the d-wave tensor from the $\hat{V}^{(1)}_\rho$ amplitude. This contains an additional $\alpha L+\beta R$ chiral vertex factor, and so the coefficient of the abnormal parity part of Eq. (4.10) comes from $\alpha^2\beta R+\beta^2\alpha L$ and gives $-1/2\alpha\beta\gamma_5$. The value of the integral is then a factor of 2 times Eq. (3.19) [with $f^2-1/2g^2$ replaced by f^2 in Eq. (3.19) since we are now considering the order-f^2 terms only]. After computing the $\partial/\partial y_\rho$ divergence, as in Eqs. (3.20) and (3.21), the abnormal parity part of Eq. (4.10) reads, after dropping the factor ig on both sides,

$$-\alpha\beta\frac{3f^2}{64\pi^2}\gamma_5\not{\partial}\delta^4(y)=(\alpha^3R+\beta^3L)\hat{\Sigma}(y)\big|_{\substack{\text{abn}\\ \text{par}}}.\tag{4.11}$$

This equation gives γ_5 times the effective self-energy including chiral factors from the corners of the two-loop triangle. With a little thought we can then see that each self-energy graph of the general chiral theory is related to Eq. (3.23) by

$$\hat{\Sigma}_{\mu\nu\rho}(z,x,y) = -4\alpha\beta\Sigma_{\mu\nu\rho}(z,x,y). \qquad (4.12)$$

[An extra negative sign has been gained by moving the γ_5 past the propagator $S(z-v)$ to its original position in the trace of the two-loop graphs.]

The results (4.2), (4.5), and (4.12) show that all contributions to the abnormal parity part of the $\langle J_\mu J_\nu J_\rho \rangle$ correlators in the U(1) Higgs theory with general chiral couplings are a uniform factor of $-4\alpha\beta$ times the corresponding contributions in Sec. III. Thus the sum of all order $g^3 f^2$ terms in the correlation function vanishes.

We now discuss the order g^5 virtual photon contributions. We see from Eqs. (4.1) and (3.15) that the Landau gauge is a true finite gauge which makes the one-loop vertex and self-energy graphs entirely finite. This makes the argument simpler. We first note that all order g^5 graphs contain the chiral factor $(\alpha L+\beta R)^5$, whose abnormal parity part is just $1/2(\beta^5-\alpha^5)$ times the corresponding graph in the QED correlator studied in [12]. Further, when chiral factors are extracted from the Ward identity (4.9) one can see that it coincides with the QED Ward identity used in [12] to determine the local part of the self-energy. So the analysis of [12] applies in its entirety and shows that the two-loop $\langle J_\mu J_\nu J_\rho \rangle$ correlator also vanishes in the chiral U(1) model.

V. STANDARD MODEL ANOMALIES

We next calculate the two-loop anomalies in the $SU(3)\times SU(2)\times U(1)$ gauge theory with one generation of quarks and leptons. All fields are massless.

The Euclidean Lagrangian is

$$\mathcal{L} = \tfrac{1}{4}\overrightarrow{F^{\mu\nu}}\cdot\overrightarrow{F^{\mu\nu}} + \sum_{\substack{\text{quarks}\\ \text{leptons}}} \overline{\psi_{iI}}\gamma^\mu\left[\partial_\mu - ig_3\delta_{ij}\delta_{\psi,\text{quark}}T^A_{IJ}G^A_\mu - \frac{ig_2}{2}\delta_{IJ}\tau^a_{ij}W^a_\mu L - \frac{ig_1}{2}\delta_{ij}\delta_{IJ}(Y_L L + Y_R R)B_\mu\right]\psi_{jJ}$$
$$+\phi^\dagger\left(\overleftarrow{\partial}_\mu + \frac{ig_2}{2}\vec{\tau}\cdot\vec{W}_\mu + \frac{ig_1}{2}B_\mu\right)\left(\partial_\mu - \frac{ig_2}{2}\vec{\tau}\cdot\vec{W}_\mu - \frac{ig_1}{2}B_\mu\right)\phi - f_l(\overline{l_i}\phi_i Re + \overline{e}\phi_i^\dagger L l_i) - f_d(\overline{q_i}\phi_i Rd + \overline{d}\phi_i^\dagger L q_i)$$
$$-f_u(\overline{q_i}(i\tau^2)_{ij}\phi_j^\dagger Ru + \overline{u}\phi_j(i\tau^2)_{ji}Lq_i) - \tfrac{1}{4}\lambda(\phi^\dagger\phi)^2. \qquad (5.1)$$

τ^a_{ij} are the Pauli matrices and T^A_{IJ} the Gell-Mann matrices. B_μ is the Abelian gauge boson. The non-Abelian gauge bosons W^a_μ and G^A_μ will be referred to collectively as gluons. Lowercase Latin indices (i,j,a) refer to SU(2); uppercase Latin indices (I,J,A) refer to SU(3). l_i is the lepton SU(2) doublet (ν,e); q_i is the quark doublet (u,d); ϕ_i is the Higgs doublet (ϕ^+,ϕ^0). The leptons and Higgs bosons are singlets under SU(3), while the quarks are triplets. The U(1) hypercharges of the standard model matter fields are tabulated below (Table I). The one-loop anomalies are easily shown to be absent with this assignment.

The diagrams which contribute to the two-loop anomalies are essentially the same as those considered in Sec. IV, except that the fermion and gauge lines now carry group indices. We structure the calculation by comparison to the previous Abelian case with pure axial gauge coupling. There are in addition several diagrams [Figs. 2(b), 2(c), 2(d)] not considered previously involving vertices with three non-Abelian gauge bosons, but we show that they do not contribute because the anomaly vanishes at the one-loop level. The extension to N generations of quarks and leptons with unitary Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix M_{ij} for the quarks adds no new complications. Each gauge boson exchange diagram gets the factor $\mathrm{Tr}M^\dagger M = N$, and so each two-loop contribution to the correlator is identical to that given below times N.

In the standard model there is a potential anomaly for each choice of the three currents in the correlator. However, each diagram is proportional to the trace of the product of the three corresponding group generators. Since the trace of any [non-U(1)] generator is zero, the only three-current correlators which are potentially nonvanishing are $\langle J_1J_1J_1\rangle$, $\langle J_2J_2J_2\rangle$, $\langle J_2J_2J_1\rangle$, and $\langle J_3J_3J_1\rangle$. The SU(3) current is nonchiral; hence, the $\langle J_3J_3J_3\rangle$ correlator is not anomalous. Furthermore, after summing over the two directions of fermion flow, each diagram is proportional to the group theory d symbols, $\mathrm{Tr}T^a\{T^b,T^c\}$. Since the d symbols vanish for SU(2), the correlator $\langle J_2J_2J_2\rangle$ vanishes, as well.

The contributions to the abnormal parity part of the two-loop correlators from Abelian and non-Abelian gauge boson exchange vanish in the Landau gauge, as follows from the results of [12,22]. We keep the non-Abelian gauge bosons in the Landau gauge and take the gauge parameter Γ for the U(1) vector boson as the Landau gauge value 1 plus a term proportional to the Yukawa couplings which we will discuss below.

The determination of the ambiguous local part of the self-energy from the Ward identities will be discussed in a different (but equivalent) way from Sec. IV. We always check that in each gauge-invariant sector of the calculation there is a choice of the U(1) vector boson gauge parameter Γ which makes the would be divergent s-wave vertex contributions and $1/(u-v)^5$ self-energy contributions to the abnormal parity parts of the correlators simultaneously finite (and zero). It is then justified to keep only the d-wave parts of the once integrated $\int d^4v'$ vertex subgraphs, and the same for the integrated Ward identity used to determine the local part of the self-energy.

TABLE I. Fermion and Higgs hypercharges.

Y_L		Y_R	
ν	-1	0	
e	-1	-2	ϕ_i: $Y=1$
u	1/3	4/3	
d	1/3	$-2/3$	

A. $\langle J_1 J_1 J_1\rangle$

The $\langle J_1J_1J_1\rangle$ calculation in the standard model is directly analogous to that of the $\alpha L+\beta R$ theory considered in Sec. IV. The only difference is the sum over the SU(2) and SU(3) indices. We first check that the effective finite gauge mechanism works as in the previous case. It is sufficient that such a gauge exist for each gauge-invariant sector of the calculation. Since the leptons and quarks do not mix to this order, we find separate values for the gauge parameter Γ which make the lepton and quark vertex and self-energy insertions finite. We calculate the lepton contribution below. The quark calculation is analogous, with the charges replaced appropriately.

Including the relevant chirality factors and summing the contributions from internal electron and neutrino propagation, we easily find that the abnormal parity parts of $\hat{V}^{(1)}$, $\hat{V}^{(2)}$, and $\hat{V}^{(3)}$ [Fig. 1(b), 1(c), 1(d)] are, with the point z taken to zero,

$$\hat{V}^{(1)}_{\mu\nu\rho}(0,x,y)=\left(\frac{-1}{2}\right)^3\times 2\times(-4)Y_L^{(e)}Y_R^{(e)}V^{(1)}_{\mu\nu\rho}(0,x,y), \tag{5.2}$$

$$\hat{V}^{(2)}_{\mu\nu\rho}(0,x,y)=\left(\frac{-1}{2}\right)^3\times 2\times 2(Y_L^{(e)2}+Y_R^{(e)2})V^{(2)}_{\mu\nu\rho}(0,x,y), \tag{5.3}$$

$$\hat{V}^{(3)}_{\mu\nu\rho}(0,x,y)=\left(\frac{-1}{2}\right)^5(-16)\sum_{\substack{\text{neutrino}\\ \text{electron}}}(Y_R^5-Y_L^5)V^{(3)}_{\mu\nu\rho}. \tag{5.4}$$

where $V^{(1)},V^{(2)}$, and $V^{(3)}$ are the same diagrams in the pure axial U(1) theory considered in Sec. III. The left-handed electron and neutrino form an SU(2) doublet, and so they have the same hypercharge $Y_L^{(e)}$. The superscript (e) on the hypercharges denotes explicitly that only the right-handed electron hypercharge contributes. Of course, this comment is trivial since there is no right-handed neutrino in the theory, but Eqs. (5.2)–(5.4) are valid for the quark contributions as well with the appropriate replacement of lepton labels by quark labels. For example, the contribution from the f_d Yukawa coupling is obtained from Eqs. (5.2) and (5.3) by the replacement $Y_{L,R}^{(e)}\to Y_{L,R}^{(d)}$. The factors of $(-1/2)^3$ and $(-1/2)^5$ come from the difference in the definition of charge from the previous case, and one factor of 2 in Eqs. (5.2) and (5.3) is due to the SU(2) trace over the electron and neutrino. The condition for cancellation of divergent s-wave integrals in the sum of $V^{(1)}_{\mu\nu\rho},V^{(2)}_{\mu\nu\rho}$, and $V^{(3)}_{\mu\nu\rho}$ can now be written as the sum of the hypercharge coupling factors in Eqs. (5.2)–(5.4), each weighted by the factors $-1{:}2{:}-\gamma/2$ of the s-wave integrals in Eqs. (3.12)–(3.14). This gives the effective finite gauge condition

$$Y_L^{(e)}Y_R^{(e)}+(Y_R^{(e)2}+Y_L^{(e)\,2})+\frac{\gamma}{4}\sum(Y_R^5-Y_L^5)=0\,. \tag{5.5}$$

We checked that this condition agrees with that obtained from insisting that the sum of the self-energy diagrams be finite and indeed zero up to the ambiguity of local terms.

It is easy to check that the nonplanar diagram is multiplied by the same factor as $\hat{V}^{(1)}$ when compared to the pure axial U(1) case, Eq. (3.1):

$$\hat{N}_{\mu\nu\rho}(0,x,y)=Y_L^{(e)}Y_R^{(e)}N_{\mu\nu\rho}(0,x,y). \tag{5.6}$$

Next we calculate the local part of the self-energy which contributes when inserted in the fermion triangle [Fig. 1(g)] in the effective finite gauge. If we denote by $V_i^\mu(z,u,v)$ the lepton vertex (i=electron or neutrino) with charge flowing from v to u and by $\Sigma_1(u-v)$ the self-energy with charge flowing in the same direction, then the relevant Ward identity is, as in Eq. (4.8),

$$\partial_\mu^z V_i^\mu(z,u,v)=\frac{ig_1}{2}[\delta^4(z-v)-\delta^4(z-u)]\times(Y_R^iL+Y_L^iR)\Sigma_1^i(u-v). \tag{5.7}$$

This can be integrated to give

$$\int d^4u\,\partial_\mu^z V_i^\mu=-\frac{ig_1}{2}(Y_R^iL+Y_L^iR)\Sigma_1(z-v). \tag{5.8}$$

We calculate the d-wave (traceless) part of the integrated vertex as before, since the s-wave contributions and the corresponding $1/z^5$ part of the self-energy vanish in the effective finite gauge. Again, the only d-wave contribution is from Fig. 1(b) with a single Higgs boson exchanged. We find that

$$\int d^4u\,\partial_\mu^z V_i^\mu(z,u,v)\overset{d\text{ wave}}{=}\frac{3ig_1f_l^2}{16(4\pi^2)}(2Y_L^{(e)}L\,\delta_{i(e)}+Y_R^{(e)}R)\,\not{\partial}_z\delta^4(z-v). \tag{5.9}$$

Projecting out the left- and right-handed pieces of Eq. (5.9) and comparing with the Ward identity (5.8), we find for the relevant contribution of the local part of the self-energy:

$$\Sigma_1^i(z-v)=\frac{-3f_l^2}{16(4\pi^2)}\left(2\frac{Y_L^{(e)}}{Y_R^{(e)}}L\,\delta_{i(e)}+\frac{Y_R^{(e)}}{Y_L^{(e)}}R\right)\not{\partial}\delta^4(z-v). \tag{5.10}$$

The $\delta_{i(e)}$ means that the right-handed electron (but not neutrino) flows through Σ.

We insert this self-energy into the fermion triangle [Fig. 1(g)]. After pulling together chiral factors and summing over the electron and neutrino we indeed find a multiplicative factor $Y_L^{(e)}Y_R^{(e)}$ times the result in the pure axial case considered in Sec. III:

$$\hat{\Sigma}_{\mu\nu\rho}(0,x,y)=Y_L^{(e)}Y_R^{(e)}\Sigma_{\mu\nu\rho}(0,x,y), \qquad (5.11)$$

with $(f^2-1/2g^2)\rightarrow f^2$ in $\Sigma_{\mu\nu\rho}$, Eq. (3.23).

Thus the vertex, nonplanar, and self-energy diagrams are each multiplied by an overall factor of $Y_L^{(e)}Y_R^{(e)}$ times the corresponding diagrams in the pure axial U(1) case treated in Sec. III. Note that with $\alpha,\beta\rightarrow Y_L^{(e)},Y_R^{(e)}$, this result is identical to that of Sec. IV up to a factor of (-4) which comes from the difference in definition of U(1) charge and the sum over electron and neutrino. So the sum of all virtual lepton contributions to the correlator $\langle J_1^\mu J_1^\rho J_1^\nu\rangle$ vanishes as in the previous case.

The quark contributions vanish similarly. Note that while the quarks have two different Yukawa couplings, to this order these couplings do not mix because of the relative chirality flip between the two couplings. Furthermore, while the up type Yukawa coupling f_u in the Lagrangian (5.1) seems more complicated than the down or electron type, we can redefine the Higgs field to be the SU(2) conjugate $\phi_i'=(i\tau_2)_{ij}\bar{\phi}_j$, and then the kinetic term in terms of the conjugated Higgs field looks identical to the previous case except for a change of sign of the Higgs U(1) charge. The calculation is almost identical to the previous case, except for the change from lepton notation to quark notation [e.g., the superscript $(e)\rightarrow(u)$] and sign changes from the new U(1) charges. For example, in Eqs. (5.2), (5.3), (5.6), and (5.11) there is an extra negative sign from the opposite signs of the Higgs U(1) charge and $Y_L^{(u)}-Y_R^{(u)}$.

Hence the total contribution from the gauge fields and each of the three Yukawa couplings to the two-loop correlator vanishes, as expected.

B. $\langle J_2J_2J_2\rangle$

We noted earlier that the correlator $\langle J_2J_2J_2\rangle$ vanishes by virtue of the fact that the SU(2) d symbols are all zero. However, we expect that the vanishing of the anomaly should not depend on the gauge group, as long as the quarks and leptons are in a representation for which the one-loop anomaly vanishes. We therefore check that even if we neglect the fact that the group theoretic d symbols are zero the correlator still vanishes. We only require that the trace of the d symbols over the left- handed fermion representations in the theory vanish.

The calculation is remarkably simple as a result of the left-handedness of the SU(2) current. Since the Yukawa couplings in the Lagrangian (5.1) always connect a left-handed field to a right-handed one, it is easy to see that both the nonplanar diagram and the vertex insertion [Fig. 1(b)] with a single Higgs boson exchanged vanish. The contribution from the self-energy is determined via the SU(2) Ward identity and vanishes because the d-wave part of the vertex contribution vanishes. Since these are the only contributions to the correlator in the effective finite gauge, which we also checked to exist, the correlator vanishes to two-loops.

It appears at first that there are additional diagrams [Figs. 2(b), 2(c), 2(d)] which we have not calculated, but they are all proportional to the one-loop anomaly. Figure 2(b) contains the one-loop triangle, and so it is immediately proportional to the one-loop anomaly. The group theory of the other diagrams is easy to work through. The three-gauge vertex is proportional to the SU(2) structure constant $f^{abc}(=\varepsilon^{abc})$. Summing over the two directions of fermion flow, we see that the abnormal parity part of each of the diagrams [Figs. 2(c), 2(d)] is proportional to $\mathrm{Tr}f^{ade}(\tau^b\tau^c\tau^e\tau^d-\tau^d\tau^e\tau^c\tau^b)=\mathrm{Tr}f^{ade}f^{edg}\tau^b\{\tau^c,\tau^g\}=-2\mathrm{Tr}d^{abc}$. So each two-loop diagram in question contains the factor $\mathrm{Tr}d^{abc}$, where the trace sums over all left-handed fermions in the theory. The condition $\mathrm{Tr}d^{abc}=0$ also makes the one-loop anomaly vanish, as promised above. One can easily check that the same effect occurs for the contribution of the analogous diagrams to the other correlators discussed below. In both of the cases treated below in Secs. V C and V C, Figs. 2(c), 2(d) are proportional to the factor which makes the corresponding one-loop anomaly vanish.

C. $\langle J_1(z)J_2(x)J_2(y)\rangle$

The calculation of the $\langle J_1J_2J_2\rangle$ and $\langle J_1J_3J_3\rangle$ correlators is complicated by the fact that the local part of the self-energy calculated from the U(1) Ward identity and the d-wave part of the U(1) vertex is not consistent with that calculated via the SU(2) [SU(3)] Ward identity and the d-wave part of the SU(2) [SU(3)] vertex. This is not surprising since the vertex is also ambiguous up to a local term as a result of renormalization. We are free to fix the arbitrary coefficient of the local self-energy, as long as we concurrently add local parts to the vertices to make them consistent with the Ward identities.

We will use the local part of the self-energy calculated in Sec. V A and modify the local SU(2) vertex to make it consistent with this choice. The U(1) vertex is unmodified. Again, we calculate only the lepton contribution here. The quark contribution is analogous.

First we find the effective finite gauge for this calculation. Note that the d-wave contribution to the integrated SU(2) vertex vanishes. The left-handed fields do not contribute because at the Yukawa vertex they become right-handed fields which do not couple to the SU(2) current. The right-handed fields do not contribute because they are SU(2) singlets; alternatively, their contribution is proportional to $\mathrm{Tr}\tau^a=0$. The only nonvanishing two-loop diagram of the form Fig. 1(b) is from the U(1) vertex, which is inserted at the point $z=0$ in the two-loop triangle. It contributes

$$\hat{V}_{\mu\nu\rho}^{(1\,z)ab}=\tfrac{1}{2}Y_R^{(e)}\mathrm{Tr}\tau^a\tau^bV_{\mu\nu\rho}^{(1)}. \qquad (5.12)$$

The label z on $\hat{V}^{(1\,z)}$ denotes that the vertex is placed at the U(1) corner, which is chosen to lie at point z.

In diagrams of the form Fig. 1(c), the two-Higgs-boson vertex can lie at the U(1) corner or either of the SU(2) corners of the triangle. When it lies at the U(1) corner it contributes

$$\hat{V}_{\mu\nu\rho}^{(2\,z)ab}=-\tfrac{1}{4}\mathrm{Tr}\tau^a\tau^bV_{\mu\nu\rho}^{(2)}. \qquad (5.13)$$

When it lies at one of the SU(2) vertices it contributes

$$\hat{V}_{\mu\nu\rho}^{(2\,x)ab}=\hat{V}_{\mu\nu\rho}^{(2\,y)ab}=-\tfrac{1}{4}Y_L^{(e)}\mathrm{Tr}\tau^a\tau^bV_{\mu\nu\rho}^{(2)}. \qquad (5.14)$$

The contribution to the vertex diagrams from the effective U(1) gauge boson propagator is the same at each corner of the triangle. It is

$$\hat{V}^{(3z)ab}_{\mu\nu\rho}=\hat{V}^{(3x)ab}_{\mu\nu\rho}=\hat{V}^{(3y)ab}_{\mu\nu\rho}=-Y_L^{(e)3}\mathrm{Tr}\tau^a\tau^b V^{(3)}_{\mu\nu\rho}. \tag{5.15}$$

Including the relative coefficients of $V^{(1)},V^{(2)}$, and $V^{(3)}$ in Eqs. (5.12)–(5.15) in the ratio $-1:2:-\gamma/2$ as before and summing over the three corners of the triangle, we find the condition for the divergent (s-wave) parts of the vertex diagrams to cancel:

$$1-\frac{\gamma}{2}Y_L^{(e)\,2}=0. \tag{5.16}$$

Although we obtained this result by summing over the three corners of the triangle, the same condition makes the U(1) and SU(2) vertex contributions separately finite. This is expected because they are related to the same self-energy insertion graphs by Ward identities, as in Eq. (4.10).

We next consider the contribution from the local SU(2) vertex correction to the two-loop correlator. The relevant SU(2) Ward identity is

$$\partial^z_\mu V^{a\mu}_{ji}(z,u,v)=\left(\frac{-ig_1}{2}\right)\tau^a_{ji}[\delta^4(z-u)-\delta^4(z-v)]\times\Sigma(u-v)L, \tag{5.17}$$

where the SU(2) charge flows from i to j and $\Sigma(u-v)$ is the left-handed part of the self-energy (which is the same for the electron and neutrino).

In the effective finite gauge the sum of the electron and neutrino contributions to the vertex and self-energy insertions is finite. Equivalently, there are no s-wave parts of vertex insertions, and so we can confine ourselves to just the d-wave part of the Ward identity and speak separately about the electron and neutrino. However, the SU(2) vertex insertion has no Higgs boson exchange diagram [Fig. 1(b)] and thus no d-wave part. Thus, if we were to use the SU(2) Ward identity to derive the consistent local part to the self-energy, it would vanish as well. Thus the self-energy (5.10) is not consistent with Eq. (5.17). Hence we modify the vertex by a local contribution as discussed. We add to the SU(2) vertex a local part of the form

$$\Delta V^{a\mu}_{ji}(z,u,v)=\frac{ig_2}{2}Z\tau^a_{ji}\gamma^\mu L\,\delta^4(z-u)\,\delta^4(z-v). \tag{5.18}$$

Its divergence is easily calculated to be

$$\frac{\partial}{\partial z^\mu}\Delta V^{a\mu}_{ji}(z,u,v)=\frac{ig_2}{2}Z\tau^a_{ji}[\delta^4(z-v)-\delta^4(z-u)]\times R\,\partial\!\!\!/_v\,\delta^4(v-u). \tag{5.19}$$

With the self energy given in Eq. (5.10), the Ward identity (5.17) determines the parameter Z to be

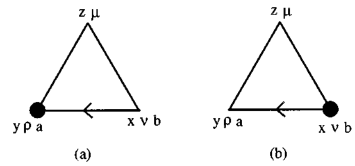

FIG. 3. Contributions to $\langle J_1^\mu(z)J_2^{a\rho}(y)J_2^{b\nu}(x)\rangle$ from the SU(2) vertex renormalization. In Sec. V D we instead renormalize the U(1) vertex.

$$Z=\frac{-3f^2}{16(4\pi^2)}\frac{Y_R^{(e)}}{Y_L^{(e)}}. \tag{5.20}$$

Inserting this vertex renormalization at either of the SU(2) corners of the one-loop lepton triangle [Figs. 3(a), 3(b)], we immediately find a result proportional to the one-loop amplitude, whose abnormal parity part $\Delta V^{ab}_{\mu\nu\rho}$ is

$$\Delta V^{ab}_{\mu\nu\rho}(0,x,y)=\frac{1}{2}\frac{(-3f_l^2)}{16(4\pi^2)}Y_R^{(e)}\mathrm{Tr}\tau^a\tau^b B_{\mu\nu\rho}(0,x,y)$$
$$=-\frac{1}{2}Y_R^{(e)}\mathrm{Tr}\tau^a\tau^b\Sigma_{\mu\nu\rho}(0,x,y), \tag{5.21}$$

where $B_{\mu\nu\rho}$ and $\Sigma_{\mu\nu\rho}$ are given in Eqs. (3.5) and (3.23) [with $(f^2-1/2g^2)\to f^2$], respectively.

The contribution from the diagram [Fig. 1(g)] with the self-energy Eq. (5.10) inserted at any of the three legs of the triangle is easily calculated to be

$$\hat{\Sigma}^{ab}_{\mu\nu\rho}=\tfrac{1}{2}Y_R^{(e)}\mathrm{Tr}\tau^a\tau^b\Sigma_{\mu\nu\rho}. \tag{5.22}$$

The nonplanar diagram with the Higgs current placed at the U(1) vertex vanishes because the chiral projectors at the Yukawa vertices annihilate the propagating fermions. The abnormal parity contribution from each of the two remaining nonplanar diagrams is

$$\hat{N}^{(x)ab}_{\mu\nu\rho}=\hat{N}^{(y)ab}_{\mu\nu\rho}=\tfrac{1}{4}Y_R^{(e)}\mathrm{Tr}\tau^a\tau^b N_{\mu\nu\rho}. \tag{5.23}$$

Recall again that in the pure axial case the relative contributions of the self-energy, vertex and nonplanar graphs were in the ratio $3:1:-4$, summing to zero. Adding the contributions in the effective finite gauge from Eqs. (5.12) and (5.21)–(5.23) gives the now familiar result

$$\Big[\underbrace{\left(-\tfrac{1}{2}\cdot 2+\tfrac{1}{2}\cdot 3\right)\cdot 3}_{\substack{\text{vertex correction}\\ \text{+self energy}}}+\underbrace{\tfrac{1}{2}\cdot 1}_{\text{vertex}}+\underbrace{\tfrac{1}{4}\cdot 2\cdot(-4)}_{\text{nonplanar}}\Big]\,Y_R^{(e)}\mathrm{Tr}\tau^a\tau^b=0. \tag{5.24}$$

D. $\langle J_1(z)J_3(x)J_3(y)\rangle$

We proceed as in Sec. V C. The difference here is that SU(3) couples only to quarks and the coupling is nonchiral. First we check that the effective finite gauge mechanism works in this case. Again, the two quark Yukawa couplings f_d and f_u contribute independently at the two-loop level. (They do not mix because of the relative chirality flip between the two couplings, as can be seen by trying to draw a two-loop Higgs boson exchange diagram with both Yukawa couplings.) We calculate the contribution from the f_d coupling here and note, as before, that the f_u coupling contributes analogously with an overall sign change from the U(1) charges.

With the one-loop Higgs boson exchange vertex inserted at the U(1) corner [Fig. 1(b)] we get

$$\hat{V}^{(1\,z)AB}_{\mu\nu\rho}=2\cdot 2\ \mathrm{Tr}T^AT^B(Y_R^{(d)}-Y_L^{(d)})V^{(1)}_{\mu\nu\rho}=-4\ \mathrm{Tr}T^AT^BV^{(1)}_{\mu\nu\rho}. \tag{5.25}$$

At each of the SU(3) vertices it contributes the same except for a negative sign which can be traced to the vector nature of the SU(3) coupling,

$$\hat{V}^{(1\,x,y)AB}_{\mu\nu\rho}=-2\times 2\ \mathrm{Tr}T^AT^B(Y_R^{(d)}-Y_L^{(d)})V^{(1)}_{\mu\nu\rho}=4\ \mathrm{Tr}T^AT^BV^{(1)}_{\mu\nu\rho}. \tag{5.26}$$

Since the Higgs boson is an SU(3) singlet, the only vertex diagram including the Higgs current [Fig. 1(c)] is from the U(1) corner of the triangle. It contributes

$$\hat{V}^{(2\,z)AB}_{\mu\nu\rho}=-2\times 2\mathrm{Tr}T^AT^BV^{(2)}_{\mu\nu\rho}. \tag{5.27}$$

The U(1) gauge boson exchange diagram [Fig. 1(d)] contributes at each corner of the triangle

$$\hat{V}^{(3\,z,x,y)AB}_{\mu\nu\rho}=-2\sum_{\mathrm{quarks}}(Y_L^3-Y_R^3)\mathrm{Tr}T^AT^BV^{(3)}_{\mu\nu\rho}. \tag{5.28}$$

Recalling that in the previous case the diagrams of Figs. 1(b), 1(c), 1(d) appeared in the ratio $1:-2:\gamma/2$, we find the effective finite gauge condition either by summing over the three corners of the triangle or by summing the contributions at any particular corner.

$$1+\frac{\gamma}{4}\sum_{\mathrm{quarks}}(Y_R^3-Y_L^3)=0\ . \tag{5.29}$$

One can easily check that the same condition makes the singular $1/(v-u)^5$ parts of the self-energy insertion diagrams [Figs. 1(e), 1(f)] cancel.

For aesthetic reasons we choose not to introduce a parity-nonconserving correction to the SU(3) fermion vertex but rather introduce a correction to the ambiguous part of the U(1) vertex. Then in the self-energy insertion Fig. 1(g) we determine the ambiguous local part of the self-energy via the relevant SU(3) Ward identity and the d-wave part of the integrated SU(3) quark vertex. The relevant identity is

$$\partial^z_\mu V^{A\mu}_{JI}=ig_3[\delta^4(z-v)-\delta^4(u-z)]T^A_{JI}\Sigma_3(u-v), \tag{5.30}$$

where the notation is as before, with I and J SU(3) indices. We calculate the d-wave part of the vertex $V^{A\mu}_{JI}$ to determine the relevant ambiguous part of the self-energy which contributes when inserted in the fermion triangle. We find

$$\Sigma^i_3(z-v)=\frac{-3f_d^2}{16(4\pi^2)}(2L\,\delta_{i(d)}+R)\not\partial_z\delta^4(z-v), \tag{5.31}$$

where the $\delta_{i(d)}$ means that the right-handed down (but not up) quark receives a self-energy contribution. Inserting this self-energy in the fermion triangle, Fig. 1(g), gives

$$\hat{\Sigma}^{AB}_{\mu\nu\rho}=-4\mathrm{Tr}T^AT^B\Sigma_{\mu\nu\rho}, \tag{5.32}$$

with $\Sigma_{\mu\nu\rho}$ given by Eq. (3.23) with $(f^2-1/2g^2)\rightarrow f^2$. Next we calculate the correction to the U(1) vertex which makes it consistent with gauge invariance. Assume a correction of the form

$$\Delta V^\mu_i(z,u,v)=\left(\frac{ig_1}{2}\right)\gamma^\mu(\alpha_iL+\beta_iR)\,\delta^4(z-u)\,\delta^4(z-v). \tag{5.33}$$

Its divergence is

$$\partial^z_\mu\Delta V^\mu_i(z,u,v)=\frac{ig_1}{2}(\alpha_iR+\beta_iL)[\delta^4(z-u)-\delta^4(z-v)]\not\partial_u\delta^4(u-v). \tag{5.34}$$

Recall the U(1) Ward identity

$$\partial^z_\mu\Delta V^\mu_i(z,u,v)=-\frac{ig_1}{2}[\delta^4(z-u)-\delta^4(z-v)]\times(Y^i_RL+Y^i_LR)\Delta\Sigma^i(u-v), \tag{5.35}$$

where $\Delta\Sigma^i(u-v)$ is

$$\Delta\Sigma^i(u-v)=\Sigma^i_3(u-v)-\Sigma^i_1(u-v)=\frac{3f_d^2}{16(4\pi^2)}\left(\frac{2}{Y_R^{(d)}}\delta_{i(d)}L-\frac{1}{Y_L^{(d)}}R\right)\not\partial_u\delta^4(u-v). \tag{5.36}$$

Comparing Eq. (5.34) with Eqs. (5.35) and (5.36) determines α and β, giving

$$\Delta V^\mu_i(z,u,v)=\frac{ig_1}{2}\left(\frac{3f_d^2}{16(4\pi^2)}\right)\gamma^\mu[L-2\,\delta_{i(d)}R]\times\delta^4(z-u)\,\delta^4(z-v). \tag{5.37}$$

Inserting this at the U(1) vertex of the basic fermion triangle (and summing over the u and d quarks) gives for the abnormal parity part

$$\Delta V^{AB}_{\mu\nu\rho}=8\mathrm{Tr}T^AT^B\Sigma_{\mu\nu\rho}, \tag{5.38}$$

where again $\Sigma_{\mu\nu\rho}$ is given by Eq. (3.23) with $(f^2-1/2g^2)\rightarrow f^2$. Finally, the only nonplanar diagram contributes

$$\hat{N}^{(z)AB}_{\mu\nu\rho}=-4\mathrm{Tr}T^AT^BN_{\mu\nu\rho}. \tag{5.39}$$

Once again, summing the contributions from Eqs. (5.25), (5.26), (5.32), (5.38), and (5.39) gives zero total contribution to the axial part of the $\langle J_1J_3J_3\rangle$ correlator, which is proportional to

$$\Big[\underbrace{(8-4\cdot 3)\cdot 3}_{\substack{\text{vertex correction}\\ \text{+self energy}}}-\underbrace{4\cdot 1}_{\text{vertex}}-\underbrace{4\cdot(-4)}_{\text{nonplanar}}\Big]\, Y_R^{(d)}\,\mathrm{Tr}\,T^A T^B=0. \tag{5.40}$$

VI. SUPERSYMMETRIC GAUGE THEORIES

Techniques very similar to those of this paper were applied in a recent study of the operator product algebra of conserved currents of SUSY gauge theories [15]. The required calculations were very briefly summarized in [15], and we will discuss some aspects in more detail here.

$N=1$ SUSY gauge theories contain component fields $A_\mu^a(x)$ and $\lambda^a(x)$, gluons and gluinos, respectively, in the adjoint representation of a gauge group G, and complex scalars ϕ^i and their spinor partners ψ^i which transform in a representation of G with Hermitian generators T_j^{ai}. There are gauge interactions with gauge coupling g and a cubic superpotential with complex coupling Y_{ijk} which is totally symmetric. Using Euclidean Majorana spinors [25], the action is

$$\begin{aligned} S=\int d^4x\big[&\tfrac{1}{4}F_{\mu\nu}^2+\tfrac{1}{2}\bar{\lambda}\not{D}\lambda+\overline{D_\mu\phi}D_\mu\phi+\tfrac{1}{2}\bar{\psi}\not{D}\psi\\ &+i\sqrt{2}g(\overline{\lambda^a}\bar{\phi}_i T_j^{ai}L\psi^j-\bar{\psi}_i R T_j^{ai}\phi^j\lambda^a)\\ &-\tfrac{1}{2}(\overline{\psi^i}LY_{ijk}\phi^k\psi^j+\bar{\psi}_i R\overline{Y^{ijk}}\bar{\phi}_k\psi_j)+\tfrac{1}{2}g^2(\bar{\phi}_i T_j^{ai}\phi^j)^2\\ &+\tfrac{1}{4}Y_{ijk}\overline{Y^{ilm}}\phi^j\phi^k\overline{\phi_l\phi_m}\big]. \end{aligned} \tag{6.1}$$

The theory has two classically conserved, but anomalous, axial vector currents, the R current, and the Konishi current:

$$R_\mu(x)=\frac{1}{2}\bar{\lambda}\gamma_\mu\gamma_5\lambda-\frac{1}{6}\bar{\psi}\gamma_\mu\gamma_5\psi+\frac{2}{3}\bar{\phi}\overleftrightarrow{D}_\mu\phi,$$

$$K_\mu(x)=\frac{1}{2}\bar{\psi}_i\gamma_\mu\gamma_5\psi^i+\bar{\phi}_i\overleftrightarrow{D}_\mu\phi^i. \tag{6.2}$$

Actually K_μ is conserved classically only if $Y_{ijk}=0$. The operator product algebra of R_μ and K_μ contains two central charges c and c', and the principal result of [15] was evidence for a universality property of the interaction-dependent radiative corrections to c and c'. For c' this information was obtained from a study of two-loop contributions to the two- and three-point correlation functions of the currents, including the correlator $\langle R_\mu(x)R_\nu(y)K_\rho(z)\rangle$. Earlier work [26] could be modified to obtain the required information about c. Internal gauge and superpotential interactions can be treated separately in two-loop order.

In the gauge sector, both currents are conserved and have no anomalous dimensions, and conformal invariance holds, and so the amplitude is again a constant multiple of the unique conformal pseudotensor $A_{\mu\nu\rho}$ of Eq. (2.10). The Feynman graphs of $\langle R_\mu R_\nu K_\rho\rangle$ involve the gluon and Yukawa interactions of Eq. (6.1) after setting $Y_{ijk}=0$, and they are the same graphs considered in Secs. III–V above with different numerical coefficients. As was the case in the standard model, there are Ward identities relating vertex functions of $R_\mu(x)$ and $K_\mu(x)$ to the same self-energy $\Sigma(u-v)$, and an additional local term is required for one of the currents. There is some freedom in the assignment of local terms to the currents and self-energy, but the full correlator $\langle R_\mu R_\nu K_\rho\rangle$ is independent of the choice made. Using one convenient choice, and after careful comparison of all graphs with those of the basic U(1) model of Sec. III, it was found that the sum of all one- and two-loop contributions is

$$\langle R_\mu(x)R_\nu(y)K_\rho(z)\rangle=\left[\frac{1}{9}\mathrm{dim}T+\frac{g^2}{32\pi^2}\mathrm{Tr}T^aT^a\times\left(\frac{8}{3}+\frac{1}{3}-3\right)\right]A_{\mu\nu\rho}(x,y,z), \tag{6.3}$$

where $A_{\mu\nu\rho}$ is given in Eq. (2.10). The order-g^2 two-loop amplitude vanishes, with nonplanar, vertex, and self-energy graphs contributing in the ratio $8:1:-9$, which is different from the ratios in Eq. (3.30). The net result is an Adler-Bardeen theorem for the $\langle R_\mu R_\nu K_\rho\rangle$ correlator, since the sum of virtual gluon graphs can again be shown to vanish by previous work [12].

The effect of the superpotential interactions was also considered in [15]. However, it was simpler to replace the current $K_\rho(z)$ by its scalar superpartner $K(z)=\bar{\phi}(z)\phi(z)$, which is a scalar mass operator of canonical dimension two, and study the correlator $\langle R_\mu(x)R_\nu(y)K(z)\rangle$. The operator $K(z)$ [as well as $K_\rho(z)$] acquires an anomalous dimension of order $Y_{ijk}\overline{Y^{ijk}}$. An anomalous dimension is consistent with conformal symmetry, and the correlator can be shown to be conformal covariant through two-loop order. Inversion, conservation, and scale properties can be used to fix its tensor form up to a multiplicative constant [15]. The graphs contributing to $\langle R_\mu R_\nu K\rangle$ are typically subdivergent because they contain subdiagrams with gauge-invariant anomalous dimension. Nevertheless, in this more complicated situation the conformal inversion technique could be combined with differential regularization [13] to compute all contributing Feynman diagrams.

All results for c' obtained by the conformal methodology of this paper were verified by an alternate method of calculation in which the four-point correlation function $\langle R_\mu(x)R_\nu(y)R_\rho(z)R_\sigma(w)\rangle$ was studied in the relevant asymptotic region using regularization by dimensional reduction in intermediate stages of the calculation. The explicit use of Ward identities to determine ambiguous local terms in self-energy and vertex insertions was not required in this approach, and so agreement of the results of the two methods provides a check on this aspect of the conformal approach.

ACKNOWLEDGMENTS

We are grateful to K. Johnson for sharing his insight into the space-time approach to quantum field theory, and to R. Jackiw and H. Osborn for useful discussions and for their interest in this study. D.Z.F. thanks H. Cheng for a steady exchange of ideas over the course of this work. He also thanks D. Anselmi, M. Grisaru, and A. Johansen, his collaborators on [15], for their permission to include the material of Sec. VI. The research of D.Z.F. was supported by NSF Grant No. PHY-92-06867. The research of J.E. was supported by U.S. DOE cooperative research agreement No. DE-FC02-94ER40818.

APPENDIX A

We discuss here the convolution integrals required in Sec. III to elucidate the finite gauge mechanism and determine the local part of the self-energy, and to calculate the two-loop nonplanar and vertex insertion diagrams for the anomalous correlation function $\langle J_\mu(z) J_\nu(x) J_\rho(y)\rangle$.

We use the method [24] of Gegenbauer polynomials, which appear naturally because their generating function is the scalar propagator

$$\frac{1}{(x-y)^2} \equiv \frac{1}{x^2}\sum_{n=0}^{\infty}\left(\frac{y}{x}\right)^n C_n(\hat{x}\cdot\hat{y}), \quad |x|>|y|, \tag{A1}$$

$$\hat{x}_\mu = \frac{x_\mu}{|x|}, \quad \hat{y}_\mu = \frac{y_\mu}{|y|}, \quad \hat{x}\cdot\hat{y} = \cos\theta,$$

$$C_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{A2}$$

The orthogonality relation obeyed by these polynomials is

$$\int d\hat{x} C_n(\hat{x}\cdot\hat{y}) C_m(\hat{x}\cdot\hat{z}) = 2\pi^2 \delta_{nm}\frac{C_n(\hat{y}\cdot\hat{z})}{n+1}, \tag{A3}$$

where $d\hat{x} = \sin^2\theta \sin\phi\, d\theta\, d\phi\, d\xi$ is the angular integration measure for the $\hat{x}$ variable.

We first list the integrals, and then comment briefly on their evaluation. We use the notation $\Delta = x-y$:

$$\int \frac{d^4v}{v^2(v-x)^2} = -\pi^2 \ln\frac{x^2}{R^2}, \tag{A4}$$

$$\int d^4v \frac{(v-x)_\rho}{v^2(v-x)^4} = -\pi^2\frac{x_\rho}{x^2}, \tag{A5}$$

$$\int d^4v \frac{(v-x)_\rho (v-y)_\sigma}{(v-x)^4 (v-y)^4} = \frac{\pi^2}{2\Delta^2}\left(\delta_{\rho\sigma} - \frac{2\Delta_\rho \Delta_\sigma}{\Delta^2}\right), \tag{A6}$$

$$\int d^4v \frac{(v_\rho v_\sigma - \frac{1}{4}\delta_{\rho\sigma} v^2)}{v^4 (v-x)^2} = \frac{\pi^2}{2x^2}(x_\rho x_\sigma - \tfrac{1}{4}x^2\delta_{\rho\sigma}), \tag{A7}$$

$$\int d^4v \frac{[(v-x)_\rho (v-x)_\sigma - \frac{1}{4}\delta_{\mu\sigma}(v-x)^2](v-y)_\lambda}{(v-x)^4 (v-y)^4} = \frac{-\pi^2}{4\Delta^2}\left(\delta_{\rho\lambda}\Delta_\sigma + \delta_{\sigma\lambda}\Delta_\rho - 2\frac{\Delta_\rho\Delta_\sigma\Delta_\lambda}{\Delta^2}\right), \tag{A8}$$

$$\int d^4v \frac{(v_\rho v_\sigma - \frac{1}{4}\delta_{\rho\sigma} v^2)}{v^6 (v-x)^2} = \frac{\pi^2}{2x^4}(x_\rho x_\sigma - \tfrac{1}{4}x^2\delta_{\rho\sigma}), \tag{A9}$$

$$\int d^4v \frac{[(v-x)_\rho (v-x)_\sigma - \frac{1}{4}\delta\rho\sigma(v-x)^2](v-y)_\lambda}{(v-x)^6 (v-y)^4} = \frac{-\pi^2}{8}\left[\frac{2\delta_{\lambda\rho}\Delta_\sigma + 2\delta_{\lambda\sigma}\Delta_\rho + \delta_{\rho\sigma}\Delta_\lambda}{\Delta^4} - \frac{8\Delta_\lambda\Delta_\rho\Delta_\sigma}{\Delta^6}\right]. \tag{A10}$$

To evaluate Eq. (A4) one applies Eq. (A1) to the factor $1/(v-x)^2$ including both regions $v<x$ and $v>x$, which have different dependence on the radial variable v. The quantity R is a temporary cutoff which has no effect on the integrals used in Sec. III. The result (A5) is obtained by differentiation of Eq. (A4), and Eq. (A6) is obtained by replacing $x \to x-y=\Delta$ in Eq. (A5), changing integration variables to $v' = v+y$, and then differentiating with respect to y. To evaluate Eq. (A7) or (A9) one takes the scalar product with $x_\rho x_\sigma$, so that the integral contains the explicit Gegenbauer polynomial $C_2(\hat{x}\cdot\hat{v})$. One then applies Eq. (A1) to the factor $1/(v-x)^2$ and uses orthogonality (A3). Finally, Eqs. (A8) and (A10) are obtained from Eqs. (A7) and (A9), respectively, by replacement $x\to x-y=\Delta$, shift of integration variables, and differentiation with respect to y.

[1] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[2] J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[3] M. F. Atiyah in *Supersymmetry and Supergravity Nonperturbative QCD*, proceedings of the Winter School held in Mahabaleshwur, India, 1984, edited by P. Roy and V. Singh, Lecture Notes in Physics, Vol. 208 (Springer-Verlag, Berlin, 1984).
[4] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).
[5] A. Zee, Phys. Rev. Lett. **29**, 1198 (1972).
[6] C. Lucchesi, O. Piguet, and S. P. Sorella, Int. J. Mod. Phys. A **2**, 385 (1987); O. Piguet and S. P. Sorella, Nucl. Phys. **B381**, 373 (1992); **B395**, 661 (1993).
[7] L. L. DeRaad, Jr., K. A. Milton, and W.-Y. Tsai, Phys. Rev. D **6**, 1766 (1972); **6**, 3491 (1972).
[8] J. Schwinger, Particles, Sources and Fields (Addison-Wesley, Reading, MA, 1989), Vol. 3, Chaps. 5–9.
[9] A. A. Ansel'm and A. A. Johansen, Sov. Phys. JETP **69**, 670 (1989).
[10] B. A. Faizullaev, M. M. Musakhanov, and N. K. Pak, Phys. Lett. B **361**, 343 (1996).
[11] H. Cheng and S. P. Li, ''The radiative corrections of the triangular anomaly,'' Report No. hep-th/9603026 (unpublished); ''How to treat γ_5,'' Report No. hep-th/9608058 (unpublished).
[12] M. Baker and K. Johnson, Physica **96A**, 120 (1979).
[13] D. Z. Freedman, K. Johnson, and J. I. Latorre, Nucl. Phys. **B371**, 353 (1992).
[14] D. Z. Freedman, G. Grignani, K. Johnson, and N. Rius, Ann. Phys. (N.Y.) **218**, 75 (1992).

[15] D. Anselmi, D. Z. Freedman, M. T. Grisaru, and A. A. Johansen, ''Universality of the operator product expansions of $SCFT_4$,'' Report No. hep-th/9608125 (to appear in Phys. Lett. B).
[16] H. Osborn and A. Petkou, Ann. Phys. (N.Y.) **231**, 311 (1994).
[17] E. J. Schreier, Phys. Rev. D **3**, 980 (1971).
[18] J. Erdmenger and H. Osborn, Nucl Phys. **B483**, 431 (1997).
[19] K. Johnson, as reported by D. Z. Freedman in *Strings and Symmetries 1991*, edited by N. Berkovitz *et al.*, Proceedings, Stony Brook, 1991 (World Scientific, Singapore, 1992).
[20] H. Sonoda, Phys. Rev. D **55**, 5245 (1997).
[21] R. Jackiw, in S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten, *Current Algebra and Anomalies* (Princeton University Press, Princeton, 1985).
[22] S. L. Adler, R. W. Brown, T. F. Wong, and B.-L. Young, Phys. Rev. D **4**, 1787 (1971).
[23] H. Cheng (private communication).
[24] J. I. Rosner, Ann. Phys. (N.Y.) **44**, 11 (1967).
[25] H. Nicolai, Nucl. Phys. **B140**, 294 (1978).
[26] I. Jack, Nucl. Phys. **B253**, 323 (1985).