Nonequilibrium dynamics of symmetry breaking in $\lambda\Phi^4$ theory

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The time evolution of O(N) symmetric $\lambda(\Phi^2)^2$ scalar field theory is studied in the large N limit. In this limit the $\langle\mathbf{\Phi}_i\rangle$ mean field and two-point correlation function $\langle\mathbf{\Phi}_i\mathbf{\Phi}_j\rangle$ evolve together as a self-consistent closed Hamiltonian system, characterized by a Gaussian density matrix. The static part of the effective Hamiltonian defines the true effective potential $U_{\rm eff}$ for configurations far from thermal equilibrium. Numerically solving the time evolution equations for energy densities corresponding to a quench in the unstable spinodal region, we find results quite different from what might be inferred from the equilibrium free energy potential F. Typical time evolutions show effectively irreversible energy flow from the coherent mean fields to the quantum fluctuating modes, due to the creation of massless Goldstone bosons near threshold. The plasma frequency and collisionless damping rate of the mean fields are calculated in terms of the particle number density by a linear response analysis and compared with the numerical results. Dephasing of the fluctuations leads also to the growth of an effective entropy and the transition from quantum to classical behavior of the ensemble. In addition to casting some light on fundamental issues of nonequilibrium quantum statistical mechanics, the general framework presented in this work may be applied to a study of the dynamics of second order phase transitions in a wide variety of Landau-Ginsburg systems described by a scalar order parameter. [S0556-2821(97)02610-6]

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I. INTRODUCTION

Spontaneous symmetry breaking by a scalar order parameter occurs in many different physical systems as diverse as liquid ^{4}He at temperatures of order 2 K to the standard model of electroweak interactions at temperatures of order 250 GeV$=3\times10^{15}$ K. The prototype renormalizable quantum field theory describing this symmetry breaking is a scalar field (or set of scalar fields) with a $\lambda\Phi^4$ self-interaction. The behavior of the finite temperature effective potential or Landau-Ginsburg-Helmholtz free energy in this theory is well known and forms the usual basis for discussion of the symmetry restoration at high temperature. On the other hand, until recently very little effort had been devoted to the nonequilibrium or time-dependent aspects of the symmetry-breaking phase transition. With the development of practical general techniques for studying time-dependent problems in quantum field theory, as well as the advent of high speed supercomputers, it has become possible to address these dynamical issues systematically for the first time. It is clear that a detailed description of the time-dependent dynamics will be necessary to calculate nonequilibrium properties of the phase transition, such as the formation and evolution of defects in the ^{4}He system after a rapid quench or the efficiency of baryogenesis in the electroweak phase transition. Other examples requiring the detailed time evolution of scalar fields are the chiral phase transition of the strong interactions such as may soon be probed in relativistic heavy-ion colliders and the problem of reheating the very early universe after it has passed through an epoch of rapid expansion and cooling.

This class of problems requires a consistent treatment of time-dependent mean fields, such as the expectation value of the scalar field, in interaction with its own quantum and/or thermal fluctuations. In this paper we consider the model where the scalar field is an N-component vector Φ_i, $i=1,\ldots,N$, with the O(N)-symmetric quartic interaction $\lambda(\Phi_i\Phi_i)^2$. When the O(N) symmetry is spontaneously broken by the nonzero expectation value of the quantum mean field $\langle\mathbf{\Phi}_i\rangle=\phi_i$, massless (i.e., gapless) excitations appear, and it is their dynamical effects on the time evolution of the mean field which we wish to take into account. This can be done in a systematic way by computing the quantum effective action in a power series expansion in the parameter $1/N$ [1–3]. Variation of this effective action yields equations of motion for the mean fields coupled to the higher point Green's functions of the theory which are suitable for implementation on a computer. The leading order in large N corresponds to a self-consistent mean field approximation, i.e., a truncation of the infinite hierarchy of Schwinger-Dyson equations for the n-point correlation functions to a closed Hamiltonian system of just the one-point function $\langle\mathbf{\Phi}_i(x)\rangle$ and two-point function $\langle\mathbf{\Phi}_i(x)\mathbf{\Phi}_j(x')\rangle$. Because no irreducible correlators higher than these appear in the leading order of the large N expansion, it is equivalent to a Gaussian approximation to the time-dependent density matrix of the system. As will become apparent, the approximation allows automatically for a mixed-state Gaussian density matrix, ρ, and is therefore more general than a Gaussian ansatz for the pure state wave function in the Schrödinger picture. The mixed-state Gaussian ensemble ρ describes at once and on the same footing both the quantum and classical statistical fluctuations of the Φ_i field about its mean value, arbitrarily far from thermal equilibrium.

An important property of the evolution equations in the large N limit is that they are Hamilton's equations for an effective *classical* Hamiltonian $H_{\rm eff}$ (in which $\hbar$ appears as a

parameter). This effective Hamiltonian turns out to be nothing else than the expectation value of the full quantum Hamiltonian **H** in the general mixed state described by the time-dependent Gaussian density matrix ρ, i.e.,

$$H_{\rm eff} = {\rm Tr}(\rho \mathbf{H}) \ . \tag{1.1}$$

The canonical variables of $H_{\rm eff}$ are in one-to-one correspondence with the parameters needed to specify the general Gaussian density matrix in the Schrödinger picture, or the one- and two-point functions of the Schwinger-Dyson hierarchy in the Heisenberg picture. In order to highlight the Hamiltonian structure of the mean field equations, our first purpose in this paper will be to establish these various correspondences in detail for the O(N)-symmetric $\lambda(\Phi_i\Phi_i)^2$ theory. The existence of an effective Hamiltonian makes it clear from the outset that the leading order large N approximation is self-consistent and energy conserving, and hence does not introduce any time irreversibility or dissipation ''by hand'' into the system. In this connection let us be very clear that in this paper we are discussing a *closed* Hamiltonian system, without any external fluctuations which can provide sources or sinks of energy.

A corollary of the identification of the effective Hamiltonian $H_{\rm eff}$ in the large N limit is that its static piece $U_{\rm eff}$ (obtained by setting all the canonical momenta to zero) is the *true* effective potential which governs the nonequilibrium evolution of the system. This true effective potential (TEP) is real, and completely well defined for states far from thermal equilibrium. In the special case of thermal equilibrium it becomes the *internal* energy U of the closed Hamiltonian system described by $H_{\rm eff}$. In contrast, the Helmholtz free energy F which is very often called the ''effective'' potential is not defined away from thermal equilibrium and becomes complex in the unstable spinodal coexistence region between the two spontaneously broken vacua in simple approximation schemes. This is easily understood in the more general nonequilibrium context of this paper: It means simply that there is no stable equilibrium state in the coexistence region with a fixed constant value of $\langle \mathbf{\Phi}_i \rangle$. In other words, if we start at $t=0$ with initial conditions corresponding to this putative thermal equilibrium state, the system immediately begins to evolve in time away from it, and this is the physical meaning of the imaginary part of F [4].

We shall see that the free energy function F (or its real part) is a very poor guide to the time evolution of the system far from thermal equilibrium. In particular, the oscillations of the time-dependent expectation value $\langle \mathbf{\Phi}_i(t) \rangle$ about the spontaneously broken minima are characterized by a frequency (the plasma frequency) which is *not* the second derivative of F at its minimum (which turns out to be zero). Moreover, even the *location* of the minima of F is in general different from the stationary points of the mean field evolution. Explicitly solving for the actual nonequilibrium mean field dynamics of a closed field theory system and demonstrating that it is quite different from what might be inferred from an uncritical use of the equilibrium free energy function F is the second major emphasis of this work. In this we corroborate the similar conclusions reached in Ref. [5].

The spontaneous breakdown of the global O(N) symmetry in the model leads to the existence of $N-1$ massless Goldstone bosons (in $d>1$ spatial dimensions), which dominate the dynamics in the large N limit. In $d\leqslant 1$ spatial dimension there is no symmetry breaking and the system is inevitably driven into the symmetric phase, no matter what the initial state or energy density. The one-dimensional case is of more than passing interest in showing how the O(N) symmetry is restored dynamically and the Mermin-Wagner-Coleman theorem [6] is satisfied in real time. For $d>1$, the absence of a finite mass threshold means that an arbitrarily small amount of energy in the mean field can create massless Goldstone boson pairs nearly at rest. This open channel provides an efficient mechanism for the mean field to continuously transfer its kinetic energy to the massless particle modes over time. The existence of degrees of freedom with zero mass or infinite correlation length is characteristic of second order phase transitions in general. Hence, the real time dynamics of spinodal decomposition in such second order transitions can be studied in our approach. The presence of a symmetry which requires massless particles is also a feature which the O(N) model shares with other physically interesting theories such as non-Abelian gauge theories and gravity. The O(N) scalar theory provides an instructive example of mean field dissipation by means of massless particle creation, which should be applicable in other quite diverse contexts, such as gluon production in relativistic heavy-ion collisions or graviton creation in early universe phase transitions. Developing techniques and gaining some valuable intuition for these more challenging problems is our third reason for presenting a study of $\lambda(\Phi_i\Phi_i)^2$ theory in some detail.

It is remarkable that despite the explicitly Hamiltonian structure of the mean field equations, we observe quasidissipative features in the evolution, in the sense that energy flows from the mean field $\langle \mathbf{\Phi}_i \rangle$ into the fluctuating particle modes without returning over times of physical interest. In other words, although the underlying equations are fully time-reversal invariant, typical evolutions beginning with energy concentrated in the mean fields are *effectively irreversible*, at least over very long intervals of time. This apparent irreversibility is quantifiable, first, in terms of the increase in particle entropy obtained by averaging over the rapidly varying phases of the fluctuating modes and, second, by the effective damping rate of the collective motion which we calculate by a standard linear response analysis. Since the mean field Gaussian approximation contains no collision terms, the particles interacting with each other only through the mean fields, this effective relaxation to a quasistationary (but nonthermal) state is a form of collisionless damping, similar to Landau damping in nonrelativistic electromagnetic plasmas. We call this collisionless damping due to effective loss of phase information in the fluctuating quantum modes *dephasing*, and present numerical results which substantiate the theoretical discussion of damping of the mean field(s) due to massless particle creation in the continuum.

Dephasing of the fluctuations is both an extremely general and efficient mechanism for introducing effective dissipation into the reversible Hamiltonian dynamics of mean field evolution. Hence, even in this relatively simple collisionless approximation to a quantum many-body theory one can begin to see how irreversibility and the second law of thermodynamics emerge from a consistent treatment of fluctuations in

a closed Hamiltonian system. Collisions which first appear at one order beyond the mean field limit in the large N expansion would be expected to make the dephasing and dissipation found at lowest order still more efficient.

In addition to the effective dissipation of energy from the collective plasmon mode to the fluctuations, dephasing is also responsible for quantum decoherence, in the sense of suppression with time of the off-diagonal elements of the Gaussian density matrix. The point is that by going to the appropriate time-dependent number basis the diagonal matrix elements of ρ are adiabatic invariants of $H_{\rm eff}$ and therefore slowly varying functions of time, while the off-diagonal elements are very rapidly varying. These rapid phase variations in the off-diagonal interference terms cancel out very efficiently when the sum over the mode momentum is performed or if the phases in a given mode are averaged in time. In either case, the particle creation and interactions have the effect of bringing the quantum system into what *effectively* looks more and more nearly like a classical mixture in which the phase information in the off-diagonal components may be discarded for most practical purposes at late times. The symmetry-breaking behavior of the density matrix and effective disappearance of quantum interference between the two classically allowed outcomes is the final result.

An important consequence of decoherence is the appearance of a diagonal effective density matrix which may be sampled to generate smooth classical field configurations. These configurations are free from spurious cutoff dependences and can be used to address issues such as, e.g., the generation of topological defects in a nonequilibrium phase transition and the modeling of individual events in heavy-ion collisions.

The time-dependent scalar theory is an excellent theoretical laboratory for the study of general nonequilibrium phenomena such as decoherence and the quantum to classical transition quite aside from specific potential applications to phase transitions in many systems of physical interest. The detailed study of effective dissipation and decoherence in an explicit field theoretic example is the fourth major focus of the present work. Some of our results on the Hamiltonian nature of the evolution and on dephasing and decoherence have been reported earlier in condensed form [7].

The paper is organized as follows. In the next section we begin by reviewing the large N expansion of $\lambda(\Phi_i\Phi_i)^2$ theory in the real time effective action formulation. The Hamiltonian structure of the equations of motion is exhibited and the effective Hamiltonian and Gaussian density matrix are identified in Sec. III. The static part of this effective Hamiltonian is the nonequilibrium true effective potential $U_{\rm eff}$ which we define and relate to the thermodynamic free energy F in Sec. IV. Numerical evolution of the actual equations of motion in both one and three dimensions shows clearly the difference with what might have been inferred from F. In Sec. V we identify the adiabatic particle number basis in which dissipation through the increase of the effective particle entropy and decoherence are described. We also present numerical evidence for the efficient dephasing of the quantum modes in the time-dependent mean fields, and show how it leads to typical classical configurations in the mixture in which quantum interference effects have been washed out. In Sec. VI we perform a linear response analysis of small perturbations away from thermal equilibrium, as well as away from the nonthermal stationary states found in the previous sections, compute the plasmon damping rate, and compare it with the numerical results. We conclude in Sec. VII with a discussion of our results and of their possible application to diverse problems of interest in the real time dynamics of second order phase transitions with a scalar order parameter.

There are three Appendixes, the first on the renormalization of the energy and pressure of the theory, the second cataloging some mathematical properties of the Gaussian density matrix used in the text, and the third containing the details of the numerical methods used in solving the equations.

II. LARGE N EFFECTIVE ACTION

The most direct method of deriving the equations of motion in the large N approximation is the method of the effective action, which also has the advantages of exhibiting the covariance properties of the theory and providing a general framework for the systematic expansion in powers of $1/N$ to any desired order beyond the mean field approximation. Here we will present only a brief synopsis of the effective action approach and refer the interested reader to the earlier work [1–3] for details of the derivation (which is quite standard). The underlying O(N)-symmetric scalar field theory with which we begin is described in d space dimensions by the classical action

$$S_{\rm cl}[\Phi_i,\chi]=\int dt\ d^dx\ L_{\rm cl}[\Phi_i,\chi] =\int dt\ d^dx\ \left\{-\frac{1}{2}\Phi_i G^{-1}[\chi]\Phi_i+\frac{N}{\lambda}\chi\left(\frac{\chi}{2}+\mu^2\right)\right\}, \tag{2.1}$$

where $i=1,\ldots,N$ and

$$G^{-1}[\chi]\equiv-\Box+\chi \tag{2.2}$$

is the inverse propagator of the N scalar fields. We take the signature of the Minkowski metric to be $(-,+,+,+)$ and use units in which the speed of light and Boltzmann's constant are unity, $c=k_B=1$, but we retain $\hbar$ in what follows in order to exhibit the semiclassical nature of the large N limit.

The form of the action (2.1) is equivalent to the more familiar Lagrangian density

$$\widetilde{L}_{\rm cl}[\Phi_i]=-\frac{1}{2}(\partial_\mu\Phi_i)(\partial^\mu\Phi_i)-V_{\rm cl} =-\frac{1}{2}(\partial_\mu\Phi_i)(\partial^\mu\Phi_i)-\frac{\lambda}{8N}\left(\Phi_i\Phi_i-\frac{2N\mu^2}{\lambda}\right)^2, \tag{2.3}$$

with the definition of the auxiliary field χ by

$$\chi=-\mu^2+\frac{\lambda}{2N}\Phi_i\Phi_i, \tag{2.4}$$

since the two Lagrangians $L_{\rm cl}$ and $\widetilde{L}_{\rm cl}$ are then equal up to a surface term. The quartic coupling in the Lagrangian has been taken to be λ/N from the outset, rather than rescaling it later by $1/N$ as is sometimes done [1].

If the parameter $\mu^2>0$, then this Lagrangian describes spontaneous symmetry breaking since the minimum of the classical potential $V_{\rm cl}$ occurs at

$$\Phi_{i\ \rm min}=\delta_{iN}\ \sqrt{N}\ v_0, \quad v_0=\sqrt{\frac{2}{\lambda}}\mu, \tag{2.5}$$

rather than at zero. The second derivative of the classical potential is

$$\frac{\partial^2 V_{\rm cl}}{\partial\Phi_i\partial\Phi_j}=\frac{\lambda}{2N}\left(\Phi_k\Phi_k-\frac{2N\mu^2}{\lambda}\right)\delta_{ij}+\frac{\lambda}{N}\Phi_i\Phi_j$$
$$=\left(\chi\delta_{ij}+\frac{\lambda}{N}\Phi_i\Phi_j\right), \tag{2.6}$$

which evaluated at $\Phi_i=\Phi_{i\ \rm min}$ becomes

$$\left.\frac{\partial^2 V_{\rm cl}}{\partial\Phi_i\partial\Phi_j}\right|_{\Phi_i=\Phi_{i\ \rm min}}=\begin{cases}0, & i\ \text{ or }\ j=1,\ldots,N-1,\\ \lambda v_0^2, & i=j=N.\end{cases} \tag{2.7}$$

At this minimum the O(N) symmetry is spontaneously broken, $\chi=0$, and there are $N-1$ massless modes. Small oscillations in the remaining $i=N$ radial direction describe a massive mode with bare mass equal to $\sqrt{2}\mu=\sqrt{\lambda}v_0$. In this standard way of describing the symmetry breaking, one direction is singled out and its vacuum expectation value is scaled with $\sqrt{N}$ as in Eq. (2.5). It is clear that in the large N limit the quantum effects of the single radial degree of freedom are down by $1/N$ compared to the $N-1$ massless modes which dominate the dynamics.

Passing now to the quantum theory, the large N mean field approximation is obtained by retaining only the leading terms in a systematic expansion of Feynman diagrams in $1/N$. To leading order these reduce to the one-loop diagram with the effective mass squared χ; i.e., the quantum effective action to leading order is just

$$\mathcal{S}_{\rm eff}[\phi_i,\chi]=S_{\rm cl}[\phi_i,\chi]+N\frac{i\hbar}{2}\ {\rm Tr}\ln G^{-1}[\chi], \tag{2.8}$$

where χ is to be treated as an independent field, determined from its own variational equation,

$$\frac{\delta\mathcal{S}_{\rm eff}[\phi_i,\chi]}{\delta\chi}=0, \tag{2.9}$$

and ϕ_i is the expectation value of the quantum operator $\mathbf{\Phi}_i$. In writing Eq. (2.8) we have ignored the difference between N and $N-1$ in the one-loop term to leading order in large N.

The classical action $S_{\rm cl}$ and Tr $\ln G^{-1}[\chi]$ term in $\mathcal{S}_{\rm eff}$ are both of order N because of the summation over $i=1,\ldots,N$ in Eq. (2.1). Once this is recognized it is somewhat more convenient to make the replacement

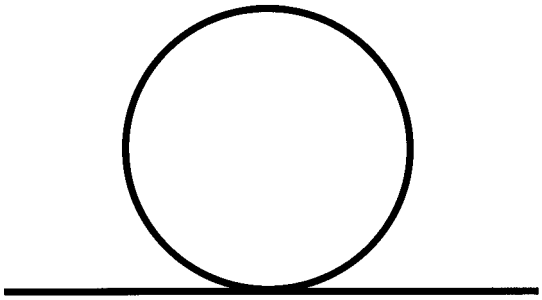

FIG. 1. The one-loop self-energy diagram.

$$\sum_{i=1}^{N}\phi_i\phi_i\to N\phi^2\ , \tag{2.10}$$

and drop the O(N) index, thereby returning to a single-component description in which ϕ is the rescaled expectation value in the symmetry-breaking direction, i.e.,

$$\phi_i=\langle\mathbf{\Phi}_i\rangle=\delta_{iN}\sqrt{N}\phi. \tag{2.11}$$

Then $S_{\rm cl}[\phi_i,\chi]=NS_{\rm cl}[\phi,\chi]$ is just N times the classical action of a single-component field and Eq. (2.8) becomes

$$\mathcal{S}_{\rm eff}[\phi,\chi]=NS_{\rm cl}[\phi,\chi]+N\frac{i\hbar}{2}\ {\rm Tr}\ln G^{-1}[\chi]\ . \tag{2.12}$$

Thus, the role of the large N limit is simply to justify the neglect of the single radial massive degree of freedom relative to the $N-1$ massless Goldstone bosons in the propagator $G[\chi]$ of the Φ field. As we shall see in Sec. VI the massive degree of freedom reappears in the guise of the χ propagator which controls the plasma oscillations around $\phi_{\rm min}$.

Before proceeding it is worth pausing at this point to compare and contrast the large N effective action (2.12) with two different but related approximations. The simplest is the standard one-loop approximation on the single-component theory, which is obtained by expanding the Φ field about its mean value ϕ. The quantum effective action in this approximation is

$$\mathcal{S}_{1\ \rm loop}[\phi]=\widetilde{\mathcal{S}}_{\rm cl}[\phi]+\frac{i\hbar}{2}\ {\rm Tr}\ln G^{-1}[V''_{\rm cl}], \tag{2.13}$$

where the effective mass appearing in the two-point function is

$$V''_{\rm cl}=\left.\frac{\partial^2 V_{\rm cl}}{\partial\Phi^2}\right|_{\Phi=\phi}=-\mu^2+\frac{3\lambda}{2}\phi^2\ . \tag{2.14}$$

The simple one-loop approximation includes only the one-loop self-energy diagram of Fig. 1, without any further resummation of diagrams. As is well known this simple perturbative expansion requires both $\hbar\lambda_R\ll 1$ *and* $\hbar\lambda_R\ln(\phi/m)\ll 1$, where m is some renormalization mass scale. The appearance of large logarithms severely limits the range of validity of the one-loop approximation, and leads also to infrared divergences in the free energy potential [8].

The approximation, usually called the Hartree or Gaussian approximation in the literature, is closely related to the large N approximation in that the second derivative of the classical potential $V''_{\rm cl}$ in Eq. (2.13) is replaced by its Gaussian mean value, viz.,

$$\mathcal{S}_{\rm Hartree}[\phi,M^2]=\mathcal{S}_{\rm cl}+\frac{i\hbar}{2}\ {\rm Tr}\ln G^{-1}[M^2], \qquad (2.15)$$

with

$$M^2_{ij}\equiv\left\langle\frac{\partial^2 V_{\rm cl}}{\partial\mathbf{\Phi}_i\partial\mathbf{\Phi}_j}\right\rangle=\left\langle-\mu^2+\frac{\lambda}{2N}\mathbf{\Phi}_k\mathbf{\Phi}_k\right\rangle\delta_{ij}+\frac{\lambda}{N}\langle\mathbf{\Phi}_i\mathbf{\Phi}_j\rangle\ . \qquad (2.16)$$

In the Hartree approximation the classical action should be expressed in terms of the appropriate set of variational parameters ϕ and M^2_{ij}. In both the Hartree and large N approximations the expectation values of bilinears obey the factorization condition

$$\langle\mathbf{\Phi}_i(x)\mathbf{\Phi}_j(y)\rangle=\langle\mathbf{\Phi}_i(x)\rangle\ \langle\mathbf{\Phi}_j(y)\rangle-iG_{ij}(x,y). \qquad (2.17)$$

Because of the sum over the repeated index $k=1,\ldots,N$ in the diagonal δ_{ij} term, it is order N^0 compared to the last, order $1/N$, term in Eq. (2.16). In the large N approximation G_{ij} is just $\delta_{ij}G[\chi]$ for $i,j\neq N$, and the last term in Eq. (2.16) is discarded at leading order as in Eqs. (2.6) and (2.12), whereas in the Hartree approximation of Eqs. (2.15) and (2.16) it is retained. An important consequence of this difference between the two approximation schemes is that unlike the Hartree approximation, the large N effective action is the leading term in a series in a well-defined expansion in powers of $1/N$. Thus, it is possible to improve on the large N limit in a systematic way by retaining higher order terms in this series [1–3]. In contrast, the Hartree approximation is simply a variational ansatz.

In terms of the ordinary rules of weak coupling perturbation theory the Feynman diagrams that contribute in both the large N limit and the Hartree approximation may be represented by the sum over the daisy and superdaisy diagrams of Fig. 2. This sum of perturbative self-energy insertions [differing in their internal O(N) index structure and combinatoric factors in the Hartree and large N cases due to the difference between $\chi\delta_{ij}$ and M^2_{ij}] simply amounts to a geometric series which shifts the mass pole of the $\langle\mathbf{\Phi}_i\mathbf{\Phi}_j\rangle$ propagator from its perturbative value (2.14) to the self-consistently determined $\chi\delta_{ij}$ or M^2_{ij}. Hence both the large N and Hartree effective actions (2.12) and (2.15) can be written in the same form as the simple one-loop effective action (2.13), but with different values of the mass parameter in the one-loop term. This is the resummation of self-energy diagrams required by a renormalization group analysis, and in fact, χ is a renormalization group invariant [1]. This is equivalent to the resummation of the leading logarithms of perturbation theory which removes the infrared divergences in the equilibrium free energy function and extends the range of validity to very small (or very large) mean fields, until the subleading logarithms, i.e., terms like $\hbar^2\lambda_R^2\ln(\phi/m)$, eventually become important.

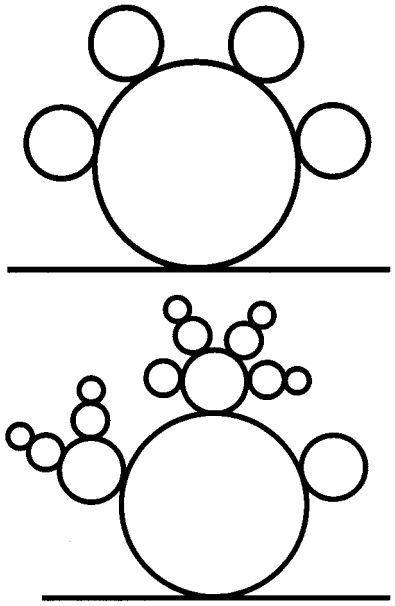

FIG. 2. Typical self-energy daisy and superdaisy diagrams included in the Hartree or leading order large N approximation.

In physical terms, the fundamental excitations in the large N limit are the $N-1$ Goldstone modes, whose masslessness is fixed by the O(N) symmetry which the $1/N$ expansion respects order by order. In contrast, the Hartree approximation is not a systematic expansion in any small parameter. The result is that although the two approximations are very similar in some respects and may be handled by the same techniques, the physics they describe is really quite different. Which approximation is more reliable depends very much on the application, and in particular whether or not the massless Goldstone modes of the large N limit actually play the lead role in the physics we wish to describe. Certainly one would have no justification in applying the large N method to the case $N=1$, for example, where no Goldstone modes exist at all. Since we wish to describe the dynamics of second order phase transitions in this paper, it is significant for our purpose that the large N expansion predicts a second order phase transition at the critical temperature given by Eq. (4.8) below, whereas the phase transition in the Hartree approximation is (weakly) first order [9]. The critical exponents of the second order transition have been calculated up to order $1/N^2$ in the large N approximation and give reasonable agreement with other methods (such as the ϵ expansion), even for moderately small N [2].

In addition to providing a practical expansion technique in equilibrium phase transitions the large N expansion has the significant conceptual advantage over the Hartree approximation of placing mean field theory in its proper context with respect to other systematic methods in nonequilibrium statistical mechanics. In fact, the systematic truncation of the Schwinger-Dyson hierarchy of connected $2n$-point functions by the large N expansion in quantum field theory is the precise analogue of the truncation of the BBGKY hierarchy of n-particle distribution functions in nonequilibrium classical statistical mechanics. The existence of an energy-conserving expansion parameter in $1/N$ which preserves all the relevant symmetries of the underlying field theory is a powerful tech-

nical tool for development of the quantum theory of nonequilibrium processes from first principles in a systematic way [3].

In the real time formulation, the effective action of Eq. (2.12) becomes the starting point for all further analysis of the large N nonequilibrium dynamics. Since $\mathcal{S}$ is constructed by taking a Legendre transform in the presence of external sources, it follows that its first variation with respect to the independent mean fields ϕ or χ is proportional to the sources for these fields. In the absence of external sources, these variations yield the equations of motion for the mean fields. The variation with respect to χ, Eq. (2.9) yields the equation of constraint or gap equation

$$\chi(x)=\left\langle -\mu^2+\frac{\lambda}{2N}\mathbf{\Phi}_i(x)\mathbf{\Phi}_i(x)\right\rangle$$
$$=-\mu^2+\frac{\lambda}{2}\phi^2(x)-\frac{i\lambda}{2}G[\chi](x,x)\ , \qquad (2.18)$$

which will be recognized as just the expectation value of the (operator) definition of the auxiliary field in Eq. (2.4) upon using the factorization condition (2.17), correct to leading order in $1/N$. Since $G[\chi]$ itself depends on χ through the definition

$$(-\Box+\chi)G[\chi](x,x')=\delta^{d+1}(x,x')\ , \qquad (2.19)$$

Eq. (2.18) is a nonlinear integral equation for the gap function χ.

The variation with respect to ϕ yields the equation of motion for the symmetry-breaking expectation value:

$$\frac{1}{N}\frac{\delta\mathcal{S}_{\rm eff}[\phi,\chi]}{\delta\phi}=G^{-1}[\chi]\phi(x)=[-\Box+\chi(x)]\phi(x)=0\ . \qquad (2.20)$$

That χ is a useful indicator of symmetry breaking should be clear from the spacetime-independent form of Eq. (2.20),

$$\chi\phi=0 \quad (\partial_\mu\phi=0), \qquad (2.21)$$

which tells us that either ϕ or χ (or both) must vanish in a uniform, stationary state. The case $\phi=0$ is the O(N)-symmetric state, while the case of nonvanishing ϕ is the spontaneously broken state in which $\chi=0$ is the vanishing Goldstone boson mass. This is an explicit realization of Goldstone's theorem, which is respected by the large N approximation. In the following the explicit functional dependence of $G[\chi]$ and its inverse on the mean field χ will be suppressed, and we adopt the simpler notation $G(x,x')$ or still more briefly G, hereafter.

Whereas the spontaneous symmetry-breaking solution $\chi=0$ can be achieved only when $\phi=\pm v_0\equiv\pm\sqrt{2/\lambda}\mu$ classically, in the large N approximation the two-point function G contributes at the same order and can even dominate the mean field ϕ in Eq. (2.18). Since G itself depends on χ, this additional term also has the effect that the χ field can undergo nonlinear collective plasmonlike oscillations as we shall see explicitly in Sec. VI. Notice also that there is nothing to prevent χ from being negative at some times which allows us to explore the dynamics of the unstable spinodal region of Fig. 3. The spinodal at a given temperature is the region where the second derivative of the potential becomes negative. In the large N approximation, this corresponds to $\chi<0$.

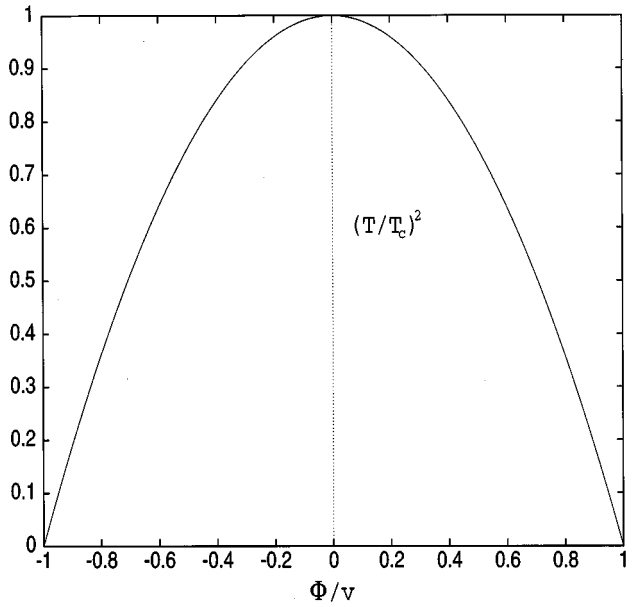

FIG. 3. The unstable spinodal region of the (ϕ,T) plane, where χ is negative. From Eq. (4.7) this is the region $\phi<\phi_T$ where $(\phi_T/v)^2=1-(T/T_c)^2$ is the parabolic curve shown in the figure.

In this paper we shall be concerned only with spatially homogeneous mean fields $\phi=\phi(t)$ and $\chi=\chi(t)$. It is not difficult to treat spatially inhomogeneous mean fields by the same methods, but as the homogeneous case is simpler and already contains much of the essential physics, we restrict ourselves to that case in this paper. As we shall see explicitly in succeeding sections this restriction corresponds to a spatially homogeneous ensemble average, although any particular member of the ensemble may be spatially inhomogeneous. Thus, information about spatial correlations is certainly contained in the two-point correlation function G, describing fluctuations away from the mean values, even when the mean values $\phi(t)$ and $\chi(t)$ are space independent. Indeed, when ϕ and χ are functions only of t, then the two-point Green's function G is a function of the spatial difference $\mathbf{x}-\mathbf{x}'$ and it is useful to introduce the Fourier transform of the corresponding Wightman function $G_>$ (or $G_<$):

$$G_>(t,\mathbf{x};t',\mathbf{x}')=-G^*_<(t,\mathbf{x};t',\mathbf{x}')$$
$$=\int[d\mathbf{k}]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}G_>(t,t';\mathbf{k}), \qquad (2.22)$$

with

$$[d\mathbf{k}]\equiv\frac{d^d\mathbf{k}}{(2\pi)^d}\ . \qquad (2.23)$$

In fact, $G_>$ is a function only of $k\equiv|\mathbf{k}|$ by rotational symmetry of the spatially homogeneous state. Information about spatial correlations resides in the k dependence of the Fourier spectrum of this two-point function.

The Wightman functions are solutions of the source-free wave equation. Explicitly, if the original quantum field operator $\mathbf{\Phi}_i$ is expanded about its mean value (2.11) in terms of the quantum modes f_k and the corresponding plane wave creation and destruction operators $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$,

$$\delta\mathbf{\Phi}_i(t,\mathbf{x}) \equiv \mathbf{\Phi}_i(t,\mathbf{x}) - \langle \mathbf{\Phi}_i \rangle$$

$$= L^{-d/2} \sum_{\mathbf{k}} \left[e^{i\mathbf{k}\cdot\mathbf{x}} f_k(t) a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}} f_k^*(t) a_{\mathbf{k}}^\dagger \right], \quad i \neq N, \tag{2.24}$$

in a d-dimensional cubical box of finite length L, then the Wightman functions are given by the usual expressions

$$\hbar G_>(t,\mathbf{x};t',\mathbf{x}') = i \langle \delta\mathbf{\Phi}(t,\mathbf{x})\, \delta\mathbf{\Phi}(t',\mathbf{x}') \rangle \ ,$$

$$\hbar G_<(t,\mathbf{x};t',\mathbf{x}') = i \langle \delta\mathbf{\Phi}(t',\mathbf{x}')\, \delta\mathbf{\Phi}(t,\mathbf{x}) \rangle \ . \tag{2.25}$$

The relation $G_> = -G_<^*$ follows immediately from the Hermiticity of the underlying quantum field $\delta\mathbf{\Phi}$ in Eq. (2.24).

The expectation value of the particle number density in this basis (not to be confused with the N of large N),

$$N(k=|\mathbf{k}|) = \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle, \tag{2.26}$$

is also a function only of k by isotropy of the spatially homogeneous state. In terms of $N(k)$ and the complex mode functions $f_k(t)$ obeying

$$\left[\frac{d^2}{dt^2} + k^2 + \chi(t) \right] f_k(t) = 0, \tag{2.27}$$

the Wightman functions may be expressed in the form

$$G_>(t,t';k) = i f_k(t) f_k^*(t') [N(k)+1] + i f_k^*(t) f_k(t') N(k) \ . \tag{2.28}$$

The commutation relation

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'} \tag{2.29}$$

implies that the complex mode functions should be chosen to satisfy the Wronskian condition

$$f_k \frac{df_k^*}{dt} - f_k^* \frac{df_k}{dt} = i\hbar \ , \tag{2.30}$$

in order for the quantum field operator $\mathbf{\Phi}$ to obey the usual canonical commutation relation:

$$\left[\mathbf{\Phi}(t,\mathbf{x}), \frac{\partial \mathbf{\Phi}}{\partial t}(t,\mathbf{x}') \right] = i\hbar\, \delta^d(\mathbf{x}-\mathbf{x}') \ . \tag{2.31}$$

The normalization of the time-independent Wronskian condition (2.30) is the only place where the constant $\hbar$ enters the mean field equations, which otherwise are classical in their time evolution dynamics.

We have not written the *a priori* possible bilinear terms $f_k f_k$ or $f_k^* f_k^*$ in Eq. (2.28), since they can always be absorbed into a redefinition of f_k and $N(k)$ under the transformation

$$f_k \to \cosh\gamma_k e^{i\theta_k + i\psi_k} f_k + \sinh\gamma_k e^{i\theta_k - i\psi_k} f_k^* \tag{2.32}$$

without affecting the mode equation (2.27) or Wronskian condition (2.30). This is equivalent to making a Bogoliubov transformation which sets to zero the expectation values of the pair densities $\langle a_{\mathbf{k}} a_{\mathbf{k}} \rangle = \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger \rangle = 0$. Hence there is a natural SU(1,1) Heisenberg group structure (for each $\mathbf{k}$) inherent in the leading order large N equations.

In the gap equation (2.18) the coincidence limit of the Green's function $G(x,x)$ appears. The coincidence limits of either $G_>, G_<$ or the Feynman propagator G are all identical, and so we obtain from Eqs. (2.22) and (2.28) the coincidence limit of any of the Green's functions in the form of an integral over $\mathbf{k}$ of the Fourier mode functions:

$$\hbar G_>(t,\mathbf{x};t,\mathbf{x}) = \hbar G_<(t,\mathbf{x};t,\mathbf{x})$$

$$= i \int [d\mathbf{k}]\ |f_k(t)|^2 [2N(k)+1]. \tag{2.33}$$

Since the mode functions obeying Eqs. (2.27) and (2.30) behave as

$$f_k(t) \to \tilde{f}_k(t) \equiv \sqrt{\frac{\hbar}{2\omega_k(t)}}\, \exp\left(-i \int^t dt'\, \omega_k(t') \right) \tag{2.34}$$

for large k, where

$$\omega_k(t) \equiv \sqrt{k^2 + \chi(t)} \ , \tag{2.35}$$

the integral in Eq. (2.33) is quadratically divergent in $d=3$ spatial dimensions. Introducing an explicit momentum cutoff Λ and performing the angular $\mathbf{k}$ integrations, the gap equation (2.18) may be rewritten in the form

$$\chi(t) = -\mu_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2(t) + \frac{\lambda_\Lambda}{4\pi^2} \int_0^\Lambda k^2 dk |f_k(t)|^2 \sigma_k \tag{2.36}$$

for $d=3$, where we have introduced the notation

$$\sigma_k \equiv 2N(k) + 1 \ . \tag{2.37}$$

The fact that the bare parameters μ_Λ and λ_Λ must depend on the cutoff in order to render the equations independent of Λ in the end has been exhibited explicitly as well. The quadratic divergence in $d=3$ is the divergence of the one-loop self-energy diagram in Fig. 1 and is absorbed into the bare mass parameter μ_Λ. One convenient way to effect this mass renormalization is to evaluate Eq. (2.36) in the time-independent spontaneously broken vacuum, where $\chi=0$, $\phi=v$, $\sigma_k=1$, and the mode functions are given by

$$f_k^{(\mathrm{vac})} = \sqrt{\frac{\hbar}{2k}} e^{-ikt}. \tag{2.38}$$

In this way the quadratic divergence in the integral of Eq. (2.36) is absorbed into the relation between the bare and physical expectation value of the field, μ_Λ^2 can be eliminated, and we obtain

$$\chi(t)=\frac{\lambda_\Lambda}{2}[\phi^2(t)-v^2] + \frac{\lambda_\Lambda}{4\pi^2}\int_0^\Lambda k^2dk\left\{|f_k(t)|^2\sigma_k-\frac{\hbar}{2k}\right\}, \quad (2.39)$$

which is free of quadratic divergences.

The remaining logarithmic divergence in the mode integral of Eq. (2.39) is removed by the logarithmic coupling constant renormalization in the usual way: i.e.,

$$\lambda_\Lambda=Z_\lambda^{-1}(\Lambda,m)\ \lambda_R(m^2), \quad (2.40)$$

with

$$Z_\lambda(\Lambda,m)=1-\frac{\hbar}{32\pi^2}\lambda_R(m^2)\ln\left(\frac{\Lambda^2}{m^2}\right) =\left[1+\frac{\hbar}{32\pi^2}\lambda_\Lambda\ln\left(\frac{\Lambda^2}{m^2}\right)\right]^{-1} \quad (2.41)$$

and $\lambda_R(m^2)$ the renormalized Φ^4 coupling defined at some finite mass scale m^2. By dividing both sides of Eq. (2.39) by λ_Λ and using Eqs. (2.40) and (2.41), it is straightforward to verify from the large k behavior of the integrand that the logarithmic dependence on Λ of the integral in Eq. (2.39) is canceled by the logarithm in Eq. (2.41). Thus, the resulting equation for $\chi(t)$ is independent of Λ for Λ large, and χ is in fact a renormalization-group-invariant physical mass squared of the theory.

The condition that $Z_\lambda>0$ prevents us from taking the cutoff strictly to infinity with $\lambda_R>0$ fixed, for otherwise Z_λ from Eq. (2.41) would eventually become negative and the theory would become unstable. This is just a reflection of the Landau ghost instability of scalar Φ^4 field theory, and means that the theory can be sensible and nontrivial only as an effective field theory equipped with a large but finite cutoff Λ. This presents no problem in practice as long as Λ is large enough that the physical time evolution, plasma oscillations, damping, etc., occur on time scales much greater than Λ^{-1}. In that case, the evolution is numerically quite insensitive to the value of Λ over a very wide range, provided λ_R is not too large [3]. We will focus in this work on moderate coupling stengths $\hbar\lambda_R\simeq 1$ where the effects of back reaction on the mean field evolution are significant. The case of very weak coupling (of order 10^{-12}–10^{-13}), of interest in inflationary models of the early universe, has been considered in Refs. [5,10,11].

There is no wave function or ϕ renormalization at lowest order in large N, and so no further renormalization is required in $d=3$ to this order, and Eqs. (2.20), (2.27), and (2.39) together with the constraint on the initial data (2.30) specify a well-defined closed system of evolution equations for the mean fields ϕ and χ in interaction with the fluctuations f_k.

Let us emphasize again that these equations differ from the purely classical tree-level approximation or the simple one-loop approximation in that the last term of Eq. (2.36) or (2.39) couples the fluctuations self-consistently and nonlinearly back on the time-dependent mass gap function $\chi(t)$, corresponding to the full sum of daisy and superdaisy diagrams in Fig. 2.

In $d=1$ space dimensions the integral in Eq. (2.18) is only logarithmically ultraviolet divergent and the one subtraction of Eq. (2.39) is sufficient without any λ renormalization. However, in lower spatial dimensions the small k or *infrared* behavior of the integral becomes more delicate, and must be treated carefully.

III. EFFECTIVE HAMILTONIAN AND DENSITY MATRIX

The presentation of the large N equations of motion of the previous section was based on the functional method, in which the extremization of the effective action $\mathcal{S}_{\rm eff}$ in Eqs. (2.20) and (2.18) takes the place of the usual Euler-Lagrange variational principle of classical mechanics. The equations so obtained are for the mean values of the field operators and their two-point Green's functions which describe fluctuations about the mean fields. This immediately raises a question: Is there a corresponding Hamilton form of the variational principle for an effective large N Hamiltonian involving both the mean fields and their fluctuations? In addition one would like to know to what distribution of field amplitudes in the Schrödinger wave function (or density matrix) do the large N equations for the one- and two-point functions correspond. It is to these questions that we turn in this section. Answering them will lead directly to the true effective potential of nonequilibrium large N mean field theory.

Let us begin by consideration of the case of $d=0$ spatial dimensions, i.e., quantum mechanics. The generalization to higher d will turn out to be straightforward. For $d=0$ the Lagrangian $L_{\rm cl}$ or $\widetilde{L}_{\rm cl}$ of Eq. (2.1) or (2.3) is that of an N-component anharmonic oscillator. If $N=1$, it reduces to the usual anharmonic double-well oscillator. In the $d=0$ case there is no $\mathbf{k}$ index to be integrated and the equations derived in the last section become simply

$$\left(\frac{d^2}{dt^2}+\chi(t)\right)\phi(t)=0,$$

$$\chi(t)=\frac{\lambda}{2}[\phi^2(t)+\xi^2(t)-v_0^2] \quad (d=0), \quad (3.1)$$

where we have introduced the notation

$$\xi^2(t)\equiv\sigma\ |f(t)|^2\equiv(2N+1)|f(t)|^2 \quad (3.2)$$

in terms of the expectation of the number operator $N=\langle a^\dagger a\rangle$. We recognize that

$$\xi^2(t)=\langle\mathbf{\Phi}(t)\mathbf{\Phi}(t)\rangle-\phi^2(t)=\langle(\mathbf{\Phi}(t)-\phi(t))^2\rangle \quad (3.3)$$

is just the quadratic variance of the quantum operator $\mathbf{\Phi}(t)$ from its mean value $\phi(t)$. By differentiating this relation and defining $\eta\equiv\dot{\xi}$ we find

$$2\xi(t)\,\eta(t)=\langle\dot{\mathbf{\Phi}}(t)\mathbf{\Phi}(t)+\mathbf{\Phi}(t)\dot{\mathbf{\Phi}}(t)-2\phi(t)\,\dot{\phi}(t)\rangle =2\sigma\,\mathrm{Re}(\dot{f}f^*)\ . \quad (3.4)$$

The advantage of defining these quantities will become apparent from the role they play in the physical interpretation of the equations of motion (3.1) in the Schrödinger picture. Indeed, by differentiating Eq. (3.4) again and using both the equation of motion (2.27) (with $k=0$ in $d=0$ spatial dimensions) and the Wronskian condition (2.30) for $f(t)$, we find

$$\dot{\eta}=\ddot{\xi}=-\chi\xi-\frac{\dot{\xi}^2}{\xi}+\frac{\sigma|\dot{f}|^2}{\xi}=-\chi\ \xi+\frac{\hbar^2\sigma^2}{4\xi^3} \tag{3.5}$$

in the new notation. This equation of motion and Eq. (3.1) are just Hamilton's equations

$$\dot{p}=-\frac{\partial H_{\rm eff}}{\partial\phi},$$

$$\dot{\eta}=-\frac{\partial H_{\rm eff}}{\partial\xi}, \tag{3.6}$$

for the effective *two*-dimensional classical Hamiltonian

$$H_{\rm eff}(p,\phi;\eta,\xi;\sigma)=\frac{1}{2}(p^2+\eta^2)+\frac{\lambda}{8}(\phi^2+\xi^2-v_0^2)^2+\frac{\hbar^2\sigma^2}{8\xi^2}, \tag{3.7}$$

with

$$p\equiv\dot{\phi}=\frac{\partial H_{\rm eff}}{\partial p},$$

$$\eta\equiv\dot{\xi}=\frac{\partial H_{\rm eff}}{\partial\eta} \tag{3.8}$$

the canonical momenta conjugate to the two generalized coordinates ϕ and ξ. A different but equivalent set of canonical variables was discussed in Ref. [12].

Hence by the simple change of notation in Eqs. (3.2) and (3.4) we have recognized that the large N equations for the quantum anharmonic oscillator with classical potential,

$$V_{\rm cl}(\Phi)=\frac{\lambda}{8N}\left(\sum_{i=1}^{N}\Phi_i\Phi_i-v_0^2\right)^2, \tag{3.9}$$

derived by the effective action technique of the last section are precisely equivalent to Hamilton's equations for the effective Hamiltonian (3.7). This answers the first question we posed at the beginning of this section in the affirmative, at least for the case of $d=0$.

An immediate corollary of the Hamiltonian structure in the extended phase space $(p,\phi;\eta,\xi)$ is that the large N evolution equations are energy conserving with $H_{\rm eff}$ the value of the conserved energy. It is also interesting to note that $\hbar\sigma$ is a constant of the motion which enters the effective potential

$$U_{\rm eff}(\phi,\xi;\sigma)=\frac{\lambda}{8}(\phi^2+\xi^2-v_0^2)^2+\frac{\hbar^2\sigma^2}{8\xi^2} \tag{3.10}$$

as a single ''centrifugal barrier'' term, whose effect is to repel the variance ξ away from zero. The physical and mathematical analogy to an angular momentum barrier is made even stronger by the fact that the three symmetric bilinears aa, $a^\dagger a^\dagger$, and $aa^\dagger+a^\dagger a$ generate the Lie algebra of su(1,1) or so(2,1) which is the noncompact version of the ordinary angular momentum algebra su(2) or so(3), and moreover, the Casimir invariant of this rank one Lie algebra is exactly $\hbar^2\sigma^2/4$, in the standard normalization. The corresponding Lie group is just the three-parameter group of homogeneous linear Bogoliubov transformations (2.32).

To answer the second question posed at the beginning of this section let us recall that both the time-dependent Hartree (TDH) and large N equations have been studied in the Schrödinger representation, and they are known to correspond to a Gaussian trial wave function ansatz [13]. Indeed, it is straightforward to verify that the Gaussian ansatz for the normalized pure state Schrödinger wave function,

$$\Psi(x;t)\equiv\langle x|\Psi(t)\rangle=[2\pi\xi^2(t)]^{-1/4}\exp\left\{i\frac{p(t)x}{\hbar}-\left(\frac{1}{4\xi^2(t)}+i\frac{\eta(t)}{2\hbar\xi(t)}\right)[x-\phi(t)]^2\right\}, \tag{3.11}$$

obeys the expectation value of the Schrödinger equation,

$$\langle\Psi(t)|\left\{-\frac{\hbar^2}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial\mathbf{\Phi}_i\partial\mathbf{\Phi}_i}+V(\mathbf{\Phi})\right\}|\Psi(t)\rangle=i\hbar\langle\Psi(t)|\frac{\partial}{\partial t}|\Psi(t)\rangle, \tag{3.12}$$

in the coordinate representation where $\mathbf{P}$ is the canonical momentum,

$$\langle\Psi(t)|\mathcal{O}(\mathbf{\Phi},\mathbf{P})|\Psi(t)\rangle=\int_{-\infty}^{\infty}dx\ \Psi^*(x;t)\mathcal{O}\left(x,-i\hbar\frac{\partial}{\partial x}\right)\times\Psi(x;t). \tag{3.13}$$

This is true provided the large N limit is taken and the equations of motion (3.1) and (3.5) are satisfied for $\sigma=1$. Thus, the Gaussian ansatz (3.11) is a special case of the general large N equations of motion, where ξ and η are related to the real and imaginary parts, respectively, of the Gaussian covariance.

In earlier work [14] it had been recognized that the Gaussian ansatz for the Schrödinger wave function(al) imposed one constraint on the *ab initio* three independent symmetrized variances

$$\langle\Psi(t)|(\mathbf{\Phi}-\phi)^2|\Psi(t)\rangle=\xi^2,$$

$$\langle\Psi(t)|(\mathbf{P}\mathbf{\Phi}+\mathbf{\Phi}\mathbf{P}-2\phi p)|\Psi(t)\rangle=2\xi\eta, \tag{3.14}$$

$$\langle\Psi(t)|(\mathbf{P}-p)^2|\Psi(t)\rangle=\eta^2+\frac{\hbar^2}{4\xi^2}\quad\text{(pure state)},$$

expressing all three in terms of only the two variables ξ and η, in the present notation. The one antisymmetrized variance is fixed by the commutation relation $[\mathbf{\Phi},\mathbf{P}]=i\hbar$. To what does the restriction to $\sigma=1$ correspond and how can it be relaxed? The answer to this question is suggested by the form of Eqs. (3.14), and definition of σ in Eq. (2.37) or (3.2), which shows that $\sigma=1$ corresponds to zero expectation of the number operator $a^\dagger a$, i.e., to the pure state vacuum an-

nihilated by a. However, the mean field equations of Sec. II allow for the more general possibility that this expectation value may take on any constant value N. For example, we might consider the finite temperature Bose-Einstein distribution

$$N_T=\left[\exp\left(\frac{\hbar\omega_0}{T}\right)-1\right]^{-1},$$

$$\sigma_T=1+2N_T=\coth\left(\frac{\hbar\omega_0}{2T}\right)>1, \tag{3.15}$$

for some ω_0 and temperature T. Since such a thermal state corresponds not to a pure state Schrödinger wave function, but rather to a *mixed*-state density matrix, it is not surprising that a pure state Gaussian wave function ansatz cannot describe this case. However, the general structure of the mean field equations involves only the one- and two-point functions of the quantum variable, and so we should expect them to correspond to a Gaussian ansatz but for a *mixed*-state density matrix instead of a pure state wave function. The most general form for this mixed-state normalized Gaussian density matrix ρ is

$$\langle x'|\rho(p,\phi;\eta,\xi;\sigma)|x\rangle$$
$$=(2\pi\xi^2)^{-1/2}\exp\left\{i\frac{p}{\hbar}(x'-x)-\frac{\sigma^2+1}{8\xi^2}[(x'-\phi)^2+(x-\phi)^2]+i\frac{\eta}{2\hbar\xi}[(x'-\phi)^2-(x-\phi)^2]+\frac{\sigma^2-1}{4\xi^2}(x'-\phi)(x-\phi)\right\} \tag{3.16}$$

in the coordinate representation. In the special case that $\sigma=1$ the last (mixed) term in the exponent vanishes and ρ reduces to the pure state product,

$$\rho(t)|_{\sigma=1}=|\Psi(t)\rangle\langle\Psi(t)|\ , \tag{3.17}$$

with $|\Psi(t)\rangle$ given by Eq. (3.11). For $\sigma>1$ the general Gaussian ρ does not decompose into a product, and

$$\mathrm{Tr}\ \rho^2(t)\equiv\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\langle x|\rho(t)|x'\rangle\langle x'|\rho(t)|x\rangle$$
$$=\sigma^{-1}<1, \tag{3.18}$$

which is characteristic of a mixed-state density matrix.

That Eq. (3.16) is indeed the correct generalization of the pure state Gaussian wave function (3.11) is easily verified by checking that $\rho(t)$ satisfies the expectation value of the quantum Liouville equation

$$\mathrm{Tr}\left(i\hbar\frac{\partial}{\partial t}\rho\right)=\mathrm{Tr}[\mathbf{H},\rho]\ , \tag{3.19}$$

provided the large N limit as before is taken and the equations of motion (3.1) and (3.5) are satisfied for *arbitrary* σ. Taking the large N limit is equivalent here to the replacement of the full anharmonic Hamiltonian by a time-dependent harmonic oscillator Hamiltonian:

$$\mathbf{H}\to\mathbf{H}_{\mathrm{osc}}=\tfrac{1}{2}[\mathbf{P}^2+\omega^2(t)\mathbf{\Phi}^2], \tag{3.20}$$

where

$$\omega^2(t)=\left\langle\frac{\partial^2V}{\partial\mathbf{\Phi}_i\partial\mathbf{\Phi}_j}\right\rangle\Bigg|_{i=j}\to\frac{\lambda}{2}[\phi^2(t)+\xi^2(t)-v_0^2]=\chi(t) \tag{3.21}$$

is the self-consistently determined frequency of the oscillator in the large N limit. With this replacement it is straightforward to verify that the Gaussian form is preserved by the time evolution under $\mathbf{H}_{\mathrm{osc}}$ [15–17]:

$$i\hbar\frac{\partial}{\partial t}\rho=[\mathbf{H}_{\mathrm{osc}},\rho]\ . \tag{3.22}$$

In fact, substitution of the Gaussian form (3.16) into this Liouville equation and equating coefficients of x, x', x^2, x'^2, and xx' gives five evolution equations for the five parameters specifying the Gaussian which are none other than the Eqs. (3.1) and (3.5) together with $\dot{\sigma}=0$.

The effective classical Hamiltonian for the large N equations, Eq. (3.7) is just the expectation value of the quantum Hamiltonian in this general mixed-state Gaussian density matrix, i.e.,

$$H_{\mathrm{eff}}(\phi,p;\xi,\eta;\sigma)=\mathrm{Tr}(\rho\mathbf{H})=\mathrm{Tr}(\rho\mathbf{H}_{\mathrm{osc}})=\varepsilon L^d\ , \tag{3.23}$$

in the large N limit where ε is the energy density defined in Eq. (A2). The three symmetrized variances are indeed now all independent with

$$\mathrm{Tr}[\rho(\mathbf{\Phi}-\phi)^2]=\xi^2,$$
$$\mathrm{Tr}[\rho(\mathbf{P}\mathbf{\Phi}+\mathbf{\Phi}\mathbf{P}-2\phi p)]=2\xi\eta, \tag{3.24}$$
$$\mathrm{Tr}[\rho(\mathbf{P}-p)^2]=\eta^2+\frac{\hbar^2\sigma^2}{4\xi^2},$$

replacing Eq. (3.14) of the pure state case. The mean values

$$\phi=\langle\mathbf{\Phi}\rangle=\mathrm{Tr}(\mathbf{\Phi}\rho)\quad\text{and}\quad p=\langle\dot{\mathbf{\Phi}}\rangle=\mathrm{Tr}(\mathbf{P}\rho) \tag{3.25}$$

remain valid for both the pure- and mixed-state cases.

The physical interpretation of the five parameters of the general large N equations $(p,\phi,\eta,\xi;\sigma)$ in terms of the general time-dependent mixed-state Gaussian density matrix of the Schrödinger picture is now explicit in $d=0$ quantum

mechanics. Since σ is the constant parameter which determines the degree of mixing and $\hbar$ and σ appear only in the combination $\hbar\sigma$, it is clear that the large N equations allow for a smooth interpolation between the quantum pure state case in which $\hbar\sigma=\hbar$ to the high temperature or classical limit where

$$\hbar\sigma_T \to \frac{2T}{\omega_0} \quad \text{as} \quad T\to\infty \quad \text{or} \quad \hbar\to 0, \tag{3.26}$$

in which $\hbar$ drops out entirely. Thus, quantum and classical thermal fluctuations are treated on the same footing in the large N limit, with the value of the constant parameter $\hbar\sigma$ determining whether the fluctuations described by ξ and η are to be regarded as predominantly quantal or thermal, or intermediate between the two. Moreover, we see that large N Gaussian dynamics is really classical dynamics of a Gaussian distribution function, except that $\hbar\sigma$ which measures the second moment of the classical distribution cannot be taken to zero as it could be classically, but instead is bounded from below by $\hbar$. This is made explicit by the form of the Wigner function corresponding to the density matrix (3.16), viz.,

$$\begin{aligned} f_W(x,p_x) &\equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy e^{-ip_x y/\hbar}\left\langle x+\frac{y}{2}\,|\rho|\,x-\frac{y}{2}\right\rangle \\ &= \frac{1}{\pi\hbar\sigma}\exp\left\{-\frac{(x-\phi)^2}{2\xi^2}-\frac{2\xi^2}{\hbar^2\sigma^2}\right. \\ &\quad \left.\times\left[p_x-\dot{\phi}-\frac{\eta}{\xi}(x-\phi)\right]^2\right\}. \end{aligned} \tag{3.27}$$

Before leaving our $d=0$ example it is instructive to examine the static solutions of the effective Hamiltonian, viz., the simultaneous vanishing of

$$\dot{p}=-\frac{\partial H_{\rm eff}}{\partial\phi}=-\frac{\lambda}{2}(\phi^2+\xi^2-v_0^2)\phi=\chi\phi=0$$

and

$$\dot{\eta}=-\frac{\partial H_{\rm eff}}{\partial\xi}=-\chi\xi+\frac{\hbar^2\sigma^2}{4\xi^3}=0. \tag{3.28}$$

If we look for a spontaneosly broken solution $\phi\neq 0$, then χ must vanish from the first of these conditions. But then we cannot satisfy the second condition for finite $\hbar\sigma$ and ξ. This is just a rederivation of the fact that there can be no spontaneous symmetry breaking in $d=0$ quantum mechanics (or in fact for any $d\leqslant 1$). We are forced instead to the symmetry-restored situation for which $\phi=0$ and $\chi=\chi_0>0$ is determined from the real positive root of the cubic equation

$$\chi_0=\frac{\lambda}{2}(\xi_0^2-v_0^2)=\frac{\hbar^2\sigma^2}{4\xi_0^4} \tag{3.29}$$

for $\xi_0^2(\sigma)$. The Gaussian density matrix centered at $\phi=0$ with variance $\xi_0^2(\sigma)$ is the solution of the time-independent Liouville equation for the anharmonic double-well oscillator with a trial variational density matrix of the form (3.16). In the limit that the height of the energy barrier between the two wells, $E_b=\lambda v_0^4/8$, is much greater than the fluctuation energy in either well, $E_f=\hbar\sigma\sqrt{\lambda}v_0/2$, the width of the Gaussian, $\xi_0^2\to v_0^2$, and its energy $E_0\to\hbar^2\sigma^2/8v_0^2=E_f^2/16E_b\ll E_f$, corresponding to a probability density spread over the entire region $(-v_0,v_0)$ in all N components of Φ_i.

The entire development of the Hamiltonian equations and Gaussian density matrix is quite easy to generalize to any number of spatial dimensions d, at least for the case of spatially homogeneous mean fields. Since in Fourier space the mode equations are just replicated at every spatial momentum $\mathbf{k}$, we have simply to introduce the subscript $\mathbf{k}$ on all of the relevant definitions in this section. For example, the density matrix for $d>0$ can be written as a product of Gaussians in Fourier space, viz.,

$$\begin{aligned} \langle\{\varphi_{\mathbf{k}}\}'|\rho|\{\varphi_{\mathbf{k}}\}\rangle &= \prod_{\mathbf{k}}\ \langle\{\varphi_{\mathbf{k}}'\}|\rho(p_{\mathbf{k}},\phi_{\mathbf{k}};\eta_{\mathbf{k}},\xi_{\mathbf{k}};\sigma_{\mathbf{k}})|\{\varphi_{\mathbf{k}}\}\rangle \\ &= \prod_{\mathbf{k}}\ (2\pi\xi_{\mathbf{k}}^2)^{-1/2}\exp\left\{ i\,\frac{p_{\mathbf{k}}}{\hbar}(\varphi_{\mathbf{k}}'-\varphi_{\mathbf{k}})-\frac{\sigma_{\mathbf{k}}^2+1}{8\xi_{\mathbf{k}}^2}[(\varphi_{\mathbf{k}}'-\phi_{\mathbf{k}})^2+(\varphi_{\mathbf{k}}-\phi_{\mathbf{k}})^2]\right. \\ &\quad \left. + i\,\frac{\eta_{\mathbf{k}}}{2\hbar\xi_{\mathbf{k}}}[(\varphi_{\mathbf{k}}'-\phi_{\mathbf{k}})^2-(\varphi_{\mathbf{k}}-\phi_{\mathbf{k}})^2]+\frac{\sigma_{\mathbf{k}}^2-1}{4\xi_{\mathbf{k}}^2}(\varphi_{\mathbf{k}}'-\phi_{\mathbf{k}})(\varphi_{\mathbf{k}}-\phi_{\mathbf{k}})\right\}, \end{aligned} \tag{3.30}$$

where $\varphi_{\mathbf{k}}$ is the generalized coordinate of the field amplitude in Fourier space and $p_{\mathbf{k}}$ the corresponding canonical momentum. The mean fields and their canonical momenta

$$\phi_{\mathbf{k}}=\delta_{\mathbf{k}0}\phi(t), \quad p_{\mathbf{k}}=\delta_{\mathbf{k}0}p(t) \tag{3.31}$$

all vanish except for $\mathbf{k}=0$ in the spatially homogeneous case. The definitions

$$\xi_k^2(t)\equiv\sigma_k|f_k(t)|^2\equiv[2N(k)+1]|f_k(t)|^2 \ ,$$

$$\eta_k\equiv\dot{\xi}_k, \tag{3.32}$$

and

$$\chi(t)=\frac{\lambda}{2}\left(\phi^2(t)+\int[d\mathbf{k}]\xi_k^2(t)-v_0^2\right)$$
$$=\frac{\lambda}{2}\left[\phi^2(t)+\int[d\mathbf{k}]\left(\xi_k^2(t)-\frac{\hbar}{2k}\right)-v^2\right] \quad (3.33)$$

have been introduced in obvious analogy to the $d=0$ case. The effective Hamiltonian density which gives rise to these equations is

$$\frac{H_{\rm eff}}{L^d}=\varepsilon=\frac{1}{2}p^2+\frac{1}{2\lambda}\chi^2+\frac{1}{2}\int[d\mathbf{k}]\left(\eta_k^2+k^2\xi_k^2+\frac{\hbar^2\sigma_k^2}{4\xi_k^2}\right), \quad (3.34)$$

with χ regarded as a dependent variable of ϕ and the ξ_k through the gap equation (3.33) above. The k^2 term arising from the spatial gradient of the field in $d>0$ dimensions is the most significant difference from the $d=0$ case considered previously. The Hamiltonian equations of motion

$$\dot\phi=p, \quad \dot p=-\frac{\partial H_{\rm eff}}{\partial\phi}=-\chi\phi,$$

$$\dot\xi_k=\eta_k, \quad \dot\eta_k=-\frac{\partial H_{\rm eff}}{\partial\xi_k}=-(k^2+\chi)\xi_k+\frac{\hbar^2\sigma_k^2}{4\xi_k^3} \quad (3.35)$$

are completely equivalent to the equations of motion for mean fields (2.18), (2.20), and the mode functions (2.27) in the spatially homogeneous case in any number of dimensions. The Wronskian condition (2.30) is incorporated automatically into this description.

This completes our demonstration that the large N mean field equations of the previous section are Hamilton's equations with $H_{\rm eff}$ given explicitly by Eq. (3.34) above, and the identification of the time-dependent Gaussian density matrix (3.30) to which the homogeneous mean field equations correspond. The case of spatially inhomogeneous mean fields involves only the straightforward generalization of Eq. (3.30) to Gaussian covariances which are off diagonal in the momentum index **k**, and a corresponding coupling between the different momentum modes in the effective Hamiltonian (3.34). Since we have no need of these expressions in the present work, we do not give the explicit formulas here, though they may be easily worked out within the present framework.

IV. NONEQUILIBRIUM TRUE EFFECTIVE POTENTIAL (TEP)

Having obtained the effective Hamiltonian which describes the evolution of the closed system of mean fields and fluctuations in the large N limit, it is natural to define the dynamical or true effective potential to be the static part of the effective Hamiltonian density (3.34): i.e.,

$$U_{\rm eff}(\phi,\{\xi_k\}\ ;\{\sigma_k\})$$
$$\equiv\frac{1}{2\lambda}\chi^2+\frac{1}{2}\int[d\mathbf{k}]\left(k^2\xi_k^2+\frac{\hbar^2\sigma_k^2}{4\xi_k^2}\right)$$
$$=\frac{\chi}{2}\left(\phi^2-v_0^2-\frac{\chi}{\lambda}\right)+\frac{1}{2}\int[d\mathbf{k}]\left((k^2+\chi)\xi_k^2+\frac{\hbar^2\sigma_k^2}{4\xi_k^2}\right). \quad (4.1)$$

This $U_{\rm eff}$ is the energy density of an initial state with a Gaussian density matrix (3.30) centered around ϕ with instantaneously zero velocities $p=\eta_k=0$ in both the mean field and the fluctuation variables. We call $U_{\rm eff}$ the ''true effective potential'' (TEP) because it determines the true out-of-equilibrium time evolution dynamics of the system according to Hamilton's equations (3.35). It must be clearly distinguished from the finite temperature free energy effective potential often discussed in the literature. The relationship between the two we would like to discuss next.

The standard method of calculating the free energy at temperature T is to continue analytically the effective action $\mathcal{S}_{\rm eff}$ to imaginary time [8] and evaluate the Tr ln G^{-1} term over fluctuations which are periodic in imaginary time with period $\beta=\hbar/T$. In this way one finds, for the case at hand, that the (unrenormalized) free energy is

$$\frac{\chi}{2}\left(\phi^2-v_0^2-\frac{\chi}{\lambda}\right)+\int[d\mathbf{k}]\left\{\frac{\hbar\omega_k}{2}+T\ln\left[1-\exp\left(-\frac{\hbar\omega_k}{T}\right)\right]\right\}, \quad (4.2)$$

which is a real function of ϕ provided

$$\omega_k^2=k^2+\chi\geq 0. \quad (4.3)$$

The cutoff dependent zero point energy in F is handled in the usual way by subtracting the energy of the zero temperature vacuum at $\chi=0$, i.e., $\int[d\mathbf{k}]\hbar k/2$. With the subtraction of this temperature-independent constant and the previous definitions of v and the logarithmic coupling renormalization by Eqs. (2.40) and (2.41), the thermodynamic free energy becomes cutoff independent for Λ large and is given by [18]

$$F(\phi,T)=\frac{\chi_T}{2}\left(\phi^2-v^2-\frac{\chi_T}{\lambda_\Lambda}\right)+\frac{\hbar}{4\pi^2}\int_0^\Lambda k^2\,dk\times\left\{\omega_k-k-\frac{\chi_T}{2k}+2\,T\ln\left[1-\exp\left(-\frac{\hbar\omega_k}{T}\right)\right]\right\}, \quad (4.4)$$

with χ_T the solution of the finite temperature gap equation:

$$\chi_T(\phi)=\frac{\lambda_\Lambda}{2}(\phi^2-v^2)+\frac{\hbar\lambda_\Lambda}{8\pi^2}\int_0^\Lambda k^2\,dk\times\left\{\frac{1}{\omega_k}\coth\left(\frac{\hbar\omega_k}{2T}\right)-\frac{1}{k}\right\}. \quad (4.5)$$

As long as $\chi_T \geq 0$ the free energy function (4.4) is real and well defined.

The first derivative of the free energy function is given by

$$\frac{dF}{d\phi} = \chi_T \phi \tag{4.6}$$

upon using the gap equation (4.5). In the spontaneously broken state defined by $\chi_T = 0$, we have

$$\phi^2 = \phi_T^2 = v^2 - \frac{\hbar}{2\pi^2}\int_0^\infty k\, dk\, \frac{1}{\exp\left(\frac{\hbar k}{T}\right) - 1} = v^2 - \frac{T^2}{12\hbar}\,. \tag{4.7}$$

The last relation informs us that the expectation value ϕ_T vanishes at the critical temperature

$$T_c = 2\sqrt{3\hbar}\; v\;. \tag{4.8}$$

At $T = T_c$, the second derivative of $F(\phi, T)$ vanishes at $\phi = 0$, and therefore the phase transition is second order in the large N approximation. We remark that this is *not* the case in the simple one-loop or Hartree approximation where the transition is weakly first order [9]. Since the Φ^4 field theory lies in the same universality class as spin models which are known to have second order phase transitions [2], the large N mean field approximation gets the order of the transition correct. Characteristic of a second order transition, the correlation length, which is given by the the inverse mass of the radial excitation,

$$d(T) = \lambda_R^{-1/2}\phi_T^{-1} \to \left(\frac{6}{\lambda_R T_c}\right)^{1/2} (T_c - T)^{-1/2}, \tag{4.9}$$

diverges as $T \to T_c$. The critical exponent $-1/2$ is that of mean field theory.

Because of the divergence of the correlation length at the critical temperature, it is sometimes said that the dynamics of a second order phase transition is classical, in the sense that

$$\hbar N_T(k) \to \frac{T}{k}\,, \tag{4.10}$$

which is independent of $\hbar$ as $k \to 0$. It is true that the very long wavelength dynamics is dominated by the classical Rayleigh-Jeans part of the Bose-Einstein distribution $N_T(k)$. However, at finite length scales and/or in nonequilibrium situations where finite time scales enter, the classical limit is not strictly justified *a priori*, and should be checked on a case-by-case basis. Certainly the space and time scales of the order of the thermal wavelength $\hbar/T$ or less are not classical at all, even in thermal equilibrium. Also, the classical limit of long wavelength modes in Eq. (4.10) is a different limit from that in which particles act like billiard balls (i.e., when their de Broglie wavelength is much *smaller* than their mean free path) and a classical Boltzmann transport description becomes appropriate. The extent to which the Gaussian quantum density matrix of the large N approximation can be replaced by a classical probability distribution at late times when a relation like Eq. (4.10) holds is discussed in the next section.

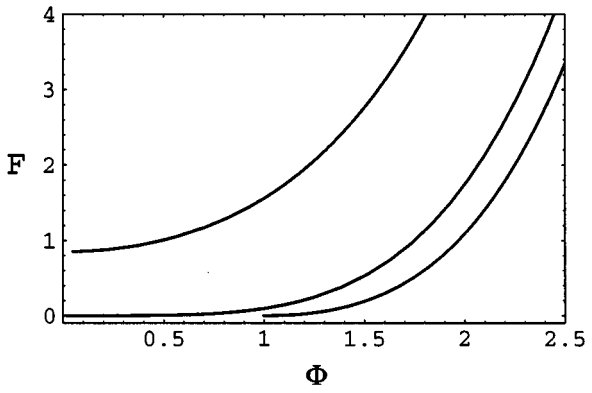

FIG. 4. The Helmholtz free energy F as a function of ϕ for three temperatures $T/T_c = 0.1, 1, 2$ (in ascending order) in units of $v = 1$ and $\lambda = 1$ in the leading order large N approximation. The ϕ-independent thermal free energy $-\pi^2 T^4/90$ has been subtracted out. The lowest curve terminates at the minimum of F at $\phi = \phi_T$, $\chi_T = 0$, given by Eq. (4.7), which is the boundary of the spinodal region; i.e., for $\phi < \phi_T$ there is no stable equilibrium state. The second derivative of F at $\phi = \phi_T$ vanishes, showing in particular that the phase transition at $T = T_c$ is second order.

As it turns out, the second derivative of the free energy function vanishes at its minimum for *any* $T \leq T_c$. This follows the fact that the derivative of the gap equation (4.5) evaluated at $\chi_T = 0$ involves an infrared linear divergence with the result that solving for the derivative gives

$$\left.\frac{d\chi_T(\phi)}{d\phi}\right|_{\chi_T = 0} = 0 \tag{4.11}$$

and

$$\frac{d^2F}{d\phi^2} = \chi_T + \frac{d\chi_T(\phi)}{d\phi}\phi = 0 \tag{4.12}$$

at $\chi_T = 0$. This perhaps surprising feature of the large N free energy is illustrated in Fig. 4. It is a direct consequence of the massless Goldstone particles present in the spontaneously broken phase. As we shall see in Sec. VI, it also makes the equilibrium free energy useless for describing the plasma oscillations about the spontaneously broken thermal minimum.

For $T < T_c$ and $\phi < \phi_T$ the gap equation (4.5) has no positive real solutions. This is the unstable spinodal region of Fig. 3. If $\chi < 0$, then the frequency ω_k becomes imaginary for $k < |\chi|^{1/2}$. This means that, strictly speaking the thermodynamic free energy F is not well defined in this region. Despite the fact that no thermal equilibrium uniform state can exist in this spinodal region, what is sometimes done is to *define* the free energy by an analytic continuation from $\chi > 0$ to $\chi = -|\chi| < 0$. By this procedure F acquires an imaginary part, while its real part is no longer a convex function of ϕ (as general theorems require). This has led to much discussion in the literature [19,20]. However, we need

not wonder at the meaning of this analytic continuation and imaginary part of F in the more general nonequilibrium context. The analytic continuation from $\chi>0$ to $\chi=-|\chi|<0$ simply defines a Gaussian density matrix centered at $\phi<\phi_T$ with instantaneously zero velocities, $\dot{\phi}=\eta_k=0$, and parameters ξ_k and $\sigma_k=2N_T(k)+1$ determined by the thermal distribution at $|\chi_T|=-\chi_T$ where the free energy is well defined. The density matrix will certainly evolve away from this unstable configuration and the imaginary part is a measure of the rate at which this evolution will occur initially [4]. However, shortly afterwards the nonlinear effects of the evolution described by our general nonequilibrium equations of motion will set in, and this imaginary part can only be at best a simple order of magnitude estimate for the evolution away from the initial state in the weak coupling limit $\hbar\lambda_R\ll 1$.

The nonconvexity of the real part of the free energy is also related to the same instability [20]. The point is that the construction of the equilibrium free energy by a loop expansion or the large N expansion tacitly assumes the existence of a stable configuration as the starting point for the expansion. It is precisely this assumption which breaks down in the unstable spinodal region. A careful definition of the ''effective potential'' F as the minimum of the free energy for fixed $\phi=\langle\mathbf{\Phi}\rangle$ independent of space and time leads to a flat constant function between the two minima at $\pm v$ which is convex in accordance with the general theorems. However, this minimization cannot be achieved with a Gaussian density matrix of the form (3.30) and the flat convex form of F defined in this manner tells us nothing about the dynamical evolution of an initial Gaussian centered at a value of ϕ in the spinodal region. For this reason we propose to drop all attempts to refine the definition of a free energy ''effective potential'' and focus instead on the potential which actually governs the nonequilibrium dynamics in a given approximation scheme. In the leading order large N scheme this dynamical potential is the true effective potential $U_{\rm eff}$.

In contrast to the free energy F, the TEP $U_{\rm eff}$ is a function(al) of all of the generalized fluctuation coordinates ξ_k as well as the mean field ϕ. It also depends on the constant parameters σ_k, which need not be that in the thermal distribution (3.15). Hence it is defined much more generally than F and is manifestly real and positive. If we wish to consider a function only of the mean field ϕ, it is possible to try first minimizing $U_{\rm eff}$ with respect to the Gaussian parameters ξ_k:

$$\frac{\partial U_{\rm eff}}{\partial\xi_k}=(k^2+\chi)\ \xi_k-\frac{\hbar^2\sigma_k^2}{4\xi_k^3}=0, \tag{4.13}$$

which has the real solution

$$\overline{\xi}_k^{\,2}=\frac{\hbar\sigma_k}{2\omega_k} \tag{4.14}$$

provided again that Eq. (4.3) holds.

If we restrict $U_{\rm eff}$ further by requiring σ_k to be the Bose-Einstein thermal distribution $\sigma_{Tk}=1+2N_T(k)$ of Eq. (3.15), we find

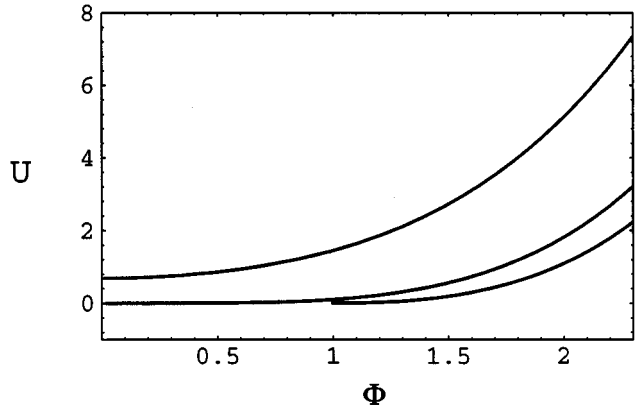

FIG. 5. The TEP $U_{\rm eff}$ as a function of ϕ and evaluated in a thermal equilibrium state for the same three temperatures as in the previous figure, with the subtraction of the thermal internal energy $\pi^2T^4/30$. The bottom curve is for $T/T_c=0.1$ and starts at $\phi=\phi_T$. The next two curves are for $T/T_c=1$ and $T/T_c=2$. The Gaussian width parameter ξ_k has been set equal to its minimum value by Eq. (4.13) so that $U_{\rm eff}$ becomes equal to the internal energy of the system.

$$\begin{aligned}U(\phi,T)&\equiv U_{\rm eff}(\phi,\{\xi_k\}=\{\overline{\xi_k}\};\sigma_k=\sigma_{T\,k})\\&=\frac{\chi_T}{2}\left(\phi^2-v^2-\frac{\chi_T}{\lambda}\right)+\frac{\hbar}{4\pi^2}\int_0^\infty k^2\,dk\\&\quad\times\left\{\omega_k-k-\frac{\chi_T}{2k}+2\omega_k\,N_T(k)\right\}.\end{aligned} \tag{4.15}$$

The first term is the classical potential energy density $V_{\rm cl}$ of the mean field ϕ and the second term is the quantum plus thermal energy of the fluctuations at the extremum (4.14). We recognize the expression (4.15) as the *internal* energy U of the Gaussian configuration specified by Eqs. (3.15) and (4.14), which differs from the Helmholtz free energy F by the standard thermodynamic relation

$$F=U-TS, \tag{4.16}$$

with

$$\begin{aligned}S=-\,{\rm Tr}\rho_T\ln\rho_T=\int[d\mathbf{k}]\{&(N_T(k)+1)\ln[N_T(k)+1]\\&-N_T(k)\ln N_T(k)\}\end{aligned} \tag{4.17}$$

the von Neumann entropy of a Gaussian thermal density matrix ρ_T. Since the Bose-Einstein number density

$$N_T(k)=\left[\exp\left(\frac{\hbar\omega_k}{T}\right)-1\right]^{-1} \tag{4.18}$$

depends on χ_T and hence on ϕ through Eqs. (4.3) and (4.5), the free energy F and internal energy U have different dependences on the mean field $\phi>\phi_T$. The free energy and internal energy become equal only at zero temperature $T=0$.

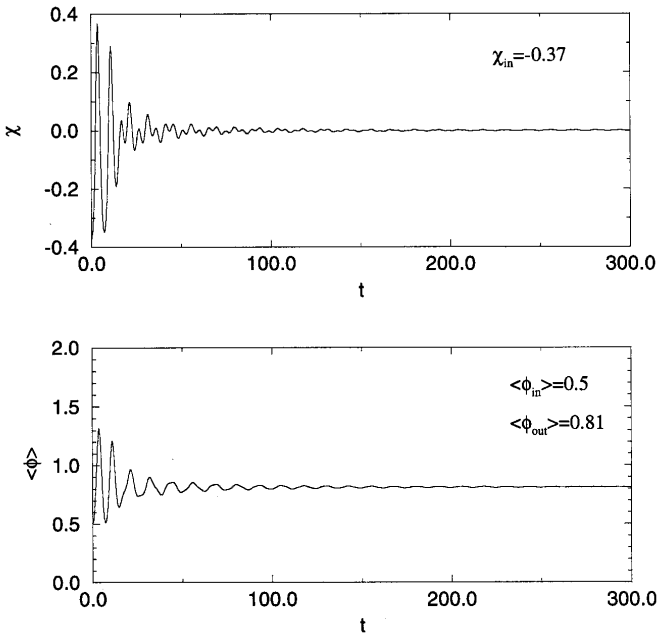

FIG. 6. A typical evolution of the mean field ϕ and the mass squared χ in the spontaneously broken phase, starting from an unstable initial state with $\chi(0)=-0.37<0$. All time and lengths are measured in units of $v=1$. Note that if we use the initial energy density $\varepsilon=0.0697$ to infer an effective temperature of massless bosons, then the expected minimum of F occurs at $\phi_{\text{out}}=0.98$ which is considerably different than the observed $\phi_{\text{out}}=0.81$. The oscillation frequency about $\phi=\phi_{\text{out}}$ also bears no relation to the second derivative of F which vanishes at its minimum.

Although F and U have qualitatively similar behaviors in the region $\chi\geqslant 0$ where they are both defined unambiguously, as shown in Figs. 4 and 5, clearly F and U have quite different physical meanings and applicability. The first is the negative of the pressure of the gas of scalar particles against which work must be done in compressing the system at fixed temperature, i.e., if the compression is performed while in contact with an external heat reservoir at temperature T. The second is the energy density of the *closed* system which is conserved if it is allowed to evolve, isolated from all external sources or sinks of energy. The free energy F can be defined only in thermal equilibrium. The internal energy U is a special case of the more general true effective potential U_{eff} which is defined for any Gaussian density matrix, equilibrium or not, and which determines the evolution of the system away from the instantaneous stationary state where $\dot{\phi}=\dot{\xi}_k=0$ according to the Hamiltonian equations (3.35). Moreover, in the spinodal region ($x<0$) where $T<T_c$ and $\phi<\phi_T$ it is clear that the minimization condition (4.13) cannot be satisfied for real ξ_k^2, which informs us that the time derivative of the canonical momentum $\dot{\eta}_k>0$ for $k^2<|\chi|$; i.e., the configuration is unstable against the fluctuation widths ξ_k growing in time for these low k. Note that by the definition (3.32), in terms of the mode function f_k, ξ_k is necessarily real. The physical interpretation is thus quite straightforward in the more general nonequilibrium framework we have laid out: There simply is no stationary spatially homogeneous Gaussian density matrix for $\chi<0$, and any such initial configuration will necessarily evolve in time by growth of the small k Fourier components of the Gaussian parameters ξ_k. Indeed, this is obvious from the mode equations (2.27) which show that all the long wavelength modes with $k^2\leqslant|\chi|$ will grow exponentially rather than oscillate when $\chi<0$ [4]. Notice that the nonexistence of a solution to the minimization condition (4.13) implies that the exponentially growing instability lies in the fluctuations ξ_k for small k. Only the nonlinear back reaction of this exponential growth of the modes in the time-dependent gap equation (2.39) can eventually bring χ to non-negative values and turn off the instability. It is clear that this time-dependent process involves many low k Fourier components of the field Φ and cannot be described adequately by a single function of one variable such as the ''effective'' potential F, much less a function of one variable defined only in thermal equilibrium in the spinodal region where no such thermal equilibrium exists.

That the time evolution as determined by U_{eff} is quite different from what might be inferred from the free energy function F is illustrated by our numerical solution of the evolution equations, presented in Fig. 6. The oscillation frequency and even the final point of the evolution of ϕ as $t\rightarrow\infty$ in general bear no simple relation to the minimum ϕ_T of the free energy ''effective'' potential. This is easily understood from the stationary points of the TEP, for a true stationary state does exist in the spontaneously broken phase if we also require

$$\frac{\partial U_{\text{eff}}}{\partial\phi}=\chi\phi=0 \tag{4.19}$$

since this is satisfied by

$$\chi=0 \quad \text{and} \quad \xi_k^2=\frac{\hbar\sigma_k}{2k}, \tag{4.20}$$

with Eq. (2.38). Notice that there is such a static solution for *any* ϕ and σ_k. The finite temperature case, Eq. (3.15), is only one of infinitely many possibilities for a static solution of the mean field equations. Correspondingly there are also an infinite number of stationary density matrices satisfying $[\rho,\mathbf{H}_{\text{osc}}]=0$, one for each choice of σ_k. If $\sigma_k=1+2N(k)$ is given, then the static value of the expectation value ϕ is determined by the static condition $\chi=0$, which from Eq. (2.39) implies

$$\phi^2|_{\chi=0}+\frac{\hbar}{2\pi^2}\int_0^\infty k dk\ N(k)=v^2\ . \tag{4.21}$$

For arbitrary $N(k)$ the relation (4.21) may be viewed as a kind of sum rule which allows us to distribute any fraction of the v^2 in the coherent mean field ϕ^2 and the remainder in the integral over particle modes. In particular, there is no reason why the sum rule cannot be saturated (or nearly saturated) by the integral alone, in which case the mean field ϕ is zero (or very small), but χ continues to vanish identically. Even though ϕ may vanish, this stationary situation should be clearly distinguished from ordinary high temperature symmetry restoration at $T>T_c$, since the number density $N(k)$ need *not* be a thermal distribution at all [11]. For example,

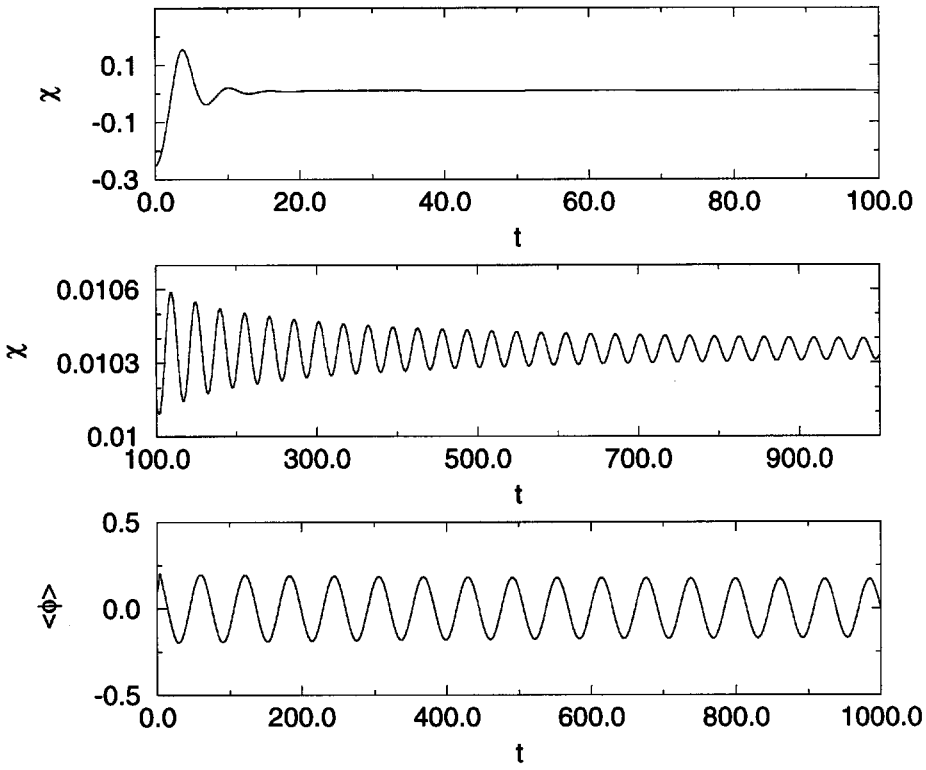

FIG. 7. Evolution of the mean field ϕ and the mass squared χ in one space dimension, starting from an unstable initial state. The middle figure shows that χ does not go to zero asymptotically, and the oscillations do not damp as effectively as in three dimensions. Both these behaviors are consequences of the Mermin-Wagner-Coleman theorem which prevents spontaneous symmetry breaking in one space dimension.

the integral in Eq. (4.21) may saturate the sum rule with the number density strongly enhanced at low momentum relative to the Bose-Einstein distribution, so that their contribution to the total energy density of the system (which involves an additional power of k^2) is very small, and much less than the equilibrium energy density at the critical temperature. In the limit $kN(k)$ approaches a δ function at $k=0$ the integral can saturate the sum rule with *no* contribution of the zero momentum particles to the energy density. Such large particle occupation numbers at zero (or very small) momenta is a kind of Bose condensation which may be studied by classical methods [11].

With the help of the sum rule (4.21), therefore, one can easily understand the perhaps initially surprising result, found numerically in Ref. [21], that the evolution of the ϕ mean field can settle to a value different from, and even much smaller than, the thermal equilibrium value ϕ_T. It simply corresponds to the fact that the distribution of particles in the final state is not at all a thermal one. Obviously such a situation cannot be described by the thermodynamic free energy F since its evaluation by continuation to imaginary time assumes from the outset a thermal distribution of particles. For this reason one could not expect the true evolution according to the mean field equations of motion to bear much relation to what one might infer from an uncritical use of F, particularly in the region $\phi<\phi_T$ where no thermodynamic uniform equilibrium state exists and the Helmholtz free energy is not even strictly defined.

The inclusion of collisions in the next order of the expansion in $1/N$ makes it more likely to bring the distribution of particles closer to a thermal one. However, since the evolution is unitary, this can happen only in some effective sense. This aspect is closely connected to the notion of dephasing which will be discussed in detail in Sec. VI.

Finally we point out that the stationary spontaneously broken solution $\chi=0$ is disallowed in $d=1$ spatial dimension (as it was in $d=0$ quantum mechanics), since the integration in the definition of χ in Eq. (3.33) diverges logarithmically as $k\to 0$ which is inconsistent with χ vanishing in Eq. (4.20). It is this infrared divergence which prevents spontaneous symmetry breaking of the global O(N) symmetry in one space dimension, consistent with the Mermin-Wagner-Coleman theorem [6]. In two or higher space dimensions there is no such divergence and the spontaneous symmetry-breaking static solution (4.20) is allowed.

The consequences of the Mermin-Wagner-Coleman theorem for nonequilibrium dynamics in one dimension are illustrated in Fig. 7 where the numerical evolution of χ and ϕ is shown beginning from an unstable initial state with negative χ. We see that χ does not go to zero at late times (no symmetry breaking) and that both the χ and ϕ oscillations damp very slowly as compared to the behavior in three dimensions. This is because of the lack of any massless particles in $d=1$.

V. DEPHASING, DISSIPATION, AND DECOHERENCE

Although we have shown by explicit construction of the effective Hamiltonian (3.34) that the large N mean field equations are completely time reversible, nevertheless one observes in typical evolutions as in Fig. 6 an *effective irreversibility*, in the sense that energy flows from the mean fields ϕ and χ to the fluctuating modes f_k without returning over times of physical interest. It is our purpose in this section to explain this apparent irreversibility in terms of *dephasing*, i.e., the dynamical averaging to zero in the sums over k of the rapidly varying phases of the fluctuations at a given time. To the extent that this phase averaging is exact

and the information in the phases cannot be recovered, the time evolution is irreversible. This is the sense in which the fluctuations f_k act as a ''heat bath'' or ''environment'' for the mean field evolution of ϕ and χ. Of course, since information is never truly ''lost'' in a closed Hamiltonian system evolved with arbitrarily high accuracy, the information in the phases can be recovered in principle and we should expect Poincaré recurrences in the mean field evolution after very long times, at least in finite volume V. As the number of modes f_k (and particularly as the number of relevant infrared modes near $k=0$) approaches infinity, we would expect the recurrence time to go to infinity. In typical evolutions we followed several tens of thousands of modes, and recurrences were never observed in practice. The precise dependence of the recurrence time on the number of modes is an interesting question which may be studied quantitatively within our mean field framework by numerical methods. However, we have not undertaken such a systematic study here, and leave the question of the recurrence time for future research. In any case, we cannot expect the leading order large N collisionless approximation to continue to be valid for times longer than the collisional relaxation time in the full theory. The study of recurrence times in the leading order approximation may be interesting nevertheless as a model for how such recurrences and rephasing can occur in more realistic situations.

To understand precisely what is meant by dephasing let us return to the Gaussian density matrix of only one degree of freedom (3.16). The coordinate representation is only one of many possible representations of the density matrix. The number basis, defined by the integer eigenvalues of

$$a^\dagger a|n\rangle = n|n\rangle, \tag{5.1}$$

defines a time-independent basis associated with the Fock decomposition of the Heisenberg operators (for $d=0$),

$$\mathbf{\Phi}(t) = \phi(t) + a f(t) + a^\dagger f^*(t),$$

$$\frac{d\mathbf{\Phi}}{dt}(t) = \mathbf{P}(t) = \dot\phi(t) + a\dot f(t) + a^\dagger \dot f^*(t), \tag{5.2}$$

in terms of its mean value and quantized fluctuations. By using the Wronskian condition (2.30) we may solve these relations for a and $a^\dagger$:

$$(\mathbf{\Phi}-\phi)\dot f^* - (\mathbf{P}-\dot\phi) f^* = i\hbar a,$$

$$(\mathbf{\Phi}-\phi)\dot f - (\mathbf{P}-\dot\phi) f = -i\hbar a^\dagger. \tag{5.3}$$

In this time-independent basis, the Hamiltonian is given by

$$\mathbf{H}_{\rm osc} = \hbar\omega(t=0)\left(a^\dagger a + \tfrac{1}{2}\right). \tag{5.4}$$

The transformation to this number basis is just a (complex) canonical transformation which on the quantum level is implemented by a unitary transformation of bases, with the transformation matrix, $\langle x|n\rangle$ in Dirac's notation. To find this transformation matrix explicitly we use the definitions (3.2) and (3.4), the equation of motion (3.5), and the Wronskian condition again to secure

$$\hbar^2\sigma\left(a^\dagger a + \frac{1}{2}\right) = (\mathbf{\Phi}-\phi)^2\left(\frac{\eta^2}{\sigma} + \frac{\hbar^2\sigma^2}{\xi^2}\right) + \xi^2(\mathbf{P}-p)^2 - \xi\eta[(\mathbf{\Phi}-\phi)(\mathbf{P}-p) + (\mathbf{P}-p)(\mathbf{\Phi}-\phi)]$$

in terms of ξ, η, and σ. Then by acting with this operator identity on the matrix $\langle x|n\rangle$ and replacing $\mathbf{\Phi}\to x$ and $\mathbf{P}\to -i\hbar d/dx$ we obtain the differential equation

$$\hbar^2\sigma\left(n+\frac{1}{2}\right)\langle x|n\rangle = \left\{(x-\phi)^2\left(\eta^2 + \frac{\hbar^2\sigma^2}{\xi^2}\right) + \xi^2\left(-i\hbar\frac{d}{dx} - p\right)^2 - 2\xi\eta\left(-i\hbar(x-\phi)\frac{d}{dx} - \frac{i\hbar}{2} - (x-\phi)p\right)\right\}\langle x|n\rangle, \tag{5.5}$$

which is easily solved in terms of the ordinary harmonic oscillator wave functions

$$\psi_n(x;\omega) = \frac{1}{(2^n n!)^{1/2}}\left(\frac{\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{\omega}{\hbar}}x\right)\exp\left(-\frac{\omega}{2\hbar}x^2\right) \tag{5.6}$$

by

$$\langle x|n\rangle = \exp\left(\frac{ip(x-\phi)}{\hbar} + \frac{i\eta(x-\phi)^2}{2\hbar\xi}\right)\psi_n\left(x-\phi;\frac{\hbar\sigma}{2\xi^2}\right). \tag{5.7}$$

With the transformation matrix element determined it is a straightforward exercise in Gaussian integration and the properties of Hermite polynomials to obtain the form of the density matrix (3.16) in the time-independent number representation:

$$\langle n'|\rho|n\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \ \langle n'|x'\rangle\langle x'|\rho|x\rangle\langle x|n\rangle = \frac{2\delta_{n'n}}{\sigma+1}\left(\frac{\sigma-1}{\sigma+1}\right)^n, \tag{5.8}$$

where σ is defined by Eq. (2.37). The derivation of these

results and some further properties on the transformation matrices are given in Appendix B.

The matrix elements of the Liouville equation in this basis are

$$i\hbar\langle n|\dot{\rho}|n'\rangle=\langle n|[\mathbf{H}_{\rm osc},\rho]|n'\rangle=(E_n-E_{n'})\langle n|\rho|n'\rangle. \quad (5.9)$$

Clearly, since the density matrix in this basis is initially diagonal, it stays diagonal and time independent. The time independence of the matrix elements of ρ in the Heisenberg number basis $|n\rangle$ is simply a reflection of the fact that the density matrix (like the state vector $|\psi\rangle$) is time independent in the Heisenberg picture where the field operator $\mathbf{\Phi}$ depends on time according to Eq. (2.24). All the time-dependent dynamics resides in the transition matrix element $\langle x|n\rangle$ which depends on the five variables $(\phi(t),p(t);\xi(t),\eta(t);\sigma)$ while the matrix elements (5.8) remain forever unchanging under the mean field evolution. Indeed, this is just another reflection of the unitary Hamiltonian nature of the evolution, since the von Neumann entropy of the Gaussian density matrix

$$S=-\mathrm{Tr}\rho\,\ln\rho=\left(\frac{\sigma+1}{2}\right)\ln\left(\frac{\sigma+1}{2}\right)-\left(\frac{\sigma-1}{2}\right)\ln\left(\frac{\sigma-1}{2}\right)$$
$$=(N+1)\ln(N+1)-N\ln N \quad (5.10)$$

is strictly constant under the time evolution. Hence, there is no information lost in the evolution in any strict sense.

There is, however, another basis which is more appropriate for discussing ''physical'' particle number. This is the time-dependent Fock basis specified by replacing the mode function f which satisfies

$$\left(\frac{d^2}{dt^2}+\omega^2(t)\right)f(t)=\left(\frac{d^2}{dt^2}+\chi(t)\right)f(t)=0 \quad (5.11)$$

by the adiabatic mode function, $\tilde{f}$ defined by Eq. (2.34) and the Fock representation (5.2) by

$$\mathbf{\Phi}(t)=\tilde{a}(t)\tilde{f}(t)+\tilde{a}^{\dagger}(t)\tilde{f}^{*}(t)\ ,$$
$$\frac{d\mathbf{\Phi}}{dt}(t)=\mathbf{P}(t)=-i\omega(t)\tilde{a}(t)\tilde{f}(t)+i\omega(t)\tilde{a}^{\dagger}(t)\tilde{f}^{*}(t)\ , \quad (5.12)$$

where the $\tilde{a}$ and $\tilde{a}^{\dagger}$ operators must now be time dependent. The corresponding time-dependent number basis is defined by

$$\tilde{a}^{\dagger}\tilde{a}\,|\tilde{n}\rangle=\tilde{n}\,|\tilde{n}\rangle\ , \quad (5.13)$$

in which

$$\mathbf{H}_{\rm osc}=\frac{\hbar\omega}{2}(\tilde{a}^{\dagger}\tilde{a}+\tilde{a}\tilde{a}^{\dagger}) \quad (5.14)$$

is diagonal for each time. In the $\tilde{a}^{\dagger}\tilde{a}$ number basis, ρ is no longer diagonal, $\langle\tilde{a}\rangle$, $\langle\tilde{a}\tilde{a}\rangle$, etc., are nonvanishing, and $\tilde{N}\equiv\langle\tilde{a}^{\dagger}\tilde{a}\rangle\neq N$ in general, except in the static case of constant ω.

In addition to diagonalizing the time-dependent harmonic oscillator Hamiltonian $\mathbf{H}_{\rm osc}$ which describes the evolution of the Gaussian density matrix, the $\tilde{n}$ basis has another important property, namely, the existence of an adiabatic invariant W. This adiabatic invariant may be constructed from the Hamilton-Jacobi equation corresponding to the effective classical Hamiltonian $H_{\rm eff}$ in the usual way:

$$\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{\omega^2}{2}x^2+\frac{1}{2}\left(\frac{\partial W}{\partial\xi}\right)^2+\frac{\omega^2}{2}\xi^2+\frac{\hbar^2\sigma^2}{8\xi^2}=E\ . \quad (5.15)$$

Since the Gaussian density matrix whose evolution is described by $H_{\rm eff}$ is equivalent to that of a time-dependent harmonic oscillator, we may regard the frequency $\omega(t)$ in Eq. (5.15) as an *arbitrary* function of time and separate the equa-

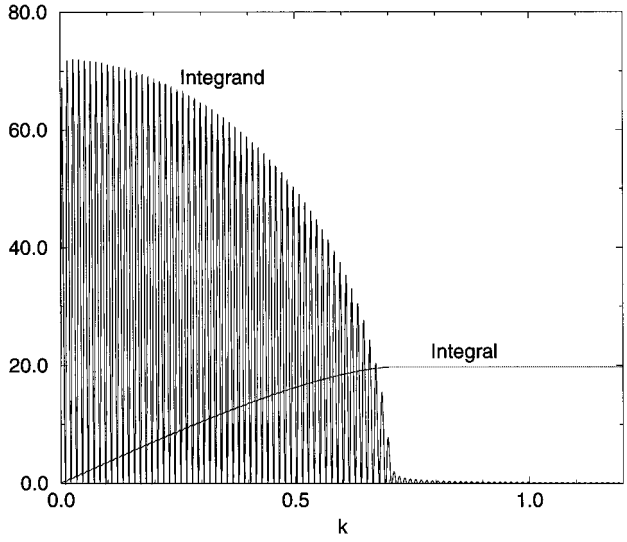

FIG. 8. Integrand and integral of the RHS of Eq. (5.22) as a function of k for fixed $t=257$. The value of the total integral is 19.74.

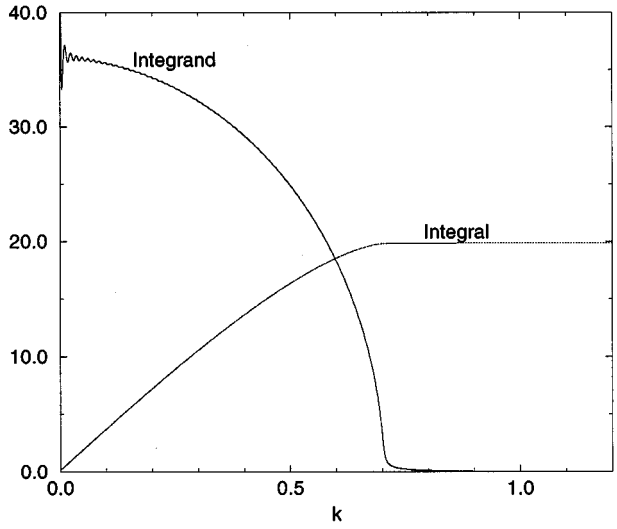

FIG. 9. Integrand and integral of the phase-averaged quantity defined by the adiabatic particle number basis as a function of k for the same fixed $t=257$. The value of the integral using the replacement (5.21) is 19.85, within 0.5% of the value in the previous figure.

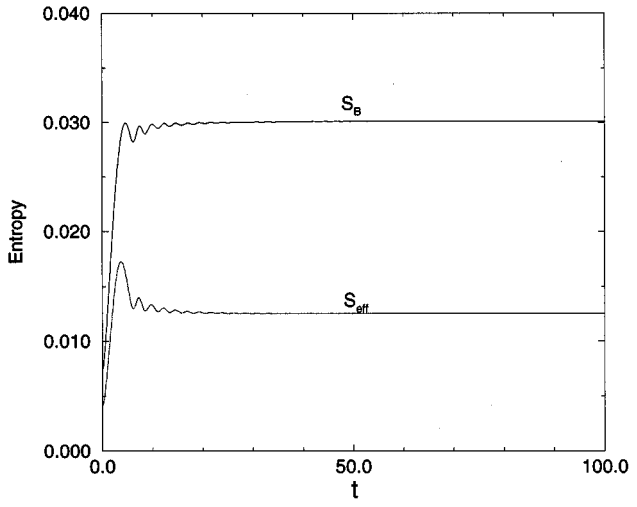

FIG. 10. Evolution of the Boltzmann and effective von Neumann entropies of the diagonal elements of the density matrix in the adiabatic particle number basis.

tion with the ansatz $W(x,\xi)=W_1(x)+W_2(\xi)$. Defining $E=E_1+E_2$ this yields, in the first variable x,

$$W_1=\oint dx\ \sqrt{2E_1-\omega^2x^2}=2\pi\frac{E_1}{\omega}, \tag{5.16}$$

while in the second variable ξ one obtains

$$W_2=\oint d\xi\sqrt{2E_2-\omega^2\xi^2-\frac{\hbar^2\sigma^2}{4\xi^2}}. \tag{5.17}$$

Under the substitution $r=\xi^2/2$ this turns out to be the same integral which occurs in the Kepler problem, again pointing out the formal similarity with the angular momentum barrier in a central potential. Using standard methods [22], one finds

$$W_2=2\pi\frac{E_2}{\omega}-\pi\hbar\sigma\ . \tag{5.18}$$

Hence the full adiabatic invariant is

$$\frac{W}{2\pi\hbar}=\frac{E}{\hbar\omega}-\frac{\sigma}{2}=\tilde{N}-N \tag{5.19}$$

upon using the definition of $\tilde{N}$, the definition of σ, and

$$E=\langle\mathbf{H}\rangle=\langle\mathbf{H}_{\rm osc}\rangle=\hbar\omega\left(\tilde{N}+\frac{1}{2}\right)\ . \tag{5.20}$$

Since N is strictly a constant of the motion, Eq. (5.19) informs us that $\tilde{N}$ is an adiabatic invariant and therefore it is slowly varying if $\omega^2=\chi$ in the actual self-consistent, nonlinear evolution is a slowly varying function of time. In the language of classical action-angle variables, W is an action variable which is slowly varying while the angle variable conjugate to it varies rapidly in time [linearly with time in the limit ω is a constant, as in the exponent in Eq. (2.34)]. This means that although the density matrix ρ is certainly not diagonal in the $\tilde{n}$ number basis, its off-diagonal elements which depend on the angle variable will be very rapidly varying functions of time, whereas its diagonal elements in this basis will be only slowly varying. This is clearly seen in the explicit form for the matrix elements of this basis in Eq. (B22) of Appendix B. Hence, if we are interested only in the motion of the mean fields which are slowly varying functions of time, we may average over the rapid phase variations in the off-diagonal matrix elements of ρ in this adiabatic number basis. For only a single quantum degree of freedom this amounts to a time averaging and can be implemented only by fiat. However, in field theory there are many momentum modes, so that this averaging is actually performed for us in the mean field evolution equations by the integrations over k at fixed t.

The term *dephasing* has a precise meaning in this adiabatic particle number representation of the density matrix. To the extent that the phases in the off-diagonal matrix elements of the density matrix in the adiabatic $\tilde{n}$ basis are rapidly varying in time, they should have little or no effect on the evolution of more slowly varying quantities such as the mean fields ϕ and χ or the mean adiabatic particle number $\tilde{N}$ itself. Thus, a natural approximation to the full density matrix ρ is immediately suggested by the existence of the adiabatic number basis, namely, to discard the off-diagonal elements [see Eq. (B22) below] of ρ in this basis, which is equivalent to time averaging the distribution of fluctuations in the exact Gaussian density matrix (3.30) over times long compared to their rapid variations. In field theory, where there are many Fourier modes, each with its own rapid phase variation, the effect of dephasing may be obtained by simply integrating over the momentum index k at fixed time. In either case, we expect this averaging procedure to scarcely affect the actual evolution of the mean fields, and the extent to which this expectation is realized is the extent to which dephasing of the fluctuations is effective, and the evolution is irreversible. In Figs. 8 and 9 we compare the actual k dependence of the function $G(k,t)$ appearing in the mean field evolution equation (2.18) with the phase averaged quantity defined by the adiabatic particle number $\tilde{N}(k)$; i.e., we make the replacement

$$-i\hbar G(k,t)=|f_k(t)|^2\rightarrow\frac{1}{2k}[2\tilde{N}(k)+1] \tag{5.21}$$

and discard the interference terms. The relevant integral in the renormalized gap equation (2.39) is

$$G(x,x,\chi)-G(x,x,\chi=0)=\frac{1}{4\pi^2}\int_0^\Lambda k^2dk\left\{|f_k(t)|^2\sigma_k-\frac{\hbar}{2k}\right\}. \tag{5.22}$$

Notice that although there are many more oscillations in the integrand of the right-hand side (RHS) of Eq. (5.22) than in the phase averaged quantity, the integrals over k of the two quantities are almost the same (19.74 and 19.85, respectively).

The adiabatic particle basis may also be used to define effective Boltzmann and von Neumann entropies [23] through the diagonal matrix elements (B22) below. Neglecting the rapidly varying off-diagonal matrix elements gives

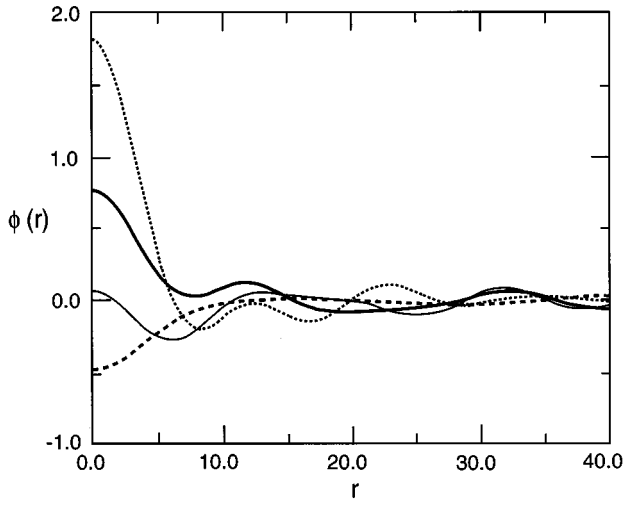

FIG. 11. Four typical field configurations drawn from the same classical distribution of probabilities in the adiabatic particle number basis, according to Eq. (5.29), for $t=490$ in the case the mean field $\phi=0$. The units of r are v^{-1} and $\lambda_\Lambda=1$.

$$S_B \equiv \sum_{\mathbf{k}} \{[\tilde{N}(\mathbf{k})+1]\ln[\tilde{N}(\mathbf{k})+1]-\tilde{N}(\mathbf{k})\ln\tilde{N}(\mathbf{k})\},$$

$$S_{\rm eff} \equiv -\,{\rm Tr}\rho_{\rm eff}\ln\rho_{\rm eff} = -\sum_{\mathbf{k}}\sum_{l=0}^{\infty} \rho_{2l}(\mathbf{k})\ln\rho_{2l}(\mathbf{k}), \tag{5.23}$$

for the Boltzmann and effective von Neumann entropy of the truncated density matrix, where $\rho_{2l}(\mathbf{k})$ is given by Eq. (5.24) below. The evolution of these two quantities for a typical solution of the mean field equations is shown in Fig. 10. Both display general increase due to continuous creation of massless Goldstone particles near threshold [24]. Neither quantity is a strictly monotonic function of time and neither obeys a strict Boltzmann H theorem. Since the particle modes f_k interact with the mean field χ but not directly with each other, the effective damping observed is certainly *collisionless*, and the dephasing here is similar to that responsible for Landau damping of collective modes in classical electromagnetic plasmas. The entropy $S_{\rm eff}$ of the effective density matrix provides a precise measure of the information lost by treating the phases as random. The Boltzmann ''entropy'' would be expected to equal $S_{\rm eff}$ only in true thermodynamic equilibrium, which is not achieved in the collisionless approximation of Eqs. (2.27). Notice the nonthermal distribution of particles in Fig. 9. In this nonthermal distribution we see from Fig. 10 that the Boltzmann entropy S_B generally overestimates the amount of information lost by phase averaging.

To the extent that the phase information in the off-diagonal matrix elements of ρ is irretrievable the system has become effectively classical, in the sense that the quantum interference effects present in the original ensemble represented by ρ are washed out as well. In that case we might as well regard the ensemble represented by the diagonal, truncated effective density matrix in the adiabatic number basis as a *classical* probability distribution with the diagonal elements of $\langle\tilde{n}=2l|\rho|\tilde{n}=2l\rangle$ giving the classical probabilities of observing l pairs in the ensemble. This probability distribution is derived in Appendix B and given by

$$\rho_{2l}(\mathbf{k}) = \langle\tilde{n}_{\mathbf{k}}=2l|\rho|\tilde{n}_{\mathbf{k}}=2l\rangle\big|_{\substack{\sigma=1\\ \phi=\dot\phi=0}}$$
$$= \frac{(2l-1)!!}{2^l l!}\,{\rm sech}\,\gamma_{\mathbf{k}}\tanh^{2l}\gamma_{\mathbf{k}}, \tag{5.24}$$

where γ is the magnitude of the Bogoliubov transformation between the a and $\tilde{a}$ bases, given explicitly in terms of the mode functions by

$$\tilde{N}(k) = \sinh^2\gamma_k = \frac{|\dot f_k + i\omega_k f_k|^2}{2\hbar\omega_k}, \tag{5.25}$$

which depends only on $|\mathbf{k}|=k$ by spatial homogeneity. Sampling this distribution with random phases will yield typical classical field amplitudes which make up the distribution. Obtaining such typical classical fields in the ensemble can give us explicit realizations of the symmetry-breaking behavior of the system, as well as providing the starting point for the study of topological defects produced during the phase transition.

The sampling of the field configurations proceeds in several steps. For a fixed late time and Fourier wave number k we calculate the Bogoliubov transformation coefficient from the static n to time-dependent adiabatic particle number $\tilde{n}$ basis from Eq. (5.25). This gives a set of numbers $\rho_{2l}=\langle\tilde{n}=2l|\rho|\tilde{n}=2l\rangle$ normalized to unit total probability,

$$\sum_{l=0}^{\infty}\rho_{2l}=1 \tag{5.26}$$

and which typically fall off very rapidly with l so that only a finite number of the ρ_{2l} need be retained. Then we sample this distribution by drawing a random number q in the unit interval $[0,1]$. Looking at the table of ρ_{2l} and the partial sums,

$$Q_l = \sum_{l'=0}^{l}\rho_{l'}, \tag{5.27}$$

we find l such that

$$Q_{l-1} < q \leqslant Q_l \tag{5.28}$$

to determine $\tilde{n}=2l$ of this random drawing for the given value of k. We then write

$$\phi(r,t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_k}}(a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}+a_{\mathbf{k}}^{*}e^{-i\mathbf{k}\cdot\mathbf{x}})$$
$$\rightarrow \sqrt{\frac{V}{\pi^2 r}}\int_0^{\infty} dk\; k\,\sin kr\,\sqrt{\frac{\tilde{n}_k}{2\omega_k}}\cos\theta_k, \tag{5.29}$$

where we have performed the angular integrations in $d=3$ dimensions and written

$$a_k = \sqrt{\tilde{n}_k}\; e^{i\theta_k} \tag{5.30}$$

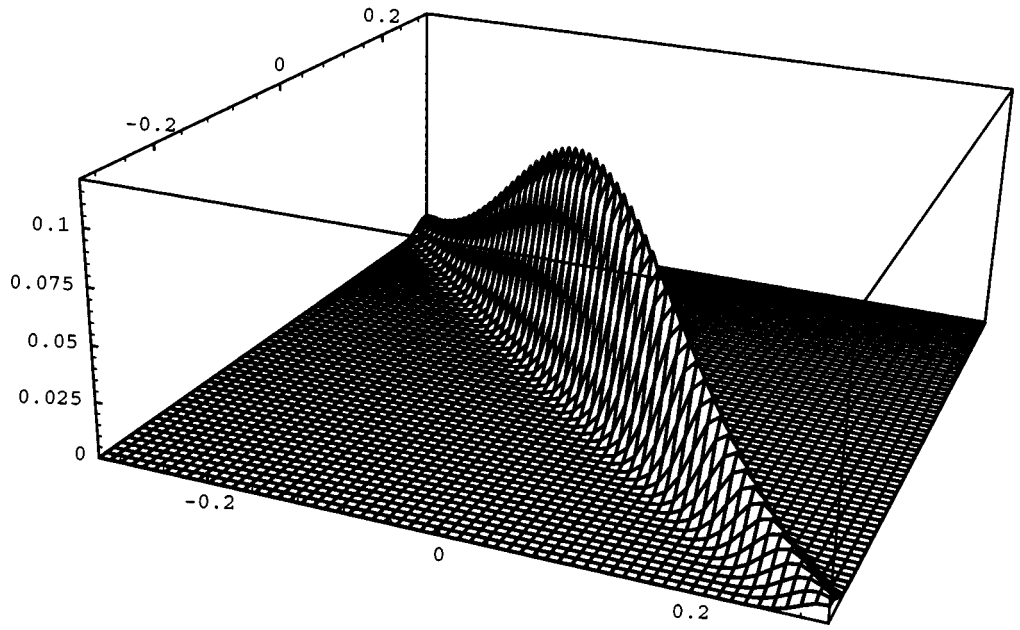

FIG. 12. The Gaussian $\rho_{\rm eff}$ for $k=0.4$ for the data and parameter values of Fig. 9 illustrating the strong suppression of off-diagonal components due to dephasing.

in terms of a random phase θ_k. After performing the Fourier transform in Eq. (5.29) the result is a typical field configuration as a function of radial r at the fixed time t. In this way we obtained the four field configurations shown in Fig. 11. We observe that although the mean field $\phi=0$ when averaged over the entire ensemble, typical field configurations in the ensemble are quite far from zero. In fact they sample values of ϕ between the two minima at v and $-v$. We observe as well that there is a typical correlation length in the classical field configurations which is of the order of the inverse momentum in which the power of the two-point function is distributed, as in Fig. 9.

It is clearly possible by such sampling techniques to generate typical field configurations with any number of components Φ_i in any number of spatial dimensions d. Then by appropriately matching the number of components to the number of dimensions we could search for different types of defectlike structures such as vortices, strings, and domain walls. This classical description of the quantum mean field theory is possible only when a classical decoherent limit exists through the diagonal density matrix in the adiabatic particle number basis. Moreover, the classical fields generated by this procedure are smooth and free of any cutoff-dependent short distance effects. The extraction of smooth classical field configurations from a quantum mean field description is a direct consequence of dephasing and the definition of defect number in these smooth configurations is free of the difficulties encountered when classical definitions of defects are applied uncritically to quantum field theories. It would be very interesting to pursue these ideas further in more realistic models of phase transitions where such topological defects are expected. This we leave for a future investigation.

The dephasing of the density matrix justifies the replacement of the exact Gaussian ρ by its diagonal elements only in the adiabatic number basis leading to the effective density matrix $\rho_{\rm eff}$. Now if this effective density matrix is transformed back into the coordinate basis, it corresponds to a density matrix of the original Gaussian form but with zero mean fields, zero momentum $\eta_k=0$, and $\widetilde{\sigma}_k\equiv 1+2\widetilde{N}(k)$ replacing σ; i.e., we obtain a product of Gaussians of the form

$$\langle \varphi'_{\mathbf{k}}|\rho_{\rm eff}|\varphi_{\mathbf{k}}\rangle=(2\pi\xi_k^2)^{-1/2}\exp\left\{-\frac{\widetilde{\sigma}_k^2}{8\xi_k^2}(\varphi'_{\mathbf{k}}-\varphi_{\mathbf{k}})^2 - \frac{1}{8\xi_k^2}(\varphi'_{\mathbf{k}}+\varphi_{\mathbf{k}})^2\right\}, \tag{5.31}$$

in all the $\mathbf{k}\neq 0$ modes where the mean field vanishes. This form shows that the Gaussian distribution off the diagonal $\varphi'_k=\varphi_k$ is strongly suppressed compared to the diagonal distribution. Indeed in the numerical evolutions we have found that $k\widetilde{N}(k)$ behaves typically like a *constant* for small k (cf. Fig. 9), so that $\widetilde{\sigma}_k\propto 1/k$. From Eq. (5.31) this implies that

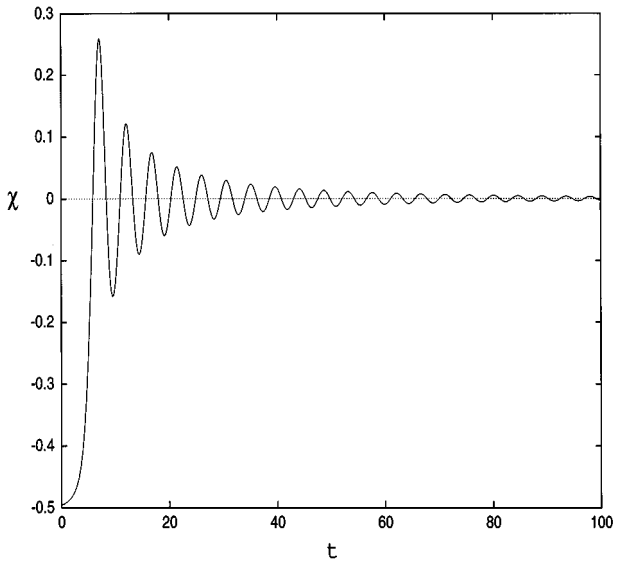

FIG. 13. Time evolution and effective damping of the χ mean field towards zero, its stationary value in the spontaneously broken phase.

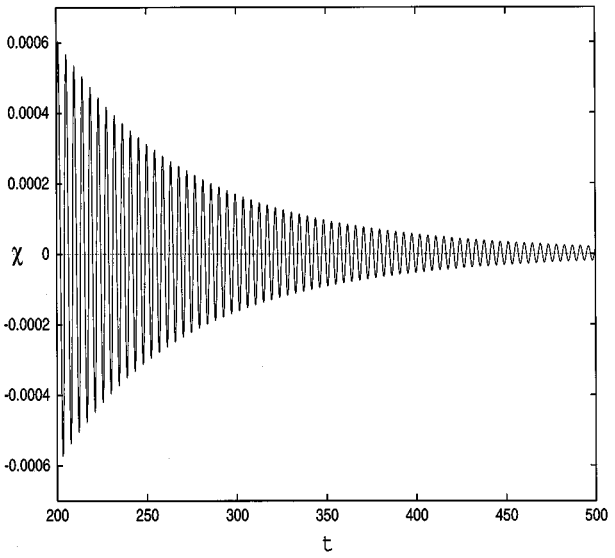

FIG. 14. Same evolution as in Fig. 13 but followed to large times and with an expanded scale for χ to show the long time behavior of the time-dependent Goldstone mass squared.

whereas the root mean square deviation from zero along the diagonal $\varphi'_{\mathbf{k}}=\varphi_{\mathbf{k}}$ is $\xi_k\cong\sqrt{\hbar N(k)/k}\propto 1/k$ and is large at small k, the root mean square deviation from zero in the orthogonal direction off the diagonal is

$$\frac{\xi_k}{\widetilde{\sigma}_k}\cong\sqrt{\frac{\hbar}{2k\widetilde{N}(k)}}, \tag{5.32}$$

which is much smaller. The result is that there is virtually no support in the distribution (5.31) for ''Schrödinger cat'' states in which quantum interference effects between the two classically allowed macroscopic states at v and $-v$ can be observed. Instead we see in a different way how dephasing produces an ensemble which may be regarded as a *classical* probability distribution over classically distinct outcomes (i.e., the diagonal field amplitude distribution) but with essentially no components in the off-diagonal quantum interference between these classical configurations. This is illustrated in Fig. 12. Hence even in this rather simple collisionless approximation the particle creation effects in the time-dependent mean field give rise to strong suppression of quantum interference effects and mediate the quantum to classical transition of the ensemble, at least for small wave numbers. This is in accordance with the usual intuition that long wavelength physics of a second order phase transition is essentially classical, where the adiabatic particle number $\widetilde{N}(k)$ has taken the place of $N(k)$ in the discussion following Eqs. (4.9) and (4.10) of the previous section.

When more realistic collisional interactions are included at the next and higher orders beyond the leading mean field limit we can expect this transition to decoherent classical behavior to be even more pronounced. As the Gaussian assumption is relaxed we expect the single peak at the origin to split into two peaks. Hence we can begin to see how a quantum phase transition leads to an effective classically broken symmetry in which large domains are in a definite classical ground state or another but not in a quantum superposition of ground states.

VI. LINEAR RESPONSE, PLASMA OSCILLATIONS, AND DAMPING RATE

The very efficient damping of the oscillations around the final state which we have studied in the density matrix formalism of the last section may also be understood as due to the continuous creation of massless Goldstone particles near threshold. In this appearance of strictly massless modes the spontaneously broken $\lambda\Phi^4$ model is qualitatively different from our previous studies of QED in the large N approximation [25]. In that case the charged particles are massive and there is finally a tunneling barrier which shuts off particle creation effects after the mean electric field has decreased below a certain critical value. Beyond this point the mean field undergoes essentially undamped plasma oscillations since there is no transfer of energy to created particles. In contrast, the present model has no such critical threshold in the mean field χ which is free to continue creating massless bosons for arbitrarily small amplitude and $\chi\to 0$ asymptotically. This very efficient asymptotic damping of the χ mean field towards its stationary spontaneously broken solution at $\chi=0$ is very well illustrated by the numerical results shown in Figs. 13 and 14 for $\lambda_\Lambda=1$ and an initial $\chi=-0.5$. We emphasize that the absence of a finite mass threshold is essential to the long time quasidissipative behavior observed in Figs. 13 and 14. In the infinite volume limit $V=L^3\to\infty$ the continuum of modes extends all the way to $k=0$ and this behavior persists to infinite time. In our simulations the length $L\sim 4\times 10^4$ which is much greater than t in these figures, so the continuum behavior is observed.

Since χ becomes very small at late times, it is possible to analyze the approach to zero by linear response methods. Given any static solution to the equations of motion with spontaneous symmetry breaking, as in Eq. (4.20), we may consider the real time linear response of the system away from this static solution. This is accomplished by linearizing the evolution equations in the deviations from the static solution

$$\phi\to\phi+\delta\phi,$$

$$\chi\to 0+\chi,$$

$$f_k\to f_k^{(\mathrm{vac})}(t)+\delta f_k(t), \tag{6.1}$$

with $f_k^{(\mathrm{vac})}$ given by Eq. (2.38), the mode functions corresponding to the static solution $\chi=0$.

The linearized mode equation

$$\left[\frac{d^2}{dt^2}+k^2\right]\delta f_k(t)=-\chi(t)f_k^{(\mathrm{vac})}(t) \tag{6.2}$$

may be solved by use of the retarded Green's function

$$G_R(t-t';k)=\frac{\sin k(t-t')}{k}\,\theta(t-t') \tag{6.3}$$

in the form

$$\delta f_k(t)=A_k f_k^{(\mathrm{vac})}(t)+B_k f_k^{(\mathrm{vac})*}(t) - \int_0^t dt'\ G_R(t-t';k)\chi(t') f_k^{(\mathrm{vac})}(t'), \quad (6.4)$$

where A_k and B_k are coefficients of the solutions to the homogeneous equation. Because the Wronskian condition (2.30) is maintained under the linearization, the A_k must have vanishing real part

$$\mathrm{Re}\ A_k=0. \quad (6.5)$$

The linearized ϕ equation gives, in the same way,

$$\delta\phi(t)=t\ \delta\dot\phi(0)+\delta\phi(0) - \int_0^t dt'\ G_R(t-t';k=0)\chi(t')\phi(t')\ . \quad (6.6)$$

At the same time the linearized gap equation reads

$$\chi=\lambda\phi\delta\phi+\lambda\int [d\mathbf{k}]\ \mathrm{Re}(\delta f_k^* f_k^{(\mathrm{vac})})[1+2N(k)]. \quad (6.7)$$

Upon substituting Eqs. (6.4)–(6.6) this becomes a linear integral equation for the perturbation, χ,

$$\chi(t)=-\lambda\int_0^t dt'\Pi(t-t')\chi(t')+\lambda B(t), \quad (6.8)$$

where

$$\Pi(t)=t\phi^2+\hbar\int [d\mathbf{k}]\frac{1+2N(k)}{2k^2}\sin kt\ \cos kt \quad (6.9)$$

is the polarization part in the static background and

$$B(t)\equiv t\phi\delta\dot\phi(0)+\phi\delta\phi(0) +\hbar\int [d\mathbf{k}]\frac{1+2N(k)}{2k}\mathrm{Re}(B_k e^{2ikt}) \quad (6.10)$$

depends only on the initial perturbation away from the static solution. The linearized integral equation (6.8) may be put in the form

$$\int_0^t dt' D^{-1}(t-t')\chi(t')=-B(t), \quad (6.11)$$

where D^{-1} is the χ inverse propagator function.

The most direct method of solving such integral equations is to make use of the Laplace transforms

$$\tilde\Pi(s)\equiv\int_0^\infty dt\ e^{-st}\Pi(t)=\frac{\phi^2}{s^2}+\hbar\int [d\mathbf{k}]\frac{1+2N(k)}{2k(s^2+4k^2)},$$

$$\tilde B(s)\equiv\int_0^\infty dt\ e^{-st}B(t) =\frac{\phi\delta\dot\phi(0)}{s^2}+\hbar\int [d\mathbf{k}]\frac{1+2N(k)}{2k}\mathrm{Re}\left(\frac{B_k}{s-2ik}\right),$$

$$\tilde\chi(s)\equiv\int_0^\infty dt\ e^{-st}\chi(t)=-\tilde D(s)\tilde B(s), \quad (6.12)$$

and $\tilde D(s)$, the simple algebraic reciprocal of the Laplace transform of $D^{-1}(t)$,

$$-\tilde D^{-1}(s)=\frac{1}{\lambda}+\tilde\Pi(s)=\frac{1}{\lambda}+\frac{\phi^2}{s^2}+\hbar\int [d\mathbf{k}]\frac{1+2N(k)}{2k(s^2+4k^2)}\ . \quad (6.13)$$

Both the bare coupling $\lambda=\lambda_\Lambda$ and the integral over $\mathbf{k}$ involving the $N(k)$-independent term are logarithmically divergent. They combine to give the ultraviolet finite contribution

$$\frac{1}{\lambda_R(s^2/4)}=\frac{1}{\lambda_\Lambda}+\frac{\hbar}{32\pi^2}\ln\left(\frac{4\Lambda^2}{s^2}\right) =\frac{1}{\lambda_R(m^2)}+\frac{\hbar}{32\pi^2}\ln\left(\frac{4m^2}{s^2}\right). \quad (6.14)$$

The renormalized sum rule (4.21) may be used in the $N(k)$-dependent term to secure, as well,

$$-\tilde D^{-1}(s)=\frac{1}{\lambda_R(s^2/4)}+\frac{v^2}{s^2}+\frac{\hbar}{2\pi^2} \times\int_0^\infty k\,dk\,N(k)\left(\frac{1}{s^2+4k^2}-\frac{1}{s^2}\right), \quad (6.15)$$

which is now independent of the static value of ϕ.

The function $\tilde D^{-1}(s)$ has a singularity at $s=0$ and, depending on the exact form of $N(k)$ near $k=0$ in general, a branch cut starting from the origin in the complex s plane which may be taken along the negative real s axis. Its physical origin is the zero mass Goldstone bosons propagating in the internal loop of the polarization Π in Fig. 15. The discontinuity across this cut arises from the nonzero probability for the time-dependent χ field to create Goldstone pairs with arbitrarily small spatial momentum k above the massless threshold. The function $\tilde D^{-1}(s)$ also may have one or more zeros in the left-half complex s plane. The zero at

$$s^2\cong\pm 4m^2\exp\left(\frac{32\pi^2}{\hbar\lambda_R(m^2)}\right)\to\pm\infty \quad (6.16)$$

for $\lambda_R\ll 1$ is the Landau ghost pole in the far ultraviolet which lies outside the range of validity of the large N expansion and which in any case does not affect the long time behavior of the inverse Laplace transform.

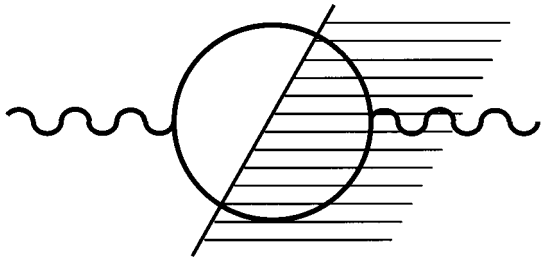

FIG. 15. One-loop diagram which contributes to the polarization $\Pi=-iGG/2$ in the leading order large N approximation. In the infinite volume continuum limit, the cut extends to $k=0$.

In the special case of vacuum initial particle density $N(k)=0$, the integral in Eq. (6.15) vanishes and $\tilde{D}_0^{-1}(s)$ is given by

$$-\tilde{D}_0^{-1}(s)=\frac{1}{\lambda_R(s^2/4)}+\frac{v^2}{s^2}\,. \tag{6.17}$$

Because of the logarithmic branch cut in Eq. (6.14), this function possesses an imaginary part when analytically continued to complex s with Re $s<0$. Taking this into account, $\tilde{D}_0^{-1}(s)$ possesses a zero at complex $s_\pm$ in the left-hand complex plane

$$s_\pm=\pm i\omega_{\rm pl}-\gamma, \tag{6.18}$$

which in the limit of weak coupling is given by

$$\omega_{\rm pl}^2\cong\lambda_{\rm pl}v^2,\quad \gamma\cong\frac{\hbar\lambda_{\rm pl}}{64\pi}\,\omega_{\rm pl}=\frac{\hbar\lambda_{\rm pl}^{3/2}}{64\pi}v, \tag{6.19}$$

where

$$\lambda_{\rm pl}=\lambda_R\left(\frac{|s|^2}{4}\cong\frac{\omega_{\rm pl}^2}{4}\right), \tag{6.20}$$

for $\lambda_{\rm pl}\ll 1$. In the frequency $\omega_{\rm pl}$ the effective mass of the radial mode, ignored in the direct quantization of the $N-1$ massless modes in Eq. (2.24), reappears in the large N limit as an oscillation in the real time linear response to perturbations about the vacuum $\phi=v$, $\chi=0$. It may be viewed either as this radial degree of freedom of the N component Φ_i field, with its effective mass dressed by the polarization effects of Π in the presence of massless Goldstone modes, or alternately and just as correctly as a genuine collective excitation of the composite field χ. The two descriptions are equivalent since the oscillations of χ and ϕ are constrained by the linearized equations (6.6) and (6.7), and there is only one degree of freedom between them. The oscillation frequency (6.19) should be compared with the second derivative of the free energy potential at $T=0$ which vanishes from Eq. (4.12). Thus the ''effective'' potential is completely ineffective at predicting the radial oscillation frequency at zero temperature. It is clear that at zero temperature the origin of the decay rate γ is the imaginary part of the two-particle cut in the polarization diagram illustrated in Fig. 15.

In the case of a thermal distribution of massless bosons Eq. (3.15), the integral in (6.15) may be performed in closed form in terms of the digamma function ψ and we find

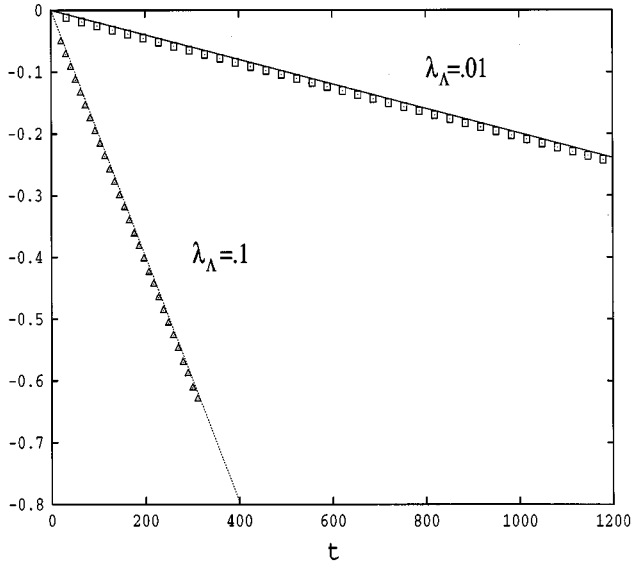

FIG. 16. The function y_t defined in Eq. (6.26) plotted for perturbed thermal initial conditions at two values of λ_Λ. The solid lines are the corresponding theoretical predictions.

$$\begin{aligned}-\tilde{D}_T^{-1}(s)&=\frac{1}{\lambda_R(s^2/4)}+\frac{\phi_T^2}{s^2}\\&\quad+\frac{\hbar}{16\pi^2}\left[\ln\left(\frac{\hbar s}{4\pi T}\right)-\frac{2\pi T}{\hbar s}-\psi\left(\frac{\hbar s}{4\pi T}\right)\right]\\&=\frac{1}{\lambda_R(4\pi^2T^2/\hbar^2)}+\frac{\phi_T^2}{s^2}-\frac{\hbar}{16\pi^2}\\&\quad\times\left[\frac{2\pi T}{\hbar s}+\psi\left(\frac{\hbar s}{4\pi T}\right)\right]\end{aligned} \tag{6.21}$$

for Re $s>0$ and $T\leqslant T_c$. At finite temperature the analytic structure of $\tilde{D}_T^{-1}(s)$ in the complex plane with Re$s<0$ is different from the zero temperature case. There is no logarithmic branch cut beginning at $s=0$, but rather a sequence of simple poles at $s_n=-4\pi Tn/\hbar$, $n=0,1,2,\ldots$, along the negative real s axis. Since s is a Laplace transform variable, these poles correspond to the Matsubara frequencies of the two-particle intermediate states in Π in the finite temperature imaginary time formalism. Only in the zero temperature limit do the poles coalesce into a logarithmic branch cut. Indeed, the digamma function of small argument behaves like

$$\psi(z)\equiv\frac{d\ln\Gamma(z)}{dz}\to-\frac{1}{z}-C\quad\text{as}\quad z\to 0, \tag{6.22}$$

where $C=0.577\,215\ldots$ is Euler's constant, so that we see from the second form of Eq. (6.21) that the logarithms cancel and the discontinuity vanishes, leaving a simple pole structure

$$-\tilde{D}_T^{-1}(s)\to\frac{1}{\lambda_R(T'^2)}+\frac{\phi_T^2}{s^2}+\frac{T}{8\pi s} \tag{6.23}$$

as $s\to 0$, with $T'\equiv 2\pi e^{-C}T/\hbar$. Since the analytic structure of the function $\tilde{D}_T^{-1}(s)$ near $s=0$ is now explicit,

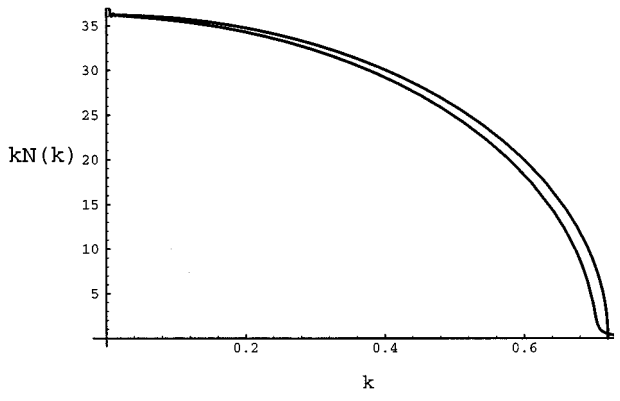

FIG. 17. Comparison of the analytic approximation to the particle number density of Eq. (6.30) (upper curve) with the adiabatic number density of the numerical calculation (bottom curve).

we can freely continue this form to $\mathrm{Re}s<0$, and find the dominant behavior of the inverse Laplace transform at late times by setting the quadratic form (6.23) equal to zero. In this way we obtain a pair of isolated poles of $\widetilde{D}_T(s)$ at $s_\pm = \pm i\omega_{\mathrm{pl}\,T} - \gamma_T$ as in Eq. (6.18) with

$$\omega_{\mathrm{pl}T}^2 \cong \lambda_R(T'^2)v^2\left[1-\frac{T^2}{T_c^2}\left(1-\frac{3\hbar\lambda_R(T'^2)}{64\pi^2}\right)\right],$$

$$\gamma_T \cong \frac{\lambda_R(T'^2)}{16\pi}\,T\,. \tag{6.24}$$

Since we have used Eq. (6.22) for the behavior of the digamma function near the origin, this formula is valid iff $|s_\pm| \ll T$, which is satisfied for weak coupling $\hbar\lambda_R/16\pi \ll 1$ and for temperatures

$$T^2 \gg \frac{\hbar\lambda_R(T'^2)}{192\pi^2}T_c^2 = \frac{\hbar^2\lambda_R(T'^2)}{16\pi^2}v^2\,. \tag{6.25}$$

We cannot take Eq. (6.24) over to the zero temperature case (6.19) since opposite limits of the digamma function are involved, and $T\to 0$ is not permitted in Eq. (6.24) without violating the assumption $|s_\pm| \ll T$. However, if we take T down to the minimum value (6.25) for which Eq. (6.24) can be valid (and the contribution of the nearby poles in the digamma function at s_n for $n=1,2,\ldots$ becomes important), we find $\omega_{\mathrm{pl}T}^2 \approx \lambda_R v^2$ and $\gamma_T \approx \hbar\lambda_R^{3/2}v/64\pi$ with the renormalized running coupling λ_R evaluated at $\omega_{\mathrm{pl}T}^2 e^{-2C}/4$ which agrees very nearly with the zero temperature results (6.19), up to small corrections of order $\hbar\lambda_R(T'^2)/16\pi^2$.

Hence the high temperature and zero temperature forms of the plasma frequency and damping rate match quite smoothly at the boundary of their domains of validity in the weak coupling limit. Notice also that as $T\to T_c$ from below the plasma frequency, $\omega_{\mathrm{pl}T}\to\lambda_R T_c/16\pi$ becomes the same as the damping rate γ_{T_c}. For $T>T_c$ the symmetry is restored and the propagator becomes massive so that formulas (6.13), (6.21), and (6.24) cease to apply. Finally we reiterate that $\widetilde{D}_T^{-1}(s)$ describes the real time oscillations of the ϕ mean field coupled to the collective plasma oscillations of the auxiliary field χ, and that the equilibrium free energy ''effective'' potential F has no such real time information at either zero or finite temperature, as conclusively demonstrated by Eq. (4.12). Finally notice that this calculation relies on the behavior of $\widetilde{D}_T^{-1}(s)$ near $s=0$, which relies in turn upon the behavior of the integrand in Eq. (6.15) near $k=0$. If the volume V is not taken to infinity first this integral is replaced by a sum, the $k=0$ mode must be handled differently, and the long time behavior of the Laplace transform will be quite different.

The linear response predictions for the oscillation frequency and the damping rate in the case of finite temperature were compared with a numerical evolution with $\lambda_\Lambda = 0.01$ and $T=1$ (in units of $\hbar$ and $v=1$). The theoretical predictions of $\omega_{\mathrm{pl}T}=0.095\,74$ and $\gamma_T = 1.9894\times 10^{-4}$ are in very good agreement with the numerical results $\omega_{\mathrm{pl}T}=0.095\,85$ and $\gamma_T = 2\times 10^{-4}$. A similar test was carried out at a larger value of the coupling $\lambda_\Lambda = 0.1$ again with excellent results: predictions of $\omega_{\mathrm{pl}T}=0.3028$ and $\gamma_T = 1.9894\times 10^{-3}$ compared with numerical values of $\omega_{\mathrm{pl}T}=0.3031$ and $\gamma_T = 1.9944\times 10^{-3}$. The extraction of the behavior of the χ envelope from the numerical data is described below.

Instead of plotting χ directly as a function of time, information may be extracted more conveniently by plotting

$$y_t \equiv \frac{\ln[\chi(t+\epsilon)]-\ln[\chi(t)]}{\ln(t+\epsilon)-\ln(t)} \tag{6.26}$$

as a function of time, where each time point t is taken at either a peak or a trough of the oscillation and ϵ is the time separation between two such neighboring points. At late times, if the frequency stabilizes, ϵ goes to a constant. Using this fact, along with the late time condition $t \gg \epsilon$, one can show from Eq. (6.26) that if the χ oscillation envelope can be fit by an expression of the form

$$At^{-\alpha}\exp(-\gamma t), \tag{6.27}$$

then, at late times,

$$y_t = -\gamma t - \alpha, \tag{6.28}$$

enabling the direct reading off of the power α and the exponent γ as the y intercept and slope of a straight line, respectively. It is important to note that this method is also a test of whether the frequency is stable as otherwise a straight line will not be obtained. The behavior of y_t against time is shown in Fig. 16 for the two values of the coupling constant mentioned above.

When the distribution $N(k)$ is nonthermal the behavior of $\widetilde{D}^{-1}(s)$ is more difficult to analyze, and there is no guarantee that it possesses a zero in general. If $N(k)$ is peaked at small k, then a reasonable first approximation is to neglect the last term in Eq. (6.15). In that case we find a zero of $\widetilde{D}^{-1}(s)$ at with

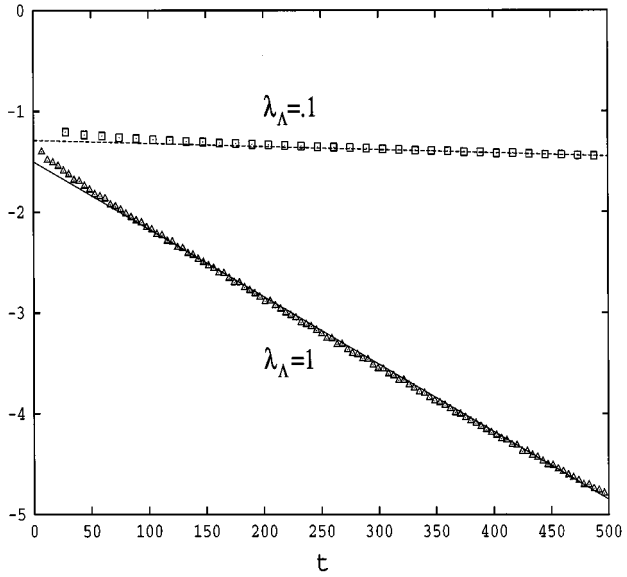

FIG. 18. The function y_t plotted for nonthermal initial conditions at two values of λ_Λ. The late time linear fits to the data are also shown with the intercepts determining the power law prefator.

$$\omega_{\rm pl}^2 \approx \lambda_{\rm pl} v^2,$$

$$\gamma \approx \frac{\lambda_{\rm pl}}{64\pi}\omega_{\rm pl} = \frac{\lambda_{\rm pl}^{3/2} v}{64\pi}. \tag{6.29}$$

If the running renormalized λ_R is not very small and/or $N(k)$ is not sharply peaked at $k=0$, then the position of this zero of $\tilde{D}^{-1}(s)$ will be shifted from Eq. (6.29), and at some values it may even cease to exist. Comparing with our numerical data for which $k\tilde{N}(k)$ is nonthermal as shown in Fig. 9, we observe that the approximations leading to the estimates (6.29) can only be order of magnitudes at best. Indeed

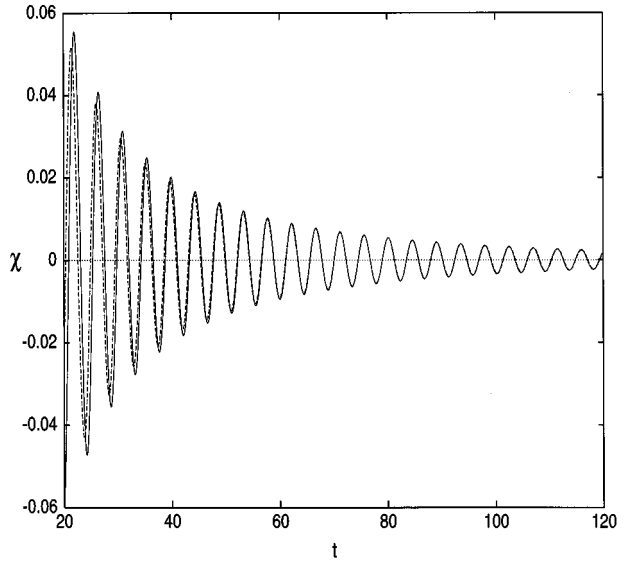

FIG. 19. The χ evolution as a function of time (dashed line) for $\lambda_\Lambda = 1$. The fit (6.27), shown by the solid curve, is $\chi(t) = 6.63 t^{-1.5}\exp(-0.0067t)\cos(1.405t + 0.5)$. This figure shows the fit over an early time range between $t=20$ and $t=120$.

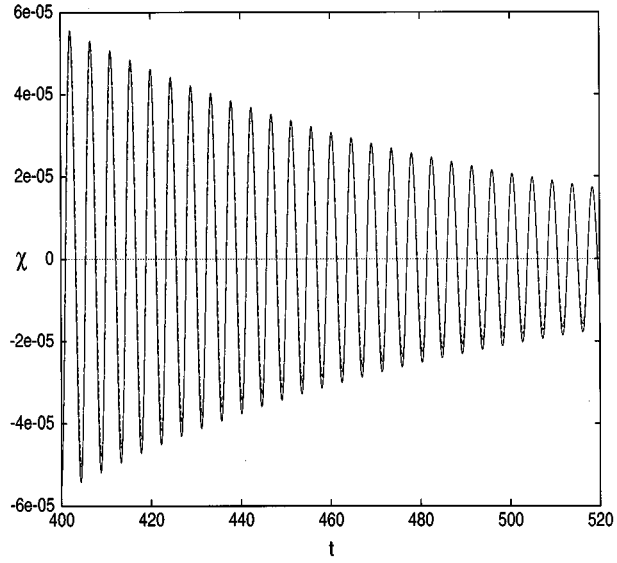

FIG. 20. The same χ evolution at late times between $t=400$ and $t=520$ showing the excellence of the simple fit over long times and along with the results of Fig. 19, very good agreement over three decades of amplitude.

the numerical value of the plasma frequency from the data at late times is $\omega_{\rm pl} = 1.405$ whereas Eq. (6.29) gives $\omega_{\rm pl} \approx 1$. The damping rate is estimated to be $\gamma = 0.005$ (or 0.007 if the correct $\omega_{\rm pl}$ is used); however, the envelope of the oscillations at late times is not even strictly exponential as we shall see below. Hence we cannot expect to obtain an accurate estimate of the damping rate by a simple pole analysis of the Laplace transform $\tilde{D}(s)$. Nevertheless, if one fits an exponential to the data at late times, a value $\gamma = 0.009$ is obtained, agreeing only roughly with the estimate (6.29).

Despite the envelope of the oscillations being nonexponential, $\omega_{\rm pl}$ is very well determined by the regular oscillations observed (to a few parts in 10^3). To obtain a more accurate approximation to the plasma frequency consider the following analytic approximation to $kN(k)$:

$$kN(k) = 8\pi\omega_0\,\theta(\omega_0^2 - 4k^2)\sqrt{1 - \frac{4k^2}{\omega_0^2}}, \tag{6.30}$$

where ω_0 is a free parameter which is close to the numerical value of the plasma frequency. The accuracy of this analytic fit to the data is shown in Fig. 17 for $\omega_0 = \omega_{\rm pl}$. Notice that because of dephasing, the $N(k)$ used in the linear response analysis may be identified with the $\tilde{N}(k)$ of the particles created at earlier times in the nonlinear evolution from the spinodal region.

Substituting the form (6.30) into Eq. (6.15) we obtain

$$-\tilde{D}^{-1}(s) = \frac{1}{\lambda_R(s^2/4)} + \frac{v^2}{s^2} + \frac{2\omega_0^2}{\pi}\int_0^1 dx (1-x^2)^{1/2} \times \left[\frac{1}{\omega_0^2 x^2 + s^2} - \frac{1}{s^2}\right]. \tag{6.31}$$

Because of the small prefactor of the logarithm in $1/\lambda_R(s^2/4)$, the running coupling constant is only a few per-

cent different from the bare λ near the zero of $\tilde{D}^{-1}(s)$ and this running with s can be neglected in lowest order. Again we set $s=\pm i\omega_{\rm pl}-\gamma$ and look for a zero of $D^{-1}(s)$ near $\omega_0^2=\omega_{\rm pl}^2$, with $\gamma/\omega\ll 1$. One could perform the integral in Eq. (6.31) exactly but we content ourselves with this simple approximation and obtain

$$0\cong\frac{1}{\lambda_{\rm pl}}-\frac{v^2}{\omega_0^2}+\frac{2}{\pi}\int_0^1 dx[(1-x^2)^{1/2}-(1-x^2)^{-1/2}]. \tag{6.32}$$

Performing the integral we find, from the real part of this equation,

$$\frac{\omega_{\rm pl}^2}{v^2}=\frac{2\lambda_{\rm pl}}{2-\lambda_{\rm pl}}. \tag{6.33}$$

In our simulations where $\lambda_\Lambda=1\simeq\lambda_{\rm pl}$, the output value of the calculated plasma oscillation frequency $\omega_{\rm pl}=1.414$ is within 0.6% of the measured value $\omega_{\rm pl}=1.405$, showing the self-consistency of the approximation and the fit (6.30), to this accuracy. We also performed the integration in Eq. (6.15) numerically using the data of Fig. 9 for $k\tilde{N}(k)$ and reconfirmed $\omega_{\rm pl}=(1.405\pm 0.001)$ from the location of the minimum of the real part of $\tilde{D}^{-1}(s)$.

The envelope of the χ oscillations obtained numerically is very well fit by $t^{-\alpha}\exp(-\gamma t)$. This general behavior has been checked for different values of the bare coupling and we have verified that it does not depend on which choice of the ω_k profiles (C4) and (C5) one makes for the initial conditions. Illustrated in Fig. 18 are the $\lambda_\Lambda=1$ and $\lambda_\Lambda=0.1$ cases, where $\alpha=1.5$, $\gamma=0.0067$ and $\alpha=1.288$, $\gamma=0.000\,32$, respectively. Figures 19 and 20 display the complete fit to the damped oscillation alongside the actual evolution for the case $\lambda_\Lambda=1$. Very good agreement is evident over long times and a large amplitude range. The case of $\lambda_\Lambda=0.1$ is very similar.

As expected from the time dependence of the envelope of the oscillations observed in Figs. 18, 19, and 20, the damping rate cannot be obtained from a simple pole analysis. Indeed the imaginary part of $\tilde{D}^{-1}(s)$ evaluated from the numerical data does not possess a zero for *any* γ with $\omega_{\rm pl}$ fixed at 1.405, although it does decrease monotonically as γ is decreased, thus favoring $\gamma<0.003$. This value is close to our momentum resolution dk and hence we could not go to lower values reliably. In any case, the decay of the envelope at late times is due to the creation of very low momentum massless bosons which have not yet dephased efficiently. Hence it is the behavior of the Laplace transform on the branch cut very close to $s=0$ which determines the late time damping, and this behavior depends in turn on the the mode functions $|f_k(t)|^2$ near $k=0$ where neither our replacement (5.21) nor our single-pole-dominance assumptions are justified.

In earlier studies of damping a $-\frac{3}{2}$ power law was encountered and identified as due to the behavior of the relevant spectral density function near threshold [26]. In the present case there is no simple argument that we are aware of which would lead to such a power law at late times.

VII. SUMMARY

In this paper we have presented a study of nonequilibrium evolution and time-dependent behavior of symmetry-breaking transitions in N-component $\lambda(\Phi^2)^2$ field theory. Starting from an effective action principle for the leading order mean field approximation we emphasized the correspondence of the equations with the Schrödinger evolution of a Gaussian density matrix according to a certain effective classical Hamiltonian. This is important for emphasizing that no fundamental irreversible behavior has been introduced by the mean field approximation. Explicit numerical time evolutions from the unstable spinodal region show effective time irreversibility in the form of the efficient averaging to zero of the phase information in the density matrix. The effective von Neumann entropy of the reduced density matrix measures the information that is lost by discarding this phase information. Its general increase with particle creation shows the close connection between dephasing and irreversibility. Thus our study of effective dissipation and decoherence casts some light on fundamental issues in quantum statistical mechanics such as the origin of irreversibility, Boltzmann's H theorem, and the quantum to classical transition. Here there is much that could be done still in the context of the framework presented in this paper, most notably to study quantitatively the Poincaré recurrence cycle(s) expected in a Hamiltonian system and their dependence on the various parameters of the model.

In the specific case at hand, O(N)-symmetric $\lambda(\Phi^2)^2$ scalar field theory is a prototype of models of spontaneous symmetry breaking in a wide variety of physical systems. It has been common to rely heavily on the thermodynamic free energy and equilibrium considerations generally in analyzing these systems. One broad and generally applicable conclusion of the present study is that this can be very misleading where nonequilibrium dynamics is concerned. We have shown that the distribution of particles created in the mean field time evolution from an initial configuration in the spinodal region is generally far from thermal. This leads to a final state in which the mean field need not be close to minimum of the free energy potential. The simple observation that there is a sum rule which the nonthermally distributed particles can saturate is sufficient to resolve this seemingly paradoxical result.

The linear response analysis of the oscillations about the allowed stationary configurations is also a new result which is quite different from what consideration of the free energy function alone might lead one to expect. We have shown that there is a collective plasma mode in the radial symmetry breaking direction whose characteristics depend on the ambient particle distribution. This also should be generally true of the various systems described by the same $(\Phi^2)^2$ field theory. While there is excellent agreement in the thermal case for both the plasmon frequency and damping rate, the nonthermal situation is more complicated. In this case, due to the lack of pole dominance, the late time behavior cannot be described by exponentially damped oscillations. Our numerical results for the plasmon oscillation envelope are well fit by an expression of the form $t^{-\alpha}\exp(-\gamma t)$. The plasmon frequency compares well with an analytical estimate. However, the precise dependence of the envelope function on the

particle density in the general case remains to be more fully investigated. We have outlined also how dephasing leads also to effectively classical field configurations which one can sample in order to extract information on the creation of topological defects during the phase transition. These classical field configurations in the Gaussian ensemble show evidence for a finite correlation length whose dynamics may be investigated in the present approach. This direction is certainly an interesting one to pursue both in the condensed matter and cosmological phase transitions.

Throughout the paper we have endeavored to bring the theoretical framework into close contact with practical numerical methods. Indeed one of the main conclusions of this work is that the real time dynamics of phase transitions can be studied in a concrete way with presently available computers. Besides the wide applicability of the spontaneously broken $\lambda(\Phi^2)^2$ theory it is interesting for the existence of massless Goldstone bosons which are created freely during the phase transition. This essential kinematics is shared by systems with an exact gauge symmetry such as QCD or general coordinate invariance such as gravity. The numerical techniques used in this work and the experience gained in treating the massless case should prove to be valuable in generalizations to these cases.

ACKNOWLEDGMENTS

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APPENDIX A: THE ENERGY-MOMENTUM TENSOR AND ITS RENORMALIZATION

The effective action (2.12) leads directly to a conserved energy-momentum tensor at any order of the large N expansion [3]. In this appendix we give some details about the energy-momentum tensor and its renormalization. Because of spatial homogeneity and isotropy, the only nontrivial components of this tensor are

$$\langle T_{00}\rangle=\varepsilon,$$

$$\langle T_{ij}\rangle=p\,\delta_{ij}, \tag{A1}$$

where the energy density ε and isotropic pressure p are given by

$$\begin{aligned}\varepsilon&=\frac{1}{2}\dot{\phi}^2+\frac{1}{2}\chi\phi^2-\frac{\chi}{\lambda}\left(\frac{\chi}{2}+\mu^2\right)\\&\quad+\frac{1}{2}\int[d\mathbf{k}]\sigma_k\{|\dot{f}_k|^2+(k^2+\chi)|f_k|^2\}\\&=\frac{1}{2}\dot{\phi}^2+\frac{1}{2\lambda_\Lambda}\chi^2\\&\quad+\frac{1}{4\pi^2}\int_0^\Lambda k^2\,dk\;\sigma_k(|\dot{f}_k|^2+k^2|f_k|^2)\end{aligned} \tag{A2}$$

and

$$\begin{aligned}p&=\frac{1}{2}\dot{\phi}^2-\frac{1}{2}\chi\phi^2+\frac{\chi}{\lambda}\left(\frac{\chi}{2}+\mu^2\right)\\&\quad+\frac{1}{2}\int[d\mathbf{k}]\sigma_k\left\{|\dot{f}_k|^2-\left(\frac{k^2}{3}+\chi\right)|f_k|^2\right\}\\&=\frac{1}{2}\dot{\phi}^2-\frac{1}{2\lambda_\Lambda}\chi^2\\&\quad+\frac{1}{4\pi^2}\int_0^\Lambda k^2\,dk\,\sigma_k\left(|\dot{f}_k|^2-\frac{k^2}{3}|f_k|^2\right).\end{aligned} \tag{A3}$$

We have used the gap equation (2.36) in passing to the latter expressions in each case.

The energy density ε contains a quartic but constant cutoff dependence (i.e., proportional to Λ^4). This is the expected vacuum energy contribution which should be subtracted. The conservation of the energy density,

$$\dot{\varepsilon}=0, \tag{A4}$$

is easily checked from the equations of motion, and is not affected by the subtraction of the constant quartic divergence. As it will turn out this single constant subtraction is sufficient to yield a cutoff-independent conserved energy density.

The cutoff dependence of the isotropic pressure p is more involved and requires a detailed understanding of how Lorentz invariance (or, more generally, coordinate invariance) is broken by the spatial momentum cutoff we have introduced in all the mode integrations. In the first forms of the two expressions (A2) and (A3), i.e., before using the gap equation, there appear mode integrals whose cutoff dependences correspond to those of a free theory with the mass m^2 replaced by $\chi(t)$. Divergences in the energy-momentum tensor expectation value $\langle T_{\mu\nu}\rangle$ have been studied in the literature by covariant methods such as dimensional regularization or covariant point splitting [27] with the result that these divergences must be proportional either to the metric tensor $g_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$ in flat Minkowski spacetime or the total derivative ''improvement term'' of Eq. (A12) below. Because we are using a regulator in the form of a cutoff in the spatial momentum integrations which is *not* covariant under Lorentz or general time-dependent coordinate transformations, the quartic and quadratic cutoff dependence we shall actually obtain will not have these covariant forms. Hence we shall have to perform the stress tensor renormalization in a noncovariant manner as well to correct for the spurious noncovariant Λ^4 and Λ^2 cutoff dependence, in order to obtain covariant results in the end.

The actual cutoff dependence in both ε and p is easily analyzed by means of an adiabatic expansion to the mode function equation (2.27), the lowest order solution of which is given by Eq. (2.34). By substituting this lowest order adiabatic approximation to the mode functions into the mode integrals in the first forms of Eqs. (A2) and (A3) we can characterize the most severe dependence on the ultraviolet cutoff Λ in the forms

$$\varepsilon_0 \equiv \frac{\hbar}{4\pi^2}\int_0^\Lambda k^2\, dk\, \sqrt{k^2+\chi}$$
$$= \frac{\hbar\Lambda^4}{16\pi^2} + \frac{\hbar\Lambda^2\chi}{16\pi^2} - \frac{\hbar\chi^2}{64\pi^2}\ln\left(\frac{4\Lambda^2}{\chi}\right) + \frac{\hbar\chi^2}{128\pi^2} + \mathcal{O}\left(\frac{\chi^3}{\Lambda^2}\right) \tag{A5}$$

and

$$p_0 \equiv \frac{\hbar}{12\pi^2}\int_0^\Lambda \frac{k^4\, dk}{\sqrt{k^2+\chi}}$$
$$= \frac{\hbar\Lambda^4}{48\pi^2} - \frac{\hbar\Lambda^2\chi}{48\pi^2} + \frac{\hbar\chi^2}{64\pi^2}\ln\left(\frac{4\Lambda^2}{\chi}\right) - \frac{7\hbar\chi^2}{384\pi^2} + \mathcal{O}\left(\frac{\chi^3}{\Lambda^2}\right). \tag{A6}$$

Clearly the Λ dependence is *not* such that $\varepsilon_0 = -p_0$, as required by considerations of general covariance. As has been known for some time the reason for this is that the spatial momentum cutoff Λ acts as a noncovariant point splitting regulator would, giving terms in the regulated $\langle T_{\mu\nu}\rangle$ proportional to $\delta^i_{(\mu}\delta^j_{\nu)}$ in which the spatial directions $i,j=1,2,3$ are distinguished. Since such terms do not appear in the $\mu=\nu=0$ time component, the energy density has the correct Λ dependence and requires no covariantizing correction. However, for the pressure we should have

$$p_0' \equiv -\varepsilon_0 \tag{A7}$$

on grounds of general covariance. This is easy to enforce *by hand* by adding the difference $p_0'-p_0=-\varepsilon_0-p_0$ to p_0 above which just corrects for the noncovariant terms induced by our momentum cutoff.

With this prescription to enforce covariance of the mode integrals in the pressure, the quartic subtraction required on both ε and p is the removal of the cosmological vacuum energy $\hbar\Lambda^4/16\pi^2$, corresponding to a subtractive renormalization of the constant cosmological vacuum term. We are *not* allowed to subtract the subleading quadratic and logarithmic divergences of ε appearing in Eq. (A5), since, for one thing, they are multiplied by the time-dependent function χ and hence would spoil energy conservation, and for another, such time-dependent terms do not correspond to a cosmological constant counterterm. Instead we recognize them to be just the correct cutoff-dependent terms needed to combine with the cutoff-dependent terms in the ''classical'' energy density

$$-\frac{\chi}{\lambda_\Lambda}\left(\frac{\chi}{2}+\mu_\Lambda^2\right) = -\frac{\chi^2}{2}\left\{\frac{1}{\lambda_R(m^2)} - \hbar\,\frac{1}{32\pi^2}\ln\left(\frac{4\Lambda^2}{m^2}\right)\right\} - \chi\left(\frac{v^2}{2}+\frac{\hbar\Lambda^2}{16\pi^2}\right) \tag{A8}$$

in order to render the total energy density cutoff independent. Because the analogous terms in p are precisely the negative of those appearing in ε after the correction $p_0'-p_0$ has been added to its quantum part, we also obtain a (partially) renormalized pressure by the same manipulations.

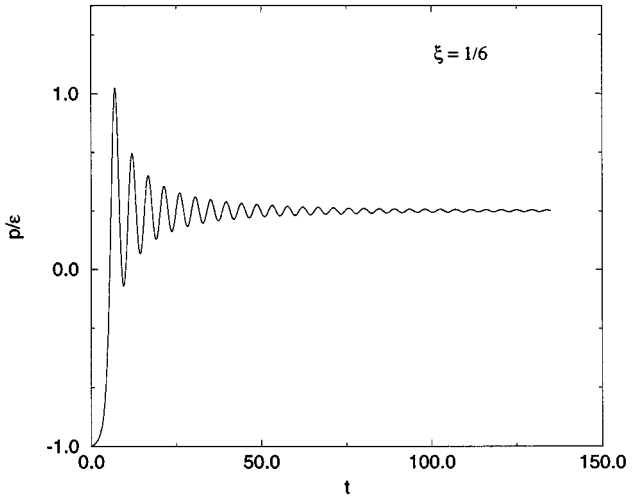

FIG. 21. Evolution of the pressure as a function of time for $\xi=1/6$. The approach to the equation of state $p=\varepsilon/3$ is clearly seen.

The reason that we must still perform one additional subtraction to obtain a fully renormalized pressure is that at the next order in the WKB adiabatic expansion of the mode equation (2.27), one obtains, for the adiabatic frequency,

$$\omega_k \to \omega_k - \frac{1}{4}\frac{\ddot{\omega}_k}{\omega_k^2} + \frac{3}{8}\frac{\dot{\omega}_k^2}{\omega_k^3} + \cdots. \tag{A9}$$

If the energy density is calculated to this adiabatic order, one finds no additional cutoff dependence in ε, and so no additional subtractions are required for it. Indeed none are permitted consistent with the principle of general coordinate invariance. However, the pressure has an additional logarithmic cutoff dependence equal to

$$-\frac{\hbar}{96\pi^2}\,\ddot{\chi}\,\ln\left(\frac{4\Lambda^2}{\chi}\right). \tag{A10}$$

Since this divergence appears only in the pressure but not in the energy, it is consistent with the generally covariant and conserved form of the ''improvement term'' [28]

$$\Delta\langle T_{\mu\nu}\rangle = \xi(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)\,\langle\mathbf{\Phi}^2\rangle \tag{A11}$$

in flat space. This term has no effect on the energy density for spatially homogeneous mean fields but adds to the pressure the total derivative

$$\Delta p = -\xi\frac{d^2}{dt^2}\left\{\phi^2 + \int [d^3\mathbf{k}]\sigma_k|f_k|^2\right\} = -2\frac{\xi}{\lambda}\ddot{\chi}, \tag{A12}$$

in which ξ is an arbitrary parameter. The fact that a divergence such as Eq. (A10) appears in the pressure means that we should introduce ξ as a free bare parameter of the theory from the very beginning, on the same footing as the mass μ_Λ^2 and coupling λ_Λ, and allow for the possibility that the bare $\xi_\Lambda\neq 0$ will renormalize in general. Indeed this is known to be the case in $\lambda\Phi^4$ theory [29] and we have the renormalization condition

$$(\xi_\Lambda - \tfrac{1}{6}) = Z_\lambda^{-1}(\Lambda,m)[\xi_R(m^2) - \tfrac{1}{6}], \tag{A13}$$

where Z_λ^{-1} is the same logarithmic renormalization constant as that for the coupling constant appearing in Eqs. (2.40) and (2.41) of the text. Thus,

$$\frac{\xi_\Lambda}{\lambda_\Lambda}=\frac{1}{6\lambda_\Lambda}+\frac{1}{\lambda_R}\left(\xi_R-\frac{1}{6}\right) \tag{A14}$$

and the additional term in the pressure (A12) becomes

$$\begin{aligned}\Delta p&=-\frac{1}{3\lambda_\Lambda}\ddot{\chi}-\frac{2}{\lambda_R}\left(\xi_R-\frac{1}{6}\right)\ddot{\chi}\\&=+\frac{1}{96\pi^2}\ln\left(\frac{4\Lambda^2}{m^2}\right)\ddot{\chi}-\frac{2\xi_R}{\lambda_R}\ddot{\chi}\ .\end{aligned} \tag{A15}$$

Comparing the latter expression with Eq. (A10) we observe that the logarithmic cutoff dependence cancels in the sum $p+\Delta p$. Hence we *must* add the improvement term and renormalize ξ in precisely this way in order to obtain a completely cutoff-independent pressure.

To summarize this discussion of the renormalization of the energy and pressure we define

$$\varepsilon_R=\varepsilon-\frac{\hbar\Lambda^4}{16\pi^2} \tag{A16}$$

to be the cutoff-independent conserved energy density and

$$p_R=p-p_0-\varepsilon_0+\frac{\hbar\Lambda^4}{16\pi^2}+\Delta p \tag{A17}$$

to be the renormalized pressure of the Φ^4 theory for the spatially homogeneous mean fields considered in this paper.

Let us also remark that the value $\xi=1/6$ for the improvement term is also the value chosen for conformal invariance of the scalar theory in the massless limit. Indeed the trace of the renormalized energy-momentum tensor is

$$\langle T^\mu_\mu\rangle_R=-\varepsilon_R+3p_R=v^2\chi+\frac{\hbar\chi^2}{32\pi^2}+\frac{1}{\lambda_R}\left(\xi_R-\frac{1}{6}\right)\ddot{\chi} \tag{A18}$$

upon using Eqs. (A5), (A6), and (A15) above. At $\xi=1/6$ the last term vanishes, the first term is the renormalized classical trace of the energy-momentum tensor, and the second term is the one-loop quantum trace anomaly

$$\frac{1}{2\lambda_R^2}\beta(\lambda_R)\chi^2=-\frac{\chi^2}{2}\Lambda\frac{d}{d\Lambda}\left(\frac{1}{\lambda_\Lambda}\right)=\frac{\hbar\chi^2}{32\pi^2} \tag{A19}$$

in terms of the β function of the coupling constant λ.

We note that the trace of the renormalized energy momentum tensor vanishes (for any value of ξ) in the static spontaneously broken vacuum where $\dot{\phi}=\chi=0$ and Eq. (2.38) holds. This implies that the relativistic equation of state

$$p_R\to\tfrac{1}{3}\varepsilon_R \tag{A20}$$

holds at late times, independently of the number density distribution. The numerically obtained approach to this equation of state is shown in Fig. 21. We emphasize that this equation of state does not imply relaxation to thermal equilibrium since collisional effects are not yet taken into account by the leading order large N approximation, and the nonthermal nature of the late time state is apparent from the created particle distribution of Fig. 9.

APPENDIX B: THE GAUSSIAN DENSITY MATRIX ρ

In this second appendix we examine some of the properties of the general mixed-state Gaussian density matrix (3.16) and its time evolution. The discussion will be restricted mainly to the case of $d=0$ spatial dimensions to simplify the notation. Generalizations to $d>0$ are straightforward. First, the time evolution of the density matrix is unitary:

$$\rho(t)=U(t)\rho(0)U^\dagger(t),\quad U(t)=\exp\left(-i\int_0^t\mathbf{H}_{\rm osc}dt\right), \tag{B1}$$

where $\mathbf{H}_{\rm osc}$ is the time-dependent harmonic oscillator Hamiltonian,

$$\mathbf{H}_{\rm osc}(\mathbf{\Phi},\mathbf{P};t)\equiv\tfrac{1}{2}[\mathbf{P}^2+\omega^2(t)\mathbf{\Phi}^2], \tag{B2}$$

which preserves the Gaussian structure of ρ under time evolution, so that the Liouville equation

$$i\hbar\frac{\partial}{\partial t}\rho=[\mathbf{H}_{\rm osc},\rho] \tag{B3}$$

is satisfied without taking the trace.

It is not difficult to find the explicit form of the unitary operator $U(t)$ in the coordinate basis

$$\begin{aligned}\langle x'|U(t)|x\rangle&=[2\pi i\hbar v(t)]^{-1/2}\\&\times\exp\left\{\frac{i}{2\hbar v(t)}[u(t)x^2+\dot{v}(t)x'^2-2xx']\right\}\end{aligned} \tag{B4}$$

in terms of the two linearly independent solutions to the classical evolution equation

$$\left(\frac{d^2}{dt^2}+\chi^2(t)\right)\begin{pmatrix}u\\v\end{pmatrix}=0,$$

$$u(0)=\dot{v}(0)=1,\quad \dot{u}(0)=v(0)=0. \tag{B5}$$

This same $U(t)$ also evolves the quantum operators

$$\begin{aligned}\mathbf{\Phi}(t)&=U^\dagger(t)\mathbf{\Phi}(0)U(t)=\phi(t)+af(t)+a^\dagger f^*(t),\\\mathbf{P}(t)&=U^\dagger(t)\mathbf{P}(0)U(t)=p(t)+a\dot{f}(t)+a^\dagger\dot{f}^*(t).\end{aligned} \tag{B6}$$

Mathematically, the three Fock space bilinear operators aa, $a^\dagger a^\dagger$, and $a^\dagger a+aa^\dagger$ generate the Lie algebra of the symplectic group $\mathrm{Sp}(2)\cong\mathrm{SU}(1,1)\cong\mathrm{SL}(2,R)$ which is the group of homogeneous linear transformations of phase space $(\mathbf{\Phi},\mathbf{P})$ which preserves the antisymmetric classical Poisson brackets $\{\mathbf{\Phi},\mathbf{P}\}=1$, i.e.,

$$\begin{pmatrix} \mathbf{\Phi} \\ \mathbf{P} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{\Phi}' \\ \mathbf{P}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{\Phi} \\ \mathbf{P} \end{pmatrix}, \tag{B7}$$

with

$$ad - bc = 1. \tag{B8}$$

Since this is one condition on four real parameters, the group Sp(2) is a three-parameter group. If the a and $a^\dagger$ Heisenberg operators are appended to these three, the algebra again closes upon itself, forming a five-parameter group, the inhomogeneous metaplectic group IMp(2) [30]. The unitary evolution (B1), (B4) of the Gaussian density matrix (3.16) is an explicit representation of this group's action.

The form of the density matrix in the time-independent Heisenberg basis is quite easy to obtain from the form of the the transition matrix element $\langle x|n\rangle$ given by Eq. (5.7) of the text. By substituting the integral representation of the Hermite polynomials

$$H_n(x) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{dz}{z^{n+1}} e^{2xz - z^2}, \tag{B9}$$

where $\mathcal{C}$ is a closed contour around the origin of the complex z plane into the expression

$$\langle n'|\rho|n\rangle = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \langle n'|x'\rangle\langle x'|\rho|x\rangle\langle x|n\rangle, \tag{B10}$$

and interchanging the orders of integration we can perform the double Gaussian integral over the shifted vector $(x-\phi, x'-\phi)$ first. Using the standard formula

$$\int_{-\infty}^{\infty} d^2x\, e^{-x\cdot A\cdot x + B\cdot x} = \frac{\pi}{(\det A)^{1/2}} e^{B^T\cdot A^{-1}\cdot B/4}, \tag{B11}$$

with A the 2×2 matrix,

$$A = \frac{\sigma+1}{8\xi^2} \begin{pmatrix} 1+\sigma & 1-\sigma \\ 1-\sigma & 1+\sigma \end{pmatrix}, \tag{B12}$$

and B the column vector,

$$B = \frac{\sqrt{2\sigma}}{\xi} \begin{pmatrix} z \\ z'^* \end{pmatrix}, \tag{B13}$$

the exponent of the resulting expression simplifies considerably. Letting $z = e^{i\theta}$ and $z'^* = e^{-i\theta'}$ we are left with

$$\langle n'|\rho|n\rangle = \frac{2}{\sigma+1} \left(\frac{n'!n!}{2^{n'+n}}\right)^{1/2} \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{in'\theta'} \int_0^{2\pi} \frac{d\theta}{2\pi} \times e^{-in\theta} \exp\left\{2\left(\frac{\sigma-1}{\sigma+1}\right) e^{i(\theta-\theta')}\right\}. \tag{B14}$$

Expanding the last exponent in a Taylor series we find that only the terms with $n' = n$ survive with Eq. (5.8) the final result. As discussed earlier in Sec. VI, in this Heisenberg basis the density matrix is time independent and diagonal. This fact allows the writing of Eq. (B14) in an operator form:

$$\rho = \frac{2}{\sigma+1} \exp\left\{\ln\left(\frac{\sigma-1}{\sigma+1}\right) a^\dagger a\right\} \tag{B15}$$

as far as computing matrix elements in this Heisenberg basis is concerned.

By making a different IMp(2) group transformation it is also possible to diagonalize Eq. (B2) at any given time, bringing the quadratic Hamiltonian into the standard harmonic oscillator form $H_{\rm osc} = \hbar\omega(\tilde{a}\tilde{a}^\dagger + \tilde{a}^\dagger\tilde{a})/2$, with $\tilde{a}$ time dependent. This adiabatic particle basis is related to the time-independent Heisenberg basis by the relations

$$f = \alpha\tilde{f} + \beta\tilde{f}^*,$$

$$a = \alpha^*\tilde{a} - \beta^*\tilde{a}^\dagger + \kappa^*, \tag{B16}$$

with $\tilde{f}$ the adiabatic mode function defined by Eq. (2.34) of the text, and

$$\alpha = \frac{i}{\hbar}\tilde{f}^*(\dot{f} - i\omega f),$$

$$\beta = -\frac{i}{\hbar}\tilde{f}(\dot{f} + i\omega f),$$

$$\kappa = \frac{i}{\hbar}(\dot{\phi}f - \phi\dot{f}). \tag{B17}$$

The Bogoliubov coefficients α and β obey

$$|\alpha|^2 - |\beta|^2 = 1, \tag{B18}$$

and therefore may be expressed in terms of a real parameter γ and two phases

$$\alpha = \cosh\gamma e^{i\psi},$$

$$\beta = -\sinh\gamma e^{i(\psi-\theta)}. \tag{B19}$$

The density matrix in the adiabatic particle number basis can be expressed in terms of these time-dependent parameters at any given time t. The direct evaluation of the expression analogous to Eq. (B10) in the $\tilde{n}$ number basis is quite tedious, and is accomplished most rapidly by use of a coherent state basis as discussed in Ref. [31]. Making the identifications

$$\alpha = \frac{1}{S_{-+}^*} = \frac{S_{+-}}{1 - |S_{++}|^2},$$

$$\beta = \frac{S_{--}}{S_{-+}} = -\frac{S_{+-}S_{++}^*}{1 - |S_{++}|^2}, \tag{B20}$$

with

$$S_{+-} = S_{+-} = \operatorname{sech}\gamma e^{i\psi},$$

$$S_{++} = \tanh\gamma e^{i\theta},$$

$$S_{--} = -\tanh\gamma e^{2i\psi - i\theta}, \tag{B21}$$

and $\chi \to \gamma$ in the notation of Ref. [31], Eq. (5.32) of that work yields the desired matrix element in the case of zero mean field, namely,

$$\begin{aligned}\langle \tilde{n}'|\rho|\tilde{n}\rangle|_{\phi=0} &= 2e^{i(\tilde{n}'-\tilde{n})\theta/2}\left(\frac{\tilde{n}'!}{\tilde{n}!}\right)^{1/2}[(\sigma^2-1)^2-4\sigma^2\sinh^2 2\gamma]^{(\tilde{n}'+\tilde{n})/4} \\ &\times[\sigma^2+1+2\sigma\cosh 2\gamma]^{-(\tilde{n}'+\tilde{n}+1)/4} \\ &\times P^{(\tilde{n}-\tilde{n}')/2}_{(\tilde{n}+\tilde{n}')/2}\left(\left[1-\frac{4\sigma^2}{(\sigma^2-1)^2}\sinh^2 2\gamma\right]^{-1/2}\right), \end{aligned} \tag{B22}$$

where P_n^m is an associated Legendre polynomial.

The first important feature of this expression for our purposes is the phase factor when $\tilde{n}\neq\tilde{n}'$. If the exact mode function f is rewritten in the form

$$f(t)=\sqrt{\frac{\hbar}{2\Omega(t)}}\exp\left(-i\int_0^t dt'\,\Omega(t')\right), \tag{B23}$$

this phase

$$\begin{aligned}\theta &= \arg\left(\frac{\alpha}{-\beta}\right)=2\int_0^t dt'\,\omega(t') \\ &+\tan^{-1}\left(\frac{\dot{\Omega}\omega/\Omega}{\Omega^2-\omega^2+\dot{\Omega}^2/4\Omega^2}\right).\end{aligned} \tag{B24}$$

Thus, even in the adiabatic limit, where $\dot{\Omega}/\Omega^2 \ll 1$, the phase angle θ depends linearly on time and the off-diagonal elements of the density matrix (B22) are rapidly varying in time. On the other hand, the diagonal elements of Eq. (B22) are independent of this phase angle and consequently much more slowly varying functions of time. Indeed, in the case of zero mean field $\phi=0$, the adiabatic invariant W of Eq. (5.19) is

$$W=\tilde{N}-N=\sigma\sinh^2\gamma \tag{B25}$$

and $\langle\tilde{n}|\rho|\tilde{n}\rangle$ depends only upon σ (a constant) and γ:

$$\begin{aligned}\langle\tilde{n}|\rho|\tilde{n}\rangle &= 2[(\sigma^2-1)^2-4\sigma^2\sinh^2 2\gamma]^{\tilde{n}/2} \\ &\times[\sigma^2+1+2\sigma\cosh 2\gamma]^{-(2\tilde{n}+1)/4} \\ &\times P_{\tilde{n}}\left(\left[1-\frac{4\sigma^2}{(\sigma^2-1)^2}\sinh^2 2\gamma\right]^{-1/2}\right).\end{aligned} \tag{B26}$$

Now, if $\sigma>1$, the Legendre polynomial is not necessarily positive which means that we cannot interpret the diagonal matrix elements of the density matrix in the adiabatic particle number basis as a classical probability distribution in the general mixed state case. However, if $\sigma=1$, then the argument of the Legendre polynomial vanishes for any $\gamma\neq 0$. Since

$$P_{2l}(0)=(-1)^l\frac{(2l-1)!!}{2^l l!} \tag{B27}$$

for $\tilde{n}=2l$ even but $P_{2l+1}(0)=0$ for $\tilde{n}=2l+1$ odd, the diagonal matrix element (B26) simplifies considerably in the pure state case:

$$\langle\tilde{n}=2l|\rho|\tilde{n}=2l\rangle\big|_{\substack{\sigma=1\\ \phi=\dot{\phi}=0}}=\frac{(2l-1)!!}{2^l l!}\operatorname{sech}\gamma\,\tanh^{2l}\gamma, \tag{B28}$$

with the mean number of created particles,

$$\tilde{N}=\sinh^2\gamma=\frac{|\dot{f}+i\omega f|^2}{2\hbar\omega}, \tag{B29}$$

which are Eqs. (5.24) and (5.25) of the text. These results were reported in [7].

In the pure state case the even $\tilde{n}$ diagonal matrix elements of the density matrix are positive definite and may be interpreted as the probabilities for observing l uncorrelated particle pairs in the adiabatic particle basis. The odd $\tilde{n}$ diagonal matrix elements vanish since particles can only be created in pairs from the vacuum. Otherwise (i.e., for $\sigma>1$), the much more complicated and nonpositive definite expression (B26) shows that there is no simple classical probability interpretation for the density matrix. The restriction to zero mean fields $\phi=\dot{\phi}=0$ is not serious for spatially homogeneous backgrounds because of Eq. (3.31) which shows that all the mean fields vanish except for the $k=0$ mode. Hence, at late times all the $k>0$ modes may be treated classically if dephasing is effective and typical classical field configurations in the ensemble can be constructed as in Eqs. (5.29) and (5.30) of the text, provided the $k=0$ mode is excluded.

APPENDIX C: NUMERICAL METHODS

In order to solve the system of equations (2.20), (2.27), and (2.39) as an initial value problem, we have to specify the initial conditions of the mean field ϕ, its time derivative, and the mode function and its time derivative. Then we have to solve the gap equation (2.39) at the initial time t_0. In order to have a finite set of renormalized equations we have to choose the mode functions so that the high momentum modes coincide with the zeroth order adiabatic vacuum described by

$$f_k(t_0)=\frac{1}{\sqrt{2\omega_k(t_0)}}, \tag{C1}$$

$$\dot{f}_k(t_0)=\left[-i\omega_k(t_0)-\frac{\dot{\omega}_k(t_0)}{2\omega_k(t_0)}\right]f_k(t_0), \tag{C2}$$

with $\omega_k^2(t_0)=k^2+\chi(t_0)$. It is easy to verify that for initial conditions with $\dot{\phi}(t_0)=0$, it is also true that $\dot{\chi}(t_0)=0$ by inspecting the time derivative of the gap equation. This simplifies the form of $\dot{f}_k(t_0)$ to

$$\dot{f}_k(t_0)=-i\omega_k(t_0)f_k(t_0). \tag{C3}$$

For an initial state with positive square effective mass χ we could use the same form of the mode functions of Eq. (C1) also for the low momentum modes. However, if we wish to investigate the case of a ''quench'' within the unstable spinodal region by initial conditions with a negative $\chi(0)$, we

have to modify the initial values of the low momentum modes in order to avoid the singularity of $f_k(t_0)=1/\sqrt{2\omega_k}$ at $k^2=-\chi(t_0)$. We have used two different profiles of the frequency ω_k for the initial mode functions:

$$\omega_k^2(t_0)=k^2+\chi(t_0)\tanh\left(\frac{k^2+\chi(t_0)}{|\chi(t_0)|}\right), \tag{C4}$$

$$\omega_k^2(t_0)=k^2+\chi(t_0)\exp(-v^4/k^4). \tag{C5}$$

At large momentum these profiles coincide with the adiabatic vacuum frequency as required. In all cases, except for thermal initial conditions, the initial number of quasiparticles, $N(k)=0$.

Numerical simulations were performed on a massively parallel computer using a momentum grid with 32 000 modes. The upper cutoff was set at $\Lambda=5v$, implying a grid resolution

$$dk=\frac{2\pi}{L}=\frac{\Lambda}{32\,000}=0.000\,156. \tag{C6}$$

The field ϕ was scaled in units of v so that k is also given in units of v, χ in units of v^2, and the time t in units of v^{-1}. Most of the results discussed in the text are for a bare coupling constant $\lambda_\Lambda=1$ but other values were also investigated. The corresponding renormalized coupling constants at the renormalization point $|\chi(t_0)|$ are given by Eq. (2.40) with m^2 replaced by $|\chi(t_0)|$. With $\lambda_\Lambda=1$ and for $\phi(t_0)/v=0$, $\lambda_R=0.990\,036$, while for $\phi(t_0)/v=0.5$, $\lambda_R=0.990\,248$. The energy densities in these cases are $\varepsilon=0.1237$ and $\varepsilon=0.069\,78$, respectively.

The mode equations were stepped forward in time using a sixth order adaptive time step Runge-Kutta integrator. The time-steps were controlled by tracking the evolution of χ. Energy conservation to a few parts per 10^6 was achieved over the temporal range of typical evolutions.

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