BPS domain wall solutions in self-dual Chern-Simons-Higgs systems

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We study domain wall solitons in the relativistic self-dual Chern-Simons-Higgs systems by the dimensional reduction method to two-dimensional spacetime. The Bogomol'nyi bound on the energy is given by two conserved quantities in a similar way that the energy bound for Bogomol'nyi-Prasad-Sommerfield (BPS) dyons is set in some Yang-Mills-Higgs systems in four dimensions. We find the explicit soliton configurations which saturate the energy bound and their nonrelativistic counterparts. We also discuss the underlying $N=2$ supersymmetry. [S0556-2821(97)01410-0]

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I. INTRODUCTION

In past years the Abelian self-dual Chern-Simons-Higgs system with an appropriate potential has been studied extensively [1–3]. One of its interesting properties is that there is a Bogomol'nyi-type bound on the energy [4], which is saturated by the self-dual solitons which carry the fractional spin and satisfy the fractional statistics. The potential of this model has two degenerate vacua, implying the existence of topological domain walls interpolating between them. It turns out that the topological domain walls satisfy another Bogomol'nyi bound as a two-dimensional model, which is different from such a bound on three-dimensional self-dual solitons [2].

These domain walls which interpolate between the symmetric and asymmetric phases have been studied to understand the behavior of the rotationally symmetric solitons in the large charge limit. In this case the energy density is concentrated on a circular ring of large radius and its radial cross section resembles the domain wall [2]. However, in the symmetric phase there exist also rotationally symmetric nontopological solitons with vortices in the center, whose large charge and vorticity limit cannot be described exactly by the topological domain walls. This suggests to us a richer structure of domain walls.

On the other hand, recently some two-dimensional nonlinear σ models with appropriate potentials [5] have been shown to have many properties similar to the four-dimensional Yang-Mills-Higgs systems which admit the Bogomol'nyi-Prasad-Sommerfield (BPS) dyons solutions. In particular, a Noether charge and a topological charge are found to set the BPS-type bound on the energy [4,6]. Since these two-dimensional self-dual models have the underlying $N=2$ supersymmetry and so do our Chern-Simons-Higgs systems, our domain walls are also expected to have the BPS bound, which is more general than the Bogomol'nyi bound.

The goal of this paper is to explore the domain wall structure of the self-dual Chern-Simons-Higgs system, by taking the dimensional reduction of the model to two dimensions. We show that the resultant two-dimensional model has a BPS-like energy bound. Our model hence forms a new class of two-dimensional models with the BPS-like energy bound, similar to the four-dimensional Yang-Mills Higgs systems. We show that the domain wall configurations saturating the energy bound consist of topological and nontopological domain walls: topological domain walls interpolating the symmetric and asymmetric phases and nontopological domain walls residing in the symmetric phase.

The plan of this paper is as follows. In Sec. II, we introduce our model and find the BPS-type energy bound. We reduce the self-dual equations to a single ordinary nonlinear differential equation. In Sec. III, we solve the differential equation to find and investigate all possible domain wall solutions. In Sec. IV, we study the underlying $N=2$ supersymmetry. In Sec. V, we study the nonrelativistic limit and its solitons in the symmetric phase. In Sec. VI, we conclude with some remarks. In the Appendix, we study another bound on the energy functional, which works only for the topological domain walls, and discuss its relation to the BPS bound.

II. MODEL

We start with the self-dual Abelian Chern-Simons-Higgs system whose Lagrangian [1] is

$$\mathcal{L}_{\rm CSH}=\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+|D_\mu\phi|^2-\frac{1}{\kappa^2}\phi^2(|\phi|^2-v^2)^2, \tag{1}$$

where $D_\mu\phi=(\partial_\mu-iA_\mu)\phi$. There are two phases in the theory: the symmetric phase where $\phi=0$ and the antisym-

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metric phase where $\phi=v$. Elementary charged excitations in the symmetric phase have mass $\mu=v^2/\kappa$ and spin $s=4\pi/\kappa$, and in the broken phase there are charge neutral particles of mass 2μ. There are also self-dual anyonic solitons in the symmetric and asymmetric phases.

We consider the modes independent of the x^2 coordinate to get the dimensionally reduced Lagrangian in two-dimensional spacetime

$$\mathcal{L}=\kappa N F_{01}+|D_\mu\phi|^2-U(N,\phi), \tag{2}$$

where the potential $U(N,\phi)$ is

$$U(\phi,N)=N^2|\phi|^2+\frac{1}{\kappa^2}\phi^2(|\phi|^2-v^2)^2. \tag{3}$$

The mass dimension of the fields and parameters become $[\phi]=[v]=M^0$, $[A_\mu]=[N]=M$, and $[\kappa]=M^{-1}$. The kinetic part of N,A_μ is that of the so-called *BF* theory. To the original Lagrangian, one can add a θ term,

$$\mathcal{L}_\theta=\theta F_{01}, \tag{4}$$

which is possible only in two dimensions. It does not play any role classically and will be neglected in this paper.

The classical vacuum structure of this theory is determined by the zeroes of the potential $U(\phi,N)$. The symmetric phase $\langle\phi\rangle=0$ is infinitely degenerate as $\langle N\rangle$ can take any value N_0. In the symmetric phase there are charged scalar bosons of mass

$$m=\sqrt{N_0^2+\mu^2} \tag{5}$$

with $\mu=v^2/\kappa$ being the three-dimensional mass. The asymmetric phase is uniquely given as $\langle N\rangle=0$, $\langle\phi\rangle=v$ up to the gauge transformation. In the asymmetric phase, there are two kinds of neutral bosons of mass 2μ. Since all the vacua are degenerate, we expect that topological domain walls exist. (We call the domain walls residing in the symmetric phase nontopological, even though they are topological in a way as they interpolate between different N vacua.)

The Gauss law constraint from the variation of A_0 is

$$\kappa N'-i(D_0\phi^*\phi-\phi^*D_0\phi)=0, \tag{6}$$

where the prime is the derivative d/dx. The variation of N yields another constraint,

$$\kappa F_{01}-2N|\phi|^2=0, \tag{7}$$

which does not play any important role here.

The theory is invariant under the local gauge transformations

$$\delta\phi=i\Lambda(x)\phi, \quad \delta A_\mu=\partial_\mu\Lambda. \tag{8}$$

Its global part is the U(1) symmetry, whose conserved current is $j_\mu=i(D_\mu\phi^*\phi-\phi^*D_\mu\phi)$. Making use of the the Gauss law, we can write the corresponding charge as

$$Q=\int dx j_0=\kappa\Delta N, \tag{9}$$

where we introduce $N_\pm=N(t,\pm\infty)$, the constant asymptotic values, and define their difference as $\Delta N=(N_+-N_-)$. Q/κ can be identified by the magnetic flux per unit length in three dimensions. There is also a conserved charge, corresponding to the translation invariance along the y axis before the dimensional reduction,

$$P_y=\int dx i(D_0\phi^*\phi-\phi^*D_0\phi)N$$
$$=\frac{\kappa}{2}(N_+^2-N_-^2)=Q\overline{N}, \tag{10}$$

where we used the Gauss law and the average of N is defined as

$$\overline{N}=(N_++N_-)/2. \tag{11}$$

Thus the $\overline{N}$ can be regarded as the momentum carried by the unit charge along the y direction. If we identify $\overline{N}=N_0$, the mass $m=\sqrt{N_0^2+\mu^2}$ of elementary charged particles in the symmetric phase can be thought to be a Lorentz-boosted one from three-dimensional mass μ by the y momentum N_0.

For the conserved P_y, there exists the corresponding transformation of the fields:

$$\delta N=0, \quad \delta\phi=iN\phi, \quad \delta A_1=j_0/\kappa, \quad \delta A_0=j_1/\kappa. \tag{12}$$

This transformation is somewhat puzzling as it does not leave the Lagrangian invariant without the gauge field equation, although the Hamiltonian is invariant modulo the Gauss law. Even though the y space has disappeared, there remains this y directional translation.

Finding the BPS energy bound begins with the explicit expression of the energy density:

$$\mathcal{E}=|D_0\phi|^2+|D_1\phi|^2+N^2|\phi|^2+\frac{1}{\kappa^2}\phi^2(|\phi|^2-v^2)^2. \tag{13}$$

Together with the Gauss law, one can put the energy density as

$$\begin{aligned}\mathcal{E}=&\left|D_0\phi+\frac{i}{\kappa}\phi(|\phi|^2-v^2)\sin\alpha-iN\phi\cos\alpha\right|^2\\&+\left|D_1\phi+\frac{1}{\kappa}\phi(|\phi|^2-v^2)\cos\alpha+N\phi\sin\alpha\right|^2\\&+\frac{1}{2\kappa}[\kappa^2N^2-(|\phi|^2-v^2)^2]'\cos\alpha\\&+[N(v^2-|\phi|^2)]'\sin\alpha,\end{aligned} \tag{14}$$

where α is an arbitrary angle variable. We introduce two charges:

$$Y=\int dx\frac{1}{2\kappa}[\kappa^2N^2-(|\phi|^2-v^2)^2]', \tag{15}$$

$$Z=\int dx[N(v^2-|\phi|^2)]'. \tag{16}$$

The boundary conditions at spatial infinities for any finite configuration should approach the ground configuration of the potential ($F_\pm=1$, $N_\pm=0$) or ($F_\pm=0$, $N_\pm=$any value), where $|\phi|^2=v^2F$. We can rewrite the above two charges as

$$Y=P_y-\frac{1}{2}\kappa\mu^2(F_+-F_-)(F_++F_--2), \tag{17}$$

$$Z=\mu Q. \tag{18}$$

Thus we can identify Y as the topological charge and Z as the Noether charge in a broad sense.

From Eq. (14), we can now put a bound on the energy for configurations of a given Y and Z as

$$E\geqslant Y\cos\alpha+Z\sin\alpha \tag{19}$$

for any α. When $\cos\alpha=Y/\sqrt{Y^2+Z^2}$ and $\sin\alpha=Z/\sqrt{Y^2+Z^2}$ are chosen, the BPS-like bound follows:

$$E\geqslant\sqrt{Y^2+Z^2}. \tag{20}$$

For any localized configuration of given values of Y and Z should satisfiy this bound. [If we chose a different α, Eq. (19) would be a weaker bound than Eq. (20) and so, would not be saturated.] The bound is saturated by the field configurations satisfying the constraint equations (6) and (7), and the self-dual equations

$$D_0\phi+\frac{i}{\kappa}\phi(|\phi|^2-v^2)\sin\alpha-iN\phi\cos\alpha=0, \tag{21}$$

$$D_1\phi+\frac{1}{\kappa}\phi(|\phi|^2-v^2)\cos\alpha+N\phi\sin\alpha=0. \tag{22}$$

The self-dual configurations are static in time as one can show easily that $\partial_0|\phi|^2=\partial_0N=\partial_x\mathrm{Arg}\phi-A_x=0$. Furthermore, the self-dual configurations also satisfy the usual second-order field equations. The reason why this works is that the Noether charges Y and Z are given by the total derivatives when the Gauss law is used and so that the action principle in the Hamiltonian formalism is satisfied by the self-dual configurations. [The additional constraint (7) can be shown to be automatically satisfied by the field configurations satisfying the self-dual equations and the Gauss law.]

The dimensional reduction of the Bogomol'nyi-type energy bound in three dimensions is identical to the above case with $\sin\alpha=\pm1$ and so is not a new bound. However, in the Appendix we will describe another Bogomol'nyi-type bound which works only for the topological domain walls.

Again with $F=|\phi|^2/v^2$, we can combine the Gauss law (6), and Eqs. (21) and (22) as

$$N'=-2\mu F\{\mu(F-1)\sin\alpha-N\cos\alpha\}, \tag{23}$$

$$F'=-2F\{\mu(F-1)\cos\alpha+N\sin\alpha\}, \tag{24}$$

where the prime is d/dx. The above equations can be put together as a single ordinary differential equation:

$$\begin{aligned}(\ln F)''&=-2(\mu F'\cos\alpha+N'\sin\alpha)\\&=-4\mu^2F(1-F).\end{aligned} \tag{25}$$

After integrating this equation once, we obtain

$$F'^2-4\mu^2F^2(F^2-2F+a^2)=0, \tag{26}$$

where a is an integration constant that lies in $[0,1]$ for any finite energy density solution. For a given F configuration Eq. (24) determines N.

There is one different feature in our model compared with the previously studied BPS-type solitons: The length scale of the above Eq. (26) is independent of the parameter α, and so is that of the self-dual domain walls.

III. SELF-DUAL SOLITONS

A. Between the symmetric and asymmetric phases

For $a=1$, the solutions of Eq. (26) are

$$F=\frac{1}{1+e^{\mp2\mu x}}, \tag{27}$$

satisfying $F'=\pm2\mu F(1-F)$. These solutions describe topological domain walls interpolating between the symmetric and asymmetric phases. The transition region from the symmetric phase to the asymmetric one has the size of order $1/\mu$. The scalar field N from Eq. (24) is

$$N=\frac{\mu(\mp1+\cos\alpha)}{\sin\alpha(1+e^{\pm2\mu x})}. \tag{28}$$

Of course the position of the solitons can be translation in space by replacing x by $x-c$. This is the only zero mode of the solution.

The solutions can be classified into two classes depending on whether F_+ is one or zero. In the first case with $(F_+,F_-)=(1,0)$, we get $N_+=0$ and $N_-=\mu(-1+\cos\alpha)/\sin\alpha$, which fixes α in terms of N_-. The total energy is

$$E=\frac{\kappa\mu^2(1-\cos\alpha)}{\sin^2\alpha}. \tag{29}$$

The topological charges for this soliton are $Y=E\cos\alpha$ and $Z=E\sin\alpha$. In addition, $Q=-\kappa N_-$ and $P_y=-\kappa N_-^2/2$. Near $\alpha\approx0$, the wall does not carry any charge and the energy takes $E=\kappa\mu^2/2$, the minimum value of Eq. (29).

The second case is with $(F_+,F_-)=(0,1)$, which is the spatial reflection of the first solution. In this case $N_+=\mu(1+\cos\alpha)/\sin\alpha$, which fixes the α in terms of N_+, and $N_-=0$. The energy of the soliton is

$$E=\frac{\kappa\mu^2(1+\cos\alpha)}{\sin^2\alpha}. \tag{30}$$

Again the charges are uniquely determined by N_+ or the angle α.

B. In the symmetric phase

For $a<1$, the general solution of Eq. (26) is given by

$$F=\frac{a^2}{1+\sqrt{1-a^2}\cosh(2a\mu x)}. \tag{31}$$

Since $F_{\pm}=0$, this domain wall lives in the symmetric phase. From Eq. (24), we get

$$N=\left(\frac{\mu\sqrt{1-a^2}}{\sin\alpha}\right)\times\left(\frac{a\sinh(2a\mu x)+[\sqrt{1-a^2}+\cosh(2a\mu x)]\cos\alpha}{1+\sqrt{1-a^2}\cosh(2a\mu x)}\right), \tag{32}$$

where $N_+=\mu(a+\cos\alpha)/\sin\alpha$ and $N_-=-\mu(a-\cos\alpha)/\sin\alpha$. The energy is given by

$$E=\frac{2a\kappa\mu^2}{\sin^2\alpha}. \tag{33}$$

Thus, for a given α and a, there is a unique solution up to the translation. Again, Y and Z are $E\cos\alpha$ and $E\sin\alpha$.

Let us reconsider the above configuration somewhat differently. The average of N is $\bar{N}=\mu\cot\alpha$. This fixes the angle α. The difference of N is given by $\Delta N=2\mu a/\sin\alpha$, which fixes a in terms of $\bar{N}$ and ΔN. Since the electric charge is $Q=\kappa\Delta N$, the energy (33) can be expressed as

$$E=Q\sqrt{\bar{N}^2+\mu^2}. \tag{34}$$

Since the mass of elementary charged particles in the symmetric phase is $m=\sqrt{N_0^2+\mu^2}$, this soliton can be regarded as a Q-ball lump of elementary particles if we identify the vacuum value N_0 by the average $\bar{N}$.

In the limit $a\rightarrow 1$, the size of this soliton gets very large. The spatial dependence is characterized by the parameter μ. As x increases, the value of F jumps from zero to a value approximately one in a wall of size $1/\mu$, then stays there for the interval of approximate size $(-\ln\sqrt{1-a^2})/2\mu$, then falls down to zero in the wall of size $1/\mu$. From the spatial dependence of F and N, one can see that in this limit the soliton looks more and more like a combination of two topological solitons considered in the previous subsection.

Especially when we choose $\bar{N}=0$ or $\cos\alpha=0$, the limit $Q=2\kappa\mu a\rightarrow 2v^2$ as $a\rightarrow 1$. For the semiclassical picture to be correct, the charge of this soliton should be much larger than that of elementary particles, or $2v^2\gg 1$.

IV. SUPERSYMMETRY

In Ref. [7] the underlying $N=2$ supersymmetry theory for the self-dual Chern-Simons Higgs systems has been found. After the dimensional reduction of this model, we get again an $N=2$ supersymmetric model. Here we use the convention that $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1)$, and $\gamma^0=\sigma_2$, $\gamma^1=i\sigma_1,\gamma^2=-i\sigma_3$. After the dimensional reduction, we introduce $\gamma^5=\gamma_5\equiv\gamma^0\gamma^1=\sigma_3$. The supersymmetric Lagrangian in two dimensions becomes

$$\mathcal{L}_{\mathrm{SUSY}}=\kappa NF_{01}+|D_\mu\phi|^2-N^2|\phi|^2-\frac{1}{\kappa^2}|\phi|^2(|\phi|^2-v^2)^2+i\bar{\psi}\gamma^\mu D_\mu\psi-iN\bar{\psi}\gamma^5\psi+\frac{1}{\kappa}(3|\phi|^2-v^2)\bar{\psi}\psi. \tag{35}$$

After the dimensional reduction, the supersymmetric transformation becomes

$$\delta A_\mu=\frac{1}{\kappa}(\bar{\zeta}\gamma_\mu\psi\phi^*+\bar{\psi}\gamma_\mu\zeta\phi),$$

$$\delta N=\frac{i}{\kappa}(\bar{\zeta}\gamma^5\psi\phi^*+\bar{\psi}\gamma^5\zeta\phi),\delta\phi=\bar{\zeta}\psi,$$

$$\delta\psi=-i\gamma^\mu\zeta D_\mu\phi+i\gamma^5\zeta N\phi+\frac{1}{\kappa}\phi(|\phi|^2-v^2)\zeta. \tag{36}$$

Since the parameter ζ is a complex Dirac spinor, it generates the $N=2$ supersymmetry transformation. [We can choose the chiral spinors $(1\pm\gamma^5)\zeta/2$ as independent parameters and so the supersymmetry is really $N=(2,2)$.]

For a given superfield Φ the supersymmetric transformation

$$\delta\Phi=[i(\bar{\zeta}\mathcal{R}+\bar{\mathcal{R}}\zeta),\Phi], \tag{37}$$

is generated by the supercharge

$$\mathcal{R}=\int dx\left\{\left[(D_\mu\phi)^*\gamma^\mu+N\phi^*\gamma^5-\frac{i}{\kappa}\phi^*(|\phi|^2-v^2)\right]\gamma^0\psi\right\}. \tag{38}$$

After quantization, we get the canonical commutation relations between the fields. These lead to the relevant $N=2$ superalgebra,

$$[\bar{\zeta}\mathcal{R},\bar{\mathcal{R}}\eta]=\bar{\zeta}\gamma^\mu\eta P_\mu-\bar{\zeta}\eta Z+i\bar{\zeta}\gamma^5\eta Y, \tag{39}$$

where two central charges are given as

$$Y=\int dx\left\{N[i(D_0\phi^*\phi-\phi^*D_0\phi)-\psi^\dagger\psi]-\frac{1}{2\kappa}\partial_x(|\phi|^2-v^2)^2\right\}, \tag{40}$$

$$Z=-\int dx\left\{N\partial_x|\phi|^2+\frac{1}{\kappa}(|\phi|^2-v^2)\times[i(D_0\phi^*\phi-\phi^*D_0\phi)-\psi^\dagger\psi)]\right\}. \tag{41}$$

These central charges become those in Eqs. (15) and (16) once the Gauss law (6) is used.

After introducing a new spinor operator,

$$\mathcal{R}'=\frac{1}{2}(1+\gamma_0 e^{i(\pi/2-\alpha)\gamma_5})\mathcal{R}$$
$$=\frac{1}{2}(1+\gamma_0 e^{i(\pi/2-\alpha)\gamma_5})\int dx$$
$$\times\left\{\psi\left[(D_0\phi)^*-\frac{i}{\kappa}\phi^*(|\phi|^2-v^2)\sin\alpha+iN\phi^*\cos\alpha\right]\right.$$
$$\left.-\gamma_5\psi\left[(D_1\phi)^*+\frac{1}{\kappa}\phi^*(|\phi|^2-v^2)\cos\alpha+N\phi^*\sin\alpha\right]\right\}, \tag{42}$$

the superalgebra (39) becomes

$$\sum_\beta \{\mathcal{R}'_\beta,\mathcal{R}'^\dagger_\beta\}=E-(Y\cos\alpha+Z\sin\alpha). \tag{43}$$

As the left-hand side of this equation is positive definite, this equation leads to the energy bound (19). The energy bound is saturated when $\langle\mathcal{R}'\rangle=0$, which in turn implies the self-dual equations.

V. THE NONRELATIVISTIC LIMIT

We expect the nonrelativistic limit of the theory can be taken in the symmetric phase as the field ϕ describes charged particles. We expect the nonrelativistic limit is reasonable if the kinetic energy is small. For this limit to make sense, it turns out that the y momentum per unit charge should be also small compared with the three-dimensional mass, or $|N_0|\ll\mu$.

In this case we take the nonrelativistic limit of the system in the three-dimensional model by letting $\phi=(1/\sqrt{2\mu})e^{-i\mu t}\psi$, where $\mu=v^2/\kappa$. The Lagrangian (1) reduces to the well-known Lagrangian [8],

$$\mathcal{L}_{\rm nonrel}=\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+i\psi^*D_t\psi-\frac{1}{2\mu}|D_\mu\psi|^2+\frac{1}{2\mu\kappa}(|\psi|^2)^2. \tag{44}$$

After dimensional reduction, it becomes

$$\mathcal{L}_{\rm nonrel}=\kappa NF_{01}+i\psi^*D_t\psi-\frac{1}{2\mu}|D_x\psi|^2-\frac{1}{2\mu}N^2\psi^2+\frac{1}{2\mu\kappa}(|\psi|^2)^2. \tag{45}$$

This is equivalent to the nonrelativistic limit of the dimensionally reduced Lagrangian if $N_0\ll\mu$. The energy functional, after dropping the boundary term which depends only on the conserved charge, is given by

$$E_{\rm nonrel}=\int dx\left\{\frac{1}{2\mu}|D_x\psi|^2+\frac{1}{2\mu}N^2|\psi|^2-\frac{1}{2\mu\kappa}(|\psi|^2)^2\right\}. \tag{46}$$

This energy is reasonable as its density is gauge invariant. We split the N field as a sum of the average $\overline{N}$ and the fluctuation δN such that $\delta N_+ +\delta N_- =0$.

By making use of the Gauss law $\kappa N'+|\psi|^2=0$, the energy functional can be rewritten as

$$E_{\rm nonrel}=\frac{1}{2\mu}\overline{N^2}|Q|+\int dx\left\{\frac{1}{2\mu}|D_x\psi-\delta N\psi|^2\right\}, \tag{47}$$

plus the vanishing boundary terms, where $Q=-\int dx|\psi|^2<0$. Since the integrand is non-negative, the nonrelativistic energy is bounded by the y directional kinetic energy, $\overline{N^2}|Q|/(2\mu)$, which can be seen as the mass correction of the Q particles by $P_y=\overline{N}$.

Introducing $|\psi|^2=2\mu v^2F$, we can simplify the Gauss law and the nonrelativistic self-dual equations to

$$\delta N'+2\mu^2F=0, \tag{48}$$

$$F'-2\delta NF=0. \tag{49}$$

They in turn lead to the equations

$$(\ln F)''+4\mu^2F=0, \tag{50}$$

$$\delta N=(\ln F)'/2. \tag{51}$$

The equation (50) is the Liouville equation in one dimension, which can be integrated to

$$F'^2+4\mu^2F^2(2F-a^2)=0, \tag{52}$$

where $0<a<1$. This is again the nonrelativistic limit of Eq. (26) with the same parameter a. The solutions are given by

$$F=\frac{a^2}{2\cosh^2(a\mu x)}, \tag{53}$$

$$N=N_0-\frac{a\mu}{2}\tanh(a\mu x). \tag{54}$$

In the large x region the above nonrelativistic solutions match with the relativistic solution (31) if $a\ll 1$, $\sin\alpha=-1$. The charge of this configuration is again $Q=2\kappa\mu a$.

VI. CONCLUDING REMARKS

Here we have explored the structure of the domain walls in the self-dual Chern-Simons-Higgs systems by the dimensional reduction method. Their energy bound is similar to the BPS bound on dyons in some Yang-Mills Higgs systems in four dimensions. We found all possible domain solutions, which are made of topological and nontopological domain walls. We studied the $N=2$ supersymmetry behind the BPS-like energy bound. We also have studied the nonrelativistic limit.

Our two-dimensional model is different from the recent attempts to describe anyons in one-dimensional space [9]. Our model does not seem to be directly related to the Calogero-Sutherland model, in contrast to the recent work [10]. Our nonrelativistic limit is simpler than the recently discovered chiral solitons [11]. However, it turns out that our nonrelativistic model also has intriguing properties as shown in a recent work [12].

There are several directions to explore from this point. There are self-dual Chern-Simons-Higgs models for any gauge group and any matter representation [13,14]. Especially with the matter in the adjoint representation, the vacuum structure is quite rich [15,16], implying the intricate domain wall structure. As in the Abelian case, domain walls in two dimensions are generally easier to understand than solitons in three dimensions, and this will eventually lead to a better understanding of solitons in three dimensions.

We studied the $N=2$ supersymmetry underlying the self-duality. However, in three dimensions with the Chern-Simons term the maximal supersymmetry is $N=3$ [18], which could be translated to the $N=3$ supersymmetric theory in two dimensions. As a pure two-dimensional system, it would be interesting to find out how the $N=3$ supersymmetry can be maximal. One may wonder whether there exists a more general set of the self-dual systems with the *BF* kinetic term in two dimensions with larger supersymmetry than the $N=3$ supersymmetry.

Along the similar line of thought, there may be a gauged version of the nonlinear σ models considered in Ref. [5]. They would lead to a richer variety of two-dimensional models with the BPS-type energy bound, which may be similar to the dimensional reduction of the self-dual CP(N) models considered in three dimensions [17].

The quantum mechanical aspect of the theory should also be interesting. In the massless limit $v=0$, there may be a quantum conformal symmetry. Also the three-dimensional solitons may appear as instantons in two dimensions, whose effect is unclear at this moment. One of the noticeable features here is that the Euclidean action is complex and so the instanton solutions should be treated carefully as in the case of monopole instantons in three-dimensional Chern-Simons Higgs systems [19].

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APPENDIX: ANOTHER BOUND

In this appendix, we briefly review another Bogomol'nyi-type bound which works only for the topological domain walls [2]. We can rewrite the energy (13) as follows:

$$\mathcal{E}=|D_0\phi\pm iN\phi|^2+\left|D_1\phi\pm\frac{1}{\kappa}\phi(|\phi|^2-v^2)\right|^2 \mp\frac{1}{2\kappa}[\kappa^2N^2+(|\phi|^2-v^2)^2]'. \tag{A1}$$

This is not identical to Eq. (14) with $\cos\alpha=\pm1$ because of the sign difference in the boundary term. The energy bound is then

$$E\geqslant\left|\int dx\frac{1}{2\kappa}[\kappa^2N^2+(|\phi|^2-v^2)^2]'\right|. \tag{A2}$$

Let us concentrate on the upper sign. The self-dual equations in this case become

$$\kappa N'=-2|\phi|^2, \tag{A3}$$

$$D_1\phi+\frac{1}{\kappa}\phi(|\phi|^2-v^2)=0, \tag{A4}$$

where we have used the Gauss law (6). The ϕ equation leads to the topological domain wall solution (27) with $(F_+,F_-)=(1,0)$. The N equation has a nontrivial solution (28) with $(N_+=0,\ N_->0)$.

The two seemingly different sets of self-dual equations are satisfied by the fields for the same topological domain wall. The new energy bound (A2) is identical to the BPS energy (29) for these configurations. We believe that the presence of this additional energy bound is due to the interdependence of two charges Y and Z for the topological domain walls. While the above bound does not lead to anything new in the Abelian self-dual case, its analogue in the non-Abelian case may be more useful.

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