

Aspects of type IIB theory on asymptotically locally Euclidean spaces

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D-brane technology and strong/weak coupling duality supplement traditional orbifold techniques by making certain background geometries more accessible. In this spirit, we consider some of the geometric properties of the type IIB theory on $\mathbb{R}^6 \times \mathcal{M}$, where \mathcal{M} is an “asymptotically locally Euclidean” (ALE) gravitational instanton. Given the self-duality of the theory, we can extract the geometry (both singular and resolved) seen by the weakly coupled IIB string by studying the physics of a D1-brane probe. The construction is both amusing and instructive, as the physics of the probe completely captures the mathematics of the construction of ALE instantons via “hyper-Kähler quotients,” as presented by Kronheimer. This relation has been noted by Douglas and Moore for the *A* series. We extend the explicit construction to the case of the *D* and *E* series—uncovering a quite beautiful structure—and highlight how all of the elements of the mathematical construction find their counterparts in the physics of the type IIB D-string. We discuss the explicit ALE metrics which may be obtained using these techniques, and comment on the role duality plays in relating gauged linear σ models to conformal field theories. [S0556-2821(97)01508-7]

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I. INTRODUCTION

A. Motivation

The self-duality of the ten-dimensional type IIB theory under the strong or weak coupling duality map [1] makes it a particularly interesting theory to study. The use of D-brane technology [2,3,4] as an aid in gaining insight into the physics of various regimes of the theory makes its study somewhat more tractable.

Of particular interest for us will be the D1-brane. This solitonlike Bogomol’ni-Prasad-Sommerfield (BPS) saturated object has the distinction that it gets exchanged with the “fundamental” type IIB string under the strong or weak coupling duality transformation, becoming the “fundamental” string of the dual theory. The self-duality of the theory is seen here by noting that the zero-mode spectra of both strings are identical [5].

One might expect then, that aspects of the type IIB theory which might seem obscure or difficult to handle by studying the spectrum of the “fundamental” string might be better addressed in studying its dual partner, the D1-brane. The converse is also to be expected. While this is true for any dual pair of theories, the novelty here is that for the *self-dual* IIB theory, these are dual descriptions of physics which is still essentially perturbative from the point of view of *either* string theory. The dual descriptions are thus rather more like a conventional change of variables than usual.¹

It is with this in mind that we examine the physics of the type IIB theory in the neighborhood of an asymptotically

locally Euclidean (ALE) gravitational instanton, using a D1-brane probe [6]. This is a particularly interesting set of backgrounds to study, for many reasons. Chief among those, in this context, are (i) their blow-down limits are simple to describe as an orbifold [7], (ii) they are completely classified [8] (falling into an *A-D-E* series), (iii) as string backgrounds they break only half of the supersymmetry, and (iv) they are noncompact spaces, thereby allowing us to study them in the context of self-dual type IIB theory in ten dimensions.

B. Orbifolds, ALE spaces, and geometry

Much of the physics of strings on ALE spaces was well understood using orbifold technology [7]. One studies the string theory on the singular space \mathbb{R}^4/Γ where Γ is some discrete subgroup of $SU(2)$ (they fall into an *A-D-E* classification [9]) acting on the space \mathbb{R}^4 . The result that string theory is naturally well behaved on such a singular space is related to the appearance (required by modular invariance) of massless states from “twisted sectors” of the orbifold [10], which correspond to precisely the moduli needed to deform the theory to the neighboring problem with a smooth target space. So even without knowledge of the detailed form of the metric for the ALE spaces it is enough to know that the stringy resolution is complete by comparing [11] to (say) a purely algebraic construction of the moduli space of the objects which form the target space.

So in principle, the string theory “knows” everything about the metric on the ALE spaces. As these spaces are quite simple and well studied (see later for a review), there is apparently not much in the way of new physics to discover, at least away from the regimes of the theory where the new (and, currently, poorly understood) phenomena first characterized in Ref. [12] arise. The description via orbifolds is very adequate in capturing the physics.

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¹This additional aspect of IIB strong/weak coupling duality does not make it any less profound than the other dualities, from a number of points of view.

However, there is a matter of both principle and practice here. First, it is difficult (at best) to extract the detailed form of the metric that the string sees, using the orbifold technology. More generally, we can usually extract the metric of a target space defined by a conformal field theory only when we have a Lagrangian definition of it, such as a (gauged or ungauged) Wess-Zumino-Witten model. Unfortunately, such path integral definitions are only known for a very small subset of the conformal field theories of interest. Furthermore, even in principle it is not clear how the string feels its way around the smooth space on which the complete orbifold suggests that it is propagating.

This is one of the issues we would like to highlight in this paper. The techniques described herein are a means of extracting directly from the string theory itself (using strong/weak coupling duality and D-branes) the details (not just algebraic, but differential) of the resolved spaces which the orbifolded fundamental string theory sees.

Most often, we study string theory on a target space of choice by putting the metric into the formalism ourselves. In practice, we usually proceed to find the metric on the target space by solving the background field equations by hand (order by order in α'), aided by the string theory only in the cases where some symmetry of the theory provides us with a solution generating technique, or some other such means of making the problem more manageable. In general though, in order to find the solutions of interest we have to employ methods which are often supplementary to string theory itself.

In the case of the ALE instantons, which fall into an A - D - E classification [8], only the metric for the A_k series is known in a closed expression:

$$ds^2 = V^{-1}(dt - \mathbf{A} \cdot d\mathbf{y})^2 + V d\mathbf{y} \cdot d\mathbf{y}$$

where

$$V = \sum_{i=0}^k \frac{1}{|\mathbf{y} - \mathbf{y}_i|} \quad (1.1)$$

and

$$\nabla V = \nabla \times \mathbf{A}.$$

These are the Gibbons-Hawking multicenter metrics [13]. The case $k=1$ is the Eguchi-Hanson metric [14]. These spaces are asymptotically flat, but Euclidean only locally: There is a global identification which makes the surface at infinity S^3/\mathbb{Z}_{k+1} instead of S^3 .

As solutions of the vacuum Einstein equations, these metrics are also solutions of the leading-order background field equations in type IIB string theory.² However, their derivation as solutions of Einstein's equations were not particularly stringy in origin.

For a time after these metrics were found, the instantons corresponding to the D and E series were not known, al-

though their existence was strongly motivated in Ref. [8]. For the A_k series, the ALE nature arises from the discrete identifications \mathbb{Z}_{k+1} at infinity. This is the cyclic subgroup of $SU(2)$. For the D_k and $E_{6,7,8}$ series, the cyclic group is replaced by the binary dihedral (\mathbb{D}_{k-2}), tetrahedral (\mathcal{T}), octahedral (\mathcal{O}), and icosahedral (\mathcal{I}) groups. [We will remind the reader of the relation [9] between the discrete subgroups of $SU(2)$ and the simply laced Lie algebras later in this paper.]

C. Hyper-Kähler quotients

In 1987, an explicit construction of the spaces (and a proof of the conjectured A - D - E classification) was presented by Kronheimer [16]. The main tool used was the “hyper-Kähler quotient” technique [17]. In short (more details later) it was shown that one can recover the ALE spaces by starting with a parent space M (a flat hyper-Kähler manifold of high dimension) acted on by some auxiliary group F . By virtue of the hyper-Kähler structure of M , the group action naturally induces a Lie-algebra-valued triplet of functions $\boldsymbol{\mu}$ called the “moment map,” which plays a central role in the construction. It in turn defines naturally a set of constraints and a coset procedure which recover a space of real dimension four, possessing the properties of the sought-after ALE spaces, for the appropriate choice of F and M .

The resulting spaces are again solutions of the IIB string theory by virtue of their being solutions of Einstein's equations for empty space. Once again, the techniques used to find the solutions lie outside (apparently, as we shall see) the realm of string theory, using as input a number of auxiliary objects such as the group F and the parent hyper-Kähler manifold M .

D. The dual picture

Amusingly, while for the traditional orbifold description of the IIB theory on an ALE space, the above paragraph is true, in the dual picture—described by the D1-brane—the physics of how the string sees the metric can be made explicit, and it maps onto precisely Kronheimer's construction. This was shown for the A series in Ref. [18]. This paper extends the result to the full A - D - E family. In particular, there is a physical role for the group F , the parent space M , and the moment map $\boldsymbol{\mu}$, which were all essential ingredients of the mathematical construction.

The world-volume theory of the fundamental IIB string on the ALE space—an $\mathcal{N}=4$ conformal field theory—is exchanged under duality for the world-volume theory of the D1-brane probe, which is an $\mathcal{N}=4$ gauged linear σ model. The gauge group is F , which acts on a set of hypermultiplet scalars parameterizing a manifold M . The allowed values of the hypermultiplets parameterize the position of the D1-brane probe in space. Solving the D -flatness conditions for the allowed values of the hypermultiplets is equivalent to solving the constraints imposed by the moment map $\boldsymbol{\mu}$. The metric on that moduli space of vacua is the space-time metric seen by the D1-brane. As the type IIB theory is self-dual, we have therefore learned how the orbifolded fundamental type IIB string actually recovers metric data about the smooth space it propagates in.

So it seems that we have come full circle: In order to find the metric that the string sees on the ALE spaces we move

²In fact, these backgrounds are solutions to all orders in the α' expansion because the corresponding world-sheet theory has $\mathcal{N}=4$ supersymmetry [15].

progressively further away from stringy techniques to solve the problem. In doing so, we “meet ourselves coming the other way,” working deep inside the string theory, probing with D-brane variables.

E. Linear σ models and D-branes

The fact that gauged linear σ models capture the essence of certain quotient constructions used in mathematics and mathematical physics has been noticed and exploited successfully in the string theory context [19]. The essential physical picture of the hyper-Kähler quotient construction and its relevance to strings propagating near ALE singularities was discussed in detail in Ref. [12] for the A_1 singularity. These discussions all took place before the recognition [3] of the role D-branes to duality and related issues.³

That D1-branes are a natural means of producing two-dimensional gauged linear σ models was noticed in Ref. [20]. Douglas [6] made use of this technique in the type I context to recover (*via* duality) the linear σ models associated to heterotic strings in Yang-Mills instanton backgrounds [21], pioneering the explicit use of D1-branes as probes of short-distance string physics. There D-brane and duality technology captured the Atiyah-Drinfeld-Hitchin-Manin (ADHM) hyper-Kähler quotient construction of Yang-Mills instantons [22]. (The Yang-Mills instantons live at the core of heterotic fivebranes, which are dual to type I’s D5-branes in the small instanton limit.)

The D1- and D5-brane constructions in Ref. [6] were further extended to include Yang-Mills instantons on the A-series ALE spaces in Ref. [18], where it was observed that the hyper-Kähler quotient construction of Kronheimer and Nakajima [23] was this time cast into a physical setting. In a different but related context, the explicit details of the D1-branes on the A-series ALE spaces were worked out in Ref. [24]. There, the exercise (carried out for the $k=1$ case in for example Ref. [25]) of choosing good coordinates on the space of hypermultiplets and deriving the form (1.1) as the metric on moduli space is carried out in a D-brane context for the full A_k series.

F. Outline

In this paper we shall extend the explicit construction of D1-brane physics on the ALE spaces to the complete (A-D-E) family of ALE spaces. Along the way we find that the D-brane physics is a remarkably clear and simple guide to the elements of the mathematical construction. At the same time, many beautiful mathematical results are united in this simple physical setting. The results make the potentially unpleasant structures present in the D and E series (due to the nonlinearities introduced by the non-Abelian nature of Γ)

more manageable in many respects, while allowing the elegance of the A-D-E classification to shine throughout.

In Sec. II we start the study of strings on ALE spaces by first considering the singular “blow-down” limit. We first introduce the discrete subgroups Γ of $SU(2)$ and their natural action on \mathbb{R}^4 . Introducing the D1-brane probe(s) we impose a projection on the Chan-Paton factors to ensure that they respect the Γ symmetry, and solve for the spectrum of the world-volume theory. Using the result of McKay [26] concerning the representations of Γ we find that the D-branes are organized according to the structure of the extended Dynkin diagrams associated to the A-D-E root systems. The resulting spectrum of the world-volume theory ($D=2$, $\mathcal{N}=4$) is described. We pause to note the relation to elements of the construction of Kronheimer. A discussion of the various branches of the classical moduli space of vacua of the world-volume theories maps out the geometry that the probes see, simultaneously introducing the essence of the hyper-Kähler quotient construction of the blown-down ALE spaces.

In Sec. III we complete the construction, by turning on the closed string fields which deform to resolved ALE spaces. After discussing how these couple in the world-volume theory (following Ref. [18]) we revisit the Higgs branch of the moduli space of vacua and complete the discussion of IIB strings on resolved ALE spaces, thus also completing the hyper-Kähler quotient discussion of Kronheimer. A discussion of the explicit metric on moduli space (i.e., explicit metrics on the ALE spaces) takes place in Sec. IV, and with a few remarks and speculations in Sec. V we end this presentation.

II. D-STRINGS ON ALE SPACES: THE BLOW-DOWN

A. Preliminaries

Starting at an arbitrary point in the moduli space of an ALE instanton, we can adjust certain moduli in order to shrink the “core” of the instanton, locating the associated curvature in a progressively smaller region of four-dimensional Euclidean space. The limit of this procedure is the “blow-down” or “orbifold limit” of the space, when all of the curvature is localized at a single singular point. The blown-down ALE instanton is essentially just the space \mathbb{R}^4/Γ , where Γ is a discrete group. (For simplicity, we’ll place the singular point at the origin of our coordinates, in our definitions which follow.)

Let us denote the Cartesian coordinates on the \mathbb{R}^4 as x^6, x^7, x^8 , and x^9 . Let us also define a set of complex coordinates, $z^1 = x^6 + ix^7$ and $z^2 = x^8 + ix^9$. Thus we will sometimes describe the space as the complex space \mathbb{C}^2 , coordinatized by the z^i . Many of the results of our analysis will inherit this structure.

Yet another natural description of $\mathbb{R}^4 \cong \mathbb{C}^2$ is as a trivial quaternionic space, where we can write the coordinate

$$q = \begin{pmatrix} z^1 & -\bar{z}^2 \\ z^2 & \bar{z}^1 \end{pmatrix}. \quad (2.1)$$

There is an $SU(2)_L \times SU(2)_R \subset SO(4)$ group action on the space given by

³The reader may also wish to consult Ref. [7] for a very useful review of Kronheimer’s hyper-Kähler quotient construction of ALE spaces. Again, the quotient construction is treated there as supplementary to the string theory. The paper’s main concern is a construction of the relevant orbifold conformal field theories and the study of their marginal deformations.

$$q \rightarrow g_L q g_R \quad \text{for } g_{L,R} \in \text{SU}(2)_{L,R}. \quad (2.2)$$

Choosing either of the $\text{SU}(2)$'s (we choose the left), the orbits of its free action on the z^i parametrize families of S^3 's, the natural invariant submanifolds of the action of $\text{SO}(4)$ on \mathbb{R}^4 . The $\text{SU}(2)_R$ will remain as the standard global symmetry associated with hyper-Kähler manifolds (eventually manifesting itself as an R -symmetry in the supersymmetric field theories which we study later).

The asymptotic group of discrete identifications Γ of the ALE spaces are subgroups of $\text{SU}(2)$. These groups have been classified by Klein [9]. They are governed by the same Diophantine equations which organize the point group symmetries of regular solids in three dimensions,⁴ and consequently fall into an A - D - E classification.

There are [28] the following.

(i) The A_k series ($k \geq 1$). This is the set of cyclic groups of order $k+1$, denoted \mathbb{Z}_{k+1} . Their action on the z^i is generated by

$$g = \begin{pmatrix} e^{2i\pi/(k+1)} & 0 \\ 0 & e^{-2i\pi/(k+1)} \end{pmatrix}. \quad (2.3)$$

(ii) The D_k series ($k \geq 4$). This is the binary extension of the dihedral group, of order $4(k-2)$, denoted \mathbb{D}_{k-2} . Their action on the z^i is generated by

$$A = \begin{pmatrix} e^{i\pi/(k-2)} & 0 \\ 0 & e^{-i\pi/(k-2)} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (2.4)$$

In this representation the central element is $\mathcal{Z} = -1 [= \mathcal{A}^2 = \mathcal{B}^2 = (\mathcal{AB})^2]$. Note that the generators \mathcal{A} form a cyclic subgroup \mathbb{Z}_{2k-4} .

(iii) The remaining groups fall into the $E_{6,7,8}$ series, and are the binary tetrahedral (\mathcal{T}), octahedral (\mathcal{O}), and icosahedral (\mathcal{I}) groups of order 24, 48, and 120, respectively.

The group \mathcal{T} is generated by taking the elements of \mathbb{D}_2 and combining them with

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}, \quad (2.5)$$

where ε is an eighth root of unity.

The group \mathcal{O} is generated by taking the elements of \mathcal{T} and combining them with

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}. \quad (2.6)$$

Finally \mathcal{I} is generated by

$$-\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}$$

and

$$\frac{1}{\eta^2 - \eta^3} \begin{pmatrix} \eta + \eta^4 & 1 \\ 1 & -\eta - \eta^4 \end{pmatrix}, \quad (2.7)$$

where η is a fifth root of unity.

More properties of all of these discrete groups will appear as we proceed. Let us postpone their introduction until such time as we need them, and now turn to the string theory.

B. The open string sector

We start by considering string propagation on $\mathbb{R}^6 \times \mathbb{R}^4 / \Gamma$. The orbifold space breaks half of the $D=10$, $\mathcal{N}=2$ supersymmetry, leaving us with (thinking of this as a compactification) $\mathcal{N}=2$ in the six dimensions of the \mathbb{R}^6 , with coordinates x^0, \dots, x^5 .

We introduce a family of parallel D1-brane probes, all lying in (say) the x^0, x^1 directions, and positioned in the x^6, \dots, x^9 space in a Γ -invariant way. These break half of the supersymmetry again, leaving us with $\mathcal{N}=1$ in six dimensions. We shall focus on the world-sheet theories of the D1-branes (D-strings) which are thus $D=2$ field theories with $\mathcal{N}=4$ supersymmetry, via dimensional reduction.

Our space time is noncompact everywhere, and so it is consistent to place an arbitrary number of these branes in the problem [4].⁵ However, we should ensure that there are enough D-branes in the problem to ensure a faithful representation of the discrete group Γ in the open string sector [30]. We therefore introduce $|\Gamma| = k+1, 4(k-2), 24, 48$, or 120 D-strings depending upon whether we are studying the A_k, D_k , or $E_{6,7,8}$ series.

We will ask that the group Γ be represented on the D-brane (Chan-Paton) indices by the action of $|\Gamma| \times |\Gamma|$ matrices γ_Γ . This is the “regular” representation. (In what follows, appearances of γ_Γ will be taken to imply the action of all of the elements of the representation of Γ .)

Placing all of the D-branes at (say) the origin of the “internal” $\mathbb{R}^4 \cong \mathbb{C}^2$, we have gauge group $U(|\Gamma|)$, arising in the familiar way from massless open strings connecting the various coincident D-branes. We then proceed by projecting this system to obtain invariance under the group Γ . This will reduce the gauge group to some subgroup of $U(|\Gamma|)$. Let us discover what this is.

There are three types of open string sector states to consider.

Vector-multiplets:

$$\lambda_V \psi_{-1/2}^\mu |0\rangle, \quad \mu = 0, 1.$$

⁴See for example Ref. [27] for a nice description of the point groups, and Ref. [28] for a review and discussion of the Kleinian singularities.

⁵See for example Refs. [29,30] for studies of D -branes near ALE singularities in the context of compact internal spacetimes, the manifold $K3$. There, the number of branes is fixed.

The Chan-Paton matrix λ_V starts out life as an arbitrary $|\Gamma| \times |\Gamma|$ Hermitian matrix, a generator of the $U(|\Gamma|)$ gauge group. Invariance under Γ means that it should satisfy additionally

$$\gamma_\Gamma \lambda_V \gamma_\Gamma^{-1} = \lambda_V. \quad (2.8)$$

This constraint will give rise to vector fields (denoted generically A_μ) in the theory, transforming in the adjoint of some subgroup of $U(|\Gamma|)$. We shall see what that subgroup is shortly.

Hypermultiplets I:

$$\lambda_H^i \psi_{-1/2}^i |0\rangle, \quad i=2,3,4,5.$$

Here again the Chan-Paton matrices begin as arbitrary $|\Gamma| \times |\Gamma|$ Hermitian matrix, giving hypermultiplets in the adjoint of $U(|\Gamma|)$ (the rest of the $D=6$, $\mathcal{N}=1$ vectors from a $D=6$ point of view). They satisfy a similar equation to (2.8) and result in hypermultiplets in the adjoint of the gauge group which results from Eq. (2.8). From the $\mathcal{N}=4$, $D=2$ point of view, they are simply the scalar parts of the gauge multiplets. These scalars, which we shall denote ϕ_H^i , parameterize motions of the branes transverse to their world volumes in the x^2 , x^3 , x^4 , x^5 directions.

Hypermultiplets II:

$$\lambda_H^m \psi_{-1/2}^m |0\rangle, \quad m=6,7,8,9.$$

In this sector, given the action of Γ on \mathbb{C}^2 as shown in the previous subsection, it is prudent to relabel our string modes and the resulting hypermultiplets to respect that structure. Our massless modes are thus $\lambda_H^1 \psi_{-1/2}^1 |0\rangle$ and $\lambda_H^2 \psi_{-1/2}^2 |0\rangle$, (and their adjoints) and our constraint equation is more complicated than previously, as these are the coordinates upon which the discrete group acts

$$\begin{pmatrix} \gamma_\Gamma \lambda_H^1 \gamma_\Gamma^{-1} \\ \gamma_\Gamma \lambda_H^2 \gamma_\Gamma^{-1} \end{pmatrix} = G_\Gamma \begin{pmatrix} \lambda_H^1 \\ \lambda_H^2 \end{pmatrix}, \quad (2.9)$$

where G_Γ is a matrix acting in the 2×2 representation of Γ , acting on the indices j of the hypermultiplets λ_H^j . We shall denote the resulting massless (complex) fields in the world-volume theory as simply ψ_H^1 and ψ_H^2 . (These are not to be confused with the standard superconformal field theory modes $\psi_{-1/2}$. We shall not refer to those in what follows.) These hypermultiplets parameterize the motion of the branes in the x^6 , x^7 , x^8 , x^9 directions. In analogy with Eq. (2.1), these hypermultiplets are naturally gathered together a quaternionic form as

$$\Psi = \begin{pmatrix} \psi_H^1 & -\psi_H^{2\dagger} \\ \psi_H^2 & \psi_H^{1\dagger} \end{pmatrix}. \quad (2.10)$$

C. The $D=2$, $\mathcal{N}=4$ world-volume gauge theory

The quickest way to begin to see the solution to the Eq. (2.8) is to note a little more of the structure of the discrete group Γ and its irreducible representations. In particular, recall that elementary group theory tells us that we can always find a basis in which the $|\Gamma| \times |\Gamma|$ “regular” representation of

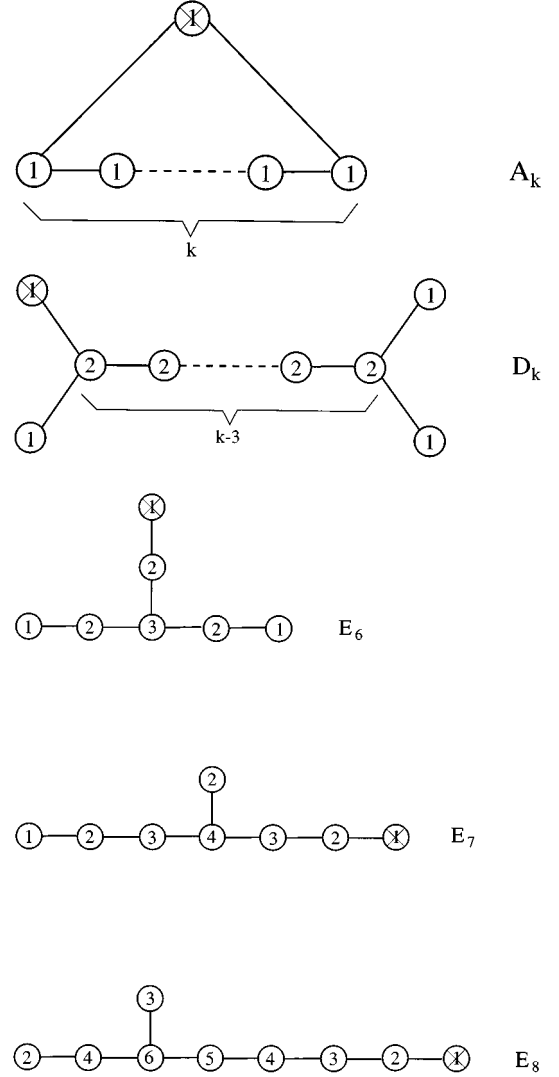


FIG. 1. The Dynkin graphs of the extended A - D - E root systems. They are isomorphic to the structure of irreducible representations of the discrete subgroups of $SU(2)$. They encode the arrangement and resulting spectrum of the open string sectors (D-branes), and also organized the twisted spectors of the closed string spectrum. See text for details.

Γ is of block diagonal form [27]. Each of the representations, R_n (of dimension n), appears as an $n \times n$ block n times in the decomposition.

This information about the representations (and hence conjugacy classes) of Γ is succinctly encapsulated in the extended Dynkin diagrams for the associated A - D - E root systems [26] depicted in Fig. 1.

In the Dynkin diagrams, each vertex represents an irreducible representation of Γ . The integer in the vertex denotes its dimension. The special vertex with the “X” sign is the trivial representation, the one-dimensional conjugacy class containing only the identity. The specific connectivity of each graph encodes the information about the following decomposition:

$$Q \otimes R_i = \bigoplus_j a_{ij} R_j, \quad (2.11)$$

where R_i is the i th irreducible representation and Q is the

TABLE I. The open string spectrum of the D -brane world-volume theory. It is an $\mathcal{N}=4$ supersymmetric gauge theory in $D=2$. In addition, the hypermultiplets I transform in the adjoint of the gauge group F appearing in the vector-multiplets column. See text for details.

Γ	Vector multiplets	Hypermultiplets II
Z_{k+1}	$U(1)^{k+1}$	$\{k+1\} \times (1,1)$
D_{k-2}	$U(1)^4 \times U(2)^{k-3}$	$4 \times (1,2) + \{k-4\} \times (2,2)$
\mathcal{T}	$U(1)^3 \times U(2)^3 \times U(3)$	$3 \times (1,2) + 3 \times (2,3)$
\mathcal{O}	$U(1)^2 \times U(2)^3 \times U(3)^2 \times U(4)$	$2 \times \{(1,2) + (2,3) + (3,4)\} + (2,4)$
\mathcal{I}	$U(1) \times U(2)^2 \times U(3)^2 \times U(4)^2 \times U(5) \times U(6)$	$(1,2) + (2,3) + (3,4) + (4,5) + (5,6) + (2,4) + (4,6) + (3,6)$

defining two dimensional representation. Here, the a_{ij} are the elements of the adjacency matrix A of the simply laced extended Dynkin diagrams.

The process of solving Eqs. (2.8) and (2.9) to find the gauge content of our world-volume theory is made much simpler with the knowledge of these decompositions. In the block-diagonal basis alluded to above, we can see (for example) that the only nonvanishing off-diagonal blocks in λ_V are those connecting different copies of a given representation R_n . In the end, exactly enough conditions are imposed so as to retain only the content of an arbitrary $n \times n$ Hermitian matrix for each irreducible representation R_n (repeated n times). This means that the gauge group is

$$F = \prod_i U(n_i), \quad (2.12)$$

where i labels the irreducible representations R_i of dimension n_i . Pictorially, the gauge group associated with a D -string on a ALE singularity is simply a product of unitary groups associated to the extended A - D - E Dynkin diagram, with a unitary group coming from each vertex. (See Fig. 1).

Turning to the hypermultiplets, as stated before, we trivially have $\dim(F)$ hypermultiplets transforming in the adjoint of F . These come from the 2, 3, 4, 5 sector. Equivalently, they are simply the internal components of the six-dimensional vectors after dimensional reduction.

More interestingly, we have hypermultiplets coming from the 6, 7, 8, 9 sector. To solve the Eq. (2.9) in the block-diagonal basis for γ_Γ is almost as simple as it was for the vectors. The results is best written with a pair of matrices,⁶ λ_H^1 and λ_H^2 in which nonzero entries appear in off-diagonal blocks which connect different unitary groups making up F . The structure of these nonzero elements is isomorphic to the adjacency matrix A of the extended Dynkin diagram.

In other words, these hypermultiplets transform in the fundamentals of the unitary groups, according to the representations

$$\bigoplus_i a_{ij}(\mathbf{n}_i, \bar{\mathbf{n}}_j). \quad (2.13)$$

⁶As the two dimensional representation Q of Γ contains off-diagonal elements in general, components of λ_H^1 and λ_H^2 are related. The exception is the case of the A_k series.

Pictorially, the hypermultiplets are simply the links of the extended Dynkin diagrams. The complete picture with all of the D -strings sitting at the singular point (the ‘‘origin’’) shows some interesting structure: In this ‘‘blown-down’’ limit of the ALE space, the Γ projection arranges the $|\Gamma|$ $D1$ -brane probes on the vertices of the associated extended Dynkin diagram for A , D , or E simple groups, depending upon Γ . On the i th vertex labeled n_i , there are n_i^2 $D1$ -branes. There are massless vector and hypermultiplets in the $D1$ -brane world-volume theory arising from the massless fundamental strings which connect the various branes. The world-volume spectrum is summarised in Table I.

D. The moduli space of vacua: Part I

So on the two-dimensional world-volume of the D strings, we have an $\mathcal{N}=4$ gauge theory with gauge group F given in the table. The hypermultiplets transform under the gauge symmetry as

$$\delta\phi_H = i\varepsilon_a[\lambda_V^a, \phi_H], \quad \delta\psi_H = i\varepsilon_a[\lambda_V^a, \psi_H] \quad (2.14)$$

and the explicit form of the λ matrices yields the charges under F described in the table. The diagonal $U(1)$ gauge group acts trivially in all cases, and so the nontrivial gauge group actually is $F/U(1)$. Similarly there is a hypermultiplet, which we will distinguish as $\tilde{\phi}_H$, with a Chan-Paton matrix proportional to the identity and hence neutral with respect to the entire gauge group.

It is worth noting here that this physical structure has already been presented in the mathematics literature [16] in an existence proof for the metrics on the ALE spaces. In Ref. [16], there is a space M , a flat hyper-Kähler manifold, defined as the Γ -invariant subspace of a space $P = Q \otimes \text{End}(R)$. R is the regular representation of Γ and Q is the defining two-dimensional representation. There is a group of unitary transformations $U(|\Gamma|)$ naturally acting on R , and hence also on the space P by construction. The subgroup of these transformations which commutes with the action of Γ on P is a group denoted F . The group $F/U(1)$ acts nontrivially on the manifold M , preserving the three complex structures \mathcal{I} , \mathcal{J} , and \mathcal{K} of the hyper-Kähler manifold.

Returning to the physics, we see that remarkably these mathematical tools have a clear physical origin. The space P is exactly equivalent to our space of ψ_H hypermultiplets before projection, and the restriction to M is the projection (2.9). The quaternionic space M is the space of hypermultiplets displayed in the table. These hypermultiplets, with their

content of four scalars each, naturally display the quaternionic structure crucial to the construction of Ref. [16]. Physically the origin of this quaternionic structure lies in the form appearing in the four-dimensional background \mathbb{R}^4 [see Eq. (2.1)]. The positions of the D1-branes in this space are described by these hypermultiplets. Notice that the number of hypermultiplets is equal to $|\Gamma|$. There are also $|\Gamma|$ D1-branes present at the origin of the singular space. The gauge group F corresponds precisely to the unitary transformations appearing in the construction of Ref. [16]. (Indeed, we use the same notation, for simplicity.)

The scalar potential of our $D=2$, $\mathcal{N}=4$ world-volume theory can be written as

$$\sum_{a,b} \text{Tr}[\psi_H^a, \psi_H^b]^2 + 2 \sum_{a,i} \text{Tr}[\psi_H^a, \phi_H^i]^2 + \sum_{i,j} \text{Tr}[\phi_H^i, \phi_H^j]^2, \quad (2.15)$$

where in these sums $a, b = 1, 2, \bar{1}, \bar{2}$ and $i, j = 2, 3, 4, 5$. Here this commutator structure is inherited through the dimensional reduction (and Γ projection) of the ten-dimensional $U(|\Gamma|)$ Yang-Mills action.

Broadly speaking, there are two very distinct branches of the moduli space of vacua of the theory.⁷ That with $\psi_H = 0$ and $\phi_H \neq 0$ is the ‘‘Coulomb’’ branch, and the branch with $\psi_H \neq 0$ and $\phi_H = 0$, the ‘‘Higgs’’ branch.⁸

The Coulomb branch is the branch of moduli space where the gauge symmetry is generically $U(1)^r$, where $r = \sum_i n_i$ and n_i is the dimension of the i th irreducible representation R_i of Γ . This represents a family of D1-branes all living on the singular orbifold point in \mathbb{R}^4/Γ while moving independently in the x^2, x^3, x^4, x^5 directions.⁹ This generic situation comes from giving nonzero expectation values to the scalars ϕ_H in the Cartan subalgebra of $\text{Lie}(F)$ [ensuring that the third term in the potential (2.15) vanishes]. This produces mass terms for the vectors filling out the gauge group F , and breaking it to $U(1)^r$. Similarly most of the scalars become massive leaving massless those ϕ_H carrying only Abelian charges, as well as the neutral $\tilde{\phi}_H$. These scalars correspond to the

D1-brane positions in the x^2, x^3, x^4, x^5 directions.¹⁰

The Higgs branch will be the main interest for us in the rest of the paper as it is the branch concerned with the resolution of the singularity. In the generic situation when we give expectation values to the hypermultiplets in $\psi_H^{1,2}$, we will Higgs away most of the gauge group. Naively, as the dimension of the gauge group is also $|\Gamma|$, it may seem that we can Higgs everything away. This is not quite correct, as it is the group $F/U(1)$ which acts nontrivially on the hypermultiplets. So the diagonal $U(1)$ remains unbroken, and as well the corresponding $\tilde{\phi}_H$ multiplet remains massless. The unbroken $U(1)$ is the familiar gauge group for a single brane, and the $\tilde{\phi}_H$ hypermultiplet encodes its position in the x^2, x^3, x^4, x^5 directions. Similarly the space of allowed values for the $\psi_H^{1,2}$ hypermultiplets corresponds to the positions that the single D1-brane can occupy in the x^6, x^7, x^8, x^9 directions.

To see that the allowed space in which the D1-brane can propagate is indeed four dimensional is relatively easy. On this branch of moduli space, asking that the potential (2.15) vanishes for generic $\psi_H^{1,2} \neq 0$ will first of all require $\phi_H = 0$ except for the uncharged hypermultiplet (i.e., that with a Chan-Paton matrix proportional to the identity). With some algebra, one can rewrite the first term in the potential as a sum of three D terms:

$$(\text{Tr}[\lambda_V^a \{ \Psi_H^{1\dagger} \Sigma \Psi_H^1 + \Psi_H^{2\dagger} \Sigma \Psi_H^2 \}])^2, \quad (2.16)$$

where

$$\Psi_H^{1\dagger} = (\psi_H^1, -\psi_H^{2\dagger})$$

and

$$\Psi_H^{2\dagger} = (\psi_H^2, \psi_H^{1\dagger}) \quad (2.17)$$

are the natural $SU(2)_R$ doublets appearing in the quaternionic form (2.10). The λ_V^a are generators of $F/U(1)$. The three components, σ^i , of Σ are the Pauli matrices acting on the $SU(2)_R$ doublet space. Thus the vanishing of the individual D terms is a set of $3(|\Gamma|-1)$ real constraints on the available $4|\Gamma|$ -dimensional space parameterized by the hypermultiplets. This restriction gives us a space of overcounted vacua, since we have not accounted for the gauge symmetries. So we must impose another $|\Gamma|-1$ gauge-fixing conditions, which leaves us with a space of four real dimensions.

For these vacua on the Higgs branch then, the D1-branes have moved off the orbifold point, leaving a single D1-brane in open space. Equivalently, we can say that the $|\Gamma|$ D1-branes are now all away from the origin of \mathbb{R}^4 , but that the Γ projection relates them as images of one another, leaving only one independent D1-brane.

There is a natural way to characterize this space of vacua algebraically [28]. It is possible to determine a family of polynomials of the coordinates z^1 and z^2 which are invariant

⁷We are grateful to M. Strassler for a conversation which helped to clarify the field theory terminology.

⁸It is interesting that in the orbifold limit, there are no intermediate situations with both $\psi_H \neq 0$ and $\phi_H \neq 0$. This will no longer hold when the singularity is resolved as in the following section.

⁹Actually a complete exploration of the Coulomb branch would lead us to singularities, the discussion of which will take us beyond the scope of this paper. A complete discussion involves the study of the scalars coming from the closed string sector (see later). These include a family of theta angles Θ_i , one for every $U(1)$ in the gauge group, and families of other scalars. In leaving them out of the discussion so far, we have tacitly assumed that we have set them all to zero. In fact, it has been shown that the orbifold retains nonzero values for the θ angles [31]. It is sufficient to avoid the singularities in the Coulomb branch this way by keeping [12] the $\Theta_i \neq 0$. Tuning the theory further by setting all of the closed string scalars to zero takes us to regions of moduli space containing singularities which are of interest (e.g., the strings become tensionless), but are not the subject of this paper.

¹⁰Note that with the A_k series, for which the gauge group F is entirely Abelian, on the Coulomb branch we are giving expectation values to all of the hypermultiplets ϕ_H and $\tilde{\phi}_H$, and none of the gauge symmetries are broken.

under the action of the group Γ on the space \mathbb{C}^2 . In each case (A , D , or E) there are three such distinguished invariants, x , y , and z . These invariants satisfy a well-known polynomial relation in each case $\mathcal{W}(x,y,z)=0$, where

$$\begin{aligned}\mathcal{W}_{A_k} &= xy + z^{k+1}; \\ \mathcal{W}_{D_k} &= x^2 + y^2z + z^{k-1}; \\ \mathcal{W}_{E_6} &= x^2 + y^3 + z^4; \\ \mathcal{W}_{E_7} &= x^2 + y^3 + yz^3; \\ \mathcal{W}_{E_8} &= x^2 + y^3 + z^5.\end{aligned}\tag{2.18}$$

Algebraically, the singular ALE space \mathbb{C}^2/Γ is described as a variety \mathcal{V}_0 in \mathbb{C}^3 described by the vanishing locus $\mathcal{W}(x,y,z)=0$, where x , y , and z are coordinates on \mathbb{C}^3 .

The space of hypermultiplets $\psi_H^{1,2}$, as acted on by the gauge group F , inherits much of the structure just described. Indeed, the procedure of finding the gauge invariant family of allowed vacua defines three combinations of hypermultiplets X , Y , and Z which are isomorphic to the x , y and z of the algebraic discussion.¹¹

In this way we make our first contact here with a large body of classic results concerning the mathematical nature of the A - D - E singularities. Also, we have actually described a special case of the “hyper-Kähler quotient” technique [17], to recover the singular \mathbb{C}^2/Γ spaces. The next step is to introduce the remaining elements of the construction—arising in the closed string sector—which will allow us to construct the deformed spaces, complete the description of the hyper-Kähler quotient, and discuss metrics on the ALE spaces.

III. D-STRINGS ON ALE SPACES: THE BLOW-UP

The closed string sector provides the remaining fields of relevance. Our discussion in the previous section was restricted to the case when their expectation values were set to zero.

A. The closed string sector

The closed string spectrum is much more familiar in this context. The traditional route to the spectrum is via orbifold techniques.

In general, the orbifold procedure is complicated somewhat by the non-Abelian nature of the group Γ (only for the A_k series is it Abelian). Using the naive twisted sectors would not lead to a modular invariant theory, as sectors twisted by elements in the same conjugacy class can mix [10]. The procedure for computing twisted sectors thus takes into account the structure of the conjugacy classes discussed above. Each conjugacy class ultimately contributes a space-time field with well-defined transformation properties under the Lorentz group.

We are interested in that part of the closed string spectrum which consists of the scalars corresponding to the moduli we

use to blow up the orbifold. We can identify these scalars quite quickly using knowledge of the algebraic resolution of the ALE spaces.

In the resolved space, there is an algebraic description (see, for example, Ref. [28]) of the region which will become the singular point in the blow-down in terms of a family of two-cycles. We can deduce what scalars arise in the closed string theory by contracting the various rank two tensor fields on these cycles. These will be the fields which arise in the orbifold limit if we chose to enumerate the spectrum directly.

Deformation of \mathcal{V}_0 is done via elements of the ring of polynomials $\mathcal{R}=\mathbb{C}^3[x,y,z]/\partial\mathcal{W}$. Just away from the singular limit, this deformation defines a map ρ from the smooth “minimally resolved” variety \mathcal{V} to the singular variety \mathcal{V}_0 which is an isomorphism everywhere except at the singular point itself $\{0\}$. The neighborhood which maps to the singular point, $\rho^{-1}(\mathcal{V}_0-\{0\})$, (the “exceptional divisor”) is described as a series of algebraic curves \mathbb{P}^1 , or two-cycles \mathbf{c}_i (essentially two spheres S^2) which have an intersection matrix which is the (negative) Cartan matrix: $\mathbf{c}_i \cdot \mathbf{c}_j = -2\delta_{ij} + a_{ij}$, where a_{ij} is the Dynkin adjacency matrix we encountered earlier. Here, i and j run over only the $(n-1=k, k, 6, 7, \text{ or } 8)$ nontrivial representations of Γ , i.e., there is no two cycle \mathbf{c}_0 corresponding to the extended vertex of the Dynkin graph. Therefore, the two cycles have the structure of the Dynkin diagrams we found earlier. Each nontrivial conjugacy class has a two cycle associated with it.

The ten-dimensional type IIB theory contains the following rank two tensors: There is the metric G and the antisymmetric tensor field $B^{(2)}$, from the Neveu-Schwarz–Neveu-Schwarz NS-NS sector. The R - R sector supplies the two form $A^{(2)}$. Therefore, from each conjugacy class, we get (in the flat six dimensions) two scalars (corresponding to θ angles) arising from contracting the forms $A^{(2)}$ and $B^{(2)}$ with the two cycles. Meanwhile, three scalars come from the metric G . This is the scalar content of a tensor multiplet of $\mathcal{N}=2$ supersymmetry¹² in $D=6$: $(\mathbf{1},\mathbf{3})+5(\mathbf{1},\mathbf{1})$. The self-dual antisymmetric tensors in these multiplets arise from contracting the R - R sector self-dual four form $A^{(4)}$ on the two cycles. In this way, each nontrivial conjugacy class gives rise to a complete tensor multiplet.

Another, perhaps more contemporary way of seeing how to obtain this result is to realize that those twist fields are precisely the set of closed string fields which couple to the various string solitons arising from wrapping the self-dual D3-brane of ten-dimensional IIB theory on the two cycles. (See Ref. [12] for the A_1 case of this.) Here, this reduction procedure gives rise to $n-1$ different types of D-string in the theory, plus one more type corresponding to the “pure” D1-brane (i.e., the familiar string soliton which couples directly to $A^{(2)}$).

Continuing the organization of our physics by the representation theory of Γ , we see that there is exactly one species of D-string in the problem for every conjugacy class. Returning to the discussion in Sec. II D, it is natural that the $|\Gamma|$

¹¹This is a result of geometric invariant theory [32].

¹²We denote the transformation properties under the $SU(2)\times SU(2)$ little group.

D1-branes living on the fixed point in the Coulomb branch can be labeled according to which species they belong to, resulting in multiplicities which furnish a physical version of the “regular” $|\Gamma|$ -dimensional representation R of Γ .

This arrangement is the minimum requirement for there to be a flat direction in the potential corresponding to moving off the fixed point and into open space \mathbb{R}/Γ . The $|\Gamma|$ strings coalesce into one string which carries zero charge under the twisted sector fields. This single string is to be identified with the “pure” D1-brane, as the other types of string cannot exist in isolation off the fixed point given that they couple to twisted sector closed string fields, which are localized there. As mentioned earlier it would be consistent to place more D-strings on the singularity, but only multiples of $|\Gamma|$ can move off the singularity [29,30], in the manner just described.

Of the twisted sector tensor multiplet fields which we just discussed, it is the trio of NS-NS scalars which will interest us, for the purposes of performing the blow-up. They transform as a triplet under the $SU(2)_R$ symmetry acting on the x^6, x^7, x^8, x^9 space. We shall denote them \mathbf{D}^i , the label i running over the different conjugacy classes [i.e., the vertices in the (unextended) Dynkin diagram]. Here the $SU(2)_R$ symmetry has become the R symmetry of the $D=2, \mathcal{N}=4$ world-volume theory (see Sec. II A).

B. D -terms, D -flatness, and the moment map

The closed string fields \mathbf{D}^i couple into the world-volume theory via Fayet-Illiopoulos (FI) terms [18]. The R -symmetry $SU(2)_R$ of the $D=2$ theory with $\mathcal{N}=4$ supersymmetry requires that the D terms appear in triplets as in Eq. (2.16). Gauge invariant FI terms could be written for every $U(1)$ in the theory. However, from the closed strings there is a trio of NS-NS scalars \mathbf{D}^i for every nontrivial conjugacy class of Γ , and each class is associated with a vertex in the Dynkin diagram (the extended vertex is not included here). Hence an FI term appears only for the $U(1)$ subgroups of each unitary group appearing for every vertex of the Dynkin diagram [i.e., each of the $U(1)$ ’s in $F/U(1)$].

The relevant terms in the potential (2.16) containing the FI terms are

$$(\text{Tr}[\lambda_V^i \{\Psi_H^{1\dagger} \Sigma \Psi_H^1 + \Psi_H^{2\dagger} \Sigma \Psi_H^2\}] - \mathbf{D}^i)^2, \quad (3.1)$$

where the λ_V^i are generators of the $U(1)^r$ subgroup of $F/U(1)$ where $r=k, 6, 7$, or 8 , depending upon Γ . Of course, the terms in Eq. (2.16) corresponding to the generators λ_V^a not in the Abelian part of the group $F/U(1)$ are still present unchanged in the full potential. Again, this structure appears in the mathematics literature. The existence of a triholomorphic¹³ symmetry $[F/U(1)]$ acting on the hyper-Kähler manifold M guarantees the existence of a map μ from M to $\mathbb{R}^3 \otimes \text{Lie}[F/U(1)]$. This map is called the “moment map.” The \mathbb{R}^3 valuedness is simply the projection onto the three complex structures \mathcal{I}, \mathcal{J} , and \mathcal{K} of the manifold M , and

so μ has three components which can be arranged as a vector of $SU(2)_R$. The moment map can be written as [16]

$$\begin{aligned} \mu_1(\psi_H^1, \psi_H^2) &= [\psi_H^2, \psi_H^1] + [\psi_H^{1\dagger}, \psi_H^{2\dagger}], \\ \mu_2(\psi_H^1, \psi_H^2) &= i[\psi_H^2, \psi_H^1] - i[\psi_H^{1\dagger}, \psi_H^{2\dagger}], \\ \mu_3(\psi_H^1, \psi_H^2) &= [\psi_H^1, \psi_H^{1\dagger}] + [\psi_H^2, \psi_H^{2\dagger}]. \end{aligned} \quad (3.2)$$

Hence we have $\mu = \Psi_H^{1\dagger} \Sigma \Psi_H^1 + \Psi_H^{2\dagger} \Sigma \Psi_H^2$. As defined, μ is a vector with components in $\text{Lie}[F/U(1)]$ which we can project onto a chosen basis. In Eqs. (2.16) and (3.1), with the trace we project it explicitly onto the basis vectors λ_V^a .

C. The moduli space of vacua: Part II

As stated earlier, the Higgs branch of vacua concerns us here.¹⁴ These vacua are now characterized by the equations

$$\text{Tr}[\lambda_V^a \{\Psi_H^{1\dagger} \Sigma \Psi_H^1 + \Psi_H^{2\dagger} \Sigma \Psi_H^2\}] = 0 \quad (3.3)$$

and

$$\text{Tr}[\lambda_V^i \{\Psi_H^{1\dagger} \Sigma \Psi_H^1 + \Psi_H^{2\dagger} \Sigma \Psi_H^2\}] = \mathbf{D}^i. \quad (3.4)$$

This set of equations has a natural interpretation in the mathematics literature. The hyper-Kähler quotient construction [17] involves a choice of numbers $\zeta^i \in \mathbb{R}^3 \otimes \mathcal{Z}$, where \mathcal{Z} is the center of the Lie algebra in which the moment map μ takes its values $[\text{Lie}(F/U(1))]$ for us. The space $\mu^{-1}(\zeta^i) \subset M$ is an invariant submanifold under F , by virtue of the fact that the center \mathcal{Z} is defined as the set of F -invariant elements. The quotient space $\mathcal{M} = \mu^{-1}(\zeta^i)/[F/U(1)]$ is the hyper-Kähler quotient obtained from F and M . In Ref. [8], it is proven that \mathcal{M} , as obtained in this way, is indeed hyper-Kähler.

Our equations (3.3) and (3.4), characterizing the Higgs branch of the moduli space are simply equations telling us the restriction of the hypermultiplet space M to the subspace $\mu^{-1}(\zeta^i)$. The ζ^i parameterizing the center of $\text{Lie}[F/U(1)]$ are simply the closed string scalar fields \mathbf{D}^i which enter into the construction via FI terms.

The rest of the construction continues as follows. By imposing the potential vanishing conditions we have restricted ourselves to an $F/U(1)$ invariant subspace of M , our flat space of hypermultiplet values. Finally we restrict ourselves to the vacua which are not related by gauge transformations. This completes for us the hyper-Kähler quotient.

To make sure that the counting works out, we can check that the right number of conditions have been imposed: The real dimension of the space M is $4|\Gamma|$ which is four times the number of hypermultiplets. The moment map is valued in $\mathbb{R}^3 \otimes \text{Lie}[F/U(1)]$ and the dimension of F is $|\Gamma|$, so imposing

¹³Here, “triholomorphic” simply means that it preserves the three complex structures \mathcal{I}, \mathcal{J} , and \mathcal{K} associated with the hyper-Kähler structure of M .

¹⁴Note that for generic values of the \mathbf{D}^i solving the following two equations will lead to setting $\phi_H = 0$ in order that the full potential vanish. However, for a partial resolution of the singularity in which some of the \mathbf{D}^i vanish, the vacua may have both $\psi_H \neq 0$ and $\phi_H \neq 0$.

the D -flatness conditions gives $3 \times (|\Gamma| - 1)$ conditions. The final gauge-fixing restricts us with another $|\Gamma| - 1$ conditions, leaving a $4|\Gamma| - 4(|\Gamma| - 1) = 4$ -dimensional space.

This four-dimensional hyper-Kähler space is the resolved ALE manifold. We see explicitly how the amount of the resolution is controlled by the expectation values of the closed string twisted sector fields \mathbf{D}^i , a fact that we knew algebraically from orbifold techniques.

To summarize, we now have a clear understanding of the mechanics of how fundamental type IIB string theory probes the metric geometry of an orbifold, as the problem maps (under strong or weak coupling duality) to one of finding vacua of a field theory associated to a D1-brane's world sheet. The metric on the field theory's space of vacua, parameterized by the hypermultiplets, is precisely the space-time metric that the D1-brane sees as it moves around in the ALE space.

IV. ON FINDING EXPLICIT METRICS

In the previous sections we have seen the mechanism by which we can directly extract the details of the geometry of the space that strings propagate in. The $4|\Gamma|$ -dimensional metric on the (unconstrained) space of hypermultiplets is simply

$$ds_M^2 = \text{Tr}\{d\psi_H^{1\dagger} d\psi_H^1 + d\psi_H^{2\dagger} d\psi_H^2\}. \quad (4.1)$$

The next natural step is to find the metric on the gauge invariant submanifold. This is done by replacing the derivatives “ d ” above by covariant derivatives, minimally coupling the ψ_H to the two-dimensional gauge fields $A = A_a \lambda_V^a$:

$$d\psi_H \rightarrow D\psi_H = d\psi_H + iA_a[\lambda_V^a, \psi_H]. \quad (4.2)$$

The coset metric can now be obtained by simply choosing a gauge and integrating out¹⁵ the gauge fields A . After imposing the $3(|\Gamma| - 1)$ constraints from the moment map, we have the final metric in which the ($r=k$, k , 6 , 7 or 8) arbitrary vectors \mathbf{D}^i appear as parameters. The freedom to make different gauge choices translates into (a subset of) the freedom to make coordinate redefinitions in the final metric. Failure to completely fix the gauge will result in N redundant coordinates in the metric where N is the dimension of the *continuous* subgroup left unfixed. We stress continuous because there is always the possibility that there is some *discrete* subgroup of the gauge symmetry left unfixed. Then there

will be no redundant coordinates, but it will be necessary to make discrete identifications on the space defined by the final choice of coordinates. Notice that the most natural discrete subgroup of $F/U(1)$ is Γ . We expect that the discrete identification by Γ present in the ALE spaces will arise in the explicit metrics by precisely such an incomplete gauge fixing. (See Ref. [33] for such a coset example.)

In this way, we can in principle write down all of the metrics on the ALE spaces. Indeed, this is by now a well-known exercise for the Eguchi-Hanson metric, and it was explicitly carried out in this context in Ref. [24] for the A_k series. In this case, a number of fortuitous occurrences make the problem relatively straightforward.

For example, the Abelian nature of the problem translates into a simple structure for the relations imposed by the moment map: A basis can be found in which there is a *gauge invariant* combination of hypermultiplets of the form $(\psi_H^\dagger \Sigma \psi_H)^i$, associated to the i th vertex in the A_k Dynkin diagram, as the superscript denotes. The $k+1$ moment map relations are then simply of the form: $(\psi_H^\dagger \Sigma \psi_H)^i - (\psi_H^\dagger \Sigma \psi_H)^{i-1} = \mathbf{D}^i$, where $i=1, \dots, k+1$, reflecting the cyclic structure of the extended Dynkin graph for A_k .

Further to this, the “integrating out of the gauge fields” procedure produces a metric which is already partially written in terms of those same gauge invariant combinations of hypermultiplets in terms of which the moment map is written. This makes straightforward the step of fixing a gauge (i.e., there is no need to) and finding coordinates simultaneously adapted to the metric and the subsequent substitution of the moment map constraints into it. Specifically [24], to obtain the standard form (1.1), the coordinate $\mathbf{y} = (\psi_H^\dagger \Sigma \psi_H)^0$, and the coordinates of the multicenters are $\mathbf{y}_i = \sum_{a=1}^i \mathbf{D}^a$.

Even without those pleasant structures, the problem is made easier by the fact that the simplest example A_1 is relatively easy to handle algebraically (the starting metric is only eight dimensional), and so it would be possible to work by hand to discover the gauge and coordinate choices had they been difficult to find. The cyclic nature of the Dynkin diagrams means that those choices then generalise easily to the full A_k series, as above.

Turning to the D and E series, it is easy to see why there are currently no (to our knowledge) closed expressions for their metrics in the literature. Hardly any of the simplifying facts mentioned above appear to be true in these cases, as one finds after some exploration. The moment map does not suggest an obvious set of choices for gauge invariant coordinates on the final space. Furthermore, it is not clear what gauge slice to choose, in the absence of such coordinates. From experience with the A_k series, one might expect that there might be clues from the metric one gets after integrating out the gauge fields. This may be true, but given that for the simplest example, D_4 , that metric is 25 dimensional,¹⁶ it is a daunting task to find the correct choices by eye. However, it might turn out that there is some symmetry [such as $SU(2)_R$] which might serve as guidance, organizing the alge-

¹⁵In general, there are of course α' (inverse D-string tension) corrections to this saddle point procedure. However, there are no corrections from higher orders in string coupling. There are two ways to see this. (i) In the equivalent $D=6$, $\mathcal{N}=1$ system, the dilaton arises in a tensor multiplet, which becomes a vector multiplet in lower dimensions. As our metric is on the hypermultiplet moduli space, string loops (controlled by the dilaton) do not affect it. (ii) The strong or weak self-duality of the theory assures us that since the background metric which the fundamental string sees is not corrected at higher order in α' (it's a $D=2$, $\mathcal{N}=4$ nonlinear σ model), then the metric seen by the D-string is not corrected by string loops.

¹⁶To obtain this metric requires an inversion of a 7×7 matrix of algebraic expressions. This already a large amount of algebra.

bra and help solve the problem. One might hope that the coordinates would then generalize to the full D_k series, once found.¹⁷

The astute reader by now should have begun to suspect one of two possibilities: Either (i) we have managed to surpass all of the difficulties described above, and are only emphasizing the magnitude of the problem to make the presentation of an elegant solution more dramatic or (ii) we have failed to find the explicit form of the metrics, and are reporting our observations in the hope that they may help others interested in the problem.

Unfortunately, it is the latter (ii) which is the case. Although perhaps only of aesthetic interest, finding closed forms for the metrics generalizing Eq. (1.1) is an intriguing problem, and we still hope that a solution can be found in the near future.

V. CLOSING REMARKS

This addition of D-brane technology and strong/weak coupling duality to traditional closed string methods in order to understand nontrivial string theory backgrounds is quite satisfying. The duality map allowed us to relate the question of how strings probe the geometry of the ALE spaces to the study of vacua of two dimensional $\mathcal{N}=4$ supersymmetric field theories on D1-brane world volumes. The appropriate two-dimensional theories derived here had a spectrum directly related to the work of Kronheimer.

The discussion presented in this paper was restricted to the vacua of two-dimensional field theories because the probes were D1-branes. In Ref. [34], a study is presented of the vacua of $\mathcal{N}=4$ supersymmetric *three-dimensional* field theories with the (Kronheimer) spectrum discussed here.¹⁸ Although the context of that paper is field theory, one should be able to obtain a D-brane realisation of that scenario *via* T -dualizing our present situation along the x^2 , x^3 , x^4 , or x^5 coordinate. This will result in type IIA theory with D2-brane probes, whose three-dimensional world volumes will provide a natural setting for the discussion of Ref. [34]. The subsequent interpretation of the “mirror symmetry” duality found in Ref. [34] for those three-dimensional models should be very interesting.

Returning to the two-dimensional (world-sheet) setting of this paper, the most direct approach to discussing a full (perturbative) string theory solution representing a closed string propagating in a certain background is via the conformal field theory on the world sheet. However, knowledge of the full conformal field theory is not always a luxury which is available to us. Furthermore, even the knowledge of a conformal field theory description of a background at some point in the moduli space is not always enough to discover important global aspects of the moduli space of backgrounds. To circumvent these limitations, Witten studied how we can ex-

tract useful information about the string propagation on a nontrivial background (and often a whole moduli space of backgrounds) by studying two-dimensional gauged linear σ models [19]. These linear σ models do not represent solutions to the closed string equations of motion (i.e., conformal field theories), but are connected to such solutions by renormalisation group flow to an infrared fixed point.

Many characteristic properties of the spacetimes we wish to study (interpreted as properties of the two-dimensional world-sheet theories) can be incorporated easily by hand into the linear σ model. Under renormalization group flow, these properties survive to become the required properties of the conformal field theory representing the solution. In this indirect way, we can study many aspects of these nontrivial string theory backgrounds by manipulations of the linear σ model.

It is now clear that strong/weak coupling duality and D-branes shed new light upon the role of these linear σ models: They tell us the precise circumstances under which the gauged linear σ models, as world-sheet theories of dual strings *are* solutions representing string theory backgrounds. The new ingredient is the possibility of adding open string sectors (D-branes). So duality knows how to move along the renormalization group trajectories the existence of which we relied upon previously.

The question arises as to whether these observations have taught us anything about the “larger picture” concerning the theory underlying string theory and eleven-dimensional supergravity. On the one hand, we can take the conservative point of view that D-brane probes simply give us a new means of building linear σ models, which is already progress.

On the other hand, let us suggest the following possibility: As strong/weak coupling duality seems to relate different points on the renormalization group flow trajectory, we might interpret this as a new clue as to the nature of duality itself, or at least a new handle on it. For every conformal field theory (representing a closed string background, say), there is a whole universality class of σ models which flow to it in the infrared. Perhaps for every one of these models, there is implied a duality transformation to a new theory in which the σ model spectrum represents a valid configuration. The new dual theory could be either the same string theory, a new one, or something else.

Let us briefly engage in a little revisionist history to make the point: The knowledge [12] that the linear σ model containing Kronheimer’s data flows to the conformal field theory representing the type IIB string in an ALE background would imply the existence of a duality transformation to a new theory where the linear σ model is a solution. Researchers examining the σ model in that light would then discover that it was the theory of a D1-brane of the same type IIB theory on the ALE background. A similar story might be told for the SO(32) theory: The knowledge [21] that the linear σ model with the ADHM data flows to the conformal field theory representing the SO(32) heterotic string in a Yang-Mills instanton background would suggest to physicists the existence of a dual theory giving rise to that linear σ model. This eventually turns out to be [6] the SO(32) type I theory with D1- and D5-branes. It would certainly be interesting to investigate this suggestion further.

¹⁷In this instance, it might be that D_4 is too simple a first example, as it has none of the (2,2) hypermultiplets, the main structure which generalizes in the higher k examples. Note, however, that starting metric of D_5 is 48 dimensional.

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