Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories

F. Lizzi*
Dipartimento di Scienze Fisiche, Università di Napoli Federico II and INFN,Sezione di Napoli, Italy and Theoretical Physics, 1 Keble Road, Oxford, OX1 3NP, United Kingdom

G. Mangano†
Dipartimento di Scienze Fisiche, Università di Napoli Federico II and INFN, Sezione di Napoli, Italy

G. Miele‡
Dipartimento di Scienze Fisiche, Università di Napoli Federico II and INFN, Sezione di Napoli, Italy and Theoretical Physics, 1 Keble Road, Oxford, OX1 3NP, United Kingdom

G. Sparano§
Dipartimento di Scienze Fisiche, Universitá di Napoli Federico II and INFN, Sezione di Napoli, Italy
(Received 4 October 1996)

In this paper we study the structure of the Hilbert space for the recent noncommutative geometry models of gauge theories. We point out the presence of unphysical degrees of freedom similar to the ones appearing in lattice gauge theories (fermion doubling). We investigate the possibility of projecting out these states at the various levels in the construction, but we find that the results of these attempts are either physically unacceptable or geometrically unappealing. [S0556-2821(97)02310-2]

PACS number(s): 11.15.Tk. 02.40.−k

I. INTRODUCTION

Noncommutative geometry [1] provides a powerful algebraic scheme to handle a large variety of geometrical frameworks. Its application to gauge theories, and in particular to the standard model (SM) of strong and electroweak forces [1–4], is a unique original way to fully geometrize the interaction of elementary particles. More recently, attempts have been made towards a unification with gravity as well [5,6]. In noncommutative geometry the role that is classically played by a manifold, seen as an ensemble of points, is taken by a $*$-algebra, which in the *commutative* case is just the algebra of continuous complex valued functions, but in general can be a generic non-Abelian $*$-algebra. This algebra is then represented as bounded operators on a Hilbert space on which a generalized Dirac operator D also acts, providing all information usually carried by a metric structure [1].

A very appealing aspect of the Connes-Lott (CL) version of the SM and of his subsequent versions and improvements (for a review see [7]) is that the Hilbert space on which the algebra and the generalized Dirac operators act is the space of physical fermions. In the model the fermionic action corresponds to a generalization of the (interaction) Dirac action $\bar{\psi}(\partial\!\!\!/+A\!\!\!/)\psi$ while the bosonic one is obtained by taking the trace of the squared curvature two-form which is constructed out of the algebra $C^\infty(M,\mathbb{C})\otimes\mathcal{A}_F$, where $C^\infty(M,\mathbb{C})$ is the algebra of smooth complex valued function on the (Euclidean) four-dimensional space-time manifold M, and $\mathcal{A}_F=M_3\oplus\mathbb{H}\oplus\mathbb{C}$, with M_3 and $\mathbb{H}$ the algebra of 3×3 complex matrices and of quaternions, respectively. The result generalizes the Yang-Mills Euclidean action $(1/4)F^{\mu\nu}F_{\mu\nu}$. Most remarkably, in this framework, the scalar Higgs field appears as the connection in the internal (noncommutative) space while its action, including the usually *ad hoc* quartic potential, naturally appears as the square of the curvature in the internal space. This model is quite constrained and, with the choice of the Hilbert space composed by the known fermions only, seems to point to some unique features of the SM, forbidding, for example, standard grand unified theories [8].

Quite recently Chamseddine and Connes (CC) [6] have also proposed a different definition of the bosonic action, based on the so-called *spectral action principle*, which from the generalized Dirac operator only, now including also the gravitational spin connections, produces the SM action coupled to Einstein plus Weyl gravity.

In both approaches the Hilbert space of fermions $\mathcal{H}$ seems to play a crucial role. On the one side it is necessary in order to represent the algebra, which gives the topology of space, though this last feature can be recovered independently of whether an explicit representation is assigned or not. On the other side $\mathcal{H}$ is definitely necessary for the introduction of the D operator, which encodes, as mentioned, all information on the metric.

The structure chosen for $\mathcal{H}$ is the one of a tensor product of a continuous infinite dimensional factor, the space $L^2(S_M)$ of square integrable Dirac spinor over M, which is related to space-time, times a finite dimensional space, which describes the physical particle degrees of freedom, *including*

*Electronic address: Lizzi@na.infn.it
†Electronic address: Mangano@na.infn.it
‡Electronic address: Miele@na.infn.it
§Electronic address: Sparano@na.infn.it

helicity. In this paper we point out that, for a chiral gauge theory built up in both the CL and CC approaches, this choice for $\mathcal{H}$, though imposed by the initial ansatz for the algebra, gives rise to two problems. One due to an overcounting of the physical degrees of freedom, the other, more serious, is connected to the presence of unphysical degrees of freedom. These problems are basically due to the fact that the helicity degrees of freedom are contained both in the spinor, and in the finite dimensional space.

As we will see the states responsible for the overcounting are a consequence of the *duplication* of degrees of freedom in the finite part of the Hilbert space. In all gauge models, in fact, in which more than one factor of the group acts on some fermion multiplet, in order to correctly represent the algebra, it is necessary to include for all fermions the corresponding charge-conjugated states. Once the action is obtained, a simple *identification* procedure, which is to say, a quotient of the initial Hilbert space, is sufficient to remove this unwanted duplication of particles.[1]

On the other side, the unphysical degrees of freedom look like mirror fermions, namely states which couple to the chiral factors of the gauge group but which have, however, wrong chiral quantum numbers. For the case of the minimal SM, for example, we will see how right-handed Dirac spinors would be coupled in general to the $SU(2)_L$ gauge bosons. The presence of these unphysical fermions seems more closely related to the choice for $\mathcal{H}$ in the form of a tensor product. What is implicitly done in the literature is to throw away at the very end the unwanted terms from the action. One wonders therefore if there is a more geometrical way of getting rid of the unphysical fermions. Actually they can be eliminated by a *projection* onto a proper subspace of $\mathcal{H}$. This projection, however, may lead to different results if performed at different stages of the noncommutative geometric construction of a gauge theory, since the corresponding projector operators do not commute with all the intermediate steps of such a construction. For example, since they project onto definite chirality states in $L^2(S_M)$, they do not commute with the Dirac operator, which contains Dirac γ matrices, as well as with the connection one-forms.

As we will see in the following, using the projection at the level of the algebra leads to a trivial result. It is also possible to perform the projection, for the bosonic part, at the level of the curvature two-form. In this way the action is obtained by tracing the squared curvature over the physical Hilbert subspace only. As far as the fermionic action is concerned, this procedure gives the same result obtained in the literature. Instead, in the bosonic sector, in addition to the usual kinetic term, topological terms of the form $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ will generally appear. However, the combination of the two terms in the curvature is such that only the self-dual or anti-self-dual components of the field survive. This means that half of the gauge physical degrees of freedom has been projected out as well.

It seems therefore that the only consistent procedure to obtain the action of the SM and, more generally, of any chiral gauge theory, is to just neglect the unwanted mirror states in the action. This *ad hoc* restriction of the action just at the very end of the powerful noncommutative construction is quite unsatisfactory and probably, a different, less trivial choice for the structure of the fermion Hilbert space is required, possibly on the lines of a supersymmetric generalization [10] or some even more radical changes [11].

The paper is organized as follows. In Sec. II we review the CL construction and the concept of spectral triple. We also discuss the problem of redundant fermionic degrees of freedom and, in particular, of mirror fermions. In Sec. III we first discuss a simple model based on a spontaneously broken $SU(2)_L \otimes SU(2)_R$ gauge symmetry, showing how the possible projections work. The case of the standard model is then considered. Section IV is devoted to a similar study in the CC model for the simple $SU(2)_L \otimes SU(2)_R$ unbroken case. Finally, in Sec. V we give our conclusions and outlook.

II. SPECTRAL TRIPLE, THE CONSTRUCTION OF GAUGE THEORIES AND FERMION HILBERT SPACE

The basic ingredient of the noncommutative geometry construction is the so-called *spectral triple*, denoted by $(\mathcal{A},\mathcal{H},D)$, where $\mathcal{A}$ is an involutive $*$-algebra faithfully represented by bounded operators on the Hilbert space $\mathcal{H}$, and D is a self-adjoint operator with compact resolvent (generalized Dirac operator). The spectral triple becomes a *real spectral triple* if an antilinear isometry J of $\mathcal{H}$, obeying suitable relations, is introduced [3]. Note that J can be seen as a generalized *CPT* operator.

In this framework a gauge theory, with group of invariance G, is fully geometrized and it is on the same footing as gravity. The former, in fact, emerges as the gauge theory of the inner automorphisms of the algebra

$$\mathcal{A}=C^\infty(M,\mathbb{C})\otimes\mathcal{A}_F, \tag{2.1}$$

where $\mathcal{A}_F$ is the smallest $*$-algebra containing G as the group of its unitary elements. Analogously, the latter can be seen as the gauge theory of diffeomorphisms of M, which are nothing but the outer automorphisms of $\mathcal{A}$.

As far as the Hilbert space $\mathcal{H}$ is concerned a suitable choice is to take

$$\mathcal{H}=L^2(S_M)\otimes\mathcal{H}_F, \tag{2.2}$$

where $L^2(S_M)$ is the space of square integrable spinors defined on M, and $\mathcal{H}_F$ is a finite dimensional linear space corresponding to all discrete degrees of freedom, like chirality, flavor, charge, etc. Finally, the generalized Dirac operator is

$$D=\partial\!\!\!/\otimes\mathbb{I}+\gamma_5\otimes D_F, \tag{2.3}$$

with D_F denoting the selfadjoint fermion mass matrix. In this way, the real spectral triples $(\mathcal{A},\mathcal{H},D)$ is the tensor product of two real triples, one which is the continuous (space-time) part $(C^\infty(M,\mathbb{C}),L^2(S_M),\partial\!\!\!/)$ and another for the internal part $(\mathcal{A}_F,\mathcal{H}_F,D_F)$.

Given a generalized Dirac operator D, the gauge connection is written as

$$A=\sum_i \beta_i[D,\alpha_i]\equiv\sum_i \beta_i d\alpha_i, \tag{2.4}$$

where α_i and β_i are elements of $\mathcal{A}$ such that A is Hermitian, and the differential d is defined by $d\alpha\equiv[D,\alpha]$. From the

[1]It is worth noticing at this point that such a problem is absent in the so called "old version" of the model [1,9].

connection A, one defines the curvature θ as[2]

$$\theta \equiv dA + A^2, \tag{2.5}$$

and thus the bosonic action is obtained as

$$\mathcal{S}_{\mathcal{B}} = \mathrm{Tr}\ \theta^2. \tag{2.6}$$

Note that the trace includes the integration over M. For the fermionic action one has instead

$$\mathcal{S}_{\mathcal{F}} = \langle \psi, (D + A + JAJ)\psi \rangle. \tag{2.7}$$

It is worth noticing that exterior algebra emerges from the Clifford algebra as its antisymmetric part, and thus it is crucial that the Hilbert space contains as its continuous part the space of Dirac spinors $L^2(S_M)$. In fact, the use of Weyl spinors on a four-dimensional space-time M would lead to an incorrect result, since the algebra generated by the Pauli matrices plus identity is only four-dimensional, and thus is not sufficient to faithfully represent the corresponding 16-dimensional exterior algebra. In particular, all the two forms would be junk forms and thus θ would be trivial.

The choice (2.2) for the Hilbert space has problems in the case of theories, like the standard model, where fermions with different chirality transform independently under the gauge group. In the new formulation of the SM in the manner of Connes and Lott the discrete part of the Hilbert space $\mathcal{H}_F$ results:

$$\mathcal{H}_F = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c, \tag{2.8}$$

where

$$\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}), \tag{2.9}$$

$$\mathcal{H}_R = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}), \tag{2.10}$$

and $\mathcal{H}_{L,R}^c$ are the corresponding spaces for antiparticles,

$$\mathcal{H}_R^c = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}), \tag{2.11}$$

$$\mathcal{H}_L^c = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}). \tag{2.12}$$

In this framework a natural basis is given by

$$\begin{pmatrix} u_\alpha \\ d_\alpha \end{pmatrix}_L, \quad \begin{pmatrix} c_\alpha \\ s_\alpha \end{pmatrix}_L, \quad \begin{pmatrix} t_\alpha \\ b_\alpha \end{pmatrix}_L, \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \tag{2.13}$$

$$\begin{matrix} (u_\alpha)_R, & (c_\alpha)_R, & (t_\alpha)_R \\ (d_\alpha)_R, & (s_\alpha)_R, & (b_\alpha)_R, \end{matrix} \quad (e)_R, \ (\mu)_R, \ (\tau)_R, \tag{2.14}$$

$$\begin{pmatrix} u_\alpha^c \\ d_\alpha^c \end{pmatrix}_R, \quad \begin{pmatrix} c_\alpha^c \\ s_\alpha^c \end{pmatrix}_R, \quad \begin{pmatrix} t_\alpha^c \\ b_\alpha^c \end{pmatrix}_R, \quad \begin{pmatrix} \nu_e^c \\ e^c \end{pmatrix}_R, \quad \begin{pmatrix} \nu_\mu^c \\ \mu^c \end{pmatrix}_R, \quad \begin{pmatrix} \nu_\tau^c \\ \tau^c \end{pmatrix}_R, \tag{2.15}$$

[2]Note that the d operator so defined is not nilpotent and hence a quotient is necessary in order to obtain the correct differential algebras [1,9]. The forms quotiented out are the so-called junk forms.

$$\begin{matrix} (u_\alpha^c)_L, & (c_\alpha^c)_L, & (t_\alpha^c)_L \\ (d_\alpha^c)_L, & (s_\alpha^c)_L, & (b_\alpha^c)_L, \end{matrix} \quad (e^c)_L, \ (\mu^c)_L, \ (\tau^c)_L, \tag{2.16}$$

where $\alpha = 1,2,3$ is the color index. This exhausts all of the 90 physical fermionic degrees of freedom. However, when $\mathcal{H}_F$ is tensored with $L^2(S_M)$, the number of degrees of freedom becomes redundant. In particular, an element $h_F \in \mathcal{H}_F$ can be decomposed as

$$h_F = h_L + h_R + h_L^c + h_R^c, \tag{2.17}$$

where the four vectors on RHS of Eq. (2.17) belong to the corresponding Hilbert spaces $\mathcal{H}_L$, $\mathcal{H}_R$, $\mathcal{H}_L^c$, and $\mathcal{H}_R^c$, respectively. Furthermore, for each $x \in M$ a generic spinor ψ can be decomposed as

$$\psi(x) = \psi_L + \psi_R + \psi_R^c + \psi_L^c. \tag{2.18}$$

Thus, by tensoring Eq. (2.18) with Eq. (2.17) we have 16 possible combinations, namely four times the needed ones. In order to analyze the physical meaning of these combinations it is useful to divide the tensor product $\psi \otimes h_F$ in three parts:

$$(\psi_L \otimes h_L + \psi_R \otimes h_R + \psi_R^c \otimes h_R^c + \psi_L^c \otimes h_L^c), \tag{2.19}$$

$$(\psi_L \otimes h_R^c + \psi_R \otimes h_L^c + \psi_R^c \otimes h_L + \psi_L^c \otimes h_R), \tag{2.20}$$

$$(\psi_L + \psi_R^c) \otimes (h_R + h_L^c) + (\psi_R + \psi_L^c) \otimes (h_L + h_R^c). \tag{2.21}$$

The fermions in this last expression behave as the mirror fermions present in lattice chiral gauge theories. In fact, if we consider, for example, the term $\psi_L \otimes h_R$ of Eq. (2.21) it corresponds to a left-handed particle (as specified by ψ_L) which behaves under the gauge group as a right-handed one (as specified by h_R). On the contrary, the other two combinations (2.19) and (2.20) have the right properties, though each of them independently is sufficient to describe all the physical particles. As it will be clear in the next section, this last redundancy is usually eliminated by identifying the degrees of freedom of Eq. (2.19) with the ones of Eq. (2.20). Concerning the unphysical part, it has to be projected out.

Let us denote with P the projector on the physical subspace corresponding to the combinations (2.19) and (2.20). Note that this projector cannot be used from the very beginning. The above subspace, in fact, would be no more a tensor product involving the space of Dirac spinors, but rather Weyl spinors, and this would lead, as stated above, to a trivial result. Furthermore, since P does not commute with the generalized Dirac operator and consequently with the gauge connection, the form of the action would depend on the particular step of the construction in which it has been used.

In literature, this problem seems to be ignored. What is implicitly done is to compute the trace of θ^2 on the whole Hilbert space to get the bosonic action, while the fermionic part is obtained with the *ad hoc* prescription of retaining in the scalar product (2.7) only the physical state contribution.

This operation can be formally viewed as follows. First we note that the scalar product (2.7) can be seen as a trace,

$$\mathcal{S}_{\mathcal{F}} = \mathrm{Tr}|\psi\rangle\langle(D+A+JAJ)\psi|, \tag{2.22}$$

and we define the map $\Phi:\mathcal{H}\otimes\Omega^1\to\mathcal{B}(\mathcal{H})$:

$$\Phi(\psi,A)=\theta^2(A)+|\psi\rangle\langle(D+A+JAJ)\psi|, \tag{2.23}$$

where Ω^1 is the space of one-form connections. The action is then the functional $\mathrm{Tr}\circ\Phi|_{\mathcal{H}_{phys}\otimes\Omega^1}$, i.e., only the physical fermions are retained, while the trace is still performed on the whole space $\mathcal{H}$. This procedure is extremely *ad hoc*.

A possibility would then be to compute the traces on the physical Hilbert space. The action is then

$$\mathcal{S} = \mathrm{Tr}P\theta^2 + \mathrm{Tr}|P\psi\rangle\langle(D+A+JAJ)P\psi|. \tag{2.24}$$

Note that the only difference with respect to the usual action is in the presence of P even for the bosonic term. Remarkably, this difference for chiral theories, as we will see in the next section, will cause the disappearance of some gauge physical degrees of freedom as well.

III. PHYSICAL HILBERT SPACE AND THE TOPOLOGICAL TERMS IN CHIRAL GAUGE MODELS

In order to illustrate the general discussion of the previous section we consider first a simple model with gauge group $SU(2)_L\otimes SU(2)_R$ and then the standard model, limiting ourselves to the one-quark family case to keep notations to a minimum.

A. $SU(2)_L\otimes SU(2)_R$ model

In this case the algebra can be chosen as $\mathcal{A}=C^\infty(M,\mathbb{C})\otimes\mathcal{A}_F$ where $C^\infty(M,\mathbb{C})$ is the algebra of smooth complex valued function on M and $\mathcal{A}_F=\mathbb{H}_L\oplus\mathbb{H}_R$, with $\mathbb{H}$ the algebra of quaternions. The Hilbert space is the tensor product $\mathcal{H}=L^2(S_M)\otimes\mathcal{H}_F$ and is the space of spinor fields of the model. In particular we take $\mathcal{H}_F=\mathcal{H}_L\oplus\mathcal{H}_R=\mathbb{C}^2\oplus\mathbb{C}^2$ corresponding to two doublets ξ_L and ξ_R under, respectively, the action of $\mathbb{H}_L$ and $\mathbb{H}_R$. Finally for the generalized Dirac operator D (2.3), D_F is chosen as the mass matrix

$$D_F=\begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^\dagger & 0\end{pmatrix}. \tag{3.1}$$

Having assigned the spectral triple $(\mathcal{A},\mathcal{H},D)$ the model is completely defined once a (faithful) representation of the algebra on $\mathcal{H}$ is specified, which we choose as follows:

$$\rho(q_L,q_R)\equiv\begin{pmatrix} q_L & 0 \\ 0 & q_R\end{pmatrix}, \tag{3.2}$$

where $q_{L,R}$ are quaternions represented as 2×2 matrices. Notice that in the general case one should also consider, as elements of the Hilbert space, the charge conjugated states $\xi^c_{L,R}$, related to the $\xi_{L,R}$ via the real structure J. This is only necessary when more than one factor of the chosen algebra is acting on each multiplet of the Hilbert space. This is actually the case of the strong and electroweak standard model, since, for example, left-handed quark doublets transform both under the algebras $\mathbb{H}_L$ and $M_3(\mathbb{C})$, whose unimodular elements correspond to the groups $SU(2)_L$ and $SU(3)_c$. In our simple example this is redundant, since a simple representation of the algebra can be achieved and no bivector potentials are required. We will come back to the more general structure later, when we will consider the case of the standard model. For the moment we are only interested in showing how projecting out the unphysical degrees of freedom corresponds to the natural appearance of topological terms in the classical action.

The gauge connection, computed according to Eq. (2.4), takes the form[3]

$$A(A_L,A_R,\phi)=\begin{pmatrix} A\!\!\!/_L & \gamma_5(\phi-\phi_0) \\ \gamma_5(\phi^\dagger-\phi_0^\dagger) & A\!\!\!/_R\end{pmatrix}, \tag{3.3}$$

where we have defined $A\!\!\!/_{L,R}=\Sigma_i A\!\!\!/^i_{L,R}\sigma^i/2\equiv\Sigma_i q'^i_{L,R}\partial\!\!\!/ q^i_{L,R}$ and $\phi-\phi_0\equiv\Sigma_i q'^i_L(\mathcal{M}q^i_R-q^i_L\mathcal{M})$, where σ^i are the Pauli matrices, with the condition $A^{\mu *}_{L,R}=-A^\mu_{L,R}$. Note that, under a unimodular element u of the algebra, $A(A_L,A_R,\phi)$ transforms as

$$A(A_L,A_R,\phi)\to u[D,u]+uA(A_L,A_R,\phi)u^*, \tag{3.4}$$

and therefore, using the representation for the algebra, it follows that $A_{L,R}$ transform, as usual, as the adjoint representation of the corresponding SU(2) factor, and the Higgs field ϕ as a doublet under both $SU(2)_L$ and $SU(2)_R$. The corresponding curvature two-form θ, once the junk forms have been subtracted out, reads

$$\theta=\begin{pmatrix} \frac{1}{2}\gamma_{\mu\nu}F_L^{\mu\nu}+(\phi^\dagger\phi-\phi_0^\dagger\phi_0) & -\gamma_5 D\!\!\!\!/\ \phi \\ \gamma_5(D\!\!\!\!/\ \phi)^\dagger & \frac{1}{2}\gamma_{\mu\nu}F_R^{\mu\nu}+(\phi\phi^\dagger-\phi_0\phi_0^\dagger)\end{pmatrix}, \tag{3.5}$$

[3]Concerning the Euclidean Dirac γ matrices we choose the Hermitian representation $\gamma_\mu^\dagger=\gamma_\mu$. Moreover by definition $\gamma^{\mu\nu}\equiv[\gamma^\mu,\gamma^\nu]/2$.

where $F^{\mu\nu}_{L,R}$ are the usual gauge field tensors and $\mathbb{D}\ \phi=(\partial\!\!\!/+A\!\!\!/_L)\phi-\phi A\!\!\!/_R$ is the covariant derivative of the Higgs field ϕ. Finally the bosonic action is calculated as $\mathrm{Tr}\theta^2$, where the trace is understood over the *internal* gauge degrees of freedom and the external ones, related to the manifold M, which produces the integration $\int d^4x$.The fermionic action is the scalar product $\langle\psi,(D+A(A_L,A_R,\phi))\psi\rangle$. Actually this last contribution can be cast, as already mentioned, in the form of a trace of an operator as well:

$$\begin{aligned}\langle\psi,&(D+A(A_L,A_R,\phi))\psi\rangle\\ &=\mathrm{Tr}[|\psi\rangle\langle(D+A(A_L,A_R,\phi))\psi|].\end{aligned} \tag{3.6}$$

It is worth observing that the choice for the signature influences the structure of the spin-invariant scalar product (3.6), and thus determines the final expression of the Lagrangian density. Since we are interested in the description of physical models, we write the classical action, S, in Minkowski space with signature $(+---)$, and correspondingly we adopt the Lorentzian spin-invariant scalar product

$$\begin{aligned}\mathcal{S}=\int d^4x\ \Big[&-\frac{1}{4}\mathrm{tr}F^{\mu\nu}_L F^L_{\mu\nu}-\frac{1}{4}\mathrm{tr}F^{\mu\nu}_R F^R_{\mu\nu}+(\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)^\dagger\\ &-(\phi^\dagger\phi-\phi_0^\dagger\phi_0)^2+\overline{\Psi}_L(i\partial\!\!\!/+A\!\!\!/_L)\Psi_L\\ &+\overline{\Psi}_R(i\partial\!\!\!/+A\!\!\!/_R)\Psi_R-[\overline{\Psi}_L(\mathcal{M}+\phi)\Psi_R+\mathrm{H.c.}]\Big],\end{aligned} \tag{3.7}$$

where $\Psi_{L,R}=\chi\otimes\xi_{L,R}$, χ a Dirac spinor of $L^2(S_M)$, ξ_i an element of $\mathcal{H}_F$, and tr denotes the trace over the gauge internal indices. According to our notation, the indices L,R on Ψ refer *to the discrete degree of freedom and not to the chiral component of* χ.

As we have already noticed in the previous section, the action obtained contains a *mirror fermion* problem, since both right-handed and left-handed spinors χ of $L^2(S_M)$ couple to the chiral gauge bosons.

Note that, as already mentioned, the structure of the fermionic terms of Eq. (3.7), which comes from Eq. (3.6), follows from the signature of the metric on M. More precisely, the explicit expression of the fermionic bilinears in Eq. (3.7) in terms of chiral components of χ depends on the signature. The latter has other profound consequences, such as the reality property of the spinor spaces. These aspects have not been investigated. However, the mirror fermion problem, since it only relies on the fact that any spinor of $L^2(S_M)$ is tensored with both ξ_L and ξ_R, seems completely unrelated to the choice of signature.

The first procedure outlined in Sec. II, which consists in simply neglecting in the action the *wrong* states of the form $\chi_L\otimes\xi_R$ and $\chi_R\otimes\xi_L$, leads to the customary result.

Let us now, instead, consider in more detail the second one, in which the trace in the action is restricted to the physical subspace via the introduction of the projector operator P:

$$P=\frac{1-\gamma_5}{2}\otimes\mathcal{P}_1\oplus\frac{1+\gamma_5}{2}\otimes\mathcal{P}_2, \tag{3.8}$$

where $\mathcal{P}_i$ is the projector onto the component ξ_i in the finite Hilbert space. In matrix form,

$$P=\frac{1}{2}\begin{pmatrix}1-\gamma_5 & 0\\ 0 & 1+\gamma_5\end{pmatrix}. \tag{3.9}$$

Note that P commutes with the curvature tensor θ, as it can be immediately checked by using the properties of γ matrices. This means that $\mathrm{Tr}(P\theta P)^2=\mathrm{Tr}P\theta^2P\ =\mathrm{Tr}\theta^2P$. After a straightforward computation we have, for the Lorentzian action of the model in this case,

$$\begin{aligned}\mathcal{S}=\int d^4x\ \Big[&-\frac{1}{4}\mathrm{tr}[F^{\mu\nu}_L F^L_{\mu\nu}-iF^{\mu\nu}_L{*}F^{\mu\nu}_L]-\frac{1}{4}\mathrm{tr}[F^{\mu\nu}_R F^R_{\mu\nu}+iF^{\mu\nu}_R{*}F^{\mu\nu}_R]+(\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)^\dagger-(\phi^\dagger\phi-\phi_0^\dagger\phi_0)^2+\overline{\psi_L}(i\partial\!\!\!/+A\!\!\!/_L)\psi_L\\ &+\overline{\psi_R}(i\partial\!\!\!/+A\!\!\!/_R)\psi_R-[\overline{\psi_L}(\mathcal{M}+\phi)\psi_R+\mathrm{H.c.}]\Big],\end{aligned} \tag{3.10}$$

where ${*}F^{\mu\nu}_{L,R}=(1/2)\epsilon^{\mu\nu\rho\sigma}F^{L,R}_{\rho\sigma}$ are the dual gauge tensor fields, and $\psi_{L,R}=\chi_{L,R}\otimes\xi_{L,R}$ are the physical fermionic states. As a result of the projection, all unphysical states in $\mathcal{S}$ disappear. On the other hand, in the gauge sector, only the anti-self-dual component of the gauge tensor field $F^{\mu\nu}_L$, satisfying (in the Minkowski space) $F^{\mu\nu}_L=-i{*}F^{\mu\nu}_L$ appears, while the self-dual has a vanishing kinetic term. Similarly for $F^{\mu\nu}_R$ only the self-dual part $F^{\mu\nu}_R=i{*}F^{\mu\nu}_R$ contributes. In other words the projection over states of definite chirality in the fermionic sector leads to the result that a similar projection is made onto the gauge fields, of which only one of the two independent components, *left-moving* or *right-moving*, remains as a physical degree of freedom, the other being projected out.

This result is of course unacceptable from a physical point of view, since the action (3.10) violates CPT symmetry. As we will see in the next section, in fact, similar results hold in the case of the standard model, and no such dramatic violation of *CPT* symmetry are allowed by the huge phenomenology on electroweak processes at low scale. It is worth noticing however that in the case of a purely vector or axial coupling of fermions to gauge fields, with $SU(2)_V$ or $SU(2)_A$ gauge symmetry, namely if one makes the identification $F_L^{\mu\nu}=\pm F_R^{\mu\nu}$, the two topological terms cancel each other and the corresponding action reduces to the usual form. This is the case, for example, for $SU(3)_c$ color interaction. For all chiral gauge models, however, this procedure of projecting out unphysical degrees of freedom in $\mathcal{H}$ at the level of curvature tensor θ gives a wrong result.

B. The standard model

For simplicity we will discuss the Connes-Lott version of the gauge theory $SU(3)_c\otimes SU(2)_L\otimes U(1)_Y$ with only one quark family. Actually the inclusion of leptons is quite crucial in order to obtain the correct assignment of hypercharge quantum number for fermions and Higgs bosons, by applying the unimodularity condition. See, for example, [7]. Our discussion, however, is quite general and the results reported can be easily generalized when leptons and the correct number of fermion generations are considered.

The algebra in this case is chosen as $C^\infty(M,\mathbb{C})\otimes\mathcal{A}_F$, with $\mathcal{A}_F=M_3\oplus\mathbb{H}\oplus\mathbb{C}$, with M_3 and $\mathbb{H}$ the algebra of 3×3 complex matrices and of quaternions, respectively. The fermion Hilbert space is again the tensor product $\mathcal{H}=L^2(S_M)\otimes\mathcal{H}_F$, where the finite factor $\mathcal{H}_F$ can be obtained from Eqs. (2.8)–(2.12) by only considering the first term in the direct sums and choosing $N=1$. In particular it has as basis elements the $SU(2)_L$ doublet q_L^α, the $SU(2)_L$ singlets u_R^α and d_R^α, which we will collectively denote by q_R^α, and the corresponding C-conjugate states. With α we denote the color index.

Finally the D_F term in the Dirac operator (2.3) is the fermion mass matrix

$$D_F=\begin{pmatrix} 0 & \mathcal{M} & 0 & 0\\ \mathcal{M}^\dagger & 0 & 0 & 0\\ 0 & 0 & 0 & \mathcal{M}^*\\ 0 & 0 & \mathcal{M}^T & 0\end{pmatrix} \tag{3.11}$$

with

$$\mathcal{M}=\begin{pmatrix} m_u\otimes\mathbb{I}_3 & 0\\ 0 & m_d\otimes\mathbb{I}_3\end{pmatrix}. \tag{3.12}$$

The Connes-Lott representation of the algebra is

$$\rho(c,q,B)\equiv\begin{pmatrix} q\otimes\mathbb{I}_3 & 0 & 0 & 0\\ 0 & B\otimes\mathbb{I}_3 & 0 & 0\\ 0 & 0 & \mathbb{I}_2\otimes c & 0\\ 0 & 0 & 0 & \mathbb{I}_2\otimes c\end{pmatrix}, \tag{3.13}$$

with c, q elements, respectively, of M_3, $\mathbb{H}$, and

$$B=\begin{pmatrix} b & 0\\ 0 & b^*\end{pmatrix} \quad b\in\mathbb{C}. \tag{3.14}$$

The calculation then goes along the same lines of the previous section. In particular for the connection A we get

$$A(G,A_L,B,\phi)=\begin{pmatrix} \not{A}_L\otimes\mathbb{I}_3 & \gamma_5(\phi-\phi_0)\otimes\mathbb{I}_3 & 0 & 0\\ \gamma_5(\phi^\dagger-\phi_0^\dagger)\otimes\mathbb{I}_3 & \not{B}\otimes\mathbb{I}_3 & 0 & 0\\ 0 & 0 & \mathbb{I}_2\otimes\not{G} & 0\\ 0 & 0 & 0 & \mathbb{I}_2\otimes\not{G}\end{pmatrix}, \tag{3.15}$$

where as before $\not{A}_L=\Sigma_i\not{A}_L^i(\sigma_i/2)$ and $\not{G}=\Sigma_{a=1}^8\not{G}^a(\lambda_a/2)+\mathbb{I}_3\not{G}^0$ with λ_a the Gell-Mann SU(3) matrices and

$$\not{B}=\begin{pmatrix} \not{\beta} & 0\\ 0 & \not{\beta}^*\end{pmatrix} \quad \beta_\mu\in\mathbb{C}. \tag{3.16}$$

From the conditions $A_L^{\mu *}=-A_L^\mu$, $G^{\mu *}=-G^\mu$ and $B^{\mu *}=-B^\mu$ in particular it follows $\beta_\mu=-\beta_\mu^*$.

For the curvature tensor, with all junk forms subtracted out, one has instead

$$\theta=\begin{pmatrix} \frac{1}{2}\gamma_{\mu\nu}F_L^{\mu\nu}+(\phi^\dagger\phi-\phi_0^\dagger\phi_0) & -\gamma_5\not{\mathcal{D}}\ \phi & 0 & 0 \\ \gamma_5(\not{\mathcal{D}}\ \phi)^\dagger & \frac{1}{2}\gamma_{\mu\nu}B^{\mu\nu}+(\phi\phi^\dagger-\phi_0\phi_0^\dagger) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\gamma_{\mu\nu}G^{\mu\nu} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\gamma_{\mu\nu}G^{\mu\nu} \end{pmatrix}, \tag{3.17}$$

with $\not{\mathcal{D}}\ \phi=(\not\partial+\not{A}_L)\phi-\phi\not{B}$. The unimodularity condition $\mathrm{Tr}(A+JAJ)=0$ removes the U(1) factor corresponding to G_μ^0 and the action, obtained by tracing over the entire Hilbert space, reads, in Minkowski,

$$\mathcal{S}=\int d^4x\ \Big[-\frac{1}{4}\mathrm{tr}F_L^{\mu\nu}F_{\mu\nu}^L-\frac{1}{4}B^{\mu\nu}B_{\mu\nu}-\frac{1}{4}\mathrm{tr}G^{\mu\nu}G_{\mu\nu}+(\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)^\dagger-(\phi^\dagger\phi-\phi_0^\dagger\phi_0)^2+\overline{Q_L}(i\not\partial+\not{A}_L+\not{G})Q_L$$
$$+\overline{Q_R^c}(i\not\partial+\not{A}_L+\not{G})Q_R^c+\overline{Q_R}(i\not\partial+\not{B}+\not{G})Q_R+\overline{Q_L^c}(i\not\partial-\not{B}+\not{G})Q_L^c-[\overline{Q_L}(\mathcal{M}+\phi)Q_R+\overline{Q_R^c}(\mathcal{M}^*+\phi)Q_L^c+\text{H. c.}]\Big], \tag{3.18}$$

where $Q_{L,R}=\chi\otimes q_{L,R}^\alpha$, with χ a Dirac spinor of $L^2(S_M)$. The redundancy of the degrees of freedom due to the presence of C-conjugate states can be eliminated by identifying the corresponding states, $Q_{L,R}\equiv Q_{R,L}^c$, namely making the quotient with respect to the equivalence relation given by the real structure J. On the other hand, as in the case discussed in the previous section, in the action appear mirror unphysical states with a chirality *mismatch*, like $\chi_L\otimes q_R$ or $\chi_R\otimes q_L$.

If we now restrict the trace of the operator $\theta^2+|\psi\rangle\langle(D+A(G,A_L,B,\phi))\psi|$ via the introduction of the projector P,

$$P=\frac{1}{2}\begin{pmatrix} 1-\gamma_5 & 0 & 0 & 0 \\ 0 & 1+\gamma_5 & 0 & 0 \\ 0 & 0 & 1+\gamma_5 & 0 \\ 0 & 0 & 0 & 1-\gamma_5 \end{pmatrix}, \tag{3.19}$$

we get

$$\mathcal{S}=\int d^4x\ \Big[-\frac{1}{4}\mathrm{tr}[F_L^{\mu\nu}F_{\mu\nu}^L-iF_L^{\mu\nu}{}^*F_{\mu\nu}^L]-\frac{1}{4}[B^{\mu\nu}B_{\mu\nu}+iB^{\mu\nu}{}^*B_{\mu\nu}]-\frac{1}{4}\mathrm{tr}G^{\mu\nu}G_{\mu\nu}+(\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)^\dagger-(\phi^\dagger\phi-\phi_0^\dagger\phi_0)^2$$
$$+\overline{q_L}(i\not\partial+\not{A}_L+\not{G})q_L+\overline{q_R^c}(i\not\partial+\not{A}_L+\not{G})q_R^c+\overline{q_R}(i\not\partial+\not{B}+\not{G})q_R+\overline{q_L^c}(i\not\partial-\not{B}+\not{G})q_L^c$$
$$-[\overline{q_L}(\mathcal{M}+\phi)q_R+\overline{q_R^c}(\mathcal{M}^*+\phi)q_L^c+\text{H. c.}]\Big], \tag{3.20}$$

where, to simplify notation, we have denoted with $q_{L,R}$ the physical states $\chi_{L,R}\otimes q_{L,R}^\alpha$. Hence the topological terms appear for both the $\mathrm{SU}(2)_L$ and the U(1) factor, while the $\mathrm{SU}(3)_c$ fields, due to its vector coupling to quarks, contribute to $\mathcal{S}$ with the usual $\mathrm{tr}G^{\mu\nu}G_{\mu\nu}$ term only. In particular the self-dual component of F_L receives no kinetic contribution and is *projected out* by the introduction of P. In the fermion sector, instead, all unphysical states are absent. As already pointed out this approach, leading to Eq. (3.20), or Eq. (3.10), implies CPT symmetry violation and is at variance with low-energy phenomenology.

IV. THE SPECTRAL ACTION

Recently Chamseddine and Connes [5,6,12] have proposed another form for the bosonic action, which includes gravity as well, while the fermionic action remains the same.

The idea behind the spectral action is that while the topology is encoded by the algebra, all other information (metric in the first instance) are encoded by the generalized covariant Dirac operator $D_A=D+A+JAJ$, where now D_A also contains the spin connection terms. Moreover, the operator D can be characterized completely by its spectrum.

This leads Connes and Chamseddine to consider as the bosonic action

$$\mathcal{S}_{\mathcal{B}}=\mathrm{Tr}\left[\chi\left(-\frac{D_A^2}{m_0^2}\right)\right], \tag{4.1}$$

where χ is a suitable cutoff function. The quantity m_0 is a cutoff with dimensions (in natural units) of a mass which indicates at which scale the theory under consideration effectively shows its noncommutative geometric nature. The action basically sums up the eigenvalues of D_A which are smaller than m_0. The trace can be evaluated with heath kernel techniques [13]. We now consider what happens in the model discussed in Sec. III A. To further simplify notations we make the additional simplification of only considering the gauge and gravitational contribution. The other contributions can be added without altering the results. This is accomplished by choosing a Dirac operator with vanishing fermion mass terms

$$D_A=\begin{pmatrix} D_L & 0 \\ 0 & D_R \end{pmatrix}, \tag{4.2}$$

where

$$D_L \equiv \nabla\!\!\!\!/ \otimes \mathbb{I}_2 - \frac{i}{2} g_L A\!\!\!/_L\,, \tag{4.3}$$

$$D_R \equiv \nabla\!\!\!\!/ \otimes \mathbb{I}_2 - \frac{i}{2} g_R A\!\!\!/_R\,. \tag{4.4}$$

Furthermore, by ∇_μ we denote the covariant derivative corresponding to the metric connection only.

According to the considerations developed in the previous sections we substitute the expression (4.1) for the action, with a similar expression in which the trace is performed over the physical states only. This can be done by using the previously defined projector P:

$$P=\begin{pmatrix} P_L & 0 \\ 0 & P_R \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\gamma_5 & 0 \\ 0 & 1+\gamma_5 \end{pmatrix}. \tag{4.5}$$

Hence in terms of Eq. (4.5) we get

$$\begin{aligned}\mathcal{S}_{\mathcal{B}}&=\mathrm{Tr}\left[\chi\left(-\frac{D_A^2}{m_0^2}\right)P\right]=\mathrm{Tr}\left[\chi\left(-\frac{D_A^2 P}{m_0^2}\right)\right]\\ &=\mathrm{Tr}\left[\chi\left(-\frac{D_L^2 P_L}{m_0^2}\right)\right]+\mathrm{Tr}\left[\chi\left(-\frac{D_R^2 P_R}{m_0^2}\right)\right],\end{aligned} \tag{4.6}$$

where to get the RHS of Eq. (4.6) we have used the property the P commutes with D^2. Note that the minus sign in Eq. (4.6) is due to our choice of Hermitian γ matrices.

The trace (4.6) is defined by using the heat-kernel expansion [13]

$$\mathrm{Tr}\left[\chi\left(-\frac{D^2 P}{m_0^2}\right)\right]\simeq\sum_{n\geqslant 0} f_n\left[a_n\left(-\frac{D_L^2 P_L}{m_0^2}\right)+a_n\left(-\frac{D_R^2 P_R}{m_0^2}\right)\right], \tag{4.7}$$

where the coefficients f_n are given by

$$f_0=\int_0^\infty \chi(u)\;\; u\;\; du, \quad f_2=\int_0^\infty \chi(u)\;\; du,$$

$$f_{2(n+2)}=(-1)^n\;\; \chi^{(n)}(0) \quad \text{with}\;\; n\geqslant 0, \tag{4.8}$$

and

$$a_n\left(-\frac{D_{L,R}^2 P_{L,R}}{m_0^2}\right)=\int_M \sqrt{g}\;\; a_n\left(x,-\frac{D_{L,R}^2 P_{L,R}}{m_0^2}\right)d^4x. \tag{4.9}$$

Note that a_n vanish for odd n.

From definitions (4.3), (4.4), and (4.6) one can compute the positive definite operator

$$\begin{aligned}-D_{L,R}^2 P_{L,R}=\Big\{&-\Box_{1/2}\otimes\mathbb{I}_2-\frac{1}{8}R_{\mu\nu\rho\sigma}\gamma^{\mu\nu}\gamma^{\rho\sigma}\otimes\mathbb{I}_2\\ &+\frac{i}{2}g_{L,R}(D_\mu^{L,R}A_{L,R}^\mu)+\frac{i}{4}g_{L,R}\gamma^{\mu\nu}F_{\mu\nu}^{L,R}\\ &+ig_{L,R}\mathbb{I}_4\otimes A_{L,R}^\mu\nabla_\mu\Big\}P_{L,R}\,,\end{aligned} \tag{4.10}$$

where $D_\mu^{L,R}A_\nu^{L,R}=\nabla_\mu A_\nu^{L,R}-i(g_{L,R}/2)[A_\mu^{L,R},A_\nu^{L,R}]$ is the complete covariant derivative and $F_{\mu\nu}^{L,R}\equiv D_\mu^{L,R}A_\nu^{L,R}-D_\nu^{L,R}A_\mu^{L,R}$. Moreover, in the previous equation $\Box_{1/2}\equiv\nabla^\mu\nabla_\mu=g^{\mu\nu}(\partial_\mu+\omega_\mu)(\partial_\nu+\omega_\nu)-\Gamma^\mu(\partial_\mu+\omega_\mu)$, where ω_μ denotes the spin connection, $\Gamma^\mu\equiv g^{\rho\sigma}\Gamma_{\rho\sigma}^\mu$, and finally, we have chosen the representation for the Riemann tensor according to which $R_{1212}=1/r^2$ on the $2d$ sphere of radius r. From these definitions we can recast Eq.(4.10) as

$$-D_{L,R}^2 P_{L,R}=-(g^{\mu\nu}P_{L,R}\otimes\mathbb{I}_2\partial_\mu\partial_\nu+C_{L,R}^\mu\partial_\mu+B_{L,R}), \tag{4.11}$$

where

$$C_{L,R}^\mu=[(2\omega^\mu-\Gamma^\mu)\otimes\mathbb{I}_2-ig_L\mathbb{I}_4\otimes A_{L,R}^\mu]P_{L,R}\,, \tag{4.12}$$

$$\begin{aligned}B_{L,R}=\Bigg[&\left(\partial_\mu\omega^\mu+\omega_\mu\omega^\mu-\frac{R}{4}\;\;\mathbb{I}_4-\Gamma^\mu\omega_\mu\right)\otimes\mathbb{I}_2-ig_{L,R}\omega_\mu A_{L,R}^\mu\\ &-\frac{i}{2}g_{L,R}\mathbb{I}_4\otimes(D_\mu^{L,R}A_{L,R}^\mu)-\frac{i}{4}g_L\gamma^{\mu\nu}F_{\mu\nu}^{L,R}\Bigg]P_{L,R}\,.\end{aligned} \tag{4.13}$$

In order to apply the formalism developed in Ref. [13] to compute the Seeley–deWitt coefficients, it is convenient to introduce the quantities

$$\xi_{L,R}^\mu\equiv\frac{1}{2}(C_{L,R}^\mu+\Gamma^\mu P_{L,R})=\left(\omega_\mu\otimes\mathbb{I}_2-\frac{i}{2}g_{L,R}\mathbb{I}_4A_{L,R}^\mu\right)P_{L,R}\,, \tag{4.14}$$

$$E_{L,R}\equiv B_{L,R}-(\partial_\mu \xi^\mu_{L,R}+\xi^\mu_{L,R}\xi^{L,R}_\mu-\Gamma^\mu \xi^{L,R}_\mu)$$
$$=-\left[\frac{R}{4}\ \mathbb{I}_4\otimes\mathbb{I}_2+\frac{i}{4}g_{L,R}\gamma^{\mu\nu}F^{L,R}_{\mu\nu}\right]P_{L,R}, \quad (4.15)$$

and

$$\Omega^{L,R}_{\mu\nu}=\left[\frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\otimes\mathbb{I}_2-\frac{i}{2}g_{L,R}\mathbb{I}_4\otimes F^{L,R}_{\mu\nu}\right]P_{L,R}. \quad (4.16)$$

Thus by following [13] we get

$$a_0\left(x,-\frac{D^2_{L,R}P_{L,R}}{m_0^2}\right)=\frac{1}{16\pi^2}\mathrm{Tr}(P_{L,R}\otimes\mathbb{I}_2)=\frac{1}{4\pi^2}, \quad (4.17)$$

$$a_2\left(x,-\frac{D^2_{L,R}P_{L,R}}{m_0^2}\right)=\frac{1}{16\pi^2}\mathrm{Tr}\left(E_{L,R}+\frac{R}{6}P_{L,R}\otimes\mathbb{I}_2\right), \quad (4.18)$$

$$a_4\left(x,-\frac{D^2_{L,R}P_{L,R}}{m_0^2}\right)=\frac{1}{16\pi^2}\frac{1}{360}[60\Box E+60RE+180E^2+30\Omega_{\mu\nu}\Omega^{\mu\nu}+(12\Box R+5R^2-2R_{\mu\nu}R^{\mu\nu}+2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})P_{L,R}\otimes\mathbb{I}_2], \quad (4.19)$$

By substituting in Eqs. (4.17)–(4.19) the expressions (4.15) and (4.16) we get

$$a_0\left(x,-\frac{D^2_LP_L}{m_0^2}\right)+a_0\left(x,-\frac{D^2_RP_R}{m_0^2}\right)=\frac{1}{2\pi^2}, \quad (4.20)$$

$$a_2\left(x,-\frac{D^2_LP_L}{m_0^2}\right)+a_2\left(x,-\frac{D^2_RP_R}{m_0^2}\right)=-\frac{1}{24\pi^2}R, \quad (4.21)$$

and

$$a_4\left(x,-\frac{D^2_LP_L}{m_0^2}\right)+a_4\left(x,-\frac{D^2_RP_R}{m_0^2}\right)=\frac{1}{16\pi^2}\frac{1}{180}\Big(-12\Box R+5R^2-8R_{\mu\nu}R^{\mu\nu}-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+15g_L^2\mathrm{Tr}(F^L_{\mu\nu}F^{\mu\nu}_L)$$
$$+15g_R^2\mathrm{Tr}(F^R_{\mu\nu}F^{\mu\nu}_R)-\frac{45}{8}g_L^2\mathrm{Tr}(F^L_{\mu\nu}{}^*F^{\mu\nu}_L)+\frac{45}{8}g_R^2\mathrm{Tr}(F^R_{\mu\nu}{}^*F^{\mu\nu}_R)\Big). \quad (4.22)$$

Note that Eq. (4.22) is in still Euclidean signature, the continuation to Lorentzian signature being straightforward. We can therefore conclude that also in the case of the spectral action the effect of considering traces over the physical Hilbert space is that of adding the topological term to the action. In this case, however, unlike what happens in the (CL) case, the two components of the gauge fields, the self-dual and anti-self-dual, are both present but their kinetic term are weighted by different factors. Again this represents a violation of CPT symmetry of the model.

V. CONCLUSIONS

In this paper we have analyzed the structure of the Hilbert space $\mathcal{H}$ adopted in the noncommutative geometry models of gauge theories of the type described[4] in Refs. [1–6]. We have pointed out that $\mathcal{H}$ contains unphysical fermionic degrees of freedom. Some of these are harmless and can be removed with a quotient. The others, behaving as mirror particles, can be eliminated only via a projection. What seems to us unsatisfactory is that the projection operator does not fit into the geometrical construction. In fact since it does not commute with the generalized Dirac operator, it leads to different results depending on the step in the construction at which it is applied. Furthermore none of these correspond to a physically acceptable model. The only way to get, for example, the correct result for the standard model is through the *ad hoc* prescription of neglecting the unphysical fermionic degrees of freedom in the action.

How serious is this problem for the noncommutative geometry approach to the standard model? One can take two extreme views. On the one hand, one can consider the fact that some extra terms in the Lagrangian appear to be irrelevant and simply ignore them without worrying too much. On the other hand, one can consider that noncommutative geometry fails in its attempt to at least reproduce the action

[4] It is worth noticing that there exist different noncommutative geometrical approaches to gauge models, see for example Refs. [14,15], and references therein.

of a gauge theory. Obviously both these positions can be easily criticized. Noncommutative geometry remains a program which is able to give very fruitful results in particle physics and gravity, but probably, in order to obtain the standard model in a fully geometrical way, some modification of the spectral triple, or of some other crucial ingredient of the theory, is needed.

ACKNOWLEDGMENTS

We would like to thank G. Bimonte, G. Esposito, and G. Landi for useful discussions. F. Lizzi and G. Miele would like to thank N. Mavromatos for hospitality in the Physics Department (Theoretical Physics) of the University of Oxford.

[1] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).
[2] A. Connes and J. Lott, in *Recent Advances in Field Theory*, Proceedings of the Workshop, Annecy, France, 1990, edited by P. Binétroy *et al.* [Nucl. Phys. B (Proc. Suppl.) **18**, 29 (1990)].
[3] A. Connes, J. Math. Phys. (N.Y.) **36**, 6194 (1995).
[4] B. Iochum and T. Schucker, Commun. Math. Phys. **178**, 1 (1996).
[5] A. Connes, Commun. Math. Phys. **182**, 155 (1996).
[6] A. Chamseddine and A. Connes, Phys. Rev. Lett. **77**, 4868 (1996).
[7] C. P. Martín, J. M. Garcia-Bondía, and J. C. Várilly, ''The Standard Model as a Noncommutative Geometry,'' Report No. hep-th09605001(unpublished).
[8] F. Lizzi, G. Mangano, G. Miele, and G. Sparano, Mod. Phys. Lett. A **11**, 2561 (1996).
[9] J. C. Várilly and J. M. Gracia-Bondía, J. Geom. Phys. **12**, 223 (1993).
[10] W. Kalau and M. Walze, ''Supersymmetry and Noncommutative Geometry,'' Report No. hep-th09604146 (unpublished).
[11] R. Wulkenhaar, ''A tour through Nonassociatie Geometry,'' Report No. hep-th9607086 (unpublished).
[12] B. Iochum, D. Kastler, and T. Schucker, ''On the Universal Chamseddine-Connes Action, 1. Details of the Action Computation,'' Report No. hep-th9607158 (unpublished).
[13] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem* (Chemical Rubber Company, Cleveland, 1995).
[14] M. Dubois-Violette, *Noncommutative Differential Geometry, Quantum Mechanics and Gauge Theory*, Proceedings of the 19th International Conference on Differential Geometric Methods in Theoretical Physics, Rapallo, Italy, 1990, Lecture Notes in Physics Vol. 375 (Springer, Berlin, 1990).
[15] G. Cammarata and R. Coquereaux, *Comments About Higgs Fields, Noncommutative Geometry and the Standard Model*, Proceedings of 34th Internationale Universitätswochen für Kern-und Teilchen-physik: Low-Dimensional Models in Statistical Physics and Quantum Field Theory, Schladming, Austria, 1995, Lecture Notes in Physics Vol. 469 (Springer, Berlin, 1996).