Thomas-Fermi approximation for gauge theories

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An effective field approximation, similar to the atomic Thomas-Fermi approach, is proposed for studying non-Abelian gauge theories which includes finite-volume effects. As applications of the formalism the equation of state for an SU(2) gauge theory with massless fermions is obtained. The extensions to realistic situations are briefly discussed. [S0556-2821(97)03510-8]

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I. INTRODUCTION

The equation of state of a quark-gluon plasma at high temperatures and/or densities is one of the most important unknowns in our current understanding of strong interaction physics [1]. The applications of such an equation of state are varied, ranging from cosmological compact objects to the physics of heavy ion collisions. Unfortunately, because of the high degree of nonlinearity present in QCD, the determination of this equation of state has proved to be a difficult task. For example, even within perturbation theory, infrared singularities require the calculation of an infinite number of graphs for the partition function beyond fifth order [2]. General expressions for Green's functions are available for the case where the internal momentum is large ($\sim\mathcal{T}$) while the external momenta are soft ($\sim g\mathcal{T}$), the so-called hard-thermal loop region [3]; using standard manipulations [4], one can then determine the partition function corresponding to all hard modes in the theory. The soft-mode contributions to the partition function have been studied using various approximations [5] and numerical calculations have also been developed (though not to the extent as in the $T=0$ case) [6].

In this paper we propose a new approximation within which the physics of a quark-gluon plasma can be studied. The formalism is based on the Thomas-Fermi model of the atom [7] and will be called Thomas-Fermi QCD (TFQCD). We consider a plasma of quarks and gluons confined to a volume $\mathcal{V}$ which we imagine subdivided into a number of subvolumes, each of which is large enough for the partons they contain to be considered a statistical ensemble. These subvolumes interact via a background gauge field whose sources are the thermally averaged non-Abelian charge densities of the subvolumes. The subvolumes are assumed to be small enough for the background field to vary very little inside them, and because of this the background field sources are essentially pointlike. The requirement of stability, together with the Yang-Mills equations for the background field, furnishes a closed set of equations which can be solved; from the solution the equation of state for the system can be derived. This program requires the evaluation of the thermally averaged non-Abelian charge densities which we obtain using perturbation theory; in this paper we will use the lowest-order approximation, but a systematic improvement is straightforward.

The atomic Thomas-Fermi approximation is useful when calculating bulk properties of an atom with a large number of electrons, such as the total ionization energy [8]; it is also useful as a starting point for a Hartree-Fock approximation. We expect the TFQCD model of a quark-gluon plasma to be reasonably accurate for bulk properties of the system, such as the equation of state. There are some differences between the atomic Thomas-Fermi and the TFQCD formalisms; in particular note that, in contrast to the atomic case, the quark-gluon plasma is not stable: if left alone it will fly apart and undergo a phase transition into a gas of hadrons. In order to study a gas of quarks and gluons we are forced to imagine the system to be enclosed in a container at sufficiently large temperature and/or density.

The presence of an external confining agency is reminiscent of the bag model [9,10]. Through most of the paper we will consider, in contrast to the usual bag models, a situation where the partons are not confined, and for which the external pressure is assumed to be generated by some physical apparatus. Despite this difference the bag boundary conditions are also relevant for the present model: the system is assumed to be confined to a spherical volume out of which neither fermion number nor color can escape, this requires we impose both the original [9] and chiral [10] bag boundary conditions. We will also briefly study a system corresponding to a hadron at zero temperature, and will show that the bag constant and strong coupling constant obtained in the present approach are consistent with those obtained using the bag model.

The volume of the system $\mathcal{V}$ will be kept finite in all computations; the results will then include finite-volume effects (such as terms in the extensive thermodynamic quantities proportional $\mathcal{V}^{2/3}$). In the infinite volume limit these surface effects can be neglected and the equation of state reduces to that of an ideal gas of gluons and quarks.

In the following section we will describe the construction of the TFQCD model and present some simple applications.

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We will concentrate on the case of an SU(2) gauge theory with a single species of massless fermions, and then describe the modification required for the important case of an SU(3) gauge theory with three (massive) fermion flavors. The remainder of the paper is organized as follows. In Sec. III we derive the equation of state for this model in the cases of zero baryon number and zero temperature. The discussion of the extension to SU(3) and to more flavors is presented in Sec. IV. Some parting comments are presented in Sec. V and a mathematical detail is given in the Appendix.

II. DESCRIPTION OF THE MODEL

The model we propose is, as mentioned above, an extension of the Thomas-Fermi model to the case of QCD. We consider a gas of partons inside a volume $\mathcal{V}$; we then imagine partitioning $\mathcal{V}$ into small subvolumes $\delta\mathcal{V}$ which are big enough so that the partons (quarks and gluons) contained in them form a statistical ensemble determined by a temperature $\mathcal{T}$ and, for the fermions, a chemical potential μ. Each subvolume is required to be in equilibrium with its environment which implies that the temperature and chemical potential are the same throughout the system (this is intuitively obvious, we present a proof in Sec. II C). The system is also assumed to be static so that no currents are present.

We assume that the subvolumes have a nonzero average color charge, which implies that the zero component of the gauge field goes to a constant $\overline{A^0}$ at its boundary [11]. We will refer to $\overline{A^0}$ as the background gauge field. The background field is assumed to vary slowly and smoothly between the $\delta\mathcal{V}$, and is determined self-consistently by requiring it to satisfy the Yang-Mills equations corresponding to the average charges of the subvolumes (which themselves depend on the background fields). This approach presupposes that the magnitude of the charge in any given $\delta\mathcal{V}$ is small, and that the background field is approximately constant within each subvolume; both these assumptions will be verified *a posteriori*.

Finally, we also assume that our system is spherically symmetric; this requirement considerably simplifies the calculations yet preserves the essential non-Abelian character of the problem. The equations obtained for the background fields are then similar to the ones derived when considering the coupling of classical, spherically symmetric Yang-Mills fields to external sources [12,13].

In the rest of this section we will treat the various ingredients of the model separately. We first review the Yang-Mills equations within the spherically symmetric ansatz. We then obtain the expression for the partonic sources for the background fields and the various thermodynamic observables. Next we derive the stability conditions for the system. Finally we combine these results in order to obtain the equations for the background fields which determine quantitatively the Thomas-Fermi-QCD (TFQCD) model.

The conventions which we use are the following. The model is based on an SU(N) Yang-Mills theory with one species of massless fermion; the (anti-Hermitian) group generators are denoted by T^a and the gauge coupling constant by g. The covariant derivative is $D_\mu=\partial_\mu+A_\mu$, where $A_\mu=gA_\mu^aT^a$. The full Lagrangian is

$$\mathcal{L}=i\overline{\psi}\not{D}\psi-\frac{1}{4}(F_{\mu\nu}^a)^2, \tag{2.1}$$

where ψ denotes the quark field, A the gauge field, and

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\epsilon_{abc}A_\mu^bA_\nu^c. \tag{2.2}$$

The sources are

$$j_\mu^a=i\overline{\psi T^a}\gamma_\mu\psi. \tag{2.3}$$

Latin indices from the beginning of the alphabet (a,b,c, etc.) correspond to color indices; Latin indices form the middle of the alphabet (i,j,k, etc.) denote space indices.

A. Spherically symmetric gauge potentials and equations

As mentioned in the previous section we will assume that the long-range forces in our system are described by a non-Abelian background gauge field generated by the average charge of each subvolume. We also assume the system to have spherical symmetry. Thus we need the most general expression for a spherically symmetric non-Abelian gauge field, which is well known [14], and is reviewed for completeness below.

The most general spherically symmetric ansatz for the gauge potentials of an Abelian theory is simply $A^0=\phi(r,t)$, $\mathbf{A}=a(r,t)\hat{\mathbf{r}}$, where $\mathbf{A}$ denotes the vector potential and $r=|\mathbf{r}|$. It is clear, however, that we can choose a gauge where $a(r,t)=0$, so we can take $\mathbf{A}=0$.

For the SU(2) non-Abelian case the structure is much richer;[1] the most general spherically symmetry ansatz is [14] (the overbar denotes the background fields)

$$\overline{A_a^0}=\mathcal{A}_0\hat{\mathbf{r}}_a,$$

$$\overline{A_a^i}=\epsilon_{iaj}\hat{\mathbf{r}}_j\left(\frac{g^{-1}+\varphi_2}{r}\right)+(\delta_{ia}-\hat{\mathbf{r}}_i\hat{\mathbf{r}}_a)\frac{\varphi_1}{r}+\hat{\mathbf{r}}_i\hat{\mathbf{r}}_a\mathcal{A}_1, \tag{2.4}$$

which exhibits spin-isospin mixing.[2] The fields $\varphi_{1,2}$ and $\mathcal{A}_{0,1}$ depend on r and t.

Within this ansatz the SU(2) Yang-Mills Lagrangian becomes

$$\frac{1}{4}F_{\mu\nu}^{a\ 2}=\frac{1}{4}f_{\mu\nu}^2-\frac{1}{r^2}|\mathcal{D}\Phi|^2+\frac{g^2}{2r^4}\left(|\Phi|^2-\frac{1}{g^2}\right)^2, \tag{2.5}$$

where the indices μ, ν, etc., equal 1 (corresponding to r) or 0 (corresponding to t); the metric is diag$(1,-1)$. We also defined $f_{\mu\nu}=\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu$, $\Phi=\varphi_1+i\varphi_2$, and $\mathcal{D}_\mu=\partial_\mu+ig\mathcal{A}_\mu$. The above expression is invariant under the gauge transformation

$$\mathcal{A}_\mu\to\mathcal{A}_\mu-\partial_\mu\Lambda,\quad \Phi\to e^{ig\Lambda}\Phi, \tag{2.6}$$

which is a remnant of the original non-Abelian invariance.

[1]The situation is similar for larger groups, see Sec. IV.

[2]In this respect the present approach differs from other investigations into spherically symmetric hadron physics; see Ref. [15].

We now consider the coupling of the above fields to a spherically symmetric charge density ρ^a, where spherical symmetry requires $\rho^a = q\hat{r}^a$. The coupling is then described by adding a term

$$\mathcal{L}_{\text{interaction}} = g\overline{A_0^a}\rho^a = gq\mathcal{A}_0 \tag{2.7}$$

to the Lagrangian.

B. The partition function for gluons and fermions

We now imagine that the volume of the system, denoted by $\mathcal{V}$, is subdivided into a large number of subvolumes $\delta\mathcal{V}$. The gauge fields inside each subvolume are separated into a background piece $\overline{A_\mu^a}$ and a fluctuation a_μ^a: $A_\mu^a = \overline{A_\mu^a} + a_\mu^a$.

In this subsection we evaluate the partition function for the partons inside $\delta\mathcal{V}$. This object, which we call $Z_{\delta\mathcal{V}}$, will depend on $\overline{A_\mu^a}$, and we can use this dependence to obtain the thermal average of the non-Abelian currents:

$$\overline{j_\mu^a} = \frac{1}{g}\left(\frac{\partial Z_{\delta\mathcal{V}}}{\partial \overline{A^{\mu a}}}\right). \tag{2.8}$$

Since the system is supposed to be in a static configuration we require $j_i^a = 0$ which implies we can take $A_i^a = 0$ inside $\delta\mathcal{V}$. Since the background fields are assumed slowly varying, we also take $\overline{A_0^a}$ constant inside $\delta\mathcal{V}$; we then choose a gauge such that $\overline{A_0^a}$ is diagonal inside $\delta\mathcal{V}$. Hence $Z_{\delta\mathcal{V}}$ will depend on the temperature $\mathcal{T}$, the fermionic chemical potential μ, and the n components of A_0 associated with the diagonal generators (n is the rank of the gauge group).

As a first approximation we will neglect the interaction between the fermions and the a_μ^a, as well as the nonlinear couplings among the a_μ^a; these interactions can be included perturbatively. Concerning the scale of $\delta\mathcal{V}$ we will assume that it is set by the fermion thermal wavelength λ that is,

$$\delta\mathcal{V} \sim \lambda^3. \tag{2.9}$$

We will for the moment restrict ourselves to the case where the gauge group is SU(2) [the extension to SU(3) will be described in Sec. IV below]. In this case the group generators are $T^a = \sigma^a/(2i)$ and, within $\delta\mathcal{V}$, $A_0 = gA_0^a T^a = g\mathcal{A}_0\sigma_3/(2i)$, so that

$$A_0^b = a_0^b + \mathcal{A}_0\delta_{b3}, \quad A_i^b = a_1^b \quad (\text{inside } \delta\mathcal{V}). \tag{2.10}$$

We first evaluate the fermionic contribution to the partition function, and then calculate the contributions from the a_μ^a.

1. Fermionic contribution

When considering the fermionic partition function we will assume only one massless fermions species (the modifications required by several species and/or nonzero masses are straightforward). Thus we look for an approximate expression for $Z_\psi = \det(i\not{D} + \mu\gamma_0)$ where the gluon fields take the form (2.10). Inside a subvolume $\delta\mathcal{V}$ it is assumed that the fermions behave as a statistical ensemble, that the interaction with the a_μ^a is small, and that the background fields $\overline{A_\mu^a}$ are essentially constant. Adopting these approximations we reduce the calculation to evaluating $\det[i\not{\partial} + (-i\overline{A_0} + \mu)\gamma_0]$ with $A_0 = g\mathcal{A}_0\sigma_3/(2i)$, $\mathcal{A}_0 = \text{const}$.

The partition function for an ideal gas of massless fermions at temperature $\mathcal{T}$ in a volume $\delta\mathcal{V}$ and with a chemical potential equal to μ is given by [4]

$$\ln Z_0 = \frac{\beta\delta\mathcal{V}}{12\pi^2}\left[\mu^4 + 2(\pi k\mathcal{T})^2\mu^2 + \frac{7}{15}(\pi k\mathcal{T})^4\right], \tag{2.11}$$

where the zero subscript indicates that no gauge fields are included. The constant background gauge fields are then included by replacing $\mu \to \mu \pm g\mathcal{A}_0/2$ in Z_0 with the sign depending on the isospin of fermion, and adding the contributions from each isospin component. Thus, within the above approximations, we obtain

$$\ln Z_\psi = \frac{\beta\delta\mathcal{V}}{6\pi^2}\left[\frac{g^4\mathcal{A}_0^4}{16} + \frac{g^2\mathcal{A}_0^2}{8\lambda^2} + \mu^4 + 2(\pi k\mathcal{T}\mu)^2 + \frac{7}{15}(\pi k\mathcal{T})^4\right], \tag{2.12}$$

where we defined the thermal wavelength

$$\lambda = \frac{1}{2\sqrt{3\mu^2 + (\pi k\mathcal{T})^2}}. \tag{2.13}$$

This approximation to Z_ψ generates the following expressions for the fermionic contribution to the (local) thermodynamic quantities

$$\mathcal{P}_\psi = \frac{\ln Z_\psi}{\beta\delta\mathcal{V}} = \frac{1}{6\pi^2}\left[\frac{g^4\mathcal{A}_0^4}{16} + \frac{g^2\mathcal{A}_0^2}{8\lambda^2} + \mu^4 + 2(\pi k\mathcal{T}\mu)^2 + \frac{7}{15}(\pi k\mathcal{T})^4\right],$$

$$s_\psi = \frac{2k}{3\pi}(\pi k\mathcal{T})\left[\frac{g^2\mathcal{A}_0^2}{4} + \mu^2 + \frac{7}{15}(\pi k\mathcal{T})^2\right],$$

$$e_\psi = \frac{1}{2\pi^2}\left[\frac{g^2\mathcal{A}_0^2}{24\lambda^2} - \frac{g^4\mathcal{A}_0^4}{48} + \mu^4 + 2(\pi k\mathcal{T}\mu)^2 + \frac{7}{15}(\pi k\mathcal{T})^4\right],$$

$$n = \frac{1}{\beta\delta\mathcal{V}}\left(\frac{\partial \ln Z_\psi}{\partial\mu}\right) = \frac{2\mu}{3\pi^2}\left[\frac{3}{4}g^2\mathcal{A}_0^2 + \mu^2 + (\pi k\mathcal{T})^2\right],$$

$$q_\psi = \frac{1}{g\beta\delta\mathcal{V}}\left(\frac{\partial \ln Z_\psi}{\partial\mathcal{A}_0}\right) = \frac{g\mathcal{A}_0}{24\pi^2}\left[g^2\mathcal{A}_0^2 + \frac{1}{\lambda^2}\right], \tag{2.14}$$

where $\mathcal{P}$ denotes the pressure and s, e, n, q, etc., denote the entropy, energy, particle, and charge *per unit volume*. Note that $e_\psi = 3\mathcal{P}_\psi - 2\mathcal{A}_0 q$ includes the energy of the interaction with the gauge fields.

2. Gluonic contribution

The gluonic contribution to the partition function is obtained in a manner similar to the one followed for the fermions. As before we will ignore the self-interactions of the fields a_μ^a, in this case the partition function, including the Faddeev-Popov determinant, reduces to $Z_{\rm gluons}=\det-\overline{D^2_{\rm adj}}$, where $\overline{D}_{\rm adj}$ denote the covariant derivative for the background fields in the adjoint representation [16].

In calculating this determinant we take into account that the partons are supposed to be in a box of side $\sim\lambda$. Moreover, the background field is supposed to include the effects from the zero (Fourier) modes in the field. It follows that we need to include only modes with energy above

$$p_0=\frac{2\pi}{\lambda}. \tag{2.15}$$

Using the gauge $\overline{A_\mu^a}=\delta_{\mu,0}\delta^{a,3}\mathcal{A}_0$ gives

$$\ln Z_{\rm gluons}=-\mathcal{V}\int_{p>p_0}\frac{d^3p}{(2\pi)^3}\,[\ln(1-e^{-\beta(p-g\mathcal{A}_0)}) +\ln(1-e^{-\beta(p+g\mathcal{A}_0)})+\ln(1-e^{-\beta p})], \tag{2.16}$$

which corresponds to a gas of massless bosons with chemical potential $\pm g\mathcal{A}_0$ and 0.

We will argue below (Sec. II D 1) that the background field $\mathcal{A}_0$ is monotonic in r and that $\lambda g\mathcal{A}_0\leqslant 3\pi/2$. Using also the fact that $\beta p_0\geqslant 4\pi^2$ we find that to a good approximation

$$\mathcal{P}_{\rm gluons}=\frac{(\pi k\mathcal{T})^4}{15\pi^2}+\frac{4(kT)^4}{\pi^2}\,(\beta^2p_0^2+2\beta p_0+2) \times e^{-\beta p_0}[\sinh(\beta g\mathcal{A}_0/2)]^2. \tag{2.17}$$

This shows that the deviations from the free-gluon values are exponentially suppressed (recall that $\beta p_0\geqslant 4\pi^2$) and can be neglected. In this case

$$\mathcal{P}_{\rm gluons}\simeq\frac{(\pi k\mathcal{T})^4}{15\pi^2},\quad e_{\rm gluons}\simeq\frac{(\pi k\mathcal{T})^4}{5\pi^2},$$

$$s_{\rm gluons}\simeq\frac{4k(\pi k\mathcal{T})^3}{15\pi},\quad q_{\rm gluons}\simeq 0, \tag{2.18}$$

where $\mathcal{P}$ denotes the pressure and e, s, and q the energy, entropy, and charge per unit volume, respectively. The errors incurred are below a few percent for the thermodynamic quantities and below 0.0075% for the charge.

C. Stability conditions

The stability criterion can be obtained from the Wong equations [17], but a more elegant argument can be gleaned from a paper by Brown and Weisberger [18]. Consider the background field contribution to the energy momentum tensor $\theta_{\rm back}$, which satisfies

$$\partial_\mu\theta^{\mu\nu}_{\rm back}=g\rho_a\overline{F_a^{\nu0}}, \tag{2.19}$$

where $\overline{F_a^{\nu0}}$ denotes the field strength for the background fields, and ρ_a the thermally averaged non-Abelian charge density.

Since the total energy momentum tensor is conserved, it follows that the averaged partonic contribution $\theta_{\rm part}$ satisfies

$$\partial_\mu\theta^{\mu\nu}_{\rm part}=-g\rho_a\overline{F_a^{\nu0}}. \tag{2.20}$$

For static situations the above equation implies

$$\partial_i\theta^{ij}_{\rm part}=-g\rho_a\overline{F_a^{j0}}. \tag{2.21}$$

If in addition we impose spherical symmetry (see Sec. II A) $\rho_a\propto\hat r^a$, $\overline{A_0^a}\propto\hat r^a$ which implies $\rho_a\overline{F_a^{j0}}=-\rho_a\partial_j\overline{A_0^a}$. For a homogeneous gas of partons the space components of the energy momentum tensor are $\theta^{ij}_{\rm part}=\mathcal{P}_{\rm part}\delta_{ij}$. Collecting these results we get $\partial_j\mathcal{P}_{\rm part}=g\rho_a\partial_j\overline{A_0^a}$ or, equivalently,

$$d\mathcal{P}_{\rm part}=g\rho_a d\overline{A_0^a}, \tag{2.22}$$

which is the desired constraint.

For the Abelian case Eq. (2.22) reduces to the usual Thomas-Fermi equilibrium condition: the pressure on $\delta\mathcal{V}$ is balanced by the electrostatic force.

This stability condition requires the chemical potential and temperature to be r independent. Indeed, $\ln Z_{\delta\mathcal{V}}$, the parton partition function for a small volume $\delta\mathcal{V}$, is a function of $\mathcal{T}$, $\mathcal{A}_0$, and μ; using Eq. (2.14) we obtain

$$d\mathcal{P}_{\rm part}=\frac{1}{\beta\mathcal{V}}\left(\frac{\partial\ln Z_{\delta\mathcal{V}}}{\partial\mathcal{A}_0}\right)d\mathcal{A}_0+\frac{1}{\beta\mathcal{V}}\left(\frac{\partial\ln Z_{\delta\mathcal{V}}}{\partial\mu}\right)d\mu +\frac{1}{\beta\mathcal{V}}\left(\frac{\partial\ln Z_{\delta\mathcal{V}}}{\partial\mathcal{T}}\right)d\mathcal{T} =gq\,d\mathcal{A}_0+n_{\rm part}\,d\mu+(e_{\rm part}-\mu n_{\rm part})\,\frac{d\mathcal{T}}{\mathcal{T}}, \tag{2.23}$$

where $n_{\rm part}$ and $e_{\rm part}$ are, respectively, the particle and energy densities of the partons and $\rho^a=q\hat r^a$. Substituting this expression for ρ_a, using Eq. (2.4), and comparing to Eq. (2.22) we obtain $d\mu=d\mathcal{T}=0$.

D. The TFQCD equations

The equations of motion are derived from the spherically symmetric Lagrangian for the background fields (2.5) when the potentials interact with a source q according to Eq. (2.7). The source, given in Eq. (2.14), is itself a function of the potentials. The resulting equations are

$$\mathcal{D}^2\Phi+\frac{g^2}{r^2}\left(|\Phi|^2-\frac{1}{g^2}\right)\Phi=0,$$

$$\partial^\mu(r^2f_{\mu\nu})+2g\ \mathrm{Im}(\Phi^*\mathcal{D}_\nu\Phi)=-gr^2q\,\delta_{\nu,0}. \tag{2.24}$$

The gauge invariance of these equations allows us to chose the $\mathcal{A}_1=0$ gauge. The second of the above equations gives, when $\nu=1$ and for static configurations, $\mathrm{Im}\Phi^*\Phi'=0$, so that we can choose Φ to be purely imaginary.

We will use the notation [12,13]

$$\mathcal{A}_0=\frac{f(r)}{rg},\quad \Phi=\frac{1}{ig}\,a(r). \tag{2.25}$$

Then, using Eq. (2.14), the above equations become

$$f''-2\left(\frac{a}{x}\right)^2 f=\frac{\alpha}{6\pi}\,f\left(\frac{f^2}{x^2}+1\right),$$

$$a''+\frac{1+f^2-a^2}{x^2}\,a=0, \tag{2.26}$$

where $\alpha=g^2/(4\pi)$ and $x=r/\lambda$ with λ defined in Eq. (2.13).

These equations determine the background self-consistently. Their solution requires the specifications of the boundary conditions to which we now turn.

1. Boundary conditions

The conditions near the origin are determined by considering the behavior of Wilson loops as $r\to 0$. We find that singularities arise unless f and a^2-1 vanish at $r=0$. Using the $a\leftrightarrow -a$ symmetry (which is a remnant of the gauge symmetry) we can then require $a\to 1$ as $r\to 0$. The precise manner in which f and $a-1$ vanish as r approaches zero is determined by requiring that the energy should have no divergences at this point. We then obtain

$$f,a-1=O(r^2)\quad \text{for } r\to 0. \tag{2.27}$$

It is easy to see that the equations of motion (2.26) require f to be concave or convex, since we can always exchange $f\to -f$ we can assume that f is concave. In particular this implies that f will not vanish for $r>0$. From Eq. (2.26) it is also easy to show that $(f/r)'>0$. In contrast a can (and will) have extrema as well as zeroes.

As mentioned in the Introduction we assume that the system is enclosed in a container which must be spherical due to the requirement of spherical symmetry; we denote by R its radius. If the system is to be confined to the region $r<R$, there should be no leakage of fermion number or color into the region $r>R$.

The first of these two conditions (fermion number confinement) requires the fermions to have zero radial component of the momentum at $r=R$. This implies that in the vicinity of $r=R$ the fermion gas becomes two dimensional. The corresponding (surface) charge density σ^a takes the form

$$\sigma^a=\vartheta\hat{r}^a \tag{2.28}$$

as mandated by spherical symmetry. Note, however, that ϑ does not have a simple analytical form:

$$\vartheta=\frac{1}{8\pi\beta^2}\,[\,\mathcal{Q}(\beta\mu+\beta g\mathcal{A}_0/2)-\mathcal{Q}(\beta\mu-\beta g\mathcal{A}_0/2)\,],$$

$$\mathcal{Q}(u)=2\int_0^\infty ds\,\ln\left[\frac{e^s+e^u}{e^s+e^{-u}}\right]\simeq\left(u^2+\frac{\pi^2}{3}\right)\tanh\left(\frac{12\ln 2}{\pi^2}\,u\right), \tag{2.29}$$

where the analytic approximation to $\mathcal{Q}$ is accurate to about 0.62%; the derivative is accurate to 0.92%.

In the examples which we consider in detail we will be interested in the limit where $\mathcal{A}_0$ is large and where $\beta\to 0$ or $\mu\to 0$. In these cases we have

$$\vartheta\simeq\frac{g^2}{16\pi}\,\mathcal{A}_0^2. \tag{2.30}$$

We will require the volume charge density in the bulk to smoothly join the surface charge density at the surface layer, that is, $\rho^a\lambda=\sigma^a$ at $r=R$. Thus we impose, $q\lambda=\vartheta$ at $r=R$ which, keeping in mind that the solutions produce large values of $\mathcal{A}_0$ at R, is equivalent to $\mathcal{A}_0^2/(4\pi)=\lambda\mathcal{A}_0^3/(3\pi^2)$, or equivalently

$$f(R)=\frac{3\pi}{2}\,\frac{R}{\lambda}. \tag{2.31}$$

It is of course possible to modify this condition by requiring only that, at $r=R$, $q\lambda=s\vartheta$ for some number $s=O(1)$, which is equivalent to replacing $\lambda\to\lambda/s$; our results are insensitive to such a replacement.

To determine the consequences of the second of the above two conditions (color confinement) we need the components of the chromoelectric and chromomagnetic fields parallel and perpendicular to $\mathbf{r}$,

$$\hat{\mathbf{r}}\cdot\mathbf{E}^a=-\frac{1}{g^2\lambda}\left(\frac{f}{x}\right)'\hat{r}^a,\quad \hat{\mathbf{r}}\cdot\mathbf{B}^a=-\frac{1}{g^2\lambda}\,\frac{a^2-1}{x^2}\,\hat{r}^a,$$

$$(\hat{\mathbf{r}}\times\mathbf{B}^a)^i=-\frac{1}{g^2\lambda}\,\frac{a'}{x}\,\epsilon_{ija}\hat{r}^j,\quad (\hat{\mathbf{r}}\times\mathbf{E}^a)^i=-\frac{1}{g^2\lambda}\,\frac{fa}{x^2}\,\epsilon_{ija}\hat{r}^j. \tag{2.32}$$

The first of these relations, together with the previously derived result $(f/r)'\neq 0$, implies that color will leak from the system unless an appropriate modification is included. The situation is identical to the one present in the bag model [9], and the solution which we adopt is the same [10]. We will couple our system at the $r=R$ boundary to a CP-odd field η' via a term proportional to the Chern-Simons term; this coupling insures that color is confined to the region $r\leqslant R$ [10]. Denoting by $F_{\eta'}$ the decay constant on the η', the coupling to this field at $r=R$ are determined by the relations

$$\hat{\mathbf{r}}\cdot\mathbf{E}^a=\frac{\alpha}{\pi F_{\eta'}}\,\hat{\mathbf{r}}\cdot\mathbf{B}^a\eta',\quad \hat{\mathbf{r}}\times\mathbf{B}^a=-\frac{\alpha}{\pi F_{\eta'}}\,\hat{\mathbf{r}}\times\mathbf{E}^a\eta' \tag{2.33}$$

from which we derive $(\hat{\mathbf{r}}\cdot\mathbf{E}^a)(\hat{\mathbf{r}}\times\mathbf{E}^a)+(\hat{\mathbf{r}}\cdot\mathbf{B}^a)(\hat{\mathbf{r}}\times\mathbf{B}^a)=0$; in terms of the a and f fields this becomes

$$fa(xf'-f\,)+xa'(a^2-1)=0,\quad \text{at } r=R, \tag{2.34}$$

which is the desired condition.[3]

[3]Concerning Eq. (2.34) we know, from the numerical integration of Eq. (2.26), that $f(xf'-f\,)$ does not vanish, we also find that it is numerically large for the situations we consider in detail. It follows that Eq. (2.34) can be approximately replaced by the simpler condition $a(R)=0$.

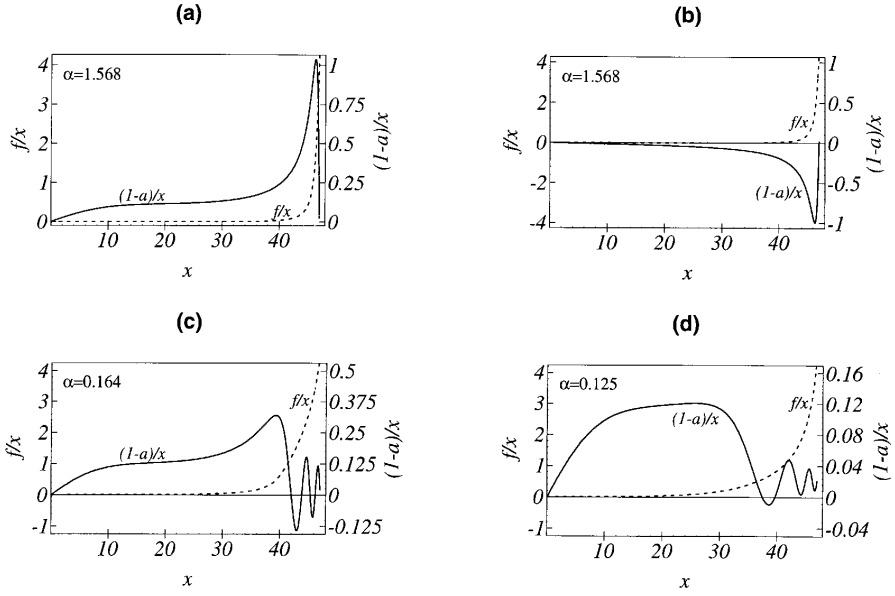

FIG. 1. Examples of solutions $f(x)$ (dashed lines) and $a(x)$ (solid lines) corresponding to $R=10$ fm, $\mathcal{E}/\mathcal{V}=4$ GeV/fm^3, and $\mathcal{T}=150$ MeV ($X=47$). Cases (a) and (b) corresponds to the solutions which minimize the thermodynamic potential. Cases (c) and (d) have larger Ω and represent unstable solutions; the values of α corresponding to each solution are indicated.

2. Character of the solutions

The TFQCD potentials f and a are then obtained by solving the equations (2.26) subject to the boundary conditions (2.27), (2.31), and (2.34). These solutions, as well as all thermodynamic variables, will depend on the parameter $\alpha=g^2/(4\pi)$. In order to specify α we first fix the thermodynamic variables of the system, such as the energy and volume; the TFQCD expresses these thermodynamic variables as functions of α, which is chosen so that the chosen values are met.

When considering Eq. (2.26) we find that, for given values of X and α, there are several solutions satisfying the boundary conditions.[4] Of these solutions there is a set (we, in fact, found two such solutions) which minimizes Ω, the thermodynamic potential at constant pressure and chemical potential [19]:

$$\Omega=-\int d^3x\mathcal{P}, \tag{2.35}$$

where $\mathcal{P}$ denotes the total pressure. Numerical studies show that there is no crossover as α changes: each member of the set of solution which minimizes Ω is a smooth function of α. Selecting the solution which minimizes Ω we then determined α by matching the specified energy and baryon number.

The explicit expressions for Ω, the total energy $\mathcal{E}$, and the total number of particles (baryon number) $\mathcal{N}$ are

$$\Omega=\frac{1}{\alpha\lambda}\int_0^X dx\left\{\frac{1}{2}\left(f'-\frac{f}{x}\right)^2+\frac{1}{2}\left(\frac{1-a^2}{x}\right)^2-(a')^2-\left(\frac{fa}{x}\right)^2-\frac{\alpha}{24\pi}f^2\left(2+\frac{f^2}{x^2}\right)\right\}-\frac{2}{9\pi}\left[\mu^4+2\mu^2(\pi k\mathcal{T})^2+\frac{13}{15}(\pi k\mathcal{T})^4\right]R^3, \tag{2.36}$$

$$\mathcal{E}=\frac{1}{\alpha\lambda}\int_0^X dx\left\{\frac{1}{2}\left(f'-\frac{f}{x}\right)^2+\frac{1}{2}\left(\frac{1-a^2}{x}\right)^2+(a')^2+\left(\frac{fa}{x}\right)^2+\frac{\alpha}{24\pi}f^2\left(2-\frac{f^2}{x^2}\right)\right\}+\frac{2}{3\pi}\left[\mu^4+2\mu^2(\pi k\mathcal{T})^2+\frac{13}{15}(\pi k\mathcal{T})^4\right]R^3, \tag{2.37}$$

$$\mathcal{N}=\frac{2\mu}{\pi}\left\{\frac{4}{9}[\mu^2+(\pi k\mathcal{T})^2]R^3+\lambda\int_0^X dx f^2\right\}, \tag{2.38}$$

where

$$X=\frac{R}{\lambda}. \tag{2.39}$$

For future reference we also provide the expression for the (total) entropy of the system

$$\frac{1}{k}\mathcal{S}=\frac{2}{3}\lambda\pi k\mathcal{T}\left\{\frac{4}{3}X^3\lambda^2\left[\mu^2+\frac{13}{15}(\pi k\mathcal{T})^2\right]+\int_0^X f^2dx\right\}. \tag{2.40}$$

[4]This is reminiscent of the situations found in the case of classical solutions to the Yang-Mills equations with external sources [13].

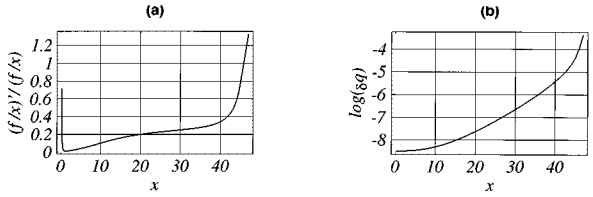

FIG. 2. Validity of the Thomas-Fermi approximation. (a): Only in the regions near the boundary at $x=47$ ($x>45.5$) and the origin ($x<1$) does the approximation break down. (b): The charge is effectively screened throughout the volume. The logarithm is to base 10.

Solutions of the equations for f and a can be obtained using standard numerical algorithms; due to the singular nature of the equations at the origin the relaxation method is best suited.

We present several examples of the solutions in Fig. 1 where we took $R=10$ fm, $\mathcal{E}/\mathcal{V}=4\ \mathrm{GeV/fm}^3$, and $\mathcal{T}=150$ MeV (which implies $X=47$). All the solutions in Fig. 1 satisfy Eq. (2.26) and the boundary conditions; the solutions which minimize the thermodynamic potential corresponds to cases (a) and (b).[5] These two solutions correspond to indistinguishable thermodynamics (within numerical errors); for the calculations below this duality presents no complications. We have not attempted to study the stability of these solutions against nonradially symmetric perturbations [20].

Given these results we must now determine whether they are consistent with the original assumptions, that is, whether f varies slowly enough to be considered constant in a region of width $\sim\lambda$. We also must determine to what extent are color charges screened. The plots presented correspond to both cases (a) and (b) in Fig. 1.

The rate of change of f is sufficiently slow provided the potential $\mathcal{A}_0$ changes little within a region of size λ, this is equivalent to

$$\frac{(f/x)'}{(f/x)}<1; \tag{2.41}$$

a plot of the left-hand side of this equation if presented in Fig. 2(a). We see that the condition (2.41) is satisfied except in the vicinity of the origin and the $r=R$ boundary. The value of Eq. (2.41) near $x=0$ presented in Fig. 2(a) is an underestimate generated by numerical errors (the equations are singular at $x=0$); for $x\to 0$, $(f/x)'/(f/x)\simeq 1/x$.

[5]The singular nature of the equations allows for the multiplicity of solutions; we have found eight solutions in total (for the given values of $\mathcal{E}$ and $\mathcal{V}$ but having different values of α), though we cannot assert that this an exhaustive list. Using the relaxation method, the solution that minimizes Ω was less sensitive to the initial trial functions, solutions with larger Ω become increasingly more difficult to find as the range of initial configurations which relax to such solutions of Eq. (2.26) becomes more and more restricted. We have not attempted to perform a complete study of the properties and number of solutions restricting ourselves to finding the one solution relevant for physical applications together with some examples of unstable solutions.

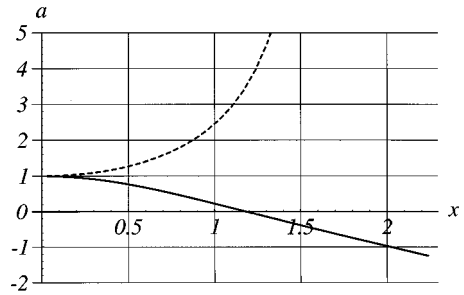

FIG. 3. Solutions for small X, solid curve: $a''(0)=-2$, dashed curve: $a''(0)=2$.

The magnitude of the charge in a subvolume $\delta\mathcal{V}\sim\lambda^3$ is obtained from Eq. (2.14), and equals

$$\delta q=\lambda^3 q=\frac{1}{24\pi^2}\frac{f}{x}\left(\frac{f^2}{x^2}+1\right). \tag{2.42}$$

A plot of this quantity is presented in Fig. 2(b). As can be seen the magnitude of the color charge inside each subvolume is quite small except near the $r=R$ boundary: the system does screen its charges quite effectively.

E. Solutions for small X

When X is small then f will be small also since it is monotonic [this follows from the boundary condition (2.31)]. In this case the equation for a decouples and so does the boundary condition (2.34):

$$x^2a''+(1-a^2)a=0,\quad a(0)=1,\quad a'(X)[a(X)^2-1]=0. \tag{2.43}$$

If we define

$$a_2=\frac{1}{2}a''(0), \tag{2.44}$$

it is easy to see that the solution to the above equation is a function of a_2x^2. It is then enough to assume $a_2=\pm 1$; the general solutions are obtained from these by rescaling x. The solutions to the above differential equation (for $a_2=\pm 1$) are presented in Fig. 3. The solutions are monotonic, so the boundary condition at $x=X$ is satisfied when $a(X)=-1$ which occurs only for $a_2<0$, numerically:

$$a_2X^2\simeq -4.1, \tag{2.45}$$

which completely specifies the solution.

Again neglecting f and evaluating numerically the integrals gives

$$\mathcal{E}\simeq\frac{5.42}{\alpha R}+4.28(k\mathcal{T})^4\mathcal{V},$$

$$\mathcal{P}\simeq\frac{2.91}{\alpha}\mathcal{V}^{-4/3}+1.43(k\mathcal{T})^4. \tag{2.46}$$

For example, at $\mathcal{T}=0$, $\mathcal{E}=1$ GeV, $R=1$ fm, $\alpha\simeq 1$, and $\mathcal{P}\simeq 85\ \mathrm{MeV/fm}^3$. Fixing $\mathcal{V}$ and $\mathcal{T}$, the coupling strength α drops as $1/\mathcal{E}$.

At zero temperature we have $\mathcal{N}\simeq 2\mu^3\mathcal{V}/(3\pi^2)$ and

$$\mathcal{E}\simeq \frac{5.42}{\alpha R}+\frac{\mu^4\mathcal{V}}{19.74},\quad \mathcal{P}=\frac{2.91}{\alpha}\,\mathcal{V}^{-4/3}\,\frac{\mu^4}{59.04},\tag{2.47}$$

so that the equation of state becomes

$$\mathcal{P}\mathcal{V}^{4/3}\simeq \frac{2.91}{\alpha}+\frac{\mathcal{N}^{4/3}}{1.62}.\tag{2.48}$$

In this case $\mathcal{N}=2$, $\mathcal{E}=1$ GeV, and $R=1$ fm imply $\alpha\simeq 2.4$ and $\mathcal{P}\simeq 77$ MeV/fm^3.

The numbers obtained for the case of small X are then quite consistent with those obtained using the bag model [9] (except perhaps for a large value for α). Note, however, that in the present calculation the contributions from the nonideal gas terms are very important and the numerical agreement is not trivial. It is also true that the present model is far from realistic [being based on an SU(2) gauge theory with a single species of massless quarks]. These results are therefore quite encouraging but not conclusive as to the physical relevance of this model.

Concerning the other thermodynamic quantities they relapse to their free-particle values up to $O(f^2)$ corrections. Note that the adiabats are, in general, defined by $X=$const which, for the case $\mathcal{N}=0$, imply $\mathcal{P}^3\mathcal{V}^4=$const just like a relativistic ideal gas.

When $\mu=0$ an approximate solution for f which satisfies the boundary conditions is

$$f\simeq 4.57\,\frac{x^2}{X},\tag{2.49}$$

in this case the entropy becomes $\mathcal{S}\simeq 1.5kX^3$ and the heat capacity equals $C_V\simeq 3\mathcal{S}$; the largest contribution to these quantities ($\sim 94\%$) comes from the $\int f^2$ term.

F. Solutions for large X

In order to study the solutions to Eq. (2.26) for x finite but $X\to\infty$ it proves convenient to define $y=x/X$. We are then interested in the small y behavior of the solutions and a power series is appropriate:

$$f=f_2\Bigg[y^2+\left(\frac{2a_2}{5}+u\right)y^4+\left(\frac{6a_2^2}{35}-\frac{f_2^2}{70}+\frac{5uf_2^2}{14X^2}+\frac{2ua_2}{7}+\frac{14u^2}{5}\right)y^6+\cdots\Bigg],$$

$$a=1+a_2y^2+\left(\frac{3a_2^2-f_2^2}{10}\right)y^4+\left(\frac{a_2^3}{10}-\frac{3a_2f_2^2}{35}-\frac{uf_2^2}{14}\right)y^6+\cdots,\tag{2.50}$$

where $u=\alpha X^2/(60\pi)$. Numerical simulations indicate that neither f_2 nor a_2 increase with X which, using Eqs. (2.31) and (2.34), leads to $f\sim 3\pi x^2/2X^2$ and $a\sim 1-x^2/X^2$ for $x\ll X$. Thus, as $X\to\infty$, $f\to 0$ and $a\to 1$ for x finite.

For $x\lesssim X$ the boundary conditions require $a\simeq 0$ and $f\gg 1$; the first of the Eqs. (2.26) can then be approximated by

$$f''\simeq\frac{\alpha}{6\pi}\,\frac{f^3}{X^2};\quad x\lesssim X,\tag{2.51}$$

whose solution [using Eq. (2.31)] reads

$$f\simeq\frac{6\pi X}{4+(X-x)\sqrt{3\pi\alpha}}.\tag{2.52}$$

Using these results we can evaluate the various thermodynamic quantities for large X. For example,

$$\mathcal{E}-\mathcal{E}_{\text{ideal gas}}\simeq\sqrt{\frac{3\pi}{16\alpha}}\,\frac{R^2}{\lambda^3};\quad X\gg 1\tag{2.53}$$

from which we find $|\mathcal{E}-\mathcal{E}_{\text{ideal gas}}|/\mathcal{E}_{\text{ideal gas}}\sim 1/X$. All other thermodynamic quantities exhibit this behavior: for large T and fixed R (corresponding to large X) the system approaches a mixture of ideal gases.

We emphasize that this is not a result of asymptotic freedom (when the running of the coupling is included the large X behavior will acquire logarithmic corrections), but a property of the solutions to the differential equations. In the infinite volume limit the charges are screened which requires $\mathcal{A}_0=0$ [see Eq. (2.42)].

It is also worth noticing that Eq. (2.53) explicitly displays the finite-volume corrections to the ideal gas results.

III. APPLICATIONS

We now consider some applications of the above formalism. We first study a system with vanishing baryon number (corresponding to $\mu=0$), and then consider the case of zero temperature.

A. Zero baryon number

This situation is believed to be of relevance in relativistic heavy ion collisions, such as those to be produced at the BNL Relativistic Heavy Ion Collider (RHIC) [21], where, in the standard picture, the nuclei will go through one another leaving behind a region of hot quark-gluon plasma with zero baryon number [22].

The requirement $\mathcal{N}=0$ in Eq. (2.38) corresponds to $\mu=0$ which simplifies some of the expressions. In particular the only scales in the system are the temperature and the volume. The plot of the equation of state is given in Figs. 4 and 5.

We have determined α by requiring the solution to minimize the thermodynamic potential Ω when the energy density equals 4 GeV/fm^3 at $T=150$ MeV, $R=10$ fm (which is consistent with the expectations for RHIC); in this case $\alpha\simeq 1.568$.

If we now allow the system to expand adiabatically, we can use the above expressions to obtain the relationship between T and R corresponding to this process. This isentropic transformation describes (in an admittedly oversimplified manner) the expansion of a quark-gluon plasma. The entropy is obtained from Eq. (2.40) by setting $\mu=0$; the result is

$$\frac{1}{k}\,\mathcal{S}=\frac{13}{135}\,X^3+\frac{1}{3}\int_0^X dx f^2(x).\tag{3.1}$$

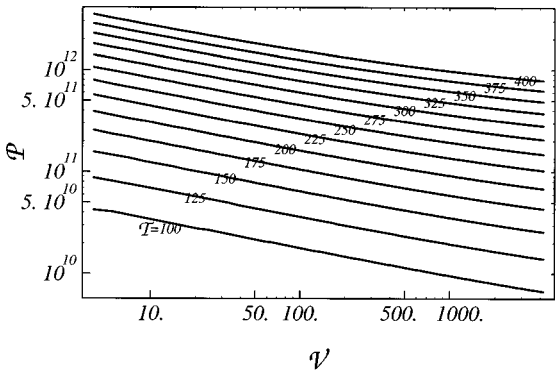

FIG. 4. Equation of state within the Thomas-Fermi approximation for the $\mathcal{N}=0$ case. The graph displays the pressure as a function of the volume for several values of the temperature (P in MeV^4, V in fm^3, T in MeV; the logarithms are base 10).

Since $\mathcal{S}$ is a function of X only (a consequence of having only two scales in the problem, R and $\mathcal{T}$), the equation for the adiabats is $X=\text{const}$ or, equivalently, $\mathcal{V}\mathcal{T}^3=\text{const}$ corresponding to an adiabatic index $\gamma=4$. Note that the $\int f^2$ term in $\mathcal{S}$ modifies the usual free fermion gas relation $S\propto\mathcal{T}^3$; the corrections are $\sim 20\%$ (which is smaller than the corresponding contributions in the case of small X, see Sec. II E).

We can also easily determine the energy density for this isentropic process. From the expression for the total energy in Eq. (2.37) it follows that $\mathcal{E}\lambda$ is a function of X only (for the $\mu=0$ case). It follows that at constant entropy $\mathcal{E}$ scales as $\mathcal{T}$. The energy density then will scale as $\mathcal{T}/R^3\propto\mathcal{T}^4$, just as for an ideal gas of massless particles.

Using the expression for $\mathcal{S}$ we obtain the heat capacity at constant volume:

$$\frac{1}{k}\,C_V=X\left(\frac{\partial\mathcal{S}}{\partial X}\right). \tag{3.2}$$

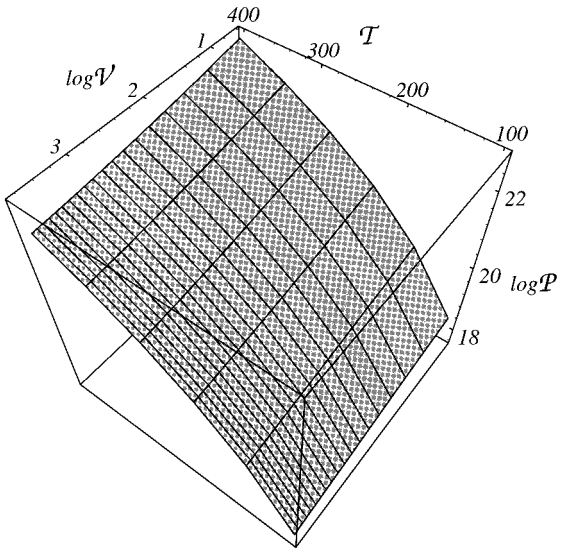

FIG. 5. Three-dimensional rendition of the equation of state for $\mathcal{N}=0$ (P in MeV^4, V in fm^3, T in MeV; the logarithms are base 10).

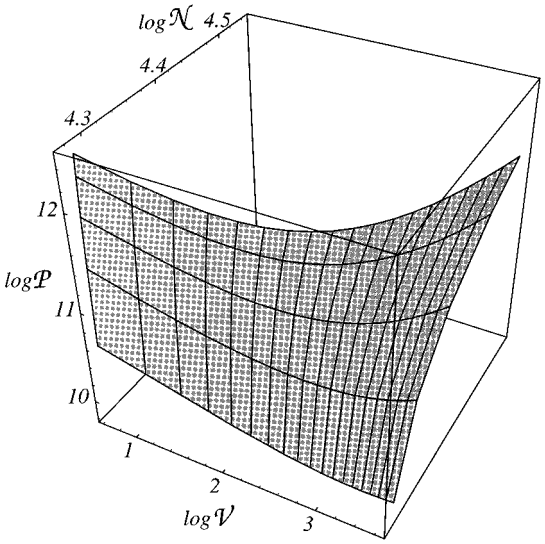

FIG. 6. The equation of state within the Thomas-Fermi approximation for the zero temperature case (P in MeV^4, V in fm^3; the logarithms are base 10).

In calculating this expression one must remember that the boundary conditions to the TFQCD equations depend on X, so that we should in fact write $f=f(X;x)$; when the partial derivative is taken in Eq. (3.2), f must also be differentiated under the integral sign.

B. Zero temperature

We now turn to the case of zero temperature; the dimensional quantities in the system are now μ and R. In this case all dimensionless quantities such as $\mathcal{E}/\mu$ will be functions of $R\mu$ only. The chemical potential is determined in terms of R and $\mathcal{N}$ using Eq. (2.38) but this must be done numerically since the nonideal gas term is significant and cannot be ignored. The plot of the equation of state for this case is given in Fig. 6.

The equivalent contour plot for various values of N is presented in Fig. 7. The equation of state (for the range of variables presented in Fig. 7) is well represented by the equation

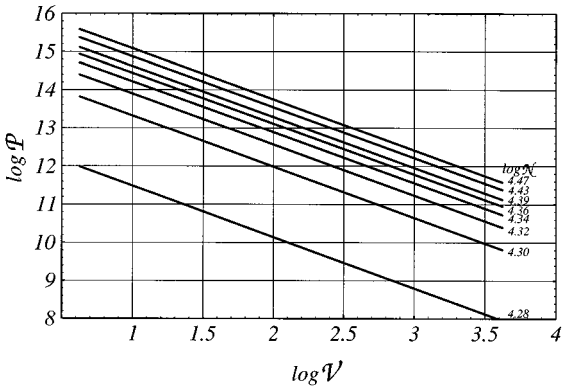

FIG. 7. Pressure as a function of volume at zero temperature, for various values of log$\mathcal{N}$ (P in Mev^4, V in fm^3; the logarithms are base 10).

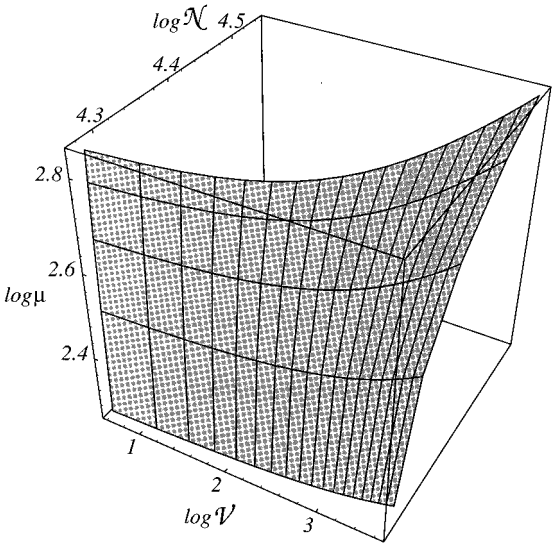

FIG. 8. The chemical potential as a function of volume and baryon number at zero temperature (P in MeV^4, V in fm^3, μ in MeV; the logarithms are base 10).

$$\mathcal{PV}^{4/3}=z(N);\quad \log z(N)\simeq 12.82+5.46\left[\ln\left(\frac{\mathcal{N}}{1.91\times 10^4}\right)\right]^{1/4}. \tag{3.3}$$

The $\mathcal{PV}^{4/3}$ behavior is a result of simple scaling arguments and is therefore present here as well as for small X. In contrast, the $\mathcal{N}$ dependence of the equation of state is radically different [cf. Eq. (2.48)].

We have also determined the chemical potential as a function of temperature and volume. The result is presented in Fig. 8. As $\mathcal{T}\to 0$ the entropy goes to zero linearly,

$$\frac{\mathcal{S}}{\pi k\mathcal{T}}\xrightarrow{\mathcal{T}\to 0}\frac{k}{\sqrt{3}\mu}\left[\left(\frac{X_0}{3}\right)^3+\frac{1}{3}\int_0^{X_0}dx f^2(x)\right],$$

$$X_0=2\sqrt{3}R\mu, \tag{3.4}$$

since the fermionic contribution dominates in this limit; we then also have $C_V=\mathcal{S}$.

IV. EXTENSIONS OF THE METHOD

The inclusion of more flavors is quite straightforward, the charges generated by each simply add. Possible computational difficulties arise when the fermion mass cannot be neglected (as is the case for the strange quark) for in this case a closed form for the fermionic partition function is not available. We will not pursue here this situation further as it involves no new concepts.

A more interesting extension is obtained by considering SU(3) as the gauge group. In this case there are two important modifications. First, within each subvolume $\delta\mathcal{V}$, though we still have $A_0=\text{const}$ and diag, this now implies $A_0=g(\mathcal{A}_0\lambda_3+\mathcal{B}_0\lambda_8)/(2i)$. In general $\mathcal{B}_0\neq 0$, so in this case we will have additional contributions depending on this new potential. The TFQCD equations are derived in the same way as for the SU(2) case. Therefore the presence of the gauge field is summarized by the replacements

$$\mu\to\mu+\frac{g}{2}(\mathcal{A}_0+\mathcal{B}_0/\sqrt{3}),$$
$$\mu+\frac{g}{2}(-\mathcal{A}_0+\mathcal{B}_0/\sqrt{3}),$$
$$\mu-g\mathcal{B}_0/\sqrt{3}, \tag{4.1}$$

in Z_0 [Eq. (2.11)]. The resulting fermionic partition function is then

$$\ln Z_\psi=\frac{\beta\delta\mathcal{V}}{4\pi^2}\left[\frac{g^4}{24}(\mathcal{A}_0^2+\mathcal{B}_0^2)^2+\frac{\mu g^3}{3\sqrt{3}}\mathcal{B}_0(3\mathcal{A}_0^2-\mathcal{B}_0^2)\right.$$
$$+\frac{g^2}{12\lambda^2}(\mathcal{A}_0^2+\mathcal{B}_0^2)+\mu^4+2\mu^2(\pi k\mathcal{T})^2$$
$$\left.+\frac{7}{15}(\pi k\mathcal{T})^4\right]. \tag{4.2}$$

Using this result we obtain the charge densities

$$q_3=\frac{g\mathcal{A}_0}{6\pi^2}\left[\frac{g^2}{4}(\mathcal{A}_0^2+\mathcal{B}_0^2)+\sqrt{3}g\mu\mathcal{B}_0+\frac{1}{4\lambda^2}\right],$$

$$q_8=\frac{g}{2\sqrt{3}\pi^2}\left[\frac{g^2}{4}\mathcal{B}_0(\mathcal{A}_0^2+\mathcal{B}_0^2)+\frac{\sqrt{3}}{2}g\mu(\mathcal{A}_0^2-\mathcal{B}_0^2)+\frac{\mathcal{B}_0}{4\lambda^2}\right]. \tag{4.3}$$

The second modification concerns the form of the spherically symmetric ansatz for the background gauge potentials. For SU(3) a possible ansatz takes the form (now including a contribution in the λ_8 direction)

$$\overline{A_a^0}=\mathcal{A}_0\hat{\mathbf{r}}_a\quad(a=1,2,3),$$

$$\overline{A_a^i}=\epsilon_{iaj}\hat{\mathbf{r}}_j\left(\frac{1+\varphi_2}{r}\right)+(\delta_{ia}-\hat{\mathbf{r}}_i\hat{\mathbf{r}}_a)\frac{\varphi_1}{r}+\hat{\mathbf{r}}_i\hat{\mathbf{r}}_a\mathcal{A}_1\quad(a=1,2,3),$$

$$\overline{A_8^0}=\mathcal{B}_0. \tag{4.4}$$

Note however that the choice of the SU(2) subgroup in which the potentials $\overline{A_a^\mu}$, $(a=1,2,3)$ reside is arbitrary, and that it costs no energy to change from one such subgroup to another; these degrees of freedom are included through a set of collective coordinates [23]. The full ansatz we use is then [we define $\overline{A^\mu}=\lambda_n\overline{A_n^\mu}/(2i)$, where the λ_n denote the usual Gell-Mann matrices]

$$\overline{A^\mu}\to U^\dagger\overline{A^\mu}U, \tag{4.5}$$

where the $\overline{A_n^\mu}$ are given in Eq. (4.4) and U is a time-dependent SU(3) matrix.

The Lagrangian for the background gauge fields then becomes

$$\frac{1}{2}\,\mathrm{tr}\overline{F_{\mu\nu}^2}\to\frac{1}{2}\,\mathrm{tr}\overline{F_{\mu\nu}^2}+2\,\mathrm{tr}F^{0i}[\overline{A_i},\mathbb{R}]+\mathrm{tr}\{[\overline{A_i},\mathbb{R}][\overline{A^i},\mathbb{R}]\}, \tag{4.6}$$

where

$$\mathbb{R}=\dot{U}U^{\dagger} \tag{4.7}$$

and $\overline{F}$ is the field strength corresponding to $\overline{A}$. When the form of the gauge potentials in the SU(2) subgroup takes the form (4.4), $\mathbb{R}$ should have no components along the generators of the SU(2) subgroup generated by $\lambda_{1,2,3}$, that is, we take

$$\mathbb{R}=\sum_{n=4}^{\infty}\frac{1}{2i}\lambda_n\mathbb{R}^n, \tag{4.8}$$

which considerably simplifies Eq. (4.6). The corresponding action is[6]

$$S=\int d^4x\,\frac{1}{2g^2}\,\mathrm{tr}F_{\mu\nu}^2+\frac{1}{2}\,c^2\int dt\,\mathrm{tr}\dot{U}^{\dagger}\dot{U};$$

$$c^2=\frac{1}{\alpha}\int_0^R dr(a-1)^2. \tag{4.9}$$

Numerically the coefficient c can be very large (for the numerical solutions presented $c\sim 3.5\times 10^3\lambda$).

We will use the notation

$$\mathcal{B}_0=\frac{h(r)}{rg}, \tag{4.10}$$

whence the TFQCD equations become (a prime denotes a derivative with respect to $x=r/\lambda$),

$$f''=\frac{2a^2}{x^2}\,f+\frac{\alpha}{6\pi}\,f\left[\frac{f^2+h^2}{x^2}+(4\sqrt{3}\lambda\mu)\,\frac{h}{x}+1\right],$$

$$a''=\frac{a^2-f^2-1}{x^2}\,a,$$

$$h''=\frac{\alpha}{6\pi}\left[\frac{h(f^2+h^2)}{x^2}+(2\sqrt{3}\lambda\mu)\,\frac{f^2-h^2}{x}+h\right], \tag{4.11}$$

which can be solved using the same methods as before. Note that $h=0$ is not allowed when $\mu\neq 0$.

For the interesting case $\mu=0$, $h=0$ is a solution to the above equations. Hence, for zero baryon number, the previous solutions also satisfy the SU(3) TFQCD equations. It does not follow, however, that these solutions again minimize the thermodynamic potential. Note also that even in the case $h=0$ there is an additional contribution to the thermodynamic functions from the collective variables U.

We will not pursue this case further in this paper. A realistic investigation of the SU(3) case requires we include mass term for the (strange) quarks, and also the contributions of the collective coordinates to the thermodynamics of the system. We will consider these issues in a forthcoming publication.

V. CONCLUSIONS

We have presented an approximate treatment of QCD based on the same ideas as the Thomas-Fermi atom. Within this framework the thermodynamics of the system can be derived and the results can be compared with the experimental results which will soon be available.

The method is based on a subdivision of the system into subvolumes which are still large enough to be considered statistical systems. These subvolumes interact through an average gauge field whose sources are the thermodynamically averaged non-Abelian charges for the subvolumes. These charges, though small, are not completely screened due to the assumed smallness of the subvolumes.

The formalism was developed in this paper for the simplified case of an SU(2) gauge group, though we did provide a brief discussion of the modifications required for and SU(3) theory. We also ignored fermion masses and all interactions between the partons inside each of the subvolumes. Nonetheless we found that the numerical values for the pressure in the small $\mathcal{N}$ case are in rough agreement with the bag-model calculations.

For large temperatures, or densities ($X\gg 1$) the solutions to the equations of motion are such that all thermodynamic quantities approach those of a mixture of ideal gases, with $1/X$ measuring the deviation from this limiting behavior. This feature is not related to asymptotic freedom but a result of screening.

In the limit $R\to\infty$ we have $f=0$ and $a=1$, and the equation of state reduces to that of an ideal gas. This model then provides an approximation to the finite-volume corrections to the ideal gas, this is explicitly demonstrated in Eq. (2.53) which gives the surface corrections to the energy of the system.

A realistic calculations must be performed for an SU(3) gauge theory with massive fermions; the partition function inside each subvolume should be evaluated to the highest order available (or possible) in perturbation theory. The inclusion of radiative corrections will induce, among other things, a dependence of the (now running) coupling constants on the temperature and chemical potential. For the present calculation no such effects were included. Finally, one should also include finite volume effects as well as the corrections induced by the gluonic partition function. We will investigate such realistic situations in a forthcoming publication.

We found two solutions to the equations of motion satisfying the boundary conditions and which minimize the thermodynamic potential Ω. Both lead to the same thermodynamics and appear indistinguishable except near the origin (at least within numerical errors). A complete study of the behavior of these solutions under nonspherical perturbations along the lines of Ref. [20] is required to determine the one which is most stable. We have not performed such an investigation since the presence of two such solutions does not alter the thermodynamics derived within the TFQCD approach.

[6]The simplicity of this result is a consequence of the fact that U is made to reside in $\mathrm{SU(3)/SU(2)}\sim S^5$, a five-dimensional sphere, where the number of invariants is very limited. The solutions to the classical equations of motion for U are geodesics representing a motion along the great circles of S^5.

The above treatment was not based on a semiclassical expansion of the partition function for the complete system. It is indeed possible to consider such an approach and use Eq. (2.12) as an approximation to the fermionic contribution. Then the integration over the gauge fields can be approximated by a saddle point method. We have not done this because the effective action which is to be minimized in the last step is, due to the Thomas-Fermi approximations used to obtain Z_ψ, unbounded from below. It is found that the solutions will soon violate the Thomas-Fermi conditions and the method is not consistent; this is displayed explicitly in the Appendix for the case of QED. In contrast, the approach described in the above is consistent with the original approximations.

APPENDIX

In this appendix we present a semiclassical calculation of the partition function of QED using the Thomas-Fermi approximation for the fermionic partition function. The general expression is

$$Z=\int [dA][d\psi][d\overline{\psi}]e^{S}, \tag{A1}$$

where $S=S_g+S_\psi$, the first term denoting the gauge contribution, the second all terms involving the fermions. By definition we have

$$Z_\psi=\int [d\psi][d\overline{\psi}]e^{S_\psi}, \tag{A2}$$

which is approximated by $Z_\psi \simeq \int d^4x \mathcal{P}_\psi$, where

$$\mathcal{P}_\psi=\frac{1}{12\pi^2}\left[(\mu+e\phi)^4+2(\pi k\mathcal{T})^2(\mu+e\phi)^2+\frac{7}{15}(\pi k\mathcal{T})^4\right], \tag{A3}$$

where ϕ denotes the electrostatic potential and e the charge of the fermions (only one flavor is considered). Note that $\mathcal{P}_\psi$ is positive definite.

Assuming spherical symmetry the gauge potentials are of the form, $\phi=\phi(r), \mathbf{A}=a(r)\hat{\mathbf{r}}$. Choosing the $a=0$ gauge gives the following expression for the partition function $Z=\int[d\phi]\exp(S_{\rm eff})$, where

$$S_{\rm eff}=4\pi\int_0^\beta dt\int_0^R dr\ r^2\left[-\frac{1}{2}(\phi')^2+\mathcal{P}_\psi\right], \tag{A4}$$

where β denotes the inverse temperature, R is the radius of the spherical vessel containing the system, a dash denotes a derivative with respect to r, the radial coordinate, and t denotes the Euclidean time variable.

The integrand in $S_{\rm eff}$ is not positive definite. Consider for example $\phi=\phi_0\cos(kr+\nu)$ for constant ν. If ϕ_0 is sufficiently small and k sufficiently large, the first term in $S_{\rm eff}$ will dominate; the larger k, the more negative $S_{\rm eff}$ becomes. The problem in this case is that these expressions for the scalar potential violate the Thomas-Fermi condition which requires $\lambda\phi'/\phi\ll 1$. This shows that a semiclassical treatment of the partition function is inconsistent with the Thomas-Fermi approximation. We have verified that the same problems arise in the non-Abelian case.

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