Vortices and domain walls in a Chern-Simons theory with magnetic moment interactions

Armando Antillón*
Laboratorio de Cuernavaca, Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 48-3, 62251 Cuernavaca, Morelos, México

Joaquín Escalona†
Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, Apdo. Postal 396-3, 62250 Cuernavaca, Morelos, México

Manuel Torres‡
Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, 01000 México, D.F., México

(Received 1 October 1996)

We study the structure and properties of vortices in a recently proposed Abelian Maxwell-Chern-Simons model in $2+1$ dimensions. The model which is described by a gauge field interacting with a complex scalar field includes two parity- and time-violating terms: the Chern-Simons and the anomalous magnetic terms. Self-dual relativistic vortices are discussed in detail. We also find one-dimensional soliton solutions of the domain wall type. The vortices are correctly described by the domain wall solutions in the large flux limit. [S0556-2821(97)01810-9]

PACS number(s): 11.15.Kc, 11.10.Lm, 11.27.+d, 13.40.Em

I. INTRODUCTION

Of the gauge field theories, the self-dual theories deserve special attention. Self-duality refers to theories in which the interactions have particular forms and special strengths such that the equations of motion reduce from second- to first-order differential equations; these configurations minimize a functional, often the energy [1]. For example, the Abelian-Higgs model admits topological solitons of the vortex type [2]. In this model the scalar potential is of the form $V(\phi)\sim(|\phi|^2-v^2)^2$ and the vortices satisfy a set of Bogomol'nyi or self-dual equations when the vector and scalar masses are chosen to be equal [3,4]. The self-dual point corresponds to the boundary between type-I and type-II superconductors. In this point the vortices become noninteracting and static multisoliton solutions may be expected [5,6]. We also notice that the self-dual structure of the gauge theories is related at a fundamental level to the existence of an extended supersymmetry [7].

Recently, considerable interest has been paid to the study of vortex solutions in (2+1)-dimensional Chern-Simons (CS) gauge theories. One common feature of the Chern-Simons solitons is that they carry electric charge as well as magnetic flux [8], in contrast with the electrically neutral Nielsen-Olesen vortices. In addition, they possess fractional spin, a property that is fundamental to consider them as candidates for anyonlike objects in quasiplanar systems. Self-dual Chern-Simons theories are known to exist if one considers pure CS theories. In these theories the Maxwell term in the Lagrangian is absent and the dynamics for the gauge field is solely provided by the Chern-Simons term [9,10]. The self-dual Chern-Simons theories enable a realization with either relativistic [11] or nonrelativistic [12,13] dynamics for the matter degrees of freedom. In the case of the relativistic theory self-dual topological and nontopological vortex solutions have been found with a particular sixth-order potential of the form $V(\phi)\sim|\phi|^2(|\phi|^2-v^2)^2$ when the vector and scalar masses are equal [12,13].

The question can be posed as to whether there are self-dual models in which the gauge field Lagrangian includes both the Maxwell and the Chern-Simons term. A self-dual Maxwell-Chern-Simons gauge theory can be constructed if a magnetic moment interaction is added between the scalar and the gauge fields [14].[1] If the interest is pursuit in a low energy effective theory containing at most second-order derivative terms, such a magnetic moment interaction has to be included. Two steps are followed to obtain the self-dual limit. First, a particular relation between the CS mass and the anomalous magnetic coupling is established whereby the equations for the gauge fields reduce from second- to first-order differential equations similar to those of the pure CS theory. Second, if the scalar potential is selected as a simple ϕ^2 potential and the scalar mass is made equal to the topological mass, the energy obeys a Bogomol'nyi-type lower bound, which is saturated by fields satisfying self-duality equations. The potential possesses a unique minimum at $\phi=0$ and topological solitons certainly do not exist, yet the theory allows nontopological vortex configurations. In this paper we examine the theory and the properties of these nontopological vortices in more detail. In addition, we find that the model admits one-dimensional soliton solutions of the domain wall type. The domain wall carries both magnetic flux and electric charge per unit length. Furthermore, we find

*Electronic address: armando@ce.ifisicam.unam.mx
†Electronic address: joaquin@servm.fc.uaem.mx
‡Electronic address: manuel@teorica1.ifisicacu.unam.mx

[1]The author of Ref. [15] analyzed the Abelian Chern-Simons model with a sixth-order potential when the Maxwell term is included. In this case it is necessary to add a neutral scalar field to obtain the self-dual condition.

that the domain wall configurations provide an approximate solution to the self-dual vortices in the large flux limit. It must be emphasized that the existence of domain wall solutions is unexpected, because the scalar potential has a unique nondegenerate minimum.

As mentioned above there are several aspects of the ϕ^2 Maxwell-Chern-Simons gauge theory which justify further consideration. In Sec. II we introduce the model in which a charged scalar field is coupled via a generalized covariant derivative to the gauge field whose dynamics includes both the Maxwell and the Chern-Simons term. Initially, we consider an arbitrary renormalizable scalar potential in 2+1 dimensions in order to discuss properties of the theory both in the symmetric and in the spontaneously broken phase. As first pointed out in Ref. [16], the nonminimal term in the covariant derivative combined with the spontaneous symmetry mechanism induces a kind of Chern-Simons term. However, we demonstrate that the induced CS term behaves in the same way as the explicit CS term only in the topological trivial sector of the theory. The properties of the two terms are different in the topological nontrivial sector, in particular the induced CS term does not contribute to the fundamental relation between charge and magnetic flux. Rather, the magnetic moment induces a contribution to the magnetization of the vortex that is proportional to the charge of the configuration. In Sec. III we study the propagating modes for the vector field, which consist of two longielliptic waves with different values for the masses. Then in Sec. IV we discuss the conditions required to reduce the original gauge field equations to equations of the pure Chern-Simons type. Section V is devoted to the derivation of the self-duality equations, and to the detailed analytical and numerical study of the cylindrically symmetric vortex solutions. In Sec. VI we discuss domain wall solutions with finite energy per unit length and use these to further examine the properties of the vortex solutions in the large flux limit. Concluding remarks comprise the final section.

II. THE MODEL

Our model possesses a local U(1) symmetry and is described by the effective Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\kappa}{4}\epsilon^{\mu\nu\alpha}A_\mu F_{\nu\alpha}+\frac{1}{2}|\mathcal{D}_\mu\phi|^2-V(|\phi|), \tag{2.1}$$

where κ is the topological mass, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. We use natural units $\hbar=c=1$ and the Minkowski-space metric is $g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$; $\mu=0,1,2$. The covariant derivative is generalized as

$$\mathcal{D}_\mu=\partial_\mu-ieA_\mu-i\frac{g}{4}\epsilon_{\mu\nu\alpha}F^{\nu\alpha}\equiv\partial_\mu-ieA_\mu-i\frac{g}{2}F_\mu, \tag{2.2}$$

where we have defined the dual field

$$F_\mu\equiv\frac{1}{2}\epsilon_{\mu\alpha\beta}F^{\alpha\beta}. \tag{2.3}$$

In terms of the dual field the CS term takes the simpler form $(\kappa/2)A_\mu F^\mu$. The introduction of an anomalous magnetic term in the covariant derivative is consistent with the Lorentz and the gauge invariance of the theory; however, it breaks the $\mathcal{P}$ and $\mathcal{T}$ symmetries. A specific feature of a $(2+1)$-dimensional world is that a Pauli-type coupling (i.e., a magnetic coupling) can be incorporated into the covariant derivative, even for spinless particles [16,17,14]. In fact, in Ref. [18] it was demonstrated that radiative corrections can induce a magnetic coupling for anyons, proportional to the fractional spin. The electromagnetic interactions of charged anyons, in particular its magnetic moment, have also been discussed for point particles in 2 +1 dimensions using the appropriate representations of the Poincaré group [19].

The most general renormalizable potential in 2+1 dimensions is of the form

$$V(\phi)=a_6|\phi|^6+a_4|\phi|^4+a_2|\phi|^2. \tag{2.4}$$

As we shall see later, the particular second-order form of the scalar potential together with its overall strength is fixed by self-duality condition. For the time being we leave the parameter in Eq. (2.4) free in order to discuss both the broken and the unbroken phases of the theory.

As Paul and Khare [16] point out a CS term can be generated by spontaneous symmetry breaking. However, the properties of this CS term are not the same as those of the explicit CS term appearing in Eq. (2.1); we would like to understand the origin of these differences. Suppose the potential is selected to have symmetry-breaking minimum at $|\phi|=v$. Then in terms of the gauge-invariant potential

$$\widetilde{A}_\mu = A_\mu-\frac{1}{e}\partial_\mu \ \mathrm{Arg}\ (\phi), \tag{2.5}$$

the contribution of the covariant derivative to the Lagrangian evaluated in the vacuum configurations ($|\phi|=v$) can be written as

$$\frac{1}{2}|\mathcal{D}_\mu\phi|^2=\frac{1}{2}\left[e^2v^2\widetilde{A}_\mu\widetilde{A}^\mu+\frac{g^2v^2}{4}F_\mu F^\mu+egv^2\widetilde{A}_\mu F^\mu\right]. \tag{2.6}$$

The first term in this expression is the usual gauge field mass $(M=ev)$ generated by the spontanous symmetry breaking. The second term modifies the coefficient of the Maxwell term in the Lagrangian. Finally, the last term is a kind of CS-type term generated by the spontaneous symmetry breaking with topological mass ev^2g. However, the explicit CS term is of the form $(\kappa/2)A_\mu F^\mu$. Instead, in the induced CS term F^μ couples to $\widetilde{A}_\mu$ rather than to A_μ. The gauge field $\widetilde{A}_\mu$ is massive so it has a finite correlation length. This does not imply that A_μ should also fall off exponentially; it can remain a pure gauge. Indeed, this is the case around a vortex where the long-range contribution, which is locally pure gauge, is globally nontrivial giving rise to a nonvanishing magnetic flux. One of the effects of the explicit CS term is that a vortex with magnetic flux Φ must also carry electric charge Q, with the two quantities related as $Q=-\kappa\Phi$. The induced CS mass term ev^2g does not contribute to this relation because of the finite correlation length of $\widetilde{A}_\mu$. Consequently, we conclude that in topologically nontrivial sector the induced term $\widetilde{A}_\mu F^\mu$ in Eq. (2.6) has not the same properties as those of the CS term; so it cannot be considered a genuine CS term. Only in the topologically trivial sector does the induced term have the same properties as those of the explicit CS term.

The equations of motion for the Lagrangian in Eq. (2.1) are

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi = -2\frac{\delta V}{\delta \phi^*}, \tag{2.7}$$

$$\epsilon_{\mu\nu\alpha}\partial^\mu\left[F^\alpha + \frac{g}{2e}J^\alpha\right] = J_\nu - \kappa F_\nu, \tag{2.8}$$

where the conserved Noether current is given as

$$J_\mu = -\frac{ie}{2}[\phi^*(\mathcal{D}_\mu\phi) - \phi(\mathcal{D}_\mu\phi)^*]. \tag{2.9}$$

From the equation of motion (2.8) it is clear that we can define a current $\mathcal{J}_\mu$ that is also conserved. $\mathcal{J}_\mu$ is defined by

$$\mathcal{J}_\mu = J_\mu + \frac{g}{2e}\epsilon_{\mu\nu\alpha}\partial^\nu J^\alpha. \tag{2.10}$$

If the current J_μ is bounded or vanishes faster than $1/r$ at spatial infinity, then the charges calculated from J_0 and $\mathcal{J}_0$ coincide [21].

Let us further examine the gauge field equations of motion (2.8) expressed in terms of the electric and magnetic fields $E_i = F_{0i}$ and $B = F^{12}$, respectively; they read

$$\nabla\cdot\vec{E} - \kappa B = \rho + \frac{g}{2e}\epsilon_{ij}\partial^i J^j,$$

$$\epsilon_{ij}(\partial^j B + \kappa E^j) = J_i + \frac{g}{2}\epsilon_{ij}\partial^j\rho + \partial^0 E_i. \tag{2.11}$$

These equations can be identified as the modified Gauss and Ampere laws, respectively. One of the most important consequences of the CS term is the fact that any object with magnetic flux $\Phi = \int d^2x B$ also carries electric charge $Q = \int d^2x\rho$, with the two quantities related as

$$Q = -\kappa\Phi. \tag{2.12}$$

Indeed, integrating the Gauss law (2.11) over all space we find that the contribution of $\nabla\cdot\vec{E}$ is zero, because of the long-distance damping produced by the ''photon'' mass. Similarly, the integral of the last term also vanishes; this is true even if the symmetry is spontaneously broken. The reason, as explained earlier, is that in the Higgs vacuum the current J_μ makes use of the massive field $\widetilde{A}_\mu$ instead of A_μ. Hence, we obtain the desired result in Eq. (2.12).

While the magnetic moment g does not have a direct effect on the fundamental relation (2.12), it does produce interesting effects, one of which can be found in the magnetization of the excitations of the system. The magnetization is found by coupling an external magnetic field and extracting the linear coupling from the Lagrangian (2.1). It is given by

$$M = \int d^2x(\epsilon_{ij}x^i J^j), \tag{2.13}$$

where $\vec{J}$ is the matter current in Eq. (2.9). Utilizing the modified Ampere law in Eq. (2.11), we find for a static configuration

$$M = \Phi - \frac{g}{2e}Q - \frac{\kappa}{2}\int d^2x(\vec{r}\cdot\vec{E}). \tag{2.14}$$

In the absence of parity-breaking terms the magnetization is equal to the magnetic flux, a result known for the neutral Nielsen-Olesen vortices [2]. Here M has two extra pieces: the magnetic moment g which induces a contribution proportional to the charge of the configuration; and the term proportional to κ which, unlike the two first contributions, depends on the structure factor of the vortex configuration so it cannot be explicitly integrated.

Finally, the energy-momentum tensor is obtained by varying the curved-space form of the action with respect to the metric

$$T_{\mu\nu} = \left(1 - \frac{g^2}{4}|\phi|^2\right)(F_\mu F_\nu - \tfrac{1}{2}g_{\mu\nu}F_\alpha F^\alpha) + \frac{1}{2}\{\nabla_\mu\phi(\nabla_\nu\phi)^* - g_{\mu\nu}[\tfrac{1}{2}|\nabla_\lambda\phi|^2 - V(|\phi|)] + \text{H. c.}\}, \tag{2.15}$$

where $\nabla_\mu = \partial_\mu - ieA_\mu$ only includes the gauge potential contribution. The Chern-Simons and linear terms in g do not appear explicitly in $T_{\mu\nu}$. This is a consequence of the fact that these terms do not make use of the space-time metric tensor $g_{\mu\nu}$, thus when $g_{\mu\nu}$ is varied to produce $T_{\mu\nu}$ no contributions arise from these terms [10]. The expression (2.15) can be considered as the energy-momentum tensors of an Abelian Higgs model in which the Maxwell term is multiplied by a particular dielectric function of the form $[1-(g^2/4)|\phi|^2]$ [20].

III. PROPAGATING MODES

The model described in the previous section has in general three propagating modes in the vacuum state, one of which corresponds to the scalar field. In order to describe the particle content of the gauge field degrees of freedom, we consider the broken phase in which the potential has a symmetry-breaking minimum at $|\phi| = v$. The plane wave solutions to the linearized Maxwell equation (2.11) with $A_\mu \propto e^{ik\cdot x}$ and $k^\mu = (\omega, \vec{k})$ lead to the dispersion relation $\omega = \sqrt{|\vec{k}|^2 + m_\pm^2}$, where the photon masses are given by

$$m_\pm = \frac{\pm(\kappa + ev^2 g) + \sqrt{(\kappa + ev^2 g)^2 + 4e^2v^2(1 - v^2g^2/4)}}{2(1 - v^2g^2/4)}. \tag{3.1}$$

The two values for the photon mass are related with two different polarizations of the electromagnetic wave. From the plane wave solution for A^{μ} and assuming that the wave propagates along the x axis we find that the the electric field can be written as

$$\vec{E} \propto \left(\pm \frac{i m_{\pm}}{\omega} \ , \ 1 \right) e^{i k \cdot x}. \tag{3.2}$$

Hence, the waves are neither transverse nor longitudinal; instead, the solutions correspond to right-handed and left-handed longielliptic waves. We notice that the two masses in Eq. (3.1) may also be deduced from the gauge propagator, or by the explicit analysis of the corresponding Maxwell-Proca equation.

From the work of Pisarski and Rao [22] it is known that the combined effect of the CS term with the mass induced by the Higgs mechanism produces two gauge modes with different masses. From Eq. (3.1) it follows that to have two distinct masses it is required that both spontaneous symmetry breaking and at least one of the $\mathcal{P}$- and $\mathcal{T}$-violating terms exist. The induced term ev^2g simply adds to the CS mass κ, as we are considering here the topologically trivial sector of the theory.

Our result in Eq. (3.1) reduces to well-known cases when the corresponding limits are considered: (i) If the parity-violating terms vanish $(\kappa = g = 0)$ we obtain the usual gauge mass $m_{\pm} = ev$ produced by the spontaneous symmetry breaking; (ii) in the symmetric phase $v = 0$ there is only one propagating mode with mass κ. The existence of massive gauge-invariant theories in $2+1$ dimension, without the Higgs mechanism, is known from the topological massive gauge theories [9]; (iii) if we cancel the magnetic moment $(g=0)$ we recover the result of Paul and Khare [8].

The stability of the model is a genuine concern and we now address this point. In terms of the electric and magnetic fields the energy density (T_{00}) can be written as

$$T_{00} = \frac{1}{2} \left(1 - \frac{g^2}{4} |\phi|^2 \right) (E_i^2 + B^2) + \tfrac{1}{2} |\nabla_0 \phi|^2 + \tfrac{1}{2} |\nabla_i \phi|^2 + V(\phi). \tag{3.3}$$

Hence, one may think that the magnetic moment contribution will in general spoil the positive definiteness of the Hamiltonian.

To investigate the conditions required to have a stable model consider small scalar and gauge field fluctuations around the vacuum solution, in such a way that only quadratic terms on the fluctuation fields are retained. First, it is straightforward to check that the model is stable for small fluctuations of the scalar field. This fact would not be true in the presence of a constant external magnetic field; however, we recall that, because of the topological mass term, a constant magnetic field is not a solution of the equations of motion. The next step is to consider gauge field fluctuations, whereas the scalar field remains at the minimum of the potential, i.e., $|\phi| = v$. Clearly, the energy remains positive definite if the following relation holds:

$$\frac{4}{g^2} \geqslant v^2. \tag{3.4}$$

If the previous relation is not satisfied then the model is no longer positive definite. In this case any gauge field fluctuation with wave vector $\vec{k}$, such that

$$|\vec{k}| > \frac{v^2 e^2}{g^2 v^2 - 4} - \frac{m_{\pm}^2}{2} \tag{3.5}$$

will render the model unstable.

For the purposes of our pursuit we have considered that condition (3.4) holds, thus expect the theory to be stable both for classical and quantum fluctuations of the fields. In particular, for a ϕ^2 potential where $v=0$ condition (3.4) is satisfied. The analysis above considered the stability of the model under the assumption of small field fluctuations. The analysis of the stability under large field fluctuations is a matter of further work. In this section we have analyzed the propagating modes and the stability of these solutions in the topological trivial sector of the theory; the stability of the soliton vortex configuration will be discussed at the end of Sec. V B.

IV. THE PURE CHERN-SIMONS LIMIT

A theory in which the Maxwell kinetic term is dropped and the dynamics is solely provided by the CS term has been frequently considered [9,10]. As mentioned in the introduction, charged vortex solutions are possible in these models. In the Maxwell-Chern-Simons theory without magnetic moment interactions the pure CS limit can be formally obtained by taking the limit e^2, $\kappa \rightarrow \infty$ keeping e^2/κ fixed and rescaling the gauge field by $A_{\mu} \rightarrow A_{\mu}/e$ [23]. This limit is shown to be equivalent to simply neglecting the $F_{\mu\nu}^2$ term in the Lagrangian. In the model described by Eq. (2.1) there is a particular relation between the CS mass and the anomalous magnetic moment for which the Eqs. (2.8) for the gauge fields reduces from second- to first-order differential equation [17,14,24], similar to that of the pure CS type. To obtain this limit we notice that if we set

$$\kappa = -\frac{2e}{g}, \tag{4.1}$$

then any solution to the Eq. (2.8) can be written as

$$F_{\mu} = \frac{1}{\kappa} J_{\mu} \ + \ \lambda G_{\mu}, \tag{4.2}$$

where λ is an arbitrary constant and G_{μ} is solution of the homogeneous field equation

$$\epsilon_{\mu\nu\alpha} \partial^{\mu} G^{\alpha} = -\kappa G_{\nu}. \tag{4.3}$$

In the vacuum the original theory possesses two gauge propagating modes with different masses. In the limit $\kappa = -2e/g$, the gauge field masses in Eq. (3.1) reduce to $m_{+} = \kappa e^2 v^2/(\kappa^2 - e^2 v^2)$ and $m_{-} = \kappa$, respectively. Clearly, the excitations described by the field G_{μ} carry a mass $m_{-} = \kappa$. However, if the nonperturbative sector with vortex solutions is considered, it is easy to see that any solution to the homogeneous equation (4.3) gives an infinite contribution to the energy of the configuration. Consequently, for the solitons to have finite energy the constant λ in Eq. (4.2) should be set to zero. Hence, when the relation (4.1) between the coupling constant holds, the gauge field equation of motion reduces to

$$F_\mu = \frac{1}{\kappa} J_\mu . \tag{4.4}$$

This is a first-order differential equation that automatically satisfies the second-order equation (2.8) when $\kappa = -2e/g$. This equation has the same structure as that of the pure CS theory [9,10]. So, we shall refer to Eqs. (4.1) and (4.4) as the pure CS limit. However, we should notice that the explicit expression for J_μ differs from that of the usual pure CS theory, because according to Eqs. (2.2) and (2.9), J_μ receives contributions from the anomalous magnetic moment.

The gauge field equation (4.4) represents a first-order differential equation, so its propagating modes should be characterized by only one mass. It is straightforward to check that this mass is given by $m_+ = \kappa e^2 v^2/(\kappa^2 - e^2 v^2)$. The other mass $m_- = \kappa$ decouples in the nontrivial topological sector because of the finite energy condition. Henceforth, we work on the limit in which Eqs. (4.1) and (4.4) are valid. Consequently, we shall consider Eq. (4.4) instead of the Eq. (2.8) as the equation of motion for the gauge fields.

In terms of the gauge-invariant potential in Eq. (2.5), the pure CS equation (4.4) reads

$$F_\mu = -\frac{\kappa e^2|\phi|^2}{(\kappa^2 - e^2|\phi|^2)} \widetilde{A}_\mu . \tag{4.5}$$

V. NONTOPOLOGICAL SOLITONS

A. Bogomol'nyi limit

In the Bogomol'nyi limit all equations of motion are known to become first-order differential equations [3]; furthermore, it is possible to recast the equations of motion as self-duality equations. In the pure CS limit (4.1), the gauge field equations have been already reduced to the first-order equations in Eq. (4.4); however, the scalar field is still governed by the second-order equations (2.7). Here we outline the necessary steps to derive the self-duality limit. The self-duality equations for the ϕ^2 model have been reported previously [25,26] but are mentioned here for completeness and also to be used in the discussion of domain wall solutions. We can exploit the pure CS equation (4.5) to eliminate A_0 and E_i from the expression of the energy density in Eq. (3.3). Hence, we can write down the energy $E = \int d^2x T_{00}$ for a static configuration as

$$E = \int d^2x \left(\frac{\kappa^2 - e^2|\phi|^2}{2e^2|\phi|^2} B^2 + \frac{1}{2}\left[|\partial_i \phi|^2 + \frac{\kappa^2 e^2 |\phi|^2}{\kappa^2 - e^2|\phi|^2} \widetilde{A}_i^2 \right] + V(\phi) \right). \tag{5.1}$$

To ensure the positivity of the energy here it is assumed that $\kappa^2 \geqslant e^2|\phi|^2$. The energy written in the previous form is similar to the expression that appears in the Nielsen-Olesen model. Thus, starting from Eq. (5.1), we can follow the usual Bogomol'nyi-type arguments in order to obtain the self-dual limit. The energy may then be rewritten, after an integration by parts, as

$$E = \frac{1}{2}\int d^2x \left[\frac{\kappa^2 - e^2|\phi|^2}{e^2|\phi|^2} \left(B \mp \frac{\kappa e|\phi|^2}{\sqrt{\kappa^2 - e^2|\phi|^2}} \right)^2 + \left(\partial_\pm |\phi| - i \frac{\kappa e|\phi|}{\sqrt{\kappa^2 - e^2|\phi|^2}} \widetilde{A}_\pm \right)^2 \right] + \int d^2x \left[V(\phi) - \frac{1}{2}\kappa^2 |\phi|^2 \right] + \frac{\kappa^2}{e}|\Phi|, \tag{5.2}$$

where $\partial_\pm = \partial_1 \pm i\partial_2$, $\widetilde{A}_\pm = \widetilde{A}_1 \pm i\widetilde{A}_2$, and Φ denotes the magnetic flux. From the previous equation we observe that the energy is bounded below; for a fixed value of the magnetic flux the lower bound is given by $E \geqslant (\kappa^2/e)\Phi$, provided that the potential is chosen as $V(\phi) = (m^2/2)|\phi|^2$ with the critical value $m = \kappa$, i.e., when the scalar and the topological masses are equal. Therefore, in this limit we are necessarily within the symmetric phase of the theory. From Eq. (5.2) we see that the lower bound for the energy

$$E = \frac{\kappa^2}{e}|\Phi| = \frac{\kappa}{e}|Q| \tag{5.3}$$

is saturated when the following self-duality equations are satisfied:

$$B = \pm \frac{\kappa e|\phi|^2}{[\kappa^2 - e^2|\phi|^2]^{1/2}}, \tag{5.4}$$

$$\frac{1}{2}\partial_\pm |\phi|^2 = \frac{ie\kappa|\phi|^2}{[\kappa^2 - e^2|\phi|^2]^{1/2}} \widetilde{A}_\pm , \tag{5.5}$$

where the upper (lower) sign corresponds to a positive (negative) value of the magnetic flux. We should remark that the present model in the self-duality limit corresponds to the bosonic part of a theory with an $(N=2)$-extended supersymmetry [27].

Equation (5.4) implies that the magnetic field vanishes whenever ϕ does. The finiteness energy condition forces the scalar field to vanish both at the center of the vortex (except for the nontopological solitons, see next section) and also at spatial infinity; consequently, the magnetic flux of the vortices lies in a ring. It is interesting to observe that Eq. (5.5) can be written as an explicit self-duality equation; indeed, if we define a new covariant derivative as $\widetilde{D}_i = \partial_i - i2e\kappa/\sqrt{\kappa^2 - e^2|\phi|^2}$, then Eq. (5.5) is equivalent to

$$\widetilde{D}_i |\phi|^2 = \mp i \epsilon_{ij} \widetilde{D}_j |\phi|^2. \tag{5.6}$$

Equations (5.4) and (5.5) can be reduced to one nonlinear second-order differential equation for a unknown function. To do this first notice that Eq. (5.5) implies that $\widetilde{A}_i$ can be determined in terms of the scalar field, on substituting the result in Eq. (5.4) we get

$$\partial_i\left[\frac{\sqrt{\kappa^2-e^2|\phi|^2}}{2e\kappa}\partial_i\ln(|\phi|^2)\right]+\frac{\kappa e|\phi|^2}{\sqrt{\kappa^2-e^2|\phi|^2}}=0. \quad (5.7)$$

B. Rotationally symmetric solutions

The self-dual limit is attained for a ϕ^2 potential; consequently, topological solitons do not exist. However, the theory allows nontopological soliton solutions. In order to look for vortex solutions we now consider static, rotationally symmetric solutions of vorticity n represented by the ansatz

$$\vec{A}(\vec{\rho})=-\hat{\theta}\,\frac{a(\rho)-n}{e\rho},\quad A_0(\vec{\rho})=\frac{\kappa}{e}h(\rho),$$

$$\phi(\vec{\rho})=\frac{\kappa}{e}f(\rho)\exp(-in\theta), \quad (5.8)$$

where ρ,θ are the polar coordinates. After substituting this ansatz the self-duality equations (5.4) and (5.5) become

$$\frac{1}{\rho}\frac{da}{d\rho}=\mp\frac{\kappa^2f^2}{(1-f^2)^{1/2}}, \quad (5.9)$$

$$\frac{df}{d\rho}=\pm\frac{fa}{\rho(1-f^2)^{1/2}}. \quad (5.10)$$

Notice that the function $h(\rho)$ can be explicitly solved using Eq. (4.4) as $h(\rho)=(1-f^2)^{1/2}$. In what follows, we select the signs (upper signs in the previous equations) corresponding to positive magnetic flux $(n>0)$. The equations for $n<0$ are obtained with the replacement $a\rightarrow -a$, $f\rightarrow f$, and $h\rightarrow -h$.

The boundary conditions are selected in such a way that the fields are nonsingular at the origin and give rise to a finite energy solution. The first condition implies that $a(0)=n$ and $n\ f(0)=0$. Whereas the finiteness of the energy implies that $a\rightarrow -\alpha_n$ and $f\rightarrow 0$ as $\rho\rightarrow\infty$. Notice that these requirements leave $a(\infty)=-\alpha_n$ undetermined. Consequently, the magnetic flux for nontopological solitons is not quantized, but rather is a continuous parameter describing the solution. Indeed, once the boundary conditions are known, the ''quantum'' numbers of the soliton can be explicitly computed. With the ansatz (5.8), the magnetic field is $B=(1/r)(da/dr)$, and so using the boundary conditions the magnetic flux and electric charge are

$$\Phi=-\frac{Q}{\kappa}=\int d^2xB=\frac{2\pi}{e}[a(0)-a(\infty)]=\frac{2\pi}{e}(n+\alpha_n). \quad (5.11)$$

We shall later see that for every value of the vorticity n the allowed values for α_n are restricted according to Eq. (5.18). The solutions are also characterized by the spin S (which in general is fractional) and the magnetic moment M. The spin is obtained from the gauge-invariant symmetric energy-momentum tensor (2.15) via $S=\int d^2x(\epsilon^{ij}x_iT_{0j})$; whereas for the magnetic moment we use Eq. (2.14). An explicit calculation yields

$$S=\frac{\pi\kappa}{e^2}(\alpha_n^2-n^2),$$

$$M=-\frac{\pi}{e}\int_0^\infty r^2\frac{dh}{dr}dr. \quad (5.12)$$

Notice that the magnetic flux, the charge, and the spin can be explicitly integrated, because they depend solely on the boundary conditions. Instead, the magnetic moment depends on the structure factor of the vortex configuration.

For the rotationally symmetric ansatz (5.8) the Eq. (5.7) reduces to

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df}{d\rho}\right]=\frac{1}{f(1-f^2)}\left[\left(\frac{df}{d\rho}\right)^2-\kappa^2f^4\right]. \quad (5.13)$$

The same result is obviously obtained if we combine Eqs. (5.9) and (5.10). If we consider the case of small f we can approximate $(1-f^2)^{-1}\approx 1$. Then, Eq. (5.13) reduces to the rotationally symmetric form of the Liouville's equation, which has the following solution:

$$f(\rho)=\frac{2N}{\kappa\rho}\left[\left(\frac{\rho}{\rho_0}\right)^N+\left(\frac{\rho_0}{\rho}\right)^N\right]^{-1}, \quad (5.14)$$

where N and ρ_0 are arbitrary constants.

As mentioned before, the finiteness of the energy implies that $f(\infty)=0$ and, therefore, the value $a(\infty)=-\alpha_n$ is not constrained. We asymptotically solve Eqs. (5.9) and (5.10) as $\rho\rightarrow\infty$:

$$f(r)=\frac{C_n}{(\kappa\rho)^\alpha}-\frac{C_n^3}{4(\alpha-1)^2\ (\kappa\rho)^{3\alpha-2}}+O((\kappa\rho)^{-5\alpha+4}),$$

$$a(r)=-\alpha+\frac{C_n^2}{2(\alpha-1)\ (\kappa\rho)^{2\alpha-2}}-O((\kappa\rho)^{-4\alpha+4}), \quad (5.15)$$

where $\alpha\equiv\alpha_n$ and C_n is a constant. Notice that $f(\rho)$ is asymptotically small so the first two terms of the previous expansion for $f(\rho)$ can be directly obtained from the Liouville approximation (5.14) if we set $N=\alpha-1$ and $(\kappa\rho_0)^{\alpha-1}=C_n/2(\alpha-1)$.

In the origin the boundary conditions are $a(0)=n$ and $n\ f(0)=0$. Hence, it is convenient to consider separately two categories of solutions: the zero vorticity and the nonzero vorticity.

1. n = 0. Nontopological solitons

In this case $a(0)$ must vanish to ensure that the solution is nonsingular at the origin, but $f(0)=f_0$ is not so constrained. These are nontopological solitons that are characterized by the value of the magnetic flux $\Phi=(2\pi/e)|\alpha_0|$. The large-distance behavior is given by Eqs. (5.15), while as $\rho\rightarrow 0$ we obtain a power-series solution

$$f(\rho)=f_0-\frac{f_0^3}{4(1-f_0^2)}(\kappa\rho)^2+\frac{f_0^5(4-f_0^2)}{64(1-f_0^2)^3}(\kappa\rho)^4+O((\kappa\rho)^6), \quad (5.16)$$

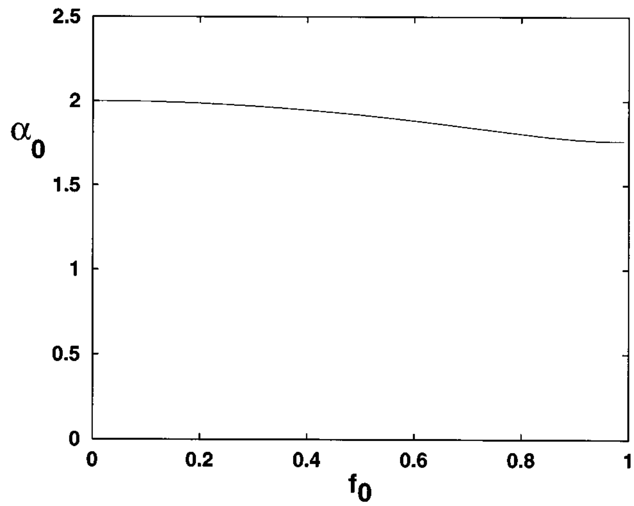


FIG. 1. Behavior of the α_0 as function of f_0 in the case of nontopological solitons $(n=0)$.

$$a(\rho)=-\frac{f_0^2}{2(1-f_0^2)^{1/2}}(\kappa\rho)^2+\frac{f_0^4(2-f_0^2)}{16(1-f_0^2)^{5/2}}(\kappa\rho)^4 +O((\kappa\rho)^6).$$

Acceptable soliton solutions exist for values of f_0 in the range $0<f_0<1$. The short- and large-distance behaviors of the solutions are related, since α_0 is a function of f_0, see Fig. 1. The extreme values on this plot are obtained as follows. If $f_0 \ll 1$ then $f(\rho)$ remains small for all ρ and can, therefore, be approximated by the Liouville solution. Comparing the expansion in Eq. (5.16) with the one obtained from the Liouville solution near the origin we see that the constant N in Eq. (5.14) should be set to 1 while $\kappa\rho_0=2/f_0$. But now the same Liouville solution is also applicable in the large-distance region, comparing with Eqs. (5.15) we obtain $\alpha=2$. Instead, as $f_0\to 1$ we find by numerical integration that $\alpha_0\to 1.755$. Thus, the magnetic flux varies continuously between $\Phi=0.877(4\pi/e)$ and $\Phi=4\pi/e$. In Fig. 2 we show profiles of the magnetic field B as function of $\kappa\rho$ for several values of the parameter f_0. For nontopological solitons the magnetic field decreases monotonically from its maximum value at the origin, so the soliton has a flux tube structure.

2. $n\neq 0$. Nontopological vortices

Following Jackiw *et al.* [13], we shall refer to these configurations with nonvanishing vorticity as nontopological vortices. In this case the boundary conditions imply that $f(0)$ must vanish and $a(0)=n$. The large-distance behavior of the fields is given by Eqs. (5.15). For small ρ, a power-series solution gives

$$f(\rho)=f_n(\kappa\rho)^n-\frac{f_n^3}{4(n+1)^2}(\kappa\rho)^{3n+2}+O((\kappa\rho)^{5n+2}),$$

$$a(\rho)=n-\frac{f_n^2}{2(n+1)}(\kappa\rho)^{2n+2}+O((\kappa\rho)^{2n+4}). \quad (5.17)$$

The constant f_n is not determined by the behavior of the field near the origin, but is instead fixed by requiring proper

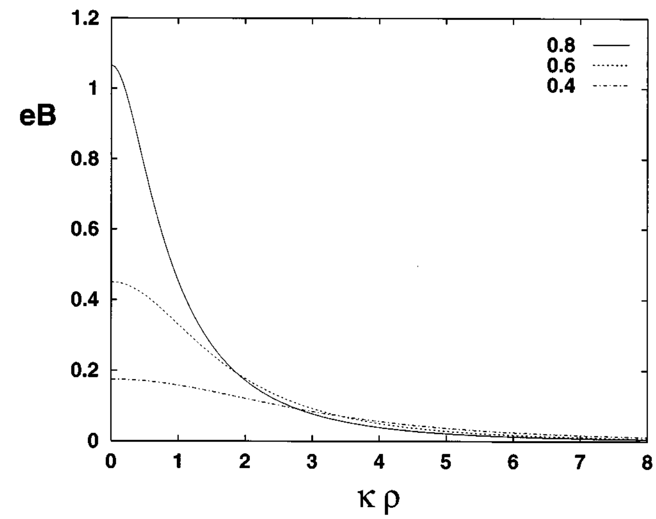


FIG. 2. The magnetic field in units of $1/e$ for the nontopological soliton solution and values of the parameter f_0 of 0.8, 0.6, and 0.4.

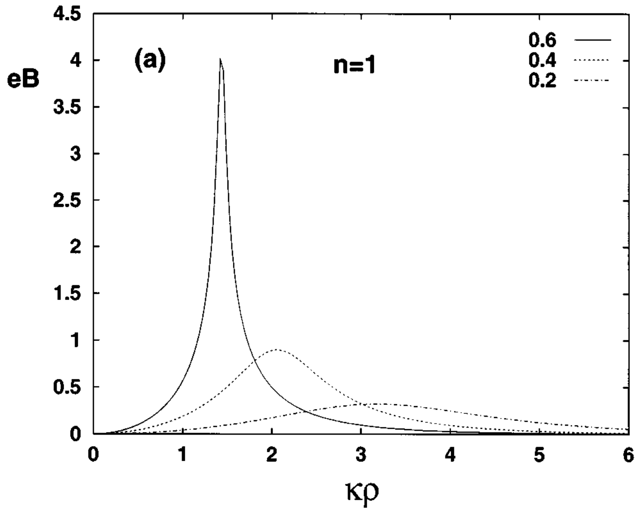
behavior at spatial infinity and that the function remains real for all ρ. For each integer n there will be a continuous set of solutions corresponding to the range $0<f_n<f_n^{\max}$. For values such that $f_n>f_n^{\max}$ there are no real solutions to the field equations (5.9) and (5.10), because the condition $f(\rho)<1$ is not satisfied for all ρ.

If we consider the case $f_n\ll 1$, then $f(\rho)$ is small for all ρ and the Liouville approximation can be used everywhere. In order to match the solution in Eq. (5.14) with both the short- and large-distance approximations in Eqs. (5.17) and (5.15) we should set: $N=n+1$, $(\kappa\rho_0)^{(n+1)}=2(n+1)/f_n$ and $\alpha_n=n+2$. Hence, the corresponding value of the magnetic flux is $\Phi=4\pi(n+1)/e$.

The value $\alpha_n=n+2$ is an upper bound. On the other hand, as $f_n\to f_n^{\max}$ we find that α_n tends to a minimum value $\alpha_n^{\min}$. In fact, it is possible to derive sum rules [28] to prove that α_n is restricted as $n<\alpha_n<n+2$. However, from a numerical analysis we find a more stringent condition on the lower bound. In Table I we present the values of the parameters $\alpha_n^{\min}$ and $f_n^{\max}$ for several vorticity numbers n. We observe that the lower bound for α_n can be taken as $n+1$; this

TABLE I. The parameters $f_n^{\max}$ and $\alpha_n^{\min}$ for the short and large expansion of the nontopological vortices as a function of the vorticity number n.

n	$f_n^{\max}$	$\alpha_n^{\min}$
1	0.607	2.142
2	0.153	3.043
3	2.262×10^{-2}	4.021
4	2.379×10^{-3}	5.013
5	1.946×10^{-4}	6.009
6	1.301×10^{-5}	7.009
7	7.372×10^{-7}	8.006
8	3.623×10^{-8}	9.003
9	1.569×10^{-9}	10.002
10	6.303×10^{-11}	11.002



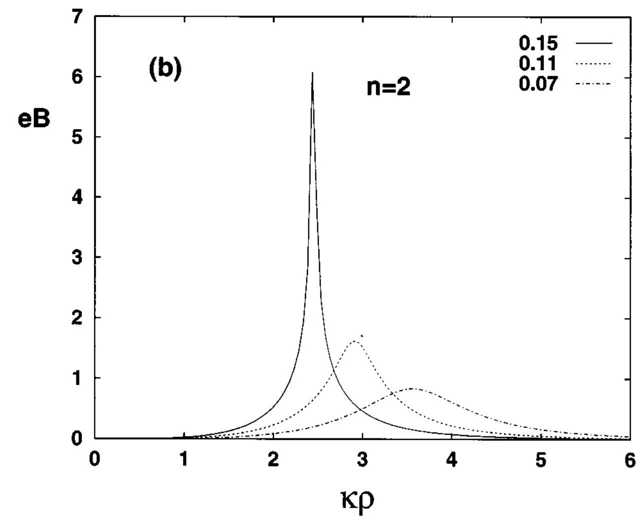


FIG. 3. The magnetic field in units of $1/e$. (a) For the nontopological vortex solution with $n=1$ and values of the parameter f_1 of 0.6, 0.4, and 0.2. (b) For the $n=2$ vorticity solution with values of f_2 of 0.15, 0.11, and 0.07.

approximation improves for larger n. Hence, we conclude that the parameter α_n satisfies the inequalities

$$n+1 < \alpha_n < n+2. \tag{5.18}$$

According to the previous results we can set $\alpha_n^{\min} \approx n+1$. Thus, for each integer n the flux varies continuously between $\Phi_n^{\min} = (4\pi/e)[n+1/2]$ and $\Phi_n^{\max} = (4\pi/e)[n+1]$. Similarly [see Eq. (5.3)] the energy spectrum consists of bands of finite width:

$$\frac{4\pi\kappa^2}{e^2}\left[n+\frac{1}{2}\right] \leqslant E_n \leqslant \frac{4\pi\kappa^2}{e^2}[n+1]. \tag{5.19}$$

In Fig. 3 we show the magnetic field $B(\rho)$ for the $n=1$ and $n=2$ solutions. For nontopological vortices the magnetic field is localized within a ring.

If we select the value of the parameter $f_n = f_n^{\max}$ in Eqs. (5.17), the function $f(\rho)$ will reach it maximum value at a given radius $\rho = R_n$, i.e., $f(R_n) = 1$. This parameter R_n can

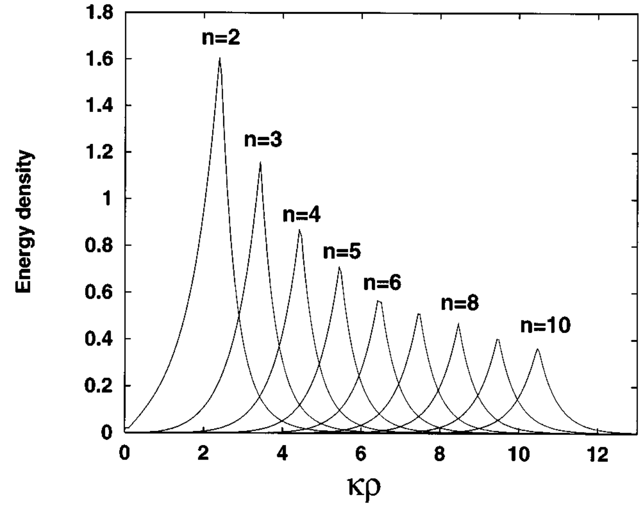


FIG. 4. Plots of energy density for the vortex solutions with several values of the vorticity number n. In all of these configurations the maximum value of the parameter $f_n = f_n^{\max}$ was selected from Table I.

be considered as the radius of the soliton. In Fig. 4 we show the profiles of the energy density as a function of $\kappa\rho$ for several values of n. The energy is indeed concentrated in the region $\rho \approx R_n$. In the next section we shall see that in the large n limit, the vortex can be considered as a ring of radius $R_n \approx n/\kappa$ and thickness $1/\kappa$. Furthermore, in the region $\rho \sim R$ the fields can be correctly approximated by a domain wall solution. The value of $f_n^{\max}$ as a function of n is plotted in Fig. 5, where the solid line indicates the prediction of Eq. (6.14) which we obtain in the next section using the domain wall approximation.

We conclude this section with some comments about the stability of the vortex solutions and also about the interaction between vortices. The vortices are neutrally stable at the self-dual point $m = \kappa$. This fact is easily demonstrated on account of the relation (5.3) between the energy and the charge: $E = \kappa|Q|/e$: The mass of the elementary excitations of the

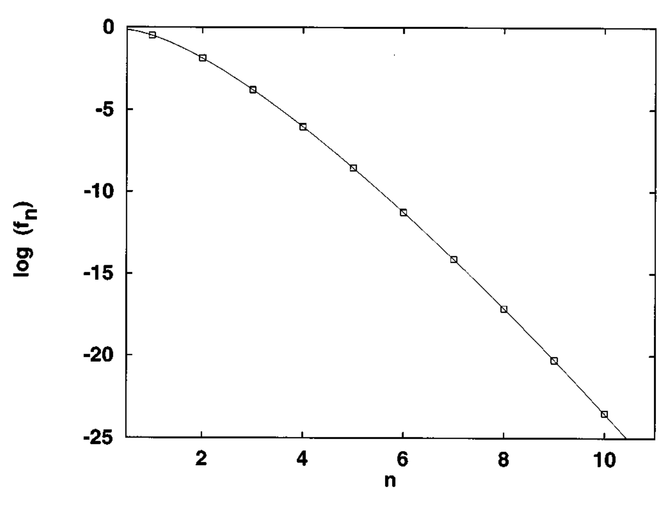


FIG. 5. Behavior of $f_n^{\max}$ as a function of n. The solid line corresponds to the asymptotic formula (6.14), while the squares represent actual data. The logarithm is to base e.

theory (scalar particles) is m and the charge e; because of charge conservation, a decaying soliton should radiate Q/e ''quanta'' of the scalar particles, thus the energy of the elementary excitations will be $\mathcal{E}=mQ/e$. This indicates that the vortices are at the threshold of stability against decay to the elementary excitations because the ratio $E_n/\mathcal{E}=\kappa/m$ is equal to 1 at the critical point $m=\kappa$. In fact, it is possible to consider a perturbative method away from the self-dual point [28] to prove that the soliton is stable against dissociation into free scalar particles when the scalar mass is bigger than the topological mass (i.e., $m>\kappa$). Instead, for $m<\kappa$ the soliton becomes unstable.

The self-dual point also corresponds to a point in which the vortices become noninteracting. Again, this property ensues from the fundamental relation (5.3). Consider two solitons of charges Q_1 and Q_2 of the same sign that are far apart. According to Eq. (5.3) their total energy is $E=(\kappa/e)(Q_1+Q_2)$. If the two vortices are superimposed at the same point, because of charge conservation the resulting configuration will represent a vortex solution of charge Q_1+Q_2. Then according to Eq. (5.3) the total energy will again be $E=(\kappa/e)(Q_1+Q_2)$. We, therefore, conclude that the vortices are noninteracting. The perturbative analysis of Ref. [28] shows that the vortex-vortex interaction is repulsive if $m>\kappa$ and attractive if $m<\kappa$. The self-dual point $m=\kappa$ represents a transition between a phase in which vortices attract and a phase in which they repel each other, similar to the transition between type-I and type-II superconductors. In fact, what we demonstrate in the ϕ^2 model is that the attraction between vortices because of the interaction through the scalar field has the same strength as the repulsion because of the interaction through the vector field. Therefore, when the range of the two interactions is the same ($m=\kappa$), the vortices become noninteracting. On the other hand, if the range of the scalar interaction is smaller than the range of the vector interaction ($m>\kappa$), the intervortex potential is repulsive; while for $m<\kappa$ the potential is attractive.

VI. DOMAIN WALLS

Domain walls appear in theories where the scalar potential possesses two or more disconnected but degenerate minima. The field configuration interpolates between two adjacent minima of the potential; the infinitely long boundary separating these two vacua states is precisely the domain wall. In $3+1$ dimensions the domain walls are planar structures, instead in $2+1$ dimensions they correspond to one-dimensional structures with finite energy per unit length. Domain wall solutions have been found in a Chern-Simons model [13] with a scalar potential of the form $V(\phi)\propto|\phi|^2(v^2-|\phi|^2)^2$.

The present ϕ^2 theory possesses a single minimum, yet it is possible to find one-dimensional soliton solutions of the domain wall type. Consider a one-dimensional structure depending only on the x variable, both at $x\to\infty$ and $x\to-\infty$ the scalar field should vanish. However, there can be an intermediate region where $\phi\neq 0$, i.e., a region of false vacuum. The maximum of ϕ determines the position of the wall. In this section we show that such solutions indeed exist for the ϕ^2 model. The domain wall carries both magnetic flux and electric charge per unit length. Furthermore, these solutions provide an approximate solution to the self-dual vortices in the large-flux limit (large n limit).

Seeking a domain wall solution parallel to the y axis, the translational invariance of the theory implies that all the fields depend only on x. By an appropriate gauge transformation the scalar field is made real everywhere $\phi=(\kappa/e)f$ and the potential $\vec{A}$ is selected along the y axis. Hence, the expression (5.1) for the energy with a potential $V(\phi)=(m^2/2)\phi^2$ can be written as

$$E=\frac{1}{2}\int d^2r\left[\frac{\kappa^2}{e^2}\left(\frac{df}{dx}\pm mf\right)^2+\left(\frac{(1-f^2)^{1/2}}{f}\frac{dA_y}{dx}\mp\frac{\kappa f A_y}{(1-f^2)^{1/2}}\right)^2\mp\frac{m\kappa^2}{e^2}\frac{df^2}{dx}\pm\kappa\frac{dA_y^2}{dx}\right]. \tag{6.1}$$

As mentioned earlier, the boundary conditions for the scalar field are $f(-\infty)=f(\infty)=0$. The magnetic flux per unit length (γ) is given by $\gamma=A_y(\infty)-A_y(-\infty)$, so $A_y(\infty)\neq A_y(-\infty)$ is required in order to get a nonvanishing magnetic flux. A configuration is sought which has a definite symmetry with respect to the position X of the domain wall, then $A_y(\infty)=-A_y(-\infty)\equiv\gamma/2$ is selected.

The static solution is obtained minimizing the energy per unit length with γ fixed. The boundary conditions cannot be satisfied if the same upper (or lower) signs in Eq. (6.1) are used for all x. Rather, the upper signs in the region to the right of the domain wall ($x>X$) are selected, whereas for $x<X$ we take the lower signs. With this selection the minimum energy per unit length becomes

$$\mathcal{E}=\frac{\kappa^2 m}{e^2}f_0^2+\frac{\kappa}{4}\gamma^2, \tag{6.2}$$

where $f_0\equiv f(X)$. This result is obtained provided that the fields satisfy the equations

$$\frac{df}{dx}=\mp mf,$$

$$\frac{dA_y}{dx}=\pm\frac{\kappa f^2}{(1-f^2)}A_y, \tag{6.3}$$

where the upper (lower) sign must be taken for $x>X$ ($x<X$). These equations are easily integrated to give

$$f(x)=e^{-m|x-X|},$$

$$A_y(x)=\mathrm{sgn}(x-X)\frac{\gamma}{2}(1-e^{-2m|x-X|})^{\kappa/2m}. \tag{6.4}$$

This is a domain wall configuration localized at $x=X$ with a width of order $1/m$. The solution to the first in Eqs. (6.3) does not restrict the value of f_0. However, $f_0=1$ has to set so the gauge field be continuous everywhere. The antikink configuration is obtained by simply reversing the signs of the fields in Eqs. (6.4).

The domain wall carries a magnetic flux and charge per unit length given by γ and $-\kappa\gamma$, respectively. Although there is a linear momentum flow along the domain wall given by

$$T_{0y}=\frac{\kappa}{2}\frac{dA_y^2}{dx}, \tag{6.5}$$

where Eqs. (2.15) and (4.5) have been used, we notice that the flow at opposite sides of the wall cancels, hence the total linear momentum of the domain wall vanishes. The magnetic field is given by

$$B=\frac{\kappa\gamma}{2}e^{-m|x-X|}(1-e^{-2m|x-X|})^{\kappa/2m-1}. \tag{6.6}$$

Notice that for $\kappa<2m$ the magnetic field is concentrated near $x=X$ and falls off rapidly away from the wall. Instead, for $\kappa\geqslant 2m$ the magnetic field vanishes at $x=X$ and the profile of B is double peaked with maximums at $x=X\pm(1/m)|\ln(\kappa/2m)|$.

The method presented in this section resembles the one used to derive the self-duality equations, so it could be questioned whether the two methods are equivalent. In general, it is not so: the self-dual limit is valid when $m=\kappa$, whereas for the domain wall solution there is no such restriction. However, for $m=\kappa$ it is straightforward to check that the fields in Eqs. (6.4) exactly solve the self-duality equations (5.4) and (5.5), with the magnetic flux and energy per unit length determined as

$$\gamma=\frac{2\kappa}{e},\quad \mathcal{E}=\frac{2\kappa^3}{e^2}. \tag{6.7}$$

The domain wall solutions in Eqs. (6.4) can also be adapted to approximate rotationally symmetric configurations. Indeed, the vortex configurations simplify in the large n limit and it is possible to utilize the domain wall as an approximated solution to the self-dual vortices. Let us consider a ring of large radius R and thickness of order $1/m$ separating two regions of vacuum. The magnetic flux is concentrated within this domain of width $\sim 1/m$ where a region of false vacuum ($\phi\neq 0$) is trapped. If $R\gg 1/m\sim 1/\kappa$, then the fields near the ring should be well approximated by the domain wall solutions (6.4). Nevertheless, in order to have a configuration with vorticity n the phase of the scalar field should vary uniformly with angle, see Eq. (5.8); hence, we gauge transform the fields in Eqs. (6.4). Thus in the region $\rho\sim R$, the fields configuration reads

$$\phi(\rho)\approx\frac{\kappa}{e}e^{-in\theta}e^{-m|\rho-R|},$$

$$\vec{A}(\rho)\approx\hat{\theta}\left[\frac{\gamma}{2}\mathrm{sgn}(\rho-R)(1-e^{-2m|\rho-R|})^{\kappa/2m}-\frac{n}{eR}\right], \tag{6.8}$$

with $\gamma=\Phi/(2\pi R)$. In first approximation the energy is dominated by the contribution near the domain wall; so according to Eq. (6.2) the energy can be approximated by

$$E\approx 2\pi R\left[\frac{m\kappa^2}{e^2}+\left(\frac{\Phi}{4\pi R}\right)^2\right]. \tag{6.9}$$

To obtain a domain wall that is stable against contraction or expansion the energy is minimized for a given magnetic flux. Minimizing the energy as a function of the radius yields

$$R\approx\frac{e\Phi}{4\pi\sqrt{m\kappa}}, \tag{6.10}$$

and thus

$$E\approx\sqrt{\frac{m}{\kappa}}\frac{\kappa^2}{e}\Phi. \tag{6.11}$$

This value for the energy saturates the Bogomol'nyi limit (5.3) when $m=\kappa$, indicating that the fields must be solutions of the self-duality equations. Indeed, we can verify that near $r\approx R$ the fields in Eq. (6.8) solve the Bogomol'nyi equations (5.9) and (5.10) if the radius R is chosen as in Eq. (6.10). Using the expression (5.11) for the magnetic flux we obtain $R=(n+\alpha_n)/(2\kappa)$ for the radius that minimizes the energy. Then, the condition $R\gg 1/\kappa$ implies $n\gg 1$, as expected for the large n limit.

In the domain wall solution the scalar field reaches its maximum value $f=1$ at $\rho=R$. Recalling the discussion of the previous section, for $n\gg 1$ the domain wall approximates the vortex solution in which the constant f_n in the short-distance expansion (5.17) is chosen as $f_n=f_n^{\max}$. But for $f_n=f_n^{\max}$ the parameter α_n reaches its minimum value; then according to Eq. (5.18) we can take $\alpha_n=\alpha_n^{\min}\approx n+1$ and so the magnetic flux and the radius become

$$\Phi\approx\frac{4\pi}{e}\left(n+\frac{1}{2}\right),$$

$$R\approx\frac{1}{\kappa}\left(n+\frac{1}{2}\right), \tag{6.12}$$

respectively. This result for R is in agreement with the one predicted by the maximum of the energy density in Fig. 4.

The domain wall solution can be combined with the Liouville approximation to find an explicit expression for the constant $f_n^{\max}$. The scalar field decays exponentially away from the domain wall, whereas the asymptotic behavior of the Liouville solution shows a power law both at large and small ρ. However, in the large n limit both approaches can be compatible. We introduce the relative coordinate $\xi=R-\rho$, the domain wall approximation for the scalar field is $f(r)\approx\exp(-\kappa|\xi|)$. At small r we can use the power-series solution (5.17) or, equivalently, the Liouville approximation (5.14) with N replaced by $n+1$ and $(\kappa\rho_0)^{(n+1)}$ by $2(n+1)/f_n$, in either case the leading contribution yields

$$f(r)\approx f_n^{\max}\kappa^n(R-\xi)^n=f_n^{\max}\, n^n\left(1+\frac{(1/2)-\kappa\xi}{n}\right)^n$$
$$\approx f_n^{\max}\, n^n\,\exp[(1/2)-\kappa|\xi|]. \tag{6.13}$$

As we are considering the large n limit the last equality was obtained using the identity $\lim_{n\to\infty}(1-z/n)^n=e^{-z}$. By comparing the previous result with the domain wall solution we find that the constant in the short-distance expansion (5.17) of the vortex solution must be

$$f_n^{\max}=\frac{e^{-1/2}}{n^n}. \tag{6.14}$$

This result is expected to provide an adequate approximation for large n. Remarkably, as shown in Fig. 5, the actual value of $f_n^{\max}$ is reproduced rather well by Eq. (6.14) for all values of n.

The Liouville approximation is also valid at large ρ; therefore, we can use the same approach to determine the coefficient C_n in the large-distance expansion (5.15). However, the large-distance expansion (5.15) and the domain wall solution can only be made compatible if we take $\alpha_n \approx n$ instead of $\alpha_n = n+1$; a sensible approximation for large n. This yields $C_n = n^n \approx f_n^{-1}$. This result suggests the existence of a relation between the large- and short-distance behavior of the vortex configuration. Indeed, comparing the leading terms in Eqs. (5.15) and (5.17) and recalling that $\alpha_n \approx n$, we find that for large n the vortex configuration is symmetric under the exchange $\kappa\rho/n \leftrightarrow n/\kappa\rho$.

VII. FINAL REMARKS

In this work we described a self-dual Maxwell-Chern-Simons model which includes an anomalous magnetic coupling between the scalar and the gauge fields. We first considered a general scalar potential in order to discuss both the symmetric and the broken phases of the theory. We found that the induced Chern-Simons term, arising from the combined effect of the spontaneous symmetry mechanism and the magnetic moment, has the same properties as those of the explicit CS term only in the topological trivial sector of the theory. We also found that in the broken phase the propagating modes of the gauge field consist of two longielliptic waves with different values for the masses.

For a particular relation between the Chern-Simons mass and the magnetic moment, Eq. (4.1), the gauge field equations reduce from second- to first-order differential equations, similar to those of the pure Chern-Simons-type. The self-dual limit occurs for a simply ϕ^2 potential when the scalar and the Chern-Simons masses are equal. Several properties of the self-dual vortices were analyzed both by analytical and numerical methods. In particular, the energy spectrum of the nontopological vortices consists of bands of constant width, $\Delta E_n = 2\pi\kappa^2/e^2$, centered at the values $E_n = (4\pi\kappa^2/e^2)(n+3/4)$, see Eq. (5.19).

We also considered one-dimensional configurations. Exact analytical domain wall solutions were found [Eq. (6.4)] for arbitrary m and κ. In general, these configurations will not be stationary. We found, however, that for $m=\kappa$, the fields in Eqs. (6.4) saturate the Bogomol'nyi limit and consequently, the configuration represent a one-dimensional stable kink with magnetic flux and energy per unit length given by Eq. (6.7). Furthermore, we found that in the large flux limit the nontopological vortices can be correctly approximated by the domain wall solution.

In the self-dual point $m=\kappa$ the vortices become noninteracting and static multisoliton solutions are expected. The index-theorem methods can be used to determine the number of independent free parameters that characterize a general n-vortex solution of the self-dual equations. The result can be obtained by counting the zero modes of the small fluctuations which preserve the self-duality equations. For nontopological vortices the calculation requires the subtraction of the continuous spectrum. The form of the fluctuations of Eqs. (5.4) and (5.5) near and far from the origin are similar to those of solitons in the ϕ^6 model considered in Ref. [13]. The result can then be taken from that paper. The number of free parameters in the general solution of nontopological vortices is $2(n+\hat{\alpha}_n-1)$ where $\hat{\alpha}_n$ is the greatest integer less than α_n. But according to Eq. (5.18) in the ϕ^2 model we have $\hat{\alpha}_n = n+1$. Consequently, the number of independent free parameters for the n-soliton solution is $4n$. The result is consistent with the fact that we require $2n$ parameters to fix the position of n solitons in the plane, while the phases and the fluxes are determined by the other $2n$ parameters.

This model raises a number of interesting questions for further investigation. In particular, a complete description of the multisoliton solution deserves to be clarified. It may also be interesting to investigate the properties of the model away from the self-dual point, including the vortex interactions. Finally will be of great interest the study of the properties of the Chern-Simons vortices upon quantization, because they can be considered as candidates for anyonlike objects in planar systems.

[1] For a review and references see, e.g., R. Jackiw and S. Y. Pi, Prog. Theor. Phys. Suppl. **107**, 1 (1992); G. Dune, Report No. hep-th/9410065, 1994 (unpublished).

[2] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).

[3] E. B. Bogomol'nyi, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)].

[4] H. J. de Vega and F. A. Schaposnik, Phys. Rev. D **14**, 1100 (1976).

[5] L. Jacobs and C. Rebbi, Phys. Rev. B **19**, 4486 (1979).

[6] C. H. Taubes, Commun. Math. Phys. **72**, 277 (1980).

[7] Z. Hlousek and D. Spector, Nucl. Phys. **B297**, 173 (1993); J. Edelstein, C. Nunez, and F. Schaposnik, Phys. Lett. B **329,** 39 (1994).

[8] S. K. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986); **182**, 414(E) (1986); C. Kumar and A. Khare, *ibid.* **178**, 395 (1986); D. Jatkar and A. Khare, *ibid.* **236**, 283 (1990).

[9] R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981); S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).

[10] R. Jackiw, in *Physics Geometry and Topology*, edited by H. C. Lee, NATO ASI Series B: Physics Vol. 238 (Plenum, New York, 1990).

[11] R. Jackiw and S. Y. Pi, Phys. Rev. Lett. **64**, 2969 (1990); Phys. Rev. D **42**, 3500 (1990).

[12] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. **64**, 2230 (1990); R. Jackiw and E. J. Weinberg, *ibid.* **64**, 2234 (1990).

[13] R. Jackiw, K. Lee, and E. J. Weinberg, Phys. Rev. D **42**, 3488 (1990).

[14] M. Torres, Phys. Rev. D **46**, R2295 (1992).

[15] C. Lee, K. Lee, and H. Min, Phys. Lett. B **252**, 79 (1990).

[16] S. K. Paul and A. Khare, Phys. Lett. B **193**, 253 (1987).

[17] J. Stern, Phys. Lett. B **265**, 119 (1991); X. G. Wen and A. Zee, Phys. Rev. B **41**, 240 (1990); J. Kapusta, M. E. Carrington, B. Bayman, D. Seibert, and C. S. Song, *ibid.* **44**, 7519 (1991); M. E. Carrington, Phys. Rev. D **51**, 4451 (1995).

[18] I. I. Kogan, Phys. Lett. B **262**, 83 (1991).

[19] C. Chou, V. P. Nair, and A. P. Polychronakos, Phys. Lett. B **304**, 105 (1993); see also J. L. Cortés, J. Gamboa, and L. Velázquez, Nucl. Phys. **B392**, 645 (1993).

[20] P. K. Gosh, Phys. Lett. B **326**, 264 (1993); Phys. Rev. D **49**, 5458 (1994).

[21] To further support the interpretation of g as magnetic moment, we consider the limit in which the scalar field contribution reduces to a point source charge; i.e., $J_\mu = eg_{\mu 0}\delta(\vec{r}\ )$. If we then combine the usual definition for the magnetic dipole moment with the expression for $\mathcal{J}_\mu$ we obtain $\int d^2x \epsilon_{ij} x^i \mathcal{J}^j = -(g/2)\epsilon^{ij}\int d^2x \epsilon_{jl} x_i \partial^l \delta(\vec{r}) = -g$.

[22] R. D. Pisarski and S. Rao, Phys. Rev. D **32**, 2081 (1985).

[23] D. Boyanowski, Nucl. Phys. **B350**, 906 (1991).

[24] S. Latinsky and D. Sorokin, Mod. Phys. Lett. A **6**, 3525 (1991).

[25] T. Lee and H. Min, Phys. Rev. D **50**, 7738 (1994).

[26] M. Torres, Phys. Rev. D **51**, 4533 (1995).

[27] P. Navratil, Phys. Lett. B **365**, 119 (1996).

[28] J. Escalona, M. Torres, and A. Antillón, Mod. Phys. Lett. A **8**, 2955 (1993).